Noncooperative Foundations of the Nucleolus in Majority Games

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Abstract

This paper studies coalition formation, payoff division and expected payoffs in a "divide the dollar by majority rule" game with random proposers. A power index is called *self-confirming* if it can be obtained as an equilibrium of the game using the index itself as probability vector. Unlike the Shapley value and other commonly used power indices, the nucleolus has this property. The proof uses a weak version of Kohlberg's (1971) balancedness result reinterpreting the balancing weights as probabilities in a mixed strategy equilibrium.

Keywords: coalition formation, bargaining, majority games, nucleolus, power indices, balancedness.

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1 Introduction

Consider the classical problem of dividing a dollar by majority rule. Suppose there are n players, player i has w_i votes and q votes ($q > \frac{\sum_{i \in N} w_i}{2}$) are needed to achieve a majority. What are the expected shares for the players in this game? If players care only about their own material payoffs and are risk-neutral this question is equivalent to "what will be the expected utility from playing the game?". From the axiomatic point of view the most widely accepted answer is the Shapley value (see Shapley, 1953; Roth, 1977a, 1977b). An alternative approach, consistent with the Nash (1953) program, is to set up a noncooperative bargaining game that we consider intuitively plausible and calculate the expected equilibrium shares.

This paper takes the latter approach, using a natural extension of the classical Baron and Ferejohn (1989) bargaining game.[1] Baron and Ferejohn consider the case of symmetric players, each of them being recognized to be the proposer with equal probability. The natural extension of this game to general weighted majority games, already hinted by Baron and Ferejohn in one of their examples, would be to select each player with a probability proportional to his number of votes. This extension has a straightforward interpretation if players are parties, different number of votes correspond to different number of representatives, and each representative is selected to be the proposer with equal probability.[2]

The first part of the paper deals with constant-sum homogeneous games without dummies. We show that the equilibria of the generalized Baron and

[1] Baron and Ferejohn (1989) build on Rubinstein (1982) and Binmore (1987).

[2] Many other papers have extended the Baron and Ferejohn game to different preferences of the players, different voting rules and/or different sets of alternatives. We consider general voting rules, but keep unchanged the set of alternatives (different divisions of a dollar) and the preferences of the players (identical and risk-neutral in money). Closely related papers are Harrington (1990), Winter (1996), Banks and Duggan (2000) and Eraslan (2002). Harrington (1990) and Eraslan (2002) both keep the assumption that each player has one vote, but allow the voting rule to be different from simple majority, and the players may be risk-averse (Harrington) or have different discount factors (Eraslan); Winter (1996) introduces veto players; Banks and Duggan (2000) show existence of stationary equilibria for arbitrary voting rules in general environments.

Ferejohn game do not generally correspond to the Shapley value. Instead, the normalized weights arise as expected equilibrium shares, vindicating the nucleolus as the value of playing the game.

One may wonder what exactly is driving this result. The first thing that comes to mind is that the recognition probabilities clearly affect the equilibrium of the noncooperative game, and we are using the weights themselves as recognition probabilities. What would be the result for other recognition probabilities? For example, suppose we make the recognition probabilities coincide with the Shapley value. Will the expected equilibrium shares also coincide with the Shapley value? In general, we will call a power index *self-confirming* if it is an equilibrium of the generalized Baron-Ferejohn game in which the power index itself is used as probability vector. We provide examples that show that neither the Shapley-Shubik index nor the Banzhaf index are self-confirming. The reason is that those values do not price the players in a competitive way (some players are obviously overpriced while others are underpriced), whereas the nucleolus does.

Another fact that seems to bias the equilibrium of the game in favor of the nucleolus is that the normalized weights coincide with the nucleolus for this class of games. However, it turns out that the nucleolus is *always* self-confirming, irrespective of whether it is a representation of the underlying majority game (or indeed of whether a representation exists), while the canonical representation of the game need not be self-confirming. The proof uses a weak version of Kohlberg's (1971) result on balanced collections of coalitions, reinterpreting the balancing weights as weights in a mixed-strategy equilibrium.

The self-confirming property is shared by all elements in a superset of the nucleolus that we will refer to as the *nucleus*. We will show that the elements of the nucleus are the most plausible self-confirming power indices, and that the random proposer game has a unique equilibrium payoff vector if the protocol is in the nucleus.

The rest of the paper is organized as follows. Section 2 serves as an introduction to weighted majority games and the nucleolus. Section 3 presents the noncooperative game. Section 4 contains the results for constant-sum homogeneous games; section 5 is devoted to the self-confirming property of the nucleolus. Section 6 contains some further discussion and section 7 concludes.

2 Preliminaries

2.1 Majority games

Let $N = \{1, ..., n\}$ be the set of players. $S \subseteq N$ represents a generic coalition of players, and $v : 2^n \to \mathbb{R}$ with $v(\varnothing) = 0$ denotes the characteristic function. The (cooperative) game (N, v) is a *simple game* iff $v(S) \in \{0, 1\}$ for all $S \subseteq N$, $v(\varnothing) = 0$, and $v(N) = 1$. It is *monotonic* if $v(S) = 1$ implies $v(T) = 1$ for any $T \supset S$. A coalition S is called *winning* iff $v(S) = 1$ and *losing* iff $v(S) = 0$. It is called *minimal winning* iff $v(S) = 1$ and $v(T) = 0$ for all T such that $T \subset S$. We will denote the set of winning coalitions by $\mathbf{W}$ and the set of minimal winning coalitions by $\mathbf{W}^m$. The set of minimal winning coalitions containing player i is denoted by $\mathbf{W}_i^m$. A player i such that $v(S \cup i) = v(S)$ for all S is called a *dummy player*.

A simple game is *proper* iff $v(S) = 1$ implies $v(T) = 0$ for all $T \subseteq N \backslash S$. It is *constant-sum* iff $v(S) + v(N \backslash S) = 1$ for all $S \subseteq N$. It is a *weighted majority game* iff there exist n nonnegative numbers (weights) $w_1, ..., w_n$ and a nonnegative number q such that $v(S) = 1$ if and only if $\sum_{i \in S} w_i := w(S) \geq q$. We will denote a weighted majority game by $(q; w_1, ..., w_n)$. The pair (q, w) is called a *representation* of the game v. A weighted majority game has many possible representations, but not all of them are equally convenient. A representation w is called *normalized* iff $\sum_{i \in N} w_i = 1$; it is *homogeneous* iff $\sum_{i \in S} w_i = q$ for all $S \in \mathbf{W}^m$. Not all weighted majority games admit a homogeneous representation. A weighted majority game admitting a homogeneous representation is called a *homogeneous game*.

