Behaviour of Dickey-Fuller Unit Root Tests Under Trend Misspecification

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Abstract

We analyse the case where a unit root test is based on a Dickey-Fuller regression whose only deterministic term is a fixed intercept. Suppose, however, as could well be the case, that the actual data generating process includes a broken linear trend. It is shown theoretically, and verified empirically, that under the $I(1)$ null and $I(0)$ alternative hypotheses the Dickey-Fuller test can display a wide range of different characteristics depending on the nature and location of the break.

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1 Introduction

Dickey-Fuller unit root tests are generally conducted through OLS estimation of regression models incorporating either an intercept or a linear trend, and on occasion there is uncertainty as to which of these specifications is appropriate, an issue considered by, for example, West (1987) and Ayat and Burridge (2000). In particular, there is concern about the consequences of an inappropriate specification. Unsurprisingly, the incorporation in an estimating model of a redundant trend term leads to a reduction in test power under the $I(0)$ alternative. More interestingly, as West (1987) has demonstrated, if a fixed trend term is incorrectly omitted, rejection probabilities are very small, irrespective of whether the $I(1)$ null or $I(0)$ alternative is true.

In this note we consider the case where the only deterministic term in the estimating model is a fixed intercept, but now allow the possibility of a *broken trend* in the data generating process (DGP). We show theoretically and empirically that the Dickey-Fuller test can display a wide range of different characteristics under both the $I(1)$ null and $I(0)$ alternative, dependent on the nature and location of the break. In particular, in the case where the DGP is $I(1)$ around a broken trend we find that rejection probabilities of the null hypothesis can be very high. In the case where the DGP is $I(0)$ around a broken trend, the null hypothesis may still be rejected very frequently. Neither of these outcomes would be anticipated from West's analysis of the case of an omitted fixed trend.

2 Trend Misspecification in the $I(1)$ Case

Consider a DGP for T observations given by

$$y_t = \begin{cases} \alpha + \beta_1 t + v_t & t \leq \tau T, \\ \alpha + \beta_1 \tau T + \beta_2 (t - \tau T) + v_t & t > \tau T \end{cases} \quad (1)$$

where

$$v_t = \rho v_{t-1} + \eta_t \quad (2)$$

with $\rho = 1$ and η_t is an IID sequence with mean zero and variance σ^2. Here, y_t is an $I(1)$ random walk process around a linear trend which changes value at observation τT.

Now suppose that

$$\begin{aligned} \beta_1 &= \sigma T^{-1/2} k_1, \\ \beta_2 &= \sigma T^{-1/2} k_2. \end{aligned} \quad (3)$$

The $t-$statistic variant of the Dickey-Fuller test, denoted DF, tests $\rho = 1$ in the fitted OLS regression model

$$y_t = \hat{\alpha} + \hat{\rho} y_{t-1} + \hat{\eta}_t \quad (4)$$

where we include an intercept but no trend term.

The following theorem gives the asymptotic null distribution of DF.

Theorem 1 *Under the DGP (1)-(3)*

$$DF \Rightarrow \frac{A}{B}$$

where

$$\begin{aligned}
A &= f_1 - f_2 f_3, \quad B = (f_4 - f_2{}^2)^{1/2}, \\
f_1 &= \frac{1}{2}\{(k_1 - k_2)\tau + k_2\}^2 + \frac{1}{2}\{W(1)^2 - 1\} + \{(k_1 - k_2)\tau + k_2\}W(1), \\
f_2 &= \frac{1}{2}\{k_1\tau(2-\tau) + k_2(1-\tau)^2\} + \int_0^1 W(r)dr, \\
f_3 &= k_1\tau + k_2(1-\tau) + W(1), \\
f_4 &= \frac{1}{3}\{k_1^2\tau^2(3-2\tau) + k_2^2(1-\tau)^3\} + k_1k_2\tau(1-\tau)^2 + \int_0^1 W(r)^2 dr \\
&\quad + 2k_1\int_0^\tau rW(r)dr + 2(k_1 - k_2)\{\tau\int_\tau^1 W(r)dr - \int_\tau^1 rW(r)dr\}.
\end{aligned}$$

