Noncooperative Bargaining in Apex Games and the Kernel*

(Running title: Apex Games and the Kernel)

Maria Montero†

This version: June 2001

Abstract

This paper studies non-cooperative bargaining with random proposers in apex games. Two different protocols are considered: the egalitarian protocol, which selects each player to be the proposer with equal probability, and the proportional protocol, which selects each player with a probability proportional to his number of votes. Expected equilibrium payoffs coincide with the kernel for the grand coalition regardless of the protocol. The equilibrium is in mixed strategies and the indifference conditions can be reinterpreted in the language of the kernel.

Keywords: noncooperative bargaining, random proposers, apex games, kernel.

JEL classification numbers: C72, C78.

*I'm indebted to Eric van Damme, Anne van den Nouweland, Alex Possajennikov, Marco Slikker and an anonymous referee for useful suggestions and discussion. Financial support from the NWO is gratefully acknowledged.

†University of Dortmund, Department of Economics, Vogelpothsweg 87, 44221 Dortmund, Germany: maria.montero@wiso.uni-dortmund.de

1 Introduction

An apex game is a simple n-player game with one major player (the *apex player*) and $n-1 \geq 3$ minor players (also called *base players*). A winning coalition can be formed by the apex player together with at least one of the minor players or by all the minor players together. Apex games can be interpreted as weighted majority games in which the major player has $n-2$ votes, each of the $n-1$ minor players has one vote, and $n-1$ votes are required for a majority.

Since the apex player only needs one of the minor players he can play them off against each other to obtain favorable terms. Each minor player has two options: try to unite with the other minor players (and run the risk that one of them accepts an offer of the apex player) or compete with them for the favor of the apex player. Apex games have received a lot of attention in the literature, starting with von Neumann and Morgenstern (1944).

This paper addresses three questions concerning apex games:

1) What coalition(s) are likely to form?

2) How will the gains from cooperation be divided for each possible coalition?

3) What are the *ex ante* expected payoffs for the players?

There are very different answers in the literature to the first question. Some papers (Cross (1967), Albers (1974), Bennett (1983), Morelli and Montero (2001)) predict that all minimal winning coalitions are possible, whereas others limit the possible outcomes to the coalition of all small players (Aumann and Myerson (1988), Hart and Kurz (1983,1984)) or to coalitions of the major player with a minor player (Chatterjee et al. (1993)). Coalitions larger than minimal winning are possible in the bargaining set literature (indeed the classical bargaining set[1] is nonempty for any possible coalition structure).

[1]We refer to the bargaining set $\mathcal{M}_1^{(i)}$ (Davis and Maschler (1967)); other variants of the bargaining set appear in Aumann and Maschler (1964).

As for the second question, equal division of gains seems indicated if all minor players form a coalition. If the apex player forms a coalition with a minor player, the division of gains is not so clear-cut. The answers given in the literature range from the "egalitarian" $\frac{1}{2} : \frac{1}{2}$ split corresponding to the kernel (Davis and Maschler (1965)) to the "proportional" (to the number of votes) $\frac{n-2}{n-1} : \frac{1}{n-1}$ split that comes from observing that a small player can not expect more than $\frac{1}{n-1}$ if all the minor players form a coalition[2]. The classical bargaining set and the Zhou (1994) bargaining set include these two extremes and all outcomes in between. The Mas-Colell (1989) bargaining set excludes the two extremes.

Most of the literature has little to say about *ex ante* payoffs. They are either very extreme (as the major player receives a payoff of zero) or undetermined (when several coalitions are possible, *ex ante* expected payoffs depend on the likelihood of each coalition, and this is left undetermined). On the other hand, *ex ante* concepts like the Shapley value give no predictions about coalitions or division of gains. The current paper attempts to provide an answer to the three questions simultaneously.

In this paper a noncooperative procedure with random proposers (see Baron and Ferejohn (1989) and Okada (1996)) is used to model bargaining in apex games[3]. Two natural protocols are examined: the "egalitarian" protocol in which each player is selected to be the proposer with equal probability, and the "proportional" protocol, in which each player is selected with a probability proportional to his number of votes[4]. The solution concept is

[2]Several solution concepts predict the proportional split, including von Neumann and Morgenstern's (1944) main simple solution and Horowitz's (1973) competitive bargaining set. Horowitz's paper was actually motivated by apex games.