2.2 The nucle(ol)us

Let (N, v) be a characteristic function game and $x = (x_1, ..., x_n)$ be an *imputation*, that is, a payoff vector with $x_i \geq v(i)$ and $x(N) = v(N)$. For any coalition S the value $e(S,x) = v(S) - x(S)$ is called the *excess* of S at x.

For any imputation x let $S_1, ..., S_{2^n}$ be an ordering of the coalitions for which $e(S_l, x) \geq e(S_{l+1}, x)$ for all $l = 1, ..., 2^n$ and let $E(x)$ be the vector of excesses defined by $E_l(x) = e(S_l, x)$ for all $l = 1, ..., 2^n$. We say that $E(x)$ is *lexicographically less* than $E(y)$ if $E_l(x) < E_l(y)$ for the smallest l for which $E_l(x) \neq E_l(y)$. The *nucleolus* is the set of imputations x for which the vector $E(x)$ is lexicographically minimal. Schmeidler (1969) shows that the nucleolus consists of a unique imputation. It is contained in the classical bargaining set (Davis and Maschler, 1967) and in the kernel (Davis and Maschler, 1965).

The following superset of the nucleolus will play an important role in the paper. Consider the set of imputations that minimize the maximum excess $E_1(x)$. For proper simple games, this set is the solution to the following linear programming problem[3]

$$\begin{aligned} & \min \ e \\ \text{s.t. } x(S) + e & \geq 1 \text{ for all } S \in \mathbf{W} \\ x(N) & = 1 \\ x_i & \geq 0 \text{ for all } i \in N. \end{aligned}$$

This set is well known in the literature but as far as I am aware it does not have a name. It seems natural to refer to this set as the **nucleus** of the game.[4]

We will use a property of the nucleus that follows from duality theory (see e.g. Vanderbei's 2001 textbook, chapter 5). It is a weak version of a

[3] This will be the case even if there is a player i with $v(i) = 1$.

[4] The nucleus is closely related to the least core introduced by Maschler et al. (1979), the difference being that the least core does not include the excess of coalition N in the program or impose $x_i \geq v(i)$ for all i. Both concepts coincide if the game is superadditive and has an empty core. If the game has a nonempty core, the nucleus coincides with the core. For simple games it coincides with the set of balanced aspirations of Cross (1967) up to a rescaling factor that changes from game to game. For constant-sum homogeneous games it coincides with the nucleolus.

much stronger result of Kohlberg (1971) that characterizes the nucleolus. For this we need some definitions.

For every simple game v and every payoff vector x, let $b_1(x, v)$ be the set of those $S \subseteq N$ for which $\max\{v(S) - x(S) : S \subseteq N\}$ is attained and $b_0(x) = \{\{i\} : x_i = 0\}$.

Let $\mathcal{C}$ be a collection of nonempty subsets of N. We say that the collection is (weakly) *balanced* iff there exist (nonnegative) positive numbers $(\lambda_S)_{S\in\mathcal{C}}$ such that, for each $i \in N$, $\sum_{S\ni i} \lambda_S = 1$.

Lemma 1 *(cf. Kohlberg, 1971) Let v be a simple game. An imputation x is in the nucleus of v if and only if $b_0(x) \cup b_1(x, v)$ is weakly balanced.*

Proof. The dual of the linear program above (after noticing that we can replace $x(N) = 1$ by $-x(N) \geq -1$ without changing the solution) is

$$\max \sum_{S\in\mathbf{W}} \lambda_S - \mu$$

$$\begin{aligned} \sum_{S\ni i} \lambda_S - \mu &\leq 0 \\ \sum_{S\in\mathbf{W}} \lambda_S - 1 &\leq 0 \\ \lambda_S &\geq 0 \text{ for all } S \in \mathbf{W},\ \mu \geq 0 \end{aligned}$$

where λ_S is the dual variable associated to the constraint $x(S) + e \geq 1$ and μ to $-x(N) \geq -1$. Complementary slackness implies that, for any coalition with $\lambda_S > 0$, we have $x(S) + e = 1$, that is, $S \in b_1(x, v)$. Moreover, $\sum_{S\ni i} \lambda_S = \mu$ for all i with $x_i > 0$. If the core is empty we have $e > 0$ and thus $\sum_{S\in\mathbf{W}} \lambda_S = 1$. It is then clear that $\mu > 0$ and we can define $\lambda'_S = \frac{\lambda_S}{\mu}$. Then $\sum_{S\ni i} \lambda'_S = 1$ for all i with $x_i > 0$ and $\sum_{S\ni i} \lambda'_S \leq 1$ for all i with $x_i = 0$, and the result follows. If the core is nonempty we have $e = 0$ as the solution of the primal, $\lambda_N = \mu = 1$ as a solution of the dual and the result follows as well.

Conversely, if we have an imputation x such that $b_0(x) \cup b_1(x, v)$ is weakly balanced with balancing weights $\left(\lambda'_S\right)_{S\in b_0(x)\cup b_1(x,v)}$, we can use this information to construct feasible solutions to the primal and to the dual satisfying complementary slackness. For the primal, simply compute the maximum excess e associated to x. For the dual, if $N \in b_1(x,v)$ then x is in the core and thus in the nucleus. Otherwise $b_1(x,v)$ only contains winning coalitions. Define $\lambda_S := \frac{\lambda'_S}{\sum_{S\in \mathbf{W}} \lambda'_S}$ if $S \in b_1(x,v)$, $\lambda_S = 0$ if $S \in \mathbf{W} \backslash b_1(x,v)$, and $\mu := \sum_{S \ni i} \lambda_S$ for some i with $x_i > 0$. One can check that (x,e) is feasible for the primal, $((\lambda_S)_{S\in \mathbf{W}}, \mu)$ is feasible for the dual and complementary slackness holds. By the complementary slackness theorem (cf. Theorem 5.3 in Vanderbei (2001)) we have an optimal solution. ∎

The dual variables $(\lambda_S)_{S\in \mathbf{W}}$ can be interpreted as the probabilities of coalition S forming (cf. Albers 1979 p. 5). Then each player with $x_i > 0$ will be in the final coalition with the same probability, $\mu = 1 - e$, and a player with $x_i = 0$ appears in the final coalition no more often than one with $x_i > 0$. Notice however that the coalition formation process is left unspecified. We will obtain the same result in a strategic model of coalition formation in which players will be free to propose any coalition with any payoff division.