Here $W(r)$ is a standard Brownian Motion process.[2,3]

Clearly the limit distribution of DF is a complicated function of k_1. k_2 and τ, but we can examine some special cases of the result that highlight the wide range of outcomes that can occur.[4] We do this by simulating the limiting functions of $W(\cdot)$, generating samples of 5,000 IID standard normal variates, over 10,000 replications. We concentrate on reporting the percentage of rejections that would be achieved by nominal 5%-level tests, based on the limiting critical values that would be appropriate for the Dickey-Fuller test if (4) were correctly specified.

(i) $k_1 = k_2$: This is the case where there is no break in trend, and so a fixed trend term has been omitted from the regression (4). This corresponds then to the situation analysed by West (1987). However, in that paper the trend magnitude was fixed, leading to convergence in probability to zero of the test statistic. Here, in view of the normalisation in (3), a proper limiting null distribution follows. We have

$$\begin{aligned}
f_1 &= \frac{1}{2}k_1^2 + \frac{1}{2}\{W(1)^2 - 1\} + k_1W(1), \\
f_2 &= \frac{1}{2}k_1 + \int_0^1 W(r)dr, \\
f_3 &= k_1 + W(1), \\
f_4 &= \frac{1}{3}k_1^2 + \int_0^1 W(r)^2 dr + 2k_1\int_0^1 rW(r)dr.
\end{aligned}$$

[2]Note that in (3) the trend magnitudes are set proportional to $T^{-1/2}$, following Leybourne and Newbold (2000). It is this approach that leads DF having a non-degenerate limiting null distribution.

[3]In the general case where η_t is generated by a stationary $AR(p)$ process $\eta_t = \sum_{j=1}^p \phi_j\eta_{t-j} + \varepsilon_t$, where ε_t is an IID sequence with mean zero and variance σ^2, the result continues to hold provided β_i, $i = 1, 2$ is defined in rescaled form as $\beta_i = \sigma(1 - \sum_{j=1}^p \phi_j)^{-1}T^{-1/2}k_i$ and (4) is augmented with p lagged terms in Δy_t.The proof of this result (and also that of Theorem 2) is straightforward and is given in Kim *et al* (2002).

[4]The most trivial special case is where $k_1 = k_2 = 0$. Here the regression model (4) is correctly specified, and the limiting null distribution of Theorem 1 of course simplifies to the usual Dickey-Fuller distribution.

As is obvious, these expressions do not depend on τ. It is also not necessary at this point to consider this case in further detail, as it corresponds precisely to extremities of two cases to be discussed later. These are $k_2 = 0, \tau = 1$ and $k_1 = 0, \tau = 0$ (though with k_2 in this latter specification playing the role of k_1 in the above equations).

(ii) $k_2 = -k_1$: The trend parameters either side of the break are equal and opposite in sign. We have

$$\begin{aligned}
f_1 &= \frac{1}{2}k_1^2(2\tau - 1)^2 + \frac{1}{2}\{W(1)^2 - 1\} + k_1(2\tau - 1)W(1), \\
f_2 &= \frac{1}{2}k_1(4\tau - 2\tau^2 - 1) + \int_0^1 W(r)dr, \\
f_3 &= k_1(2\tau - 1) + W(1), \\
f_4 &= k_1^2\{\frac{1}{3} - 2\tau(1-\tau)^2\} + \int_0^1 W(r)^2 dr \\
&\quad +2k_1 \int_0^\tau rW(r)dr + 4k_1\{\tau \int_\tau^1 W(r)dr - \int_\tau^1 rW(r)dr\}.
\end{aligned}$$

It now emerges that the value of the break fraction has a large impact on rejection probabilities. Figure 1 shows asymptotic rejection frequencies of nominal 5%-level DF tests plotted for all values of the break fraction τ, and for various values of k_1. For $\tau < 0.5$ the test is undersized. However, the picture changes for larger values of the break fraction, where very serious spurious rejections of the unit root null hypothesis can occur - the most extreme case being for $\tau \approx 0.7$. It is thus apparent that the omission of a broken trend can have quite different consequences from the omission of an unbroken trend; the latter unambiguously results in an undersized test.