[3]Other extensions of the Baron-Ferejohn model include Eraslan (1998), Eraslan and Merlo (1999) and Banks and Duggan (2000). Eraslan (1998) shows uniqueness of (stationary subgame perfect) equilibrium payoffs in symmetric majority games. Eraslan and Merlo (1999) extend Eraslan (1998) to stochastic environments and show existence of equilibrium. Banks and Duggan (2000) show existence of equilibrium in fairly general (deterministic) environments.

[4]This would be the case in a parliamentary system where the probability of a party

stationary subgame perfect equilibrium.

Intuitively, the apex player should benefit from a proportional protocol since he is chosen more often to be the proposer. However, we show that this is not the case if players are patient: expected equilibrium payoffs are proportional to the number of votes of the players for both protocols. The reason is that equilibrium strategies change so as to compensate changes in the protocol: if the protocol selects a player to be the proposer with a higher probability, the other players make offers to him with a lower probability so that his *ex ante* expected payoff remains unchanged.

We also show that all minimal winning coalitions may form, and the probability of a coalition being formed depends on the protocol (the coalition of all minor players being more frequent under a proportional rule). Ex post payoff division is rather asymmetric, with the proposer obtaining more than half of the total payoff.

The rest of the paper is organized as follows: section 2 contains the model and the results, section 3 relates the resulting expected payoffs to the kernel and section 4 concludes.

2 Bargaining with random proposers in apex games

2.1 The model

Let $N = \{1, 2, ..., n\}$ be the set of players. A game is a *weighted majority game* if there is a vector of nonnegative numbers or *weights* $w := (w_1, ..., w_n)$ and a quota q such that for all nonempty $S \subseteq N$, $v(S) = 1$ if $\sum_{i \in S} w_i \geq q$ and 0 otherwise. We will only consider weighted majority games with $q > \frac{\sum_{i \in N} w_i}{2}$. A weighted majority game can be denoted by $(q; w)$. A coalition S in a weighted majority game is called *winning* if $v(S) = 1$ and *losing* if $v(S) = 0$. A player $i \in S$ is called *pivotal* if $v(S) = 1$ and $v(S \backslash \{i\}) = 0$. If all players

being asked to form a government is proportional to the number of seats it holds.

in S are pivotal S is called a *minimal winning coalition. Apex games* are a special class of weighted majority games in which a major player has $n-2$ votes, $n-1$ players have one vote, and $q=n-1$.[5]

Bargaining in apex games is modeled as a noncooperative game with random proposers[6]. Given the underlying cooperative (apex) game (N, v), bargaining proceeds as follows: At every round $t = 1, 2, ...$ Nature selects a player randomly to be the *proposer*. This player proposes a coalition $S \subseteq N$ to which he belongs and a division of $v(S)$, denoted by $x^S = (x_i^S)_{i \in S}$. The ith component x_i^S represents a payoff for player i in S. Given a proposal, the rest of players in S (called *responders*) accept or reject sequentially (the order does not affect the results). If all players in S accept, the proposal is implemented and the game ends[7]. If at least one player rejects, the game proceeds to the next period in which nature selects a new proposer (always with the same probability distribution). Players are risk-neutral and share a discount factor $0 < \delta < 1$.[8] Thus, if a proposal x^S is accepted by all players in S at time t, each player in S receives a payoff $\delta^{t-1} x_i^S$. A player not in S remains a singleton and receives zero.

A (pure) strategy for player i is a sequence $\sigma_i = (\sigma_i^t)_{t=1}^{\infty}$, where σ_i^t, the tth round strategy of player i, prescribes

(i) A *proposal* (S, x^S).

(ii) A *response function* assigning "yes" or "no" to all possible proposals of the other players.

The solution concept is *stationary subgame perfect equilibrium (SSPE).* Stationarity requires that players follow the same strategy at every round t.