3 The noncooperative game

3.1 Description of the game

Let (N,v) be a proper simple game. We interpret this game as a transferable payoff game where n risk-neutral players decide by majority rule on the division of a (perfectly divisible) budget.

Given the underlying cooperative game, bargaining proceeds as follows: At every round $t = 1, 2, ...$ Nature selects a player randomly to be the proposer. This player proposes a coalition $S \subseteq N$ to which he belongs and a feasible division of $v(S)$, denoted by $x^S = (x_i^S)_{i\in S}$. Given a proposal, the rest of players in S (called responders) accept or reject sequentially (the order does not affect the results). If all players in S accept, the proposal is implemented and the game ends. If at least one player rejects, the game proceeds to the next period in which nature selects a new proposer (always with the same probability distribution). Players are risk-neutral and share a discount factor[5] $0 < \delta < 1$. Thus, if a proposal x^S is accepted by all players in S at time t, each player in S receives a payoff $\delta^{t-1} x_i^S$. A player not in S remains a singleton and receives zero. If no proposal is ever accepted, all players receive 0.

We will call the probability distribution Nature uses to select proposers a *protocol*. We will denote the protocol by $\theta = (\theta_i)_{i \in N}$. Two natural protocols are the *egalitarian protocol*, $\theta_i = \frac{1}{n}$ for all i in N, and (if weights are assigned to the players) the *proportional protocol*, $\theta_i = \frac{w_i}{w(N)}$ for all i in N.

A pure strategy for player i is a sequence $\sigma_i = (\sigma_i^t)_{t=1}^{\infty}$, where σ_i^t, the tth round strategy of player i, prescribes

1. A *proposal* (S, x^S).

2. A *response function* assigning "yes" or "no" to all possible proposals of the other players.

Players are free to condition their actions on the history of the game up to time t; however we will study equilibria in which they choose not to do so. The solution concept is *stationary subgame perfect equilibrium* (SSPE). Stationarity requires that players follow the same strategy at every round t: the probability that the proposer makes proposal (S, x^S) is the same for all t regardless of history, and the response function depends only on the current proposal and not on what happened in previous rounds.

We denote the noncooperative game described above by $G(v, \theta, \delta)$.

Given a SSPE σ^* we will denote the associated *expected payoff* for player i (computed at the beginning of the game, before Nature chooses the proposer) by $y_i(\sigma^*)$ - we will drop σ^* to simplify notation -. The expected payoff given that a proposal is rejected is called the *continuation value*. Continuation values play a very important role in any SSPE: because incredible threats are ruled out by subgame perfection, a responder must accept any payoff strictly higher than their continuation value. Moreover, when the equilibrium is stationary the continuation value is the same at all rounds for given σ^*: after a proposal is rejected a period elapses and the players do the

[5] If $\delta = 1$, the self-confirming property in proposition 7 still holds, but delay is not completely ruled out in equilibrium and there may be equilibria with a different expected payoff vector.

same they would do at time 1 all over again, thus player i's continuation value is simply δy_i.

We will denote the probability that player i proposes coalition S (conditional on i being the proposer) by λ_S^i and the probability that i receives a proposal by r_i. Thus, $r_i = \sum_{j \in N \setminus \{i\}} \sum_{S \supseteq \{i,j\}} \theta_j \lambda_S^j$. Like y_i, λ_S^i and r_i depend on σ^*.

3.2 Immediate agreement property of stationary subgame perfect equilibria

The first thing to notice is that no delay occurs in a SSPE provided $\delta < 1$. This is shown by Okada (1996) for $\theta = (\frac{1}{n})_{i \in N}$ and superadditive games; a direct extension for a general θ can be found in Montero (2002). Okada's proof also applies to proper simple games (not necessarily superadditive), as the following lemma shows.

Lemma 2 *Let (N, v) be a proper simple game. If σ^* is an SSPE of $G(v, \theta, \delta)$ with $\delta < 1$ and $(y_i)_{i \in N}$ the associated equilibrium expected payoffs, then σ^* is such that every player i only proposes coalitions that solve the following maximization problem*

$$\max_{S: S \ni i} v(S) - \sum_{j \in S \setminus \{i\}} \delta y_i \tag{1}$$

and all proposals are accepted.

Proof. Subgame perfection implies that a proposal that gives $\delta y_j + \epsilon$ to each responder j must be accepted. Consider the situation of player i as a proposer. If he makes an unacceptable proposal, he will receive δy_i. On the other hand, since the game is simple we have $1 = v(N) \geq \sum_{j \in N} y_j$. Thus, $v(N) > \delta \sum_{j \in N} y_j$. This means that player i can get more than δy_i by making an acceptable proposal, therefore all proposals made in equilibrium must be accepted.

Let S be a coalition with $\lambda_S^i > 0$. It must be the case that player i offers every responder j exactly δy_j (otherwise player i could reduce the payoff offered to j and we would not have an equilibrium). Moreover, S must be a solution to (1); otherwise player i would prefer to propose another coalition. ■

The arguments in the proof of the lemma lead to the following corollary.

Corollary 3 *Let (N, v) be a proper simple game. If σ^* is an SSPE of $G(v, \theta, \delta)$ with $\delta < 1$ and $(y_i)_{i \in N}$ the associated equilibrium expected payoffs, then σ^* has the following properties:*

1) There is an advantage to being the proposer:
$\max_{S:S \ni i} v(S) - \sum_{j \in S \setminus \{i\}} \delta y_j > \delta y_i$ *for all* $i \in N$.
2) $y_i \geq 0$ *for all* i, *and* $y_i > 0$ *only if* $\theta_i > 0$.
3) $\lambda_S^i > 0$ *implies* $S \in \mathbf{W}$ *for all* $i \in N$.

Proof. 1) Because δy_i is both the payoff from making unacceptable proposals and from being a responder, saying that making acceptable proposals is optimal implies an advantage of being a proposer compared to being a responder.

2) Every player can guarantee himself a zero payoff by refusing to enter a coalition. If $\theta_i > 0$, player i can always make a positive payoff as a proposer (for example, by proposing the grand coalition), thus $y_i > 0$. On the other hand, $\theta_i = 0$ implies $y_i = r_i \delta y_i$, an equation that can only hold for $y_i = 0$ given that $\delta < 1$ and $r_i \leq 1$.