(iii) $k_2 = 0$: This is the case where there is an omitted trend in the early part of the series, but none following the break. Here

$$\begin{aligned}
f_1 &= \frac{1}{2}k_1^2\tau^2 + \frac{1}{2}\{W(1)^2 - 1\} + k_1 W(1), \\
f_2 &= \frac{1}{2}k_1\tau(2 - \tau) + \int_0^1 W(r)dr, \\
f_3 &= k_1\tau + W(1), \\
f_4 &= \frac{1}{3}k_1^2\tau^2(3 - 2\tau) + \int_0^1 W(r)^2 dr \\
&\quad +2k_1\{\int_0^\tau rW(r)dr + \tau \int_\tau^1 W(r)dr - \int_\tau^1 rW(r)dr\}.
\end{aligned}$$

Asymptotic rejection frequencies of nominal 5%-level DF statistics are shown in Figure 2. The test over-rejects the unit root null hypothesis (except for very large values of τ), most severely so for $\tau \approx 0.4$. In Figure 2, $\tau = 1$ corresponds to an omitted unbroken trend and reduction in size caused by this omission is clear.

(iv) $k_1 = 0$: This is the case where there is initially no trend, but the series trends after the break. We have

$$
\begin{aligned}
f_1 &= \frac{1}{2}k_2^2(1-\tau)^2 + \frac{1}{2}\{W(1)^2 - 1\} + k_2(1-\tau)W(1), \\
f_2 &= \frac{1}{2}k_2(1-\tau)^2 + \int_0^1 W(r)dr, \\
f_3 &= k_2(1-\tau) + W(1), \\
f_4 &= \frac{1}{3}k_2^2(1-\tau)^3 + \int_0^1 W(r)^2 dr \\
&\quad -2k_2\{\tau\int_\tau^1 W(r)dr - \int_\tau^1 rW(r)dr\}.
\end{aligned}
$$

Figure 3 shows asymptotic rejection frequencies of nominal 5%-level DF tests. The contrast with Figure 2 is rather dramatic as now the test under-rejects the null for all values of τ. So, even though this case might be thought of as being similar to the previous one, its consequences for DF are actually very different. Note again that here $\tau = 0$ corresponds to the case where there is an omitted unbroken trend.

3 Trend Misspecification in the $I(0)$ Case

Suppose that y_t is generated via (1) and (2) but now $|\rho| < 1$, so that v_t is assumed to follow a stationary $AR(1)$ process. As in West (1987), β_1 and β_2 are now assumed fixed (not depending on the sample size) and defined as

$$
\begin{aligned}
\beta_1 &= \sigma k_1, \\
\beta_2 &= \sigma k_2.
\end{aligned} \tag{5}
$$

Then we have the following result for the large sample behaviour of DF.

Theorem 2 *Under the DGP (1), (2) and (5)*

$$T^{-1/2}DF \xrightarrow{p} \frac{A^*}{B^*}$$

where

$$
\begin{aligned}
A^* &= -\frac{1}{2}(k_1 - k_2)g_1, \quad B^* = g_2^{1/2}(g_3 - g_4{}^2)^{1/2}, \\
g_1 &= \tau(1-\tau)\{\tau k_1 + (1-\tau)k_2\}, \\
g_2 &= 2(1-\rho) - \frac{1}{4}\frac{(k_1-k_2)^2 g_1^2}{(g_3 - g_4{}^2)} - \{\tau k_1 + (1-\tau)k_2\}^2 + \tau k_1^2 + (1-\tau)k_2^2, \\
g_3 &= \frac{1}{3}(k_1-k_2)(k_2-2k_1)\tau^3 + (k_1-k_2)^2\tau^2 + (k_1-k_2)k_2\tau + \frac{1}{3}k_2^2, \\
g_4 &= -\frac{1}{2}(k_1-k_2)\tau^2 + (k_1-k_2)\tau + \frac{1}{2}k_2.
\end{aligned}
$$