Concerning the probability of players being selected to be proposers, we will call the probability vector used by Nature a *protocol*, and we will denote

[5]This is only one of many possible vectors of weights. We have chosen a *homogeneous representation*, in which all minimal winning coalitions have the same number of votes.

[6]The games with random proposers build on Binmore's (1987) variant of the seminal work by Rubinstein (1982).

[7]Ending the game after one coalition has been formed does not affect the results.

[8]Alternatively, after a proposal is rejected the game ends with probability $1-\delta$.

it by $\theta := (\theta_i)_{i \in N}$, where $\theta_i > 0\ \forall i \in N$ and $\sum_{i \in N} \theta_i = 1$.

Two protocols suggest themselves: the *egalitarian protocol* $\theta^E := (\frac{1}{n}, ..., \frac{1}{n})$, which selects each player with the same probability, and the proportional protocol $\theta^P := (\frac{n-2}{2n-3}, \frac{1}{2n-3}, ..., \frac{1}{2n-3})$, which selects each player with a probability proportional to his number of votes.

We will denote the noncooperative game described above $G(N, v, \theta, \delta)$, where (N, v) is an apex game unless otherwise specified.

2.2 The equilibrium

The following lemma corresponds to theorem 1 in Okada (1996). Even though the original theorem assumes the egalitarian protocol θ^E, it can be applied to any protocol θ. The proof is included for completeness.

Lemma 1 *(Okada, 1996) Consider a zero-normalized, essential and superadditive*[9] *game (N, v). In any SSPE of the game $G(N, v, \theta, \delta)$, every player i in N proposes a solution (S_i, x^{S_i}) of the maximization problem*

$$\max_{S \ni i, x} \left(v(S) - \sum_{j \in S, j \neq i} x_j\right) \tag{1}$$

$$s.t.\ x_j \geq \delta y_j\ \forall j \in S \backslash \{i\}$$

where y_j is the equilibrium expected payoff of player j. Moreover, the proposal (S_i, x^{S_i}) is accepted.

Proof. In an SSPE, any proposal that offers each responder j more than δy_j will be accepted. Since the characteristic function is superadditive we have $v(N) \geq \sum_{j \in N} y_j$. Furthermore, since in equilibrium $\sum_{j \in N} y_j > 0$, we have $v(N) > \delta \sum_{j \in N} y_j$. Thus, player i can get more than δy_i by proposing the grand coalition and offering each responder j slightly more than δy_j. Since he only gets δy_i when a proposal is rejected, it pays to always make acceptable proposals.

[9] A cooperative game (N, v) is *zero-normalized* and *essential* if $v(i) = 0\ \forall i \in N$ and $v(N) > 0$; it is *superadditive* if $v(S \cup T) \geq v(S) + v(T)\ \forall S, T \subseteq N$, $S \cap T = \varnothing$.

Finally, in an SSPE the proposer must offer each responder j exactly δy_j; moreover, only coalitions that solve (1) can be proposed. ■

Notice that all "nontrivial" weighted majority games (that is, all games with $\sum_{i \in N} w_i \geq q > w_i$ for all i) satisfy the assumptions of lemma 1.

Corollary 2 *Consider a zero-normalized, essential and superadditive game (N, v). In any SSPE of $G(N, v, \theta, \delta)$, $y_i > 0$ for all i in N. For apex games, this implies that only minimal winning coalitions are proposed.*

Proof. From the proof of Lemma 1 we know that each player gets a strictly positive expected payoff as a proposer (at least $(1 - \delta)v(N) > 0$). As a responder, he can guarantee himself a payoff of zero by rejecting all proposals. Since $\theta_i > 0$ for all i, $y_i > 0$. This implies that players will only propose winning coalitions in which all responders are pivotal. The proposer may in general not be pivotal, but in apex games there are no winning coalitions in which all players but one are pivotal. ■

Let (N, v) be a weighted majority game. We say that players i and j are of the same *type* iff $v(S \cup \{i\}) = v(S \cup \{j\})$ for all S such that $i \notin S$, $j \notin S$. The following lemma states a symmetry property of stationary subgame perfect equilibria

Lemma 3 *Let i and j be two players of the same type. If $\theta_i = \theta_j$, then any SSPE of the game $G(N, v, \theta, \delta)$ satisfies $y_i = y_j$.*

Proof. Suppose there is an SSPE with $y_i \neq y_j$. Without loss of generality let $y_i > y_j$. Expected equilibrium payoffs satisfy the following equations

$$y_i = \theta_i y_i^i + r_i \delta y_i \tag{2}$$

$$y_j = \theta_j y_j^j + r_j \delta y_j. \tag{3}$$

where y_k^k is the expected equilibrium payoff of player k conditional on being the proposer, and r_k is the probability that player k receives a proposal in the equilibrium.