3) Because proposing a losing coalition would give i at most 0, as opposed to the strictly positive payoff of proposing N. ■

3.3 The role of coalitions with maximum excess

The concept of excess plays a fundamental role in any SSPE of the game with random proposers regardless of the expected equilibrium payoffs. Because the proposer will keep the excess of any coalition that is formed, he will form a coalition with maximum excess *among the ones to which he belongs.*

Lemma 4 *Let (N, v) be a proper simple game and σ^* an SSPE of the game $G(v, \theta, \delta)$. If $\lambda_S^i > 0$ then S is a solution to the problem $max_{S \ni i} v(S) - y(S)$.*

Proof. The maximization problem in (1) can be written as

$$\max_{S \ni i} \; v(S) + \delta y_i - \sum_{j \in S} \delta y_j .$$

Because only winning coalitions are proposed in equilibrium, $v(S) + \delta y_i$ is a constant. Then the problem is equivalent to $\min_{S \ni i} \sum_{j \in S} \delta y_j$, which since $\delta > 0$ is equivalent to $\min_{S \ni i} \sum_{j \in S} y_j$, which is equivalent to maximizing $1 - \sum_{j \in S} y_j$, the excess of coalition S. ■

Note that this does not imply that only coalitions in $b_1(y, v)$ can form in equilibrium because a player may not be contained in any of the coalitions in $b_1(y, v)$.

4 Proportional Payoffs in Constant-Sum Homogeneous Games

Peleg (1968) shows that a constant-sum homogeneous game without dummies has a unique normalized homogeneous representation, coinciding with the nucleolus. It turns out that the proportional payoffs (i.e. the nucleolus) can be obtained as an equilibrium of the game with a proportional protocol. This result is a particular case of proposition 7, but I believe it is of independent interest because the proportional protocol is a natural one to use independently of whether it coincides with any solution concept and constant-sum homogeneous games are the most popular class of majority games.

Proposition 5 *Let $(q; w)$ be the normalized homogeneous representation of a constant-sum homogeneous game without dummies. Let $\theta_i = w_i$. For any $0 \leq \delta \leq 1$ there is an equilibrium σ^* of the game $G(v, \theta, \delta)$ such that $y_i = w_i$ for all i in N.*

Proof. (sketch) The set $\mathbf{W}^m$ is balanced, i.e., we can find a collection of nonnegative numbers $(\lambda_S)_{S \in \mathbf{W}^m}$ such that $\sum_{S \ni i} \lambda_S = 1$. Let each player propose each minimal winning coalition $S \ni i$ with probability λ_S and offer each responder δw_j. This is a SSPE that induces w as a payoff vector. ■

It is easy to see that other power indices cannot in general be obtained as an equilibrium of the game with the proportional protocol. Consider the apex game with four players $(3; 2111)$. Montero (2002) shows that the unique equilibrium payoffs of the game with the proportional protocol are $(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$, the nucleolus of the game. Instead, the Shapley-Shubik (1954) index -which is simply the Shapley value- and the normalized (so that the components add up to 1) Banzhaf (1965) index are both $(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$, the Deegan-Packel (1978) index is $(\frac{3}{8}, \frac{5}{24}, \frac{5}{24}, \frac{5}{24})$ and the Johnston (1978) index is $(\frac{9}{14}, \frac{5}{42}, \frac{5}{42}, \frac{5}{42})$.

5 The nucleolus as a self-confirming power index

It is well known in random proposer games that the protocol θ affects the expected payoffs of the players[6]. We have argued that a natural choice for θ in majority games is to select each player with a probability proportional to his weight. However, if the underlying game is not a weighted majority game there is no natural choice for the protocol. The protocol may be interpreted as a measure of the power the bargaining procedure assigns to each player. This power is exogenous (an input of the game) and not necessarily related to the characteristic function. On the other hand, the expected payoffs in the game can also be interpreted as a measure of the power of the players (this is what Felsenthal and Machover (1998) call P-power). This power is endogenous and in general will depend on the characteristic function as well as the bargaining procedure. It would seem to be a good idea to make exogenous and endogenous power coincide: if we use the Shapley-Shubik index in the protocol but obtain the nucleolus as expected payoff vector, then it would seem that the Shapley-Shubik index was not a good measure of the players' bargaining power in the noncooperative game.

We will call a power index *self-confirming* (in the Baron-Ferejohn game) iff it is an equilibrium of the game that uses the index itself as protocol: if we assume that the power distribution in the game corresponds to θ, there

[6] Many papers make reference to this question, including Montero (1999, 2002), Yan (2002), Gomes (2001) and Eraslan (2002).

is an equilibrium of the game that confirms this idea.

Definition 6 *Let (N, v) be a proper simple game, $\theta \in \mathbb{R}^n_+$ be a protocol and $\delta \in [0,1]$. We say that θ is **self-confirming** given v and δ iff the game $G(v, \theta, \delta)$ has an equilibrium σ^* such that $y(\sigma^*) = \theta$.*

It is easy to show that any element of the core is self-confirming, and this happens even beyond simple games (assuming the game is normalized so that $v(N) = 1$, cf. Yan (2002)). This property is not very relevant for proper simple games because the core is empty in the absence of veto players. There are also some trivial cases of self-confirming payoff vectors: all payoff vectors such that $x_i = 1$ for some i and $x_j = 0$ for $j \neq i$, and of course *any* payoff vector with $x_j \geq 0$ and $\sum_{j \in N} x_j = 1$ in the very extreme case of $\delta = 0$.

We now show that the nucleolus is self-confirming for proper simple games regardless of the discount factor. Since the property holds for any point in the nucleus we state the result in its more general form.

Proposition 7 *Let (N, v) be a proper simple game, and $\mu \in \mathbb{R}^n$ a point in the nucleus of v. Then the game $G(v, \mu, \delta)$ has an SSPE σ^* with $y(\sigma^*) = \mu$ for any $\delta \in [0,1]$.*

Proof. We will construct an equilibrium σ^* with the desired property. There are two conditions σ^* must satisfy: first, μ should be the expected payoff vector corresponding to σ^*; second, σ^* must prescribe an optimal behavior for the players (both as proposers and as responders) given the vector μ.

1. Consider the following strategy combination σ^*:

   (a) Each player accepts any proposal that gives him at least $\delta\mu_i$ (in particular, if $\mu_i = 0$ player i accepts any nonnegative payoff).