Though the probability limit is a complicated function of the parameters involved, the main issue of interest here is simply whether DF diverges to $+\infty$ or $-\infty$, that is, whether the test has asymptotic power of 0% or 100% and this is determined by the sign of the numerator term, A^*. It is straightforward to establish that $A^* > 0$ in the region $P = P_1 \cup P_2$ where

$$\begin{aligned} P_1 &= \{(k_1, k_2) : k_2 < k_1 \text{ and } k_2 < -\frac{\tau}{(1-\tau)}k_1\}, \\ P_2 &= \{(k_1, k_2) : k_2 > k_1 \text{ and } k_2 > -\frac{\tau}{(1-\tau)}k_1\}. \end{aligned}$$

and $A^* < 0$ in the region $N = N_1 \cup N_2$ where

$$\begin{aligned} N_1 &= \{(k_1, k_2) : k_2 < k_1 \text{ and } k_2 > -\frac{\tau}{(1-\tau)}k_1\}, \\ N_2 &= \{(k_1, k_2) : k_2 > k_1 \text{ and } k_2 < -\frac{\tau}{(1-\tau)}k_1\}. \end{aligned}$$

Again we can consider the special cases of this result corresponding to (i)-(iv) of the previous section.

(i′) $k_1 = k_2$: We have the same conclusion as in West (1987) that $DF \xrightarrow{p} 0$ in the omitted unbroken trend case.

(ii′) $k_2 = -k_1$: Here we find $\{k_2 = -k_1\} \subset P$ (N) for $\tau < 0.5$ $(\tau > 0.5)$ so that DF has asymptotic power of 0% for $\tau < 0.5$ and 100% for $\tau > 0.5$.

(iii′) $k_2 = 0$: In this case $\{k_2 = 0\} \subset N$, such that DF has asymptotic power of 100% for all τ.

(iv′) $k_1 = 0$: Here $\{k_1 = 0\} \subset P$, so DF has asymptotic power of 0% for all τ. Again then, even though this might be thought of as similar to (iii′), its consequences for DF are completely different.

4 Finite Sample Simulation Evidence

As a check on the reliability of the predictions of Theorems 1 and 2 in finite samples, we conducted a small simulation exercise for the situation where $k_2 = -k_1$.

In the $I(1)$ case, we generated 10,000 replications from the DGP (1)-(3) with η_t standard normal and $k_1 = 6$. Figure 1 suggests that the most serious spurious rejections of the unit root null are liable to occur for values of the break fraction τ around 0.7. Table 1 gives the rejection percentages of DF at nominal 5%-level. Here we see that, although the convergence is a little slow, the asymptotic result yields a reliable predictor of what will be found, both qualitatively and quantitatively, in moderate-sized samples.[5]

For the $I(0)$ case, we generated the DGP (1), (2) and (5) with $k_1 = 0.6$ and $\rho = 0.95$. The results are given in Table 2. Again, our asymptotic results are a good indicator of what will be

[5]The enties for $T = \infty$ are taken from Figure 1.

found in moderate-sized samples; the lack of any power when $\tau < 0.5$, and consistency of DF when $\tau > 0.5$ are both quite evident.

Finally, it should be noted that our results are specific to Dickey-Fuller tests and do not necessarily apply to other test procedures. For example, for series of $T = 100$ observations from the DGP of Table 1, we found for the test of Pantula *et al* (1994) (based on weighted symmetric estimation) and the test of Elliott *et al* (1996) (based on GLS demeaning) virtually no rejections for nominal 5%-level tests. The same conclusion was obtained for series of this length for these alternative tests for the generating process with $k_1 = 6, k_2 = 0$. However, for DF applied to data from such a process we found rejection rates of 32.2%, 44.3% and 31.9% for respective values of τ of 0.2, 0.4., and 0.6 - in close agreement with the asymptotic results of Figure 2.