First we show that $r_i \leq r_j$. Players other than i and j will propose to j at least as often as to i. How about i and j themselves? If j proposes to i with positive probability there is a winning coalition S containing i and j such that $1 - \delta y_i - \delta \sum_{k \in S \setminus \{j,i\}} y_k \geq 1 - \delta \sum_{k \in T \setminus \{j\}} y_k$ for all winning coalitions $T \ni j$. Then i must propose to j with probability 1.

Second, since i has better alternatives than j, it is clear that $y_i^i \geq y_j^j$.

Third, $y_i^i - y_j^j \leq y_i - y_j$. If $y_i^i = y_j^j$, this is obvious. If $y_i^i > y_j^j$, this means that i proposes to j. We do not know if j proposes to i as well. If j proposes to i, $y_i^i - y_j^j = \delta(y_i - y_j)$. If j does not propose to i he must have another alternative that is at least as good, thus y_j^j is at least as large as in the previous case.

Let $y_i^i - y_j^j = y_i - y_j - \epsilon$ ($\epsilon > 0$). Substracting (3) from (2) and taking into account that $\theta_i = \theta_j$, we get

$$(y_i - y_j)(1 - \theta_i - r_i\delta) = -\epsilon\theta_i - (r_j - r_i)\delta y_j.$$

Since the right-hand side is negative and the left hand side is positive, we have reached a contradiction. ■

Lemma 3 implies that if the protocol treats all minor players equally, they all must have the same expected payoffs in any equilibrium. Since we will only consider protocols with this property, we may denote expected payoffs by y_m for a minor player and y_a for the apex player.

In any SSPE the apex player makes a proposal to a minor player. The behavior of the minor players depends on how y_a compares to $(n-2)y_m$ (or, equivalently, on how y_a compares with $\frac{n-2}{2n-3}$); if $y_a < (n-2)y_m$, all minor players propose to the apex player; if $y_a > (n-2)y_m$ all minor players propose the minor player coalition; if $y_a = (n-2)y_m$, the equilibrium is in (possibly degenerated) mixed strategies. Which type of equilibrium actually arises depends on the parameters.

Proposition 4 *Let (N, v) be an apex game, θ a protocol with $\theta_i = \theta_j$ if i and j are minor players, and δ arbitrarily close to* 1. *Then*[10]

a) There is no SSPE such that all minor players propose to the apex player.

b) There is an SSPE in which all minor players propose the minor player coalition provided that $\theta_a \geq \frac{1}{2}$.

c) There is an SSPE in which the minor players are indifferent between proposing to the apex player and proposing the minor player coalition provided that $\theta_a < \frac{1}{2}$.

Proof. a) Suppose there is an SSPE such that all minor players propose to the apex player. Then the apex player's expected equilibrium payoff equals

$$y_a = \theta_a(1 - \delta y_m) + (1 - \theta_a)\delta y_a. \tag{4}$$

Since there is no delay in any SSPE and a winning coalition always forms,

$$y_a + (n - 1)y_m = 1. \tag{5}$$

The two equations together imply $y_a = \frac{\theta_a(n-1-\delta)}{n-1-\delta(n-1-(n-2)\theta_a)}$. If δ is arbitrarily close to 1, y_a is also arbitrarily close to 1, and the minor players would rather propose the minor player coalition.

b) Suppose there is an SSPE such that all minor players propose the minor player coalition. Then

$$y_a = \theta_a(1 - \delta y_m) \tag{6}$$

which, together with (5), implies $y_a = \frac{\theta_a(n-1-\delta)}{n-1-\delta\theta_a}$; y_m can then be found from (5). It is easy to check that, for δ close to 1, $y_a > (n - 2)y_m$ provided that $\theta_a \geq \frac{1}{2}$.