   (b) If a player is selected to be the proposer, he proposes only coalitions of maximum excess (that is, coalitions in $b_1(\mu, v)$) and offers each responder $\delta\mu_i$. Lemma 1 implies that each possible proposer i (each player with $\mu_i > 0$) belongs to at least one coalition in $b_1(\mu, v)$. Furthermore, since the set $b_1(\mu, v) \cup b_0(\mu)$ is weakly balanced, there exists a collection of balancing weights $(\lambda_S)_{S \in b_0(\mu) \cup b_1(\mu,v)}$. Let i be a player such that $\mu_i > 0$. We set $\lambda^i_S = \lambda_S$ for $S \in b_1(\mu, v)$, $S \ni i$ and $\lambda^i_S = 0$ otherwise. The balancedness result implies that $\sum_{S \in b_1(\mu,v), S \ni i} \lambda_S = 1$, thus player i's strategy is fully specified. If $\mu_i = 0$, player i cannot be selected to be the proposer, so we do not need to specify what would he do if he were selected.

2. We will now check that expected payoffs given that players follow strategy combination σ^* actually coincide with μ. If $\delta = 0$ or $\mu_i = 0$ this is clearly the case. Otherwise expected payoffs are given by the following equation

$$y_i = \theta_i[1 - \delta \sum_{j \in \overline{S} \setminus \{i\}} \mu_j] + r_i \delta \mu_i.$$

where $\overline{S}$ is an arbitrary coalition with $\overline{S} \in b_1(\mu, v)$, $i \in \overline{S}$, and r_i is determined by σ^*.[7] If moreover we assume $\theta = \mu$, the payoff vector corresponding to σ^* is fully determined. Substituting $\theta = \mu$ we obtain

$$y_i = \mu_i[1 - \delta \sum_{j \in \overline{S} \setminus \{i\}} \mu_j] + r_i \delta \mu_i.$$

We want to prove that $y_i = \mu_i$. This will be the case if $\sum\limits_{j \in \overline{S} \setminus \{i\}} \mu_j := \overline{\mu} - \mu_i = r_i$.

We know $r_i = \sum\limits_{S \in b_1(\mu,v), S \ni i} \sum\limits_{j \in S \setminus \{i\}} \mu_j \lambda^j_S = \sum\limits_{S \in b_1(\mu,v), S \ni i} \lambda_S \sum\limits_{j \in S \setminus \{i\}} \mu_j = \sum\limits_{S \in b_1(\mu,v), S \ni i} \lambda_S(\overline{\mu} - \mu_i) = (\overline{\mu} - \mu_i) \sum_{S \in b_1(\mu,v), S \ni i} \lambda_S = \overline{\mu} - \mu_i$.

3. So far we have constructed a strategy combination and proved that if the players follow it expected payoffs will coincide with μ. It remains

[7] In writing $1 - \delta \sum_{j \in \overline{S} \setminus \{i\}} \mu_j$ we are taking into account that only winning coalitions are proposed given σ^*. A losing coalition can only be in $b_1(\mu, v)$ if its members are getting 0 in total (and thus 0 each), but players with $\mu_j = 0$ are never selected to be proposers.

to prove that σ^* is an equilibrium. First, $\delta\mu_j$ is player j's continuation value given that the players follow σ^*, so responders are acting optimally when they accept any offers that give then at least $\delta\mu_j$. Second, coalitions with maximum excess correspond to coalitions that solve the maximization problem in (1), and proposing any coalition in $b_1(\mu, v)$ is clearly at least as good as making unacceptable proposals.

■

The strategy combination σ^* we have constructed has some nice properties if we restrict ourselves to the players with $\mu_i > 0$ (all players for the games in proposition 7). First, they show *reciprocity*: the probability that i proposes to j conditional on i being the proposer is the same as the probability that j proposes to i conditional on j being the proposer; second, they are *non-discriminatory* in the sense that the probability of entering a coalition is the same for all players and equal to what we have called $\overline{\mu}$. It is easy to see from the proof that the second property must hold in any equilibrium σ with expected payoffs equal to μ. We have thus provided an explicit model that justifies the interpretation of the dual variables in lemma 1.

The equilibrium described in proposition 7 is unique in terms of expected payoffs if $\delta < 1$. My conjecture is that the result holds for all proper simple games, but the proof covers only games with $\overline{\mu} \geq \frac{1}{2}$. This class includes all proper weighted majority games.

Proposition 8 *Let (N, v) be a simple game with $\overline{\mu} \geq \frac{1}{2}$ and $\mu \in \mathbb{R}^n$ an element of the nucleus of v. Then any SSPE σ^* of the game $G(v, \mu, \delta)$ has $y(\sigma^*) = \mu$ for any $\delta < 1$.*

Proof. See appendix. ■

The probability of each coalition forming in equilibrium is not determined in general, even if we restrict the strategies to treat symmetric players equally and to show reciprocity ($\lambda_i^S = \lambda_j^S$ for any i, $j \in S$). An example of indeterminacy would be a constant-sum homogeneous game with more types of minimal winning coalitions than of players, like $(5; 2, 2, 1, 1, 1, 1, 1)$.

It is easy to see that neither the Shapley value nor the normalized Banzhaf index are self-confirming. Consider the case of apex games with four players. Both the Shapley value of the game and the normalized Banzhaf index are $(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. If there is an equilibrium with expected payoffs $(\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$, all the minor players must propose the minor player coalition. Suppose $\theta = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$. Then the expected equilibrium payoff for the apex player equals $\frac{1}{2}[1 - \frac{\delta}{6}] \neq \frac{1}{2}$. Something analogous happens with the Johnston and Deegan-Packel indices: the corresponding payoff vectors would induce proposing behavior which is not consistent with those payoff vectors.

6 Discussion

6.1 The nucleolus versus other power indices

A property of the Shapley value and other power indices that makes them unlikely to coincide with expected equilibrium payoffs in the Baron-Ferejohn model is that they usually induce strict preferences for the players over the minimal winning coalitions they can propose. The same can be said about the kernel (Davis and Maschler, 1965)[8]. Instead, the nucleolus makes the players indifferent between several coalitions and provides us with more degrees of freedom when constructing equilibrium strategies, even if the protocol does not coincide with the nucleolus, as the following example illustrates.

Example 9 *Consider the apex game* $(3; 2, 1, 1, 1)$ *and a protocol that selects the apex player with probability* θ_1 *and each minor player with probability* $\frac{1-\theta_1}{3}$. *If* $\delta = 1$, *the nucleolus can be obtained for any* $\theta_1 \leq \frac{1}{2}$; *the Shapley value is only obtained for* $\theta_1 = \frac{3}{5}$.

