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Maria Montero and Juan J. Vidal-Puga

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Karina Whitehead
Centre for Decision Research and Experimental Economics
School of Economics
University of Nottingham
University Park
Nottingham
NG7 2RD
Tel: +44 (0) 115 95 15620
Fax: +44 (0) 115 95 14159
karina.whitehead@nottingham.ac.uk

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Demand bargaining and proportional payoffs in legislatures*

Maria Montero† Juan J. Vidal-Puga‡

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Abstract

We study a majoritarian bargaining model in which the parties make payoff demands in decreasing order of voting weight. If the game is constant-sum and homogeneous, the unique subgame perfect equilibrium is such that the minimal winning coalition of the players who move first forms and payoffs are proportional to the voting weights.

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†Corresponding author. School of Economics, University of Nottingham, University Park, Nottingham NG7 2RD, United Kingdom. Phone + 44 115 9515468, fax +44 115 951 4159, e-mail: maria.montero@nottingham.ac.uk.

‡Research Group of Economic Analysis (RGEA) and Department of Statistics and Operations Research, University of Vigo. Facultad de Ciencias Sociales, Campus A Xunqueira, 36005 Pontevedra, Spain. Phone +34 986 802014, fax +34 986 813511, e-mail vidalpuga@uvigo.es.

1 Introduction

In a parliamentary democracy, many important decisions including government formation are the outcome of bargaining between the parties in Parliament. The most influential model of legislative bargaining is the closed rule model of Baron and Ferejohn (1989), which is based on Rubinstein (1982) and Binmore (1987).[1] In this model, a party is randomly recognized to propose a complete distribution of ministerial payoffs and the remaining parties can accept or reject the proposal. This model has some properties that may be perceived as drawbacks: the proposer has a large advantage (he receives more than half of the total payoff under simple majority), and there is a multiplicity of subgame perfect equilibria. In order to single out a unique prediction, the stationary equilibrium is selected. Stationary strategies are simple but by no means uncontroversial: a stationary strategy requires a party to always make the same proposal regardless of the history of the negotiations so far. Moreover, Norman (2002) shows that sharp predictions using stationarity are only possible in the infinite horizon version of the model: in the finite horizon version there is a continuum of equilibria, all of them with history-independent strategies.

An alternative model of legislative bargaining by Morelli (1999) is based not on complete proposals but on demands.[2] Parties make individual de-

[1] The Baron-Ferejoh model has led to many applications and extensions. Recent papers based on this model include Banks and Duggan (2000), Diermeier et al. (2003) and Jackson and Moselle (2003).

[2] There have been other demand bargaining models in the literature. Binmore (1985) presents a three-player “market model” where demands are carried over to the next round and infinite plays are possible. Selten (1992) presents a general but relatively complicated model, including random draws and costs of both formulating a demand and forming a coalition. Bennett and van Damme (1991) study a simpler version in which each player selects the next one to move, and show that there may be a multiplicity of subgame perfect equilibria. Using a refinement, they select the proportional payoff division for apex games. Winter (1994), Dasgupta and Chiu (1998), and Vidal-Puga (2004) use various demand commitment procedures to implement the Shapley value in convex games.

mands for ministerial payoffs and a coalition emerges between parties making compatible demands. The Head of State chooses the first mover, and the latter chooses the order in which the parties formulate demands. Because the first mover chooses the order of moves, it may be able to play the remaining parties off against each other and obtain the whole payoff, even though the rules of the game allow the other parties to exclude the first mover (see Montero and Vidal-Puga (2006)). In this paper we study a modified bargaining procedure in which the parties must move in decreasing order of voting weight, and show that equilibrium payoffs inside the coalition that forms are proportional to the weights. The first mover has no disproportionate advantage and no refinements of subgame perfect equilibrium are needed to obtain the result.

2 The model

2.1 Weighted majority games

Consider a legislature in which n parties are represented. We denote these as $N = \{1, 2, ..., n\}$. There is a budget of size 1 to be divided by majority rule. Each party i has ω_i votes, and a quota of q is needed for a majority. The pair $[q; (w_i)_{i \in N}]$ is a *weighted majority game.* Notice that the game is not affected if weights and quota are multiplied by the same positive constant.

Given a vector $x \in \mathbb{R}^N$ and a coalition $S \subset N$, we denote as x_S the sum of the coordinates of the members of S, $x_S := \sum_{i \in S} x_i$.

A coalition $S \subset N$ is *winning* if $\omega_S \geq q$; it is *minimal winning* if it is winning and no $T \subsetneq S$ is winning. We denote as $\Omega(\omega)$ the set of all winning coalitions, and as $\Omega^m(\omega)$ the set of all minimal winning coalitions. A *dummy player is a* player who does not belong to any minimal winning coalition.