[10] It suffices to describe the equilibrium strategies of the players by a probability distribution over the coalitions they propose. Lemma 1 implies that each responder j is offered exactly δy_j and all responders accept. This fact together with the probability distribution used by the proposers determines $(y_j)_{j \in N}$.

c) Suppose there is an SSPE such that the minor players are indifferent between proposing to the apex player and proposing the minor player coalition. The indifference condition $(n-2)y_m = y_a$ together with the no delay condition (5) imply

$$y_a = \frac{n-2}{2n-3} \tag{7}$$

$$y_m = \frac{1}{2n-3} \tag{8}$$

Let λ_i ($i \neq 1$) be the probability that i proposes to the apex player conditional on i being the proposer, and $\lambda := \frac{\sum_{i \in N \setminus \{1\}} \lambda_i}{n-1}$. Then

$$y_a = \theta_a(1 - \delta y_m) + (1 - \theta_a)\lambda\delta y_a$$

Plugging the values of y_a and y_m from (7) and (8), we obtain $\lambda = \lambda(\theta) = \frac{n-2-\theta_a(2n-3-\delta)}{\delta(1-\theta_a)(n-2)}$. For δ close to 1, $\lambda > 0$ requires $\theta_a < \frac{1}{2}$.

There is a continuum of equilibria, all of them with the same value of λ. Given a collection of strategies for the minor players $(\lambda_i)_{i \in N \setminus \{1\}}$ with average λ, we can find the corresponding strategy of the major player such that each minor player has the same payoff: players with large λ_i are less likely to receive proposals from the apex player, so that all minor players have the same probability of becoming responders. The probability that the apex player proposes to player i given λ_i is $\frac{1-\lambda_i}{n-2}$ for $\theta = \theta^P$ and (for δ close to 1) about $1 - \lambda_i$ for $\theta = \theta^E$. ■

Corollary 5 *Let δ be close to 1. Since both θ^E and θ^P have $\theta_a < \frac{1}{2}$, the unique equilibrium payoffs are the same for both protocols. $\lambda(\theta^E)$ is close to $\frac{n-2}{n-1}$, whereas $\lambda(\theta^P)$ equals $\frac{1}{n-1}$.*

The reason why payoffs are the same for both protocols is that equilibrium strategies change so as to compensate changes in the protocol: if the protocol selects a player to be the proposer more often equilibrium strategies adjust so that he becomes a responder less often and his expected payoff remains unchanged.

It is easy to check that if the parameters lie outside the region of mixed-strategy equilibria the value of y_a is increasing in θ_a, so that the intuition that the apex player benefits from an increase in θ_a is confirmed[11], the reason being that the minor players are already playing an extreme strategy and cannot adjust it any further.

It can be shown that the SSPE of the apex game is unique in terms of expected payoffs regardless of whether δ is close to 1. The type of equilibrium given (N, v) and θ may depend on the value of δ. For $\theta_a = \frac{n-2}{2n-3}$, the equilibrium is in mixed strategies regardless of δ. However, for $\theta_a = \frac{1}{n}$, the equilibrium is in mixed strategies only if δ is close enough to 1. The larger the value of n, the closer to 1 δ has to be. If δ is not large enough, the equilibrium with the egalitarian protocol is such that all minor players propose to the apex player, and the apex player benefits from a proportional protocol. In the limit when δ tends to 0, the region where a mixed strategy equilibrium is played shrinks around one point ($\theta_a = \frac{n-2}{2n-3}$) and expected payoffs are strictly increasing almost everywhere in the recognition probabilities.

Remark 6 *The proposer gets at least half of the total payoff.*

In the mixed strategy region expected payoffs are proportional to the share of the total votes, and the proposer only needs to "buy" less than half of the votes. Outside this region the situation is even more favorable to the proposer.