This is because when expected payoffs coincide with the nucleolus each minor player is indifferent between proposing to the apex player and proposing to the other two minor players: in both cases the proposer pays a total of $\frac{2}{5}$. We can construct an equilibrium in which the apex player proposes to each minor player with probability $\frac{1}{3}$, and each minor player proposes to

[8] Consider the game $(6; 422111)$. The payoff vector $(\frac{14}{34}, \frac{7}{34}, \frac{7}{34}, \frac{2}{34}, \frac{2}{34}, \frac{2}{34})$ is in the kernel but is not self-confirming.

the apex player with probability λ, where λ can be found from the equation $\frac{2}{5} = \theta_1 \left[1 - \frac{1}{5}\right] + (1 - \theta_1)\lambda\frac{2}{5}$. The solution to this equation, $\lambda = \frac{1-2\theta_1}{1-\theta_1}$, is between 0 and 1 for $\theta_1 \leq \frac{1}{2}$. In contrast, given that expected payoffs coincide with the Shapley value, the apex player receives no proposals, and the expected payoff equation becomes $\frac{1}{2} = \theta_1 \left[1 - \frac{1}{6}\right]$, which has only one solution.

Another property of the Shapley value that makes it difficult to implement is that it may not be in the core, as the following example illustrates.

Example 10 *Consider the game* $(3; 211)$. *The Shapley value of this game is* $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6})$. *If* δ *is close to* 1, *there is no protocol* θ *for which expected equilibrium payoffs coincide with the Shapley value.*

Let y_1 be the expected equilibrium payoff for player 1 and y_2 the expected equilibrium payoff for 2 and 3. Then expected payoff for player 1 is given by $y_1 = \theta_1 [1 - \delta y_2] + (1 - \theta_1)\delta y_1$. For $\theta_1 > 0$, the solution of this equation together with $y_1 = 1 - 2y_2$ is $y_1 = \frac{\theta_1(2-\delta)}{2-\delta(2-\theta_1)}$, which tends to 1 when δ tends to 1.[9] For $\theta_1 = 0$ we have $y_1 = 0$. Thus for δ sufficiently close to 1 we have *no* protocol that implements the Shapley value. The same applies to all power indices based on marginal contributions or that give positive values to any player who is at least in one minimal winning coalition.

Another solution concept that makes the players indifferent over several attractive coalitions is the modiclus (Sudhölter, 1996). The modiclus is a representation of all homogeneous games, and therefore would make each player i indifferent between all coalitions in $\mathbf{W}_i^m$ in any homogeneous game. We now show by an example that the modiclus is not self-confirming, even for games with empty core.

Example 11 *Consider the game* $(5; 3, 2, 2, 1)$. *If* $\theta = (\frac{3}{8}, \frac{2}{8}, \frac{2}{8}, \frac{1}{8})$, *expected equilibrium payoffs cannot coincide with* θ.

[9] This argument can be generalized to all games with nonempty core if the Shapley value is outside the core. This is because if i is a veto player he has $r_i = 1 - \theta_i$. As δ tends to 1 the advantage of being proposer for i must vanish, but this requires that expected payoffs for all nonveto players converge to 0.

Suppose we have an equilibrium with expected payoffs $(\frac{3}{8}, \frac{2}{8}, \frac{2}{8}, \frac{1}{8})$. Player 1 is indifferent between proposing $\{1,2\}$ and $\{1,3\}$; players 2 and 3 are indifferent between proposing to player 1 and proposing $\{2,3,4\}$; player 4 prefers proposing coalition $\{2,3,4\}$. The equilibrium strategies would have to satisfy (among others) the following equations

$$y_1 = \frac{3}{8}[1-\delta\frac{2}{8}] + [\frac{2}{8}\lambda^2_{\{1,2\}} + \frac{2}{8}\lambda^3_{\{1,3\}}]\delta\frac{3}{8}$$
$$y_4 = \frac{1}{8}[1-\delta\frac{4}{8}] + [\frac{2}{8}\lambda^2_{\{2,3,4\}} + \frac{2}{8}\lambda^3_{\{2,3,4\}}]\delta\frac{1}{8}$$
$$\lambda^i_{\{1,i\}} + \lambda^i_{\{2,3,4\}} = 1;\ i = 2,3.$$

In order for y_4 to be $\frac{1}{8}$, we need $\lambda^2_{\{2,3,4\}} = \lambda^3_{\{2,3,4\}} = 1$. But then $y_1 = \frac{3}{8}[1-\delta\frac{2}{8}] \approx \frac{9}{32}$.

The nucleolus of the example above is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$. Notice that it assigns a payoff of 0 to the smallest player, even though he is not a dummy. It turns out that the self-confirming property may be incompatible with the requirement that $y_i > 0$ for all i with $\mathbf{W}^m_i \neq \varnothing$, even if the core is empty.[10]

A feature of the nucleolus that makes it attractive as a power index is that it is based on a set of attractive coalitions (the coalitions of maximum excess), whereas the Shapley value is based on *all* coalitions, including some that will probably never form. Interestingly, Evans (1996) shows that the Shapley value has a sort of self-confirming property for a bargaining procedure in which the agents are *not* strategic because coalitions form randomly and players can be forced to accept individually irrational payoffs.

6.2 Self-confirming power indices outside the nucleus

A self-confirming power index is not necessarily reasonable, as evidenced by the n "degenerated" fixed points in which a player i has $\theta_i = 1$. One should keep in mind that the self-confirming property is a circular concept in the same way as the Nash equilibrium. If we take θ as a measure of power there is an equilibrium that supports this view, but there may be no reason

[10] As will become clear from the next subsection one would need $\mathbf{W}^m$ to be weakly balanced, but as pointed by Rosenmüller and Sudhölter (1994) not all homogeneous games have this property.

to take θ as a power measure in the first place. We will argue below that payoffs vectors with a self-confirming property but not in the nucleus are not very plausible. They have undesirable properties, like giving more to losing coalitions than to winning ones, or exploitation of some of the players who are getting nothing.

Recall that, in any equilibrium with any protocol, the set $b_1(y,v)$ plays an important role (cf. lemma 4). Each player in $B_1(y)$ will propose a winning coalition in $b_1(y)$ -there is always at least one-, and $\sum_{i\in S} y_i$ equals a constant which we denote by $\overline{y}$ for any $S \in b_1(y,v)$. We can then rewrite i's payoff as a proposer as $\delta y_i + 1 - \delta \overline{y}$, and $y_i = \theta_i \left[\delta y_i + 1 - \delta \overline{y}\right] + r_i \delta y_i$.

In the case of a self-confirming power index we have $y_i = \theta_i$. If $\theta_i > 0$ we can divide by θ_i and then by δ to obtain

$$\overline{y} = \theta_i + r_i \text{ for all } i \in B_1(\theta).$$

In words, the probability of i being in the coalition that forms, which is the sum of θ_i and r_i, must be identical for any player in $B_1(\theta)$.