5 An Empirical Example

As a simple empirical illustration of one of our results, we consider monthly data on the United States M1 money stock for the period 1991.01-2002.12 ($T = 120$). We assess the properties of the log of this series, denoted y_t, using Vogelsang and Perron's (1998) additive outlier test procedure. Defining the dummy variable $d_t(\bar{\tau}) = (t - \bar{\tau}T)1[t > \bar{\tau}T]$ we fit via OLS the following model permitting a break in trend

$$y_t = \hat{\alpha} + \hat{\beta}_1 t + \hat{\beta}_2 d_t(\bar{\tau}) + \hat{v}_t. \tag{6}$$

for $0.15 \leq \bar{\tau} \leq 0.85$. The estimated trend breakpoint is then $\hat{\tau} = \arg\max |t_{\hat{\beta}_2}(\bar{\tau})|$ where $t_{\hat{\beta}_2}(\bar{\tau})$ is the $t-$statistic for testing $\beta_2 = 0$. After estimating (6) with $\hat{\tau}$ in place of $\bar{\tau}$, the unit root test is the $t-$statistic for testing $\rho = 0$ in the model

$$\hat{v}_t = \hat{\rho}\hat{v}_{t-1} + \sum_{j=1}^{p} \hat{\phi}_j \Delta \hat{v}_{t-j} + \hat{\varepsilon}_t. \tag{7}$$

For (6) we obtained the values

$\hat{\alpha}$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_1 + \hat{\beta}_2$	$\hat{\tau}$
6.6961	0.00941	-.01000	-0.000 59	0.36
(1238.9)	(47.97)	(-41.48)	(-1.41)	

($t-$statistics given in parentheses) indicating change from a significant to insignificant trend at observation $\hat{\tau}T = 43$ (1993.12). The unit root test (7) yielded a value of -2.01 (with $p = 1$, chosen from downwards testing at the 10%-level from $p_{\max} = 4$), where the 10%-level null critical value is -4.08. This analysis therefore suggests that this series might be characterized by special case (iii) of Theorem 1, that is, an $I(1)$ process with an early trend component (present up to fraction 0.36 of the series) but no trend thereafter. For such a process, Figure 2 would then predict that a Dickey-Fuller test which incorporates only an intercept term will spuriously reject the unit root null hypothesis. In fact, we find here that such a statistic (with $p = 1$ selected as above) yields a value of -3.01, which easily rejects the unit root null at the 5%-level (critical value -2.89).

6 Summary

In this note we have shown that in the presence of a broken trend, the behaviour of a Dickey-Fuller unit root test based only on a fitted intercept is highly unpredictable. Whether such a test is badly undersized or yields severe spurious rejections in the $I(1)$ case, has trivial or substantial power in the $I(0)$ case, all depends crucially on the nature, location and magnitude of the break. This is in stark contrast to the situation of an unattended fixed trend, as in this case the Dickey-Fuller test unambiguously displays under-sizing and lack of power. Indeed, it is this very feature that helps "identify" an omitted trend - if a test with an intercept does not reject the null, and one including an additional trend does reject the null, an informal decision rule is to reject the unit root null in favour of $I(0)$ about a fixed trend. In the case of a broken trend, however, our results demonstrate that such informal rules will not operate and hence they highlight the need for a rigorous approach to determining the trend properties of a series when testing for a unit root. We conjecture that a sequential trend modelling strategy, such as that advocated by Ayat and Burridge (2000) if extended to allow for a break in trend, might be fruitfully employed in this situation.

Table 1. $I(1)$ case: size of DF for nom. 5%-level tests ($k_2 = -k_1$; $k_1 = 6.0$).

	T			
τ	100	200	400	∞
0.5	3.3	4.5	4.8	5.4
0.6	17.2	21.1	23.2	25.3
0.7	36.2	41.0	43.4	46.7
0.8	31.4	34.3	36.8	38.7
0.9	9.4	10.3	11.1	10.9

Table 2. $I(0)$ case: power of DF for nom. 5%-level tests ($k_2 = -k_1$; $k_1 = 0.6$; $\rho = 0.95$).

	T		
τ	100	200	400
0.3	0.0	0.0	0.0
0.4	0.0	0.0	0.0
0.5	1.1	0.0	0.0
0.6	2.9	14.1	99.6
0.7	30.2	99.7	100.

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