Remark 7 *The coalition of all minor players is more frequent under a proportional protocol than under an egalitarian protocol.*

The payoff a player gets as a proposer and the payoff he gets as a responder do not depend on whether the protocol is egalitarian or proportional. Since the apex player is selected to be the proposer more often by θ^P, the total probability that he is in a coalition must be smaller under θ^P in order to keep

[11]A related result can be found in Eraslan (1998): for symmetric majority games, the players' expected payoffs are non-decreasing in the recognition probabilities.

y_a constant. The exact probabilities of the minor player coalition forming are found by plugging the appropriate values of θ_a and λ into equation $(1-\theta_a)(1-\lambda)$. In the limit when δ tends to 1, these values are $\frac{1}{n}$ and $\frac{n-2}{2n-3}$ respectively.

The probability that a given coalition including the apex player is formed depends on the concrete equilibrium; if we restrict ourselves to symmetric equilibria ($\lambda_i = \lambda$ for all $i \in N \backslash \{1\}$) and take the limit when δ tends to 1, then each of these coalitions forms with probability $\frac{1}{n}$ given θ^E and $\frac{1}{2n-3}$ given θ^P.

3 Apex games and the kernel

Consider a cooperative game (N, v). Assume $v(S) \geq 0 \ \ \forall S \subseteq N$ and $v(i) = 0$ $\forall i \in N$. An outcome of the game is denoted by $(x; \mathcal{B})$ where x_i denotes the payoff to the ith player and $\mathcal{B} \equiv \{B_1, ..., B_m\}$ the *coalition structure* (partition of N) that was formed. The payoff vector is assumed to satisfy

$x_i \geq 0$, $i = 1, 2, ..., n$

$\sum_{i \in B_j} x_i = v(B_j)$, $j = 1, 2, ..., m$

A payoff vector satisfying these two conditions is called an *imputation*[12]. The space of all imputations for the coalition structure $\mathcal{B}$ is denoted by $\mathrm{X}(\mathcal{B})$.

Let x be an imputation in a game (N, v) for an arbitrary coalition structure. The *excess* of a coalition S at x is $e(S, x) := v(S) - \sum_{i \in S} x_i$.

Let $(x; \mathcal{B})$ be an outcome for a cooperative game, and let k and l be two distinct players in a coalition B_j of $\mathcal{B}$. The *surplus* of k against l at x is

$$s_{k,l}(x) := \max_{\substack{k \in S, \\ l \notin S}} e(S, x)$$

Let (N, v) be a cooperative game and let $\mathcal{B}$ be a coalition structure. The kernel $\mathcal{K}(\mathcal{B})$ for $\mathcal{B}$ is

$$\mathcal{K}(\mathcal{B}) := \{x \in X(\mathcal{B}) : s_{k,l}(x) > s_{l,k}(x) \Rightarrow x_l = 0, \forall\ k, l \in B \in \mathcal{B}, k \neq l\} \quad (9)$$

[12] The terminology is taken from Maschler (1992).

Suppose (N, v) is an apex game and consider the coalition structure $\mathcal{B} = \{N\}$. The only payoff vector in the kernel is $(\frac{n-2}{2n-3}, \frac{1}{2n-3}, ..., \frac{1}{2n-3})$. If we interpret apex games as weighted majority games, the kernel predicts payoffs that are proportional to the number of votes of the players[13].

We now come back to the equilibrium of the noncooperative game described in section 2. The first condition we derived there was that all minor players must have the same expected payoffs, that is, if i and j are minor players, $y_i = y_j = y_m$. We may rewrite this condition as $1-y_a-y_i = 1-y_a-y_j$. In the language of the kernel, the surplus of i over j equals the surplus of j over i. Second, the indifference condition of the minor player, $y_a = (n-2)y_m$, implies $1 - y_a - y_m = 1 - (n-1)y_m$. In the language of the kernel, the surplus of the apex player against a minor player equals the surplus of a minor player against the apex player. Finally, because there is no delay in equilibrium (and players always propose winning coalitions), the sum of all expected payoffs equals 1, that is $y_a + (n-1)y_m = 1$. In the language of the kernel, $(y_i)_{i \in N}$ is an imputation.