If θ is in the nucleus, lemma 1 implies that any player with $\theta_i > 0$ is in B_1, but this is need not be the case for other self-confirming power indices. If $\theta_i > 0$ for some $i \notin B_1(\theta)$, coalitions belonging to $b_2(\theta)$ (and possibly other sets, depending on whether the set of players with $\theta_i > 0$ is included in $B_1(\theta) \cup B_2(\theta)$) will also play a role. Proposers in $B_2(\theta)\backslash B_1(\theta)$ only propose winning coalitions in $b_2(\theta)$, $\sum_{i\in S} y_i = \overline{y}_2 > \overline{y}$ for all $S \in b_2(y,v)$, and each proposer in $B_2(\theta)\backslash B_1(\theta)$ must be in the final coalition with probability $\overline{y}_2$.

Taking this into account, let θ be a self-confirming power index not in the nucleus. We will consider two possible cases, depending on whether all players with $\theta_i > 0$ are in $B_1(\theta)$.

Suppose $B_2(\theta)\backslash B_1(\theta)$ contains a player i with $\theta_i > 0$. As we have seen, this player must be in the final coalition more often than the players in $B_1(\theta)$. Since players in $B_1(\theta)$ will never propose to player i, the only way this can be achieved is if $\overline{y} < \frac{1}{2}$, but this means that there is a winning coalition S (any winning coalition in $b_1(\theta)$) getting less than its complement, which is a losing coalition. An example is the payoff vector $(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}, 0, 0, 0)$ for the game $(6; 4, 2, 2, 1, 1, 1)$, in which the losing coalition $\{1\}$ gets a higher payoff than the winning coalition $\{2,3,4,5,6\}$.

If all players with $\theta_i > 0$ are in $B_1(\theta)$ then (denoting the probability that S forms by λ_S) $\sum_{S\ni i}\lambda_S = \overline{y}$ for all i with $\theta_i > 0$. On the other hand, $b_0(\theta)\cup b_1(\theta,v)$ cannot be weakly balanced because then lemma 1 would imply that θ is in the nucleus. In order for $b_0(\theta)\cup b_1(\theta,v)$ not to be weakly balanced it must be the case that, for any probabilities $(\lambda_S)_{S\in b_1(\theta)}$, we can find a player i with $\theta_i = 0$ such that $\sum_{S\ni i}\lambda_S > \overline{y}$. Thus, there is always a player who is getting nothing but *must* appear in the final coalition more often than the players with $\theta_i > 0$ in order for those players to keep their payoffs. Intuitively, there is an "excess demand" for player i, who is being exploited by the rules of the game. An example would be the payoff vector $(\frac{1}{2},\frac{1}{4},\frac{1}{4},0,0,0)$ for the game $(6;4,2,2,1,1,1)$. Each of the players who are getting 0 must be in the final coalition with probability $\frac{2}{3}$ or more, whereas the players with positive payoff are in the final coalition only $\frac{1}{2}$ of the time. Instead, in the nucleolus example of $(\frac{1}{3},\frac{1}{3},\frac{1}{3},0)$ for the game $(5;3,2,2,1)$, player 4 is in the coalition with probability $\frac{1}{3}$ as opposed to a probability of $\frac{2}{3}$ for each of the other players. The weak balancedness of $b_0(\theta)\cup b_1(\theta,v)$ means that we can assign probabilities to each coalition so that the players who are getting 0 do not appear more often than the rest in the final coalition. These players may be receiving nothing for a valuable contribution, but since they are in the final coalition at most as often as the rest they are not in excess demand and thus, it may be argued, their payoff is not too low.

7 Concluding remarks

We have provided a noncooperative interpretation of the nucleolus of proper simple games. The balancing weights in Kohlberg (1971) are interpreted as mixed strategies of the players. This is quite a reasonable interpretation of the balancing weights in a balanced collection, indeed more reasonable than the interpretation of players distributing their time over several coalitions, which often requires the players to be "in two places at once[11]" besides requiring constant returns.

[11] See Garratt and Qin (2000).

The paper also makes a case for the nucleolus as a power index in divide-the-dollar games, especially in constant-sum homogeneous games. Each power index has its own advantages and indeed the Shapley-Shubik index can also be obtained as an equilibrium of a game with random proposers (see Hart and Mas-Colell 1996).[12] However, Hart and Mas-Colell (1996) assume bargaining under *unanimity rule*, though subcoalitions matter through the possibility of partial breakdown. It seems paradoxical that, while the Shapley value is usually interpreted as an expected payoff of playing the game which unlike the nucleolus does not presuppose the grand coalition, the opposite happens in the corresponding implementations: Hart and Mas-Colell require the grand coalition to form in order to obtain the Shapley value while this paper obtains the nucleolus as a "value" without giving the grand coalition a prominent role (indeed the grand coalition is never formed if the game is constant sum unless one of the players is a dictator).

Other papers have provided noncooperative interpretations of the nucleolus. Young (1978) relates the nucleolus to the equilibrium bribes in an asymmetric lobbying game played by two lobbyists. Potters and Tijs (1992) show that the nucleolus can be interpreted as the optimal strategy of one of the players in a matrix game played by auxiliary agents. Serrano (1997) provides a noncooperative interpretation of the kernel (and therefore of the nucleolus) based on the reduced game property (Davis and Maschler, 1965). This paper can be viewed as complementary to the existing literature. It is less general than some of the others but, especially for constant-sum homogeneous games, it is based on a very natural noncooperative game.

8 Appendix

Proof of proposition 8

Consider the game $G(v, \theta, \delta)$ with $\theta = \mu$. Since we have shown in the previous proposition that there is an equilibrium σ^* with $y(\sigma^*) = \mu$, it only remains to discard the existence of another equilibrium σ' with $y(\sigma') = y' \neq \mu$. We will adapt the proof of Yan (2002, p.533) for games with a

[12] Gul's (1989) implementation of the Shapley value does not apply to majority games.

nonempty core.

We first notice that, because of corollary 3 2), any payoff y' which differs from μ in the set of players getting 0 is automatically discarded, and y' and μ can only differ for players with $\theta_i > 0$, so we will ignore the set of players with $\theta_i = 0$ in what follows.