A weighted majority game is *constant-sum* if $S \in \Omega(\omega) \iff N \backslash S \notin \Omega(\omega)$ for all S. It admits an equivalent *homogeneous representation* if there exists a vector of votes $\left(\omega_1^h, ..., \omega_n^h\right)$ and a quota q^h such that $\Omega^m(\omega) = \Omega^m\left(\omega^h\right) = \left\{S \subset N : \omega_S^h = q^h\right\}$. A weighted majority game that admits an equivalent homogeneous representation is called a *homogeneous game*.

Homogeneous representations do not always exist and when they exist they may not be unique. For example, $[5; 3, 2, 2, 1]$ and $[7; 4, 3, 3, 1]$ are two homogeneous representations of the same game. Peleg (1968) shows that constant-sum homogeneous games have a unique homogeneous representation (up to multiplication by a positive constant and to the weight that is assigned to dummies, which may be 0 or a sufficiently small number).

2.2 The bargaining procedure

Let $[q; (w_i)_{i\in N}]$ be a constant-sum homogeneous weighted majority game. There is a budget of size 1 to divide. Party i's utility function is $u_i = x_i$, where x_i is i's share of the budget. Bargaining proceeds as follows. Parties move in decreasing order of weight. We label the parties in this order, so that party 1 moves first, followed by party 2, etc.

Each party i makes a demand d_i, following the order of play, where $d_i \in [0, 1]$ is the share of the budget party i claims. If, after party i makes its demand, there exists a winning coalition $S \subset \{j : j \leq i\}$ such that $d_S \leq 1$, party i has the additional choice of forming coalition S, in which case payoffs are distributed according to the demands made. If there is more than one possible S, party i decides which one is formed. If party n forms no coalition, the game ends with each party getting zero.[3]

Given $i \in N$, we denote as P_i the set of predecessors of i. Namely:

$$P_i := \{j \in N : j < i\}.$$

As it will become clear from the analysis, dummy players must get 0 in equilibrium, so for simplicity we assume there are no dummy players. We will

[3] Alternatively, we may assume a finite number of bargaining rounds T without affecting the results.

use the homogeneous representation with $\omega_n = 1$; i.e. the weakest party has exactly 1 vote. Under these circumstances, every party in a constant-sum homogeneous game has a positive integer number of votes. Furthermore:

Lemma 1 *Let $[q; (w_i)_{i \in N}]$ be a constant-sum homogeneous game. Then, $\omega_N = 2q - 1$.*

Proof. Because n is not a dummy player, there exists $S \in \Omega^m(\omega)$ such that $n \in S$. Homogeneity implies $\omega_S = q$. Because $S \in \Omega^m(\omega)$, $S \backslash \{n\}$ must be losing. Since the game is constant-sum, $(N \backslash S) \cup \{n\} \in \Omega(\omega)$. Moreover, by deleting the weakest party (i.e. party n) we obtain a losing coalition $N \backslash S$. Thus, $(N \backslash S) \cup \{n\} \in \Omega^m(\omega)$. So, $\omega_{(N \backslash S) \cup \{n\}} = q$ and $\omega_{N \backslash S} = q - 1$. Hence

$$\omega_N = \omega_S + \omega_{N \backslash S} = q + q - 1 = 2q - 1.$$

■

Corollary 1 *Let $[q; (w_i)_{i \in N}]$ be a constant-sum homogeneous game. Then, S is maximal losing (i.e. $N \backslash S \in \Omega^m(\omega)$) iff $\omega_S = q - 1$.*

Proof. Since (N, v) is constant-sum and homogeneous, S is maximal losing iff $N \backslash S \in \Omega^m(\omega)$, which means $\omega_{N \backslash S} = q$ and thus, under Lemma 1,

$$\omega_S = \omega_N - \omega_{N \backslash S} = 2q - 1 - q = q - 1.$$

■

Lemma 2 *Let $[q; (w_i)_{i \in N}]$ be a weighted majority game. Then, there is a party i such that $P_{i+1} \in \Omega^m(\omega)$.*

Proof. Suppose this was not the case. Consider the smallest index i such that $S = \{1, ..., i\}$ is a winning coalition. There is a minimal winning coalition $S' \subset S$, and S' is obtained from S by deleting at least one party $j < i$. However, this is impossible because by assumption $\{1, ..., i-1\}$ is a losing coalition, and, since $w_j \geq w_i$ for all $j < i$, this coalition has at least as many votes as S'. ■

Lemma 2 does not hold for arbitrary orders of the parties. For example, if we take the game $[3; 2, 1, 1, 1]$ and order the parties in such a way that the party with 2 votes is in the third place, no set of parties $\{1, ..., i\}$ is a minimal winning coalition. If the parties play the game in this order, the party that moves first cannot get a positive payoff for any demand, and this leads to a continuum of subgame perfect equilibria.

Theorem 1 *Let $[q; (w_i)_{i \in N}]$ be a constant-sum homogeneous game. Suppose parties play a demand commitment game in decreasing order of weight. Then in any subgame perfect equilibrium the minimal winning coalition of Lemma 2 forms with each party i demanding $\frac{\omega_i}{q}$.*

Proof. See Appendix . ■

The equilibrium strategies are roughly as follows (for a formal description see Appendix). Given the demands of the parties that have moved so far, party i determines two things: the optimal coalition to be (eventually) formed and the optimal demand to make.