4 Concluding remarks

The solution concepts based on the stability of demand vectors make intuitive predictions about the ex post payoff division in apex games, but have nothing to say about *ex ante* expected payoffs[14]. We have made intuitive predictions about the *ex ante* expected payoffs in apex games, though at the cost of very extreme *ex post* predictions (the proposer, even if he is a minor player in an apex game with very large n, gets at least half of the total payoff).

Chatterjee et al. (1993) consider a proposal-making model in which a rule of order selects the first proposer and the order in which players respond

[13]While proportional payoffs may seem only too obvious, one must take into account that neither the Shapley value nor the Banzhaf value assign proportional payoffs in an apex game.

[14]This is also the case if the situation is modeled as an extensive form game (Selten (1981), Bennett and van Damme (1991), Bennett (1997), Morelli (1999)).

to a proposal, and the first player to reject becomes the next proposer. This model predicts that a coalition of the apex player and a minor player will form and split the payoff roughly equally regardless of the number of players. The reason is that the game fails to reflect the competition between the minor players. A minor player who rejects an offer will propose to the apex player in the next period, so that his continuation value is $z_m = \delta(1 - z_a)$ regardless of the proposing strategy of the apex player and the number of players; this equation together with $z_a = \delta(1 - z_m)$ determine z_a and z_m regardless of n. In games with random proposers, the payoff of a player who rejects a proposal depends on how often other players propose to him, so that competitive pressures affect expected payoffs.

There is a significant literature focusing on the relation between noncooperative equilibria and cooperative solution concepts. Krishna and Serrano (1996) and Hart and Mas-Colell (1996) consider unanimity bargaining, and are therefore not applicable to majority games. Gul (1989) and Serrano (1997) provide noncooperative foundations of the Shapley value and the kernel respectively using bargaining procedures based on pairwise meetings. While this paper covers only apex games, the noncooperative model is very natural in the context of majority games.

We have pointed out a relation between the definition of the kernel and the indifference conditions corresponding to mixed strategy equilibria. This relation is not exclusive of apex games but extends to other majority games: if we have an equilibrium in mixed strategies of the random proposer game such that the probability that player i proposes to player j is strictly between 0 and 1 for all i and j, $i \neq j$, then expected equilibrium payoffs must belong to the kernel of the grand coalition.

The strong invariance result found for apex games generalizes to weighted majority games with one large player and $n - 1$ small players, but not to all weighted majority games. For example, expected payoffs for the egalitarian protocol and for the proportional protocol differ in a game as simple as $(4; 2, 2, 1, 1, 1)$.

Finally, we have limited ourselves to homogeneous weights. The equilibrium for the apex game given a protocol proportional to some nonhomogeneous weights may not coincide with the kernel of the grand coalition.

References

[1] ALBERS, W. (1974). "Zwei Lösungkonzepte für Kooperative Mehrpersonenspiele, die auf Anspruchsniveaus der Spieler Basieren," *OR-Verfahren* (Methods of Operations Research) **21**, 1-13.

[2] AUMANN, R. J., and M. MASCHLER (1964). "The Bargaining Set for Cooperative Games," in *Advances in Game Theory*, ed. by M. Dresher, L. Shapley and A. Tucker, 443-476. Princeton: Princeton University Press.

[3] AUMANN, R. J. and R. MYERSON (1988). "Endogenous Formation of Links Between Players and of Coalitions: an Application of the Shapley Value," in *The Shapley value, Essays in Honor of Lloyd Shapley*, ed. by A. Roth, 175-191. Cambridge: Cambridge University Press.

[4] BANKS, J.S. and J. DUGGAN (2000). "A Bargaining Model of Collective Choice," *American Political Science Review* **94**, 73-88.

[5] BARON, D. P., and J.A. FEREJOHN (1989). "Bargaining in Legislatures," *American Political Science Review*, **83**, 1181-1206.

[6] BENNETT, E. (1983). "The Aspiration Approach to Predicting Coalition Formation and Payoff Distribution in Sidepayment Games," *International Journal of Game Theory* **12**, 1-28.

[7] BENNETT, E. (1997). "Multilateral Bargaining Problems," *Games and Economic Behavior* **19**, 151-79.