1. *Notation.* We will use the symbol $'$ for all the variables corresponding to equilibrium σ' (for example, r'_i will be the probability of receiving a proposal given σ'). We will use Δ to denote the change from σ^* to σ': for example $\Delta r_i : r'_i - r_i$, and $\Delta y := y'_i - y_i$ (where $y_i = \mu_i$). We will decompose the payoff i gets as a proposer by $\delta y'_i + \pi'_i$, $\pi'_i > 0$ being the advantage of being the proposer. We will also denote by $\lambda'_i := \theta_i + r_i$ the probability that i belongs to the final coalition. Then, taking into account that $\theta_i = \mu_i$, expected equilibrium payoffs can be written as

$$y'_i = \frac{\pi'_i}{1 - \lambda'_i \delta} \mu_i \tag{2}$$

   Let $Y_{++} := \{i \in N : \Delta y_i > 0\}$, $Y_{-\,-} := \{j \in N : \Delta y_i < 0\}$, $R_+ := \{i \in N : \Delta r_i \geq 0\}$ and $R_- := \{i \in N : \Delta r_i \leq 0\}$. Since $y' \neq \mu$ and expected equilibrium payoffs always add up to 1 as shown in section 3.2, these sets will be nonempty. Notice also that, because $\lambda_i = \theta_i + r_i$ and the proposer always includes himself in the coalition, $\Delta\lambda_i = \Delta r_i$ for all $i \in N$.

   Finally, we denote by C'_i the set of players to whom i proposes with positive probability in the equilibrium σ'.

2. *Yan's claims.* Claim 1 (i) in p. 533 shows that, for any two SSPE of any random proposer game, $\sum_{i \in N} \Delta y_i \Delta r_i \leq 0$.

   Claim 1 (ii), in p. 533 shows that for any nonempty $T \subseteq Y_{++} \cap R_-$, we have $\widehat{T} \neq \varnothing$, where $\widehat{T} := \cup_{i \in T} C'_i \cap Y_{-\,-}$ and

$$\sum_{i \in T} \Delta y_i < \sum_{j \in \widehat{T}} |\Delta y_j|$$

Finally, claim 2 (p. 534) shows that $\sum_{i\in I} a_i b_i < \sum_{j\in J} a_j b_j$ where I, J are finite sets and a_i, b_i, a_j, b_j nonnegative, if there exists a correspondence $B : I \to J$ such that

(i) $a_i < a_j$ for any i ,j such that $j \in B(i)$.

(ii)$\sum_T b_i < \sum_{B(T)} b_j$ for any $T \subseteq I$.

3. *The proof.* We will show that, if $\theta = \mu$ and $y = \mu$, then for a hypothetical equilibrium payoff $y' \neq \mu$ we would have $\sum_{i\in N} \Delta y_i \Delta r_i > 0$, contradicting claim 1 (i). In order to do this we will use claim 1 (ii) and claim 2, but we will have to make sure that the conditions for claim 2 to be applicable are satisfied.

   (a) We will first show that $\sum_{i\in Y_{--}} \Delta y_i \Delta r_i \geq 0$, and $\Delta r_i \leq 0$ for all i with $\Delta y_i < 0$, so that $\sum_{i\in Y_{--}} \Delta y_i \Delta r_i = \sum_{i\in Y_{--}} |\Delta y_i||\Delta r_i|$.

   Start by the set of players who belong to at least one coalition in $b_1(y')$, -we drop the reference to v for simplicity of notation- denoted by $B_1(y')$. Players in $B_1(y', v)$ will propose winning coalitions in $b_1(y', v)$ only (there is always at least one). We have $\sum_{i\in S} y'_i := \overline{y}'$ for any winning coalition $S \in b_1(y')$. Because μ is in the nucleus, $\overline{y}' \leq \overline{\mu}$ and (since $\pi_i = 1 - \delta\overline{\mu}$ and $\pi'_i = 1 - \delta\overline{y}'$) $\pi'_i \geq \pi_i$. Looking at equation (2), it follows that $\Delta\lambda_i < 0$ (and thus $\Delta r_i < 0$) for any $i \in B_1(y')$ with $\Delta y_i < 0$. In words, because i gets at least as much as a proposer as he was getting originally, the only way in which i's payoff can go down is if he becomes a responder less often.

   If there are players in $N\backslash B_1(y')$, they do not receive any proposal from players in $B_1(y')$. Given that $\sum_{i\in B_1} \theta = \sum_{i\in B_1} \mu_i \geq \overline{\mu} \geq \frac{1}{2}$, it is clear that $\lambda'_i \leq \overline{\mu}$ for all $i \in N\backslash B_1$. Because in the equilibrium with $y = \mu$ all players have $\lambda_i = \overline{\mu}$, we have $\Delta r_i \leq 0$ for all $i \in N\backslash B_1$.

   (b) Given $i \in Y_{++}$ and $j \in Y_{--} \cap C_i$, we show that $\lambda'_j < \lambda'_i$. Because $j \in C'_i$, there is a coalition $S \supseteq \{i, j\}$ which is optimal for i. Because j can always propose S, $\pi'_i = 1 - \delta \sum_{k\in S} y'_k \leq \pi'_j$.

Looking at equation (2) we see that the question of whether $\Delta y_k \geq 0$ is equivalent to the question of whether the coefficient multiplying μ_k is greater than 1. Thus in order for $i \in Y_{++}$ and $j \in Y_{-\,-}$ we need $\frac{\pi'_i}{1-\lambda'_i\delta} > 1 > \frac{\pi'_j}{1-\lambda'_j\delta}$, which since $\pi'_i \leq \pi'_j$ requires $\lambda'_j < \lambda'_i$. Because originally $\lambda_i = \lambda_j = \overline{\mu}$, this is equivalent to say $\Delta r_j < \Delta r_i$. Note that for $\Delta r_i \leq 0$ this implies $|\Delta r_j| > |\Delta r_i|$.

(c) Finally, in order to prove $\sum_{i \in N} \Delta y_i \Delta r_i > 0$, it will be sufficient to prove $\sum_{i \in Y_{++} \cap R_-} \Delta y_i |\Delta r_i| < \sum_{j \in Y_{-\,-}} |\Delta y_j||\Delta r_j|$, since by definition $\Delta y_i \Delta r_i \geq 0$ for $i \in Y_{++} \cap R_+$. We use Yan's claim 2, with $I = Y_{++} \cap R_-$ and $J = Y_{-\,-}$. The correspondence will attach to each (singleton) set T the set $\widehat{T}$ as defined in Yan's claim 1 (ii), $a_i = |\Delta r_i|$ and $b_i = |\Delta y_i|$. We have just shown that condition (i) in claim 2 holds, and condition (ii) holds because of Yan's claim 1 (ii). This completes the proof.

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