In general, the optimal coalition S will control exactly q votes. This coalition will generally include some parties that have moved before i, as well as some parties moving after i. Since $T = S \cap P_i$ is a group of parties that have already formulated a demand, $1 - d_T$ is the benefit from buying the votes of the parties in T; this benefit will be shared by the parties in $S \backslash T$. Buying less votes leads to a higher benefit, but more votes from parties moving after i will be needed to complete a winning coalition. The coalition S is chosen such that the *average benefit per vote*, $\frac{1-d_T}{q-\omega_T}$, is maximized.

The optimal demand for party i will normally be $d_i = \omega_i \frac{1-d_T}{q-\omega_T}$, that is, party i will claim a share of the benefit proportional to its number of votes. Only in some subgames outside the equilibrium path can party i demand more than a proportional share.

Below we present a worked out example.

Example 1 *Suppose there are five parties, with* $3, 2, 2, 1$ *and* 1 *votes respectively, and the quota is* 5. *There is a unique subgame perfect equilibrium of the demand commitment game, in which coalition* $\{1, 2\}$ *forms with* $d_1 = \frac{3}{5}$ *and* $d_2 = \frac{2}{5}$.

Proof. We proceed by backward induction.

At stage 5, party 5 faces a vector of demands (d_1, d_2, d_3, d_4). It has three choices:

a) Form coalition $\{1, 4, 5\}$ and get $1 - d_1 - d_4$.

b) Form coalition $\{2, 3, 5\}$ and get $1 - d_2 - d_3$.

c) Form no coalition and get 0.

Suppose forming some coalition is optimal. Then party 5 will form coalition $\{1, 4, 5\}$ if $1 - d_1 - d_4 \geq 1 - d_2 - d_3$, or $d_4 \leq d_2 + d_3 - d_1$. Ties are broken in favor of forming the coalition that includes party 4, to guarantee that party 4 has a best response in the previous stage. Hence the maximum demand 4 can make and still get into a coalition with 5 is $d_4 = d_2 + d_3 - d_1$.

At stage 4, party 4 faces a vector of demands (d_1, d_2, d_3). It can form coalition $\{2, 3, 4\}$ or make a demand that will lead to $\{1, 4, 5\}$. It forms $\{2, 3, 4\}$ if $1 - d_2 - d_3 \geq d_2 + d_3 - d_1$, or

$$1 - d_2 - d_3 \geq \frac{1 - d_1}{2}.$$

Thus, party 4 is effectively comparing the average benefit associated to buying the votes of 2 and 3 (in which case 1 vote is enough to complete a winning coalition) or the votes of 1 (in which case 2 votes are needed to complete a winning coalition and 4 must share the benefit with 5).

From the inequality above, the maximum demand party 3 can make at the previous stage and still induce $\{2, 3, 4\}$ is

$$d_3 = \frac{1 - 2d_2 + d_1}{2}.$$

At stage 3, party 3 faces a vector of demands (d_1, d_2). It can form coalition $\{1, 3\}$ or make a demand that will induce $\{2, 3, 4\}$. It makes a demand if $\frac{1-2d_2+d_1}{2} \geq 1-d_1$ or

$$\frac{1-d_2}{3} \geq \frac{1-d_1}{2}.$$

Again, party 3 may buy the votes of party 1 (in which case 2 votes are required to complete a winning coalition), or the votes of party 2 (in which case 3 votes are required to complete a winning coalition). It chooses the alternative with the highest average benefit.

The maximum demand party 2 can make in the previous stage and still induce coalition $\{2,3,4\}$ is

$$d_2 = \frac{3d_1-1}{2}.$$

At stage 2, party 2 compares $1-d_1$ and $\frac{3d_1-1}{2}$. It forms $\{1,2\}$ if $\frac{3d_1-1}{2} \leq 1-d_1$, or $d_1 \leq \frac{3}{5}$. This inequality can be rewritten as $\frac{1-d_1}{2} \geq \frac{1}{5}$ (where $\frac{1}{5}$ is the average benefit of buying no votes).

Anticipating this, party 1 sets $d_1 = \frac{3}{5}$. Party 2 will then set $d_2 = \frac{2}{5}$ and coalition $\{1,2\}$ is formed. ■

3 Concluding remarks and discussion

We have presented a demand bargaining model that makes sharp predictions regarding coalition formation and payoff division. The model can be extended to any finite horizon, and its predictions are independent of the discount factors and the risk attitudes of the parties.

The proportional payoff prediction of our model is intuitive in the absence of policy preferences. Proportional payoffs are also predicted by many solution concepts like von Neumann-Morgenstern's (1944) main simple solution, the set of balanced aspirations (Cross, 1967), the competitive solution (McKelvey et al., 1978) and the demand bargaining set