[8] BENNETT, E. and VAN DAMME, E. (1991). "Demand Commitment Bargaining: The Case of Apex Games," in *Game Equilibrium Models*,

Vol III, *Strategic Bargaining* (R.Selten, ed.) 118-140. Berlin: Springer-Verlag.

[9] BINMORE, K. (1987). "Perfect Equilibria in Bargaining Models," in *The Economics of Bargaining,* ed by K. Binmore and P. Dasgupta, 77-105. Oxford: Blackwell.

[10] CHATTERJEE, K., DUTTA, B., RAY, D. and K. SENGUPTA (1993). "A Noncooperative Theory of Coalitional Bargaining," *Review of Economic Studies* **60**, 463-77.

[11] CROSS, J. (1967). "Some Theoretical Characteristics of Economic and Political Coalitions," *Journal of Conflict Resolution* **11**, 184-195.

[12] DAVIS, M. and M. MASCHLER (1967). "Existence of Stable Payoff Configurations for Cooperative Games," in *Essays in Mathematical Economics in Honor of Oskar Morgenstern*, ed. by M. Shubik, 39-52. Princeton University Press.

[13] ERASLAN, H. (1998) "Uniqueness of Stationary Equilibrium Payoffs in the Baron-Ferejohn Model". Typescript.

[14] ERASLAN, H. and A. MERLO (1999) "Majority Rule in a Stochastic Model of Bargaining," Typescript.

[15] GUL, F. (1989). "Bargaining Foundations of the Shapley Value," *Econometrica* **57**, 81-96.

[16] HART, S. and M. KURZ (1983). "Endogenous Formation of Coalitions," *Econometrica* **51**, 1047-1064.

[17] HART, S. and M. KURZ (1984). "Stable Coalition Structures," in *Coalitions and Collective Action*, ed. By M. J. Holler, 236-258. Würzburg: Physica-Verlag.

[18] HART, S. and A. MAS-COLELL (1996). "Bargaining and Value," *Econometrica* **64**, 357-380.

[19] HOROWITZ, A. D. (1973). "The Competitive Bargaining Set for Cooperative n-Person Games," *Journal of Mathematical Psychology* **10**, 265-289.

[20] KRISHNA, V. and R. SERRANO (1996). "Multilateral Bargaining," *Review of Economic Studies* **63**, 61-80.

[21] MASCHLER, M. (1992). "The Bargaining Set, Kernel, and Nucleolus," in *Handbook of Game Theory with Economic Applications*, *vol I*, ed. by R. J. Aumann and S. Hart, 591-667. Amsterdam: North-Holland.

[22] MAS-COLELL, A. (1989). "An Equivalence Theorem for a Bargaining Set," *Journal of Mathematical Economics* **18**, 129-139.

[23] MORELLI, M. (1999). "Demand Competition and Policy Compromise in Legislative Bargaining," *American Political Science Review* **93**, 809-20.

[24] MORELLI, M. and M. MONTERO (2001) "The Stable Demand Set: General Characterization and Application to Weighted Majority Games" Ohio State University Working Paper 0103.

[25] OKADA, A. (1996). "A Noncooperative Coalitional Bargaining Game with Random Proposers," *Games and Economic Behavior* **16**, 97-108.

[26] RUBINSTEIN, A. (1982) "Perfect Equilibrium in a Bargaining Model," *Econometrica* **50** , 97-108.

[27] SELTEN, R. (1981). "A Noncooperative Model of Characteristic Function Bargaining," in *Essays in Game Theory and Mathematical Economics*, ed. by V. Böhm and H. H. Nachtkamp, 131-151. Mannheim: Bibliographisches Institut.

[28] SERRANO, R. (1997) "Reinterpreting the Kernel," *Journal of Economic Theory* **77**, 58-80.

[29] VON NEUMANN, J. and O. MORGENSTERN (1944). *Theory of Games and Economic Behavior*. Princeton: Princeton University Press.

[30] ZHOU, L. (1994). "A New Bargaining Set of an N-Person Game and Endogenous Coalition Formation," *Games and Economic Behavior* **6**, 512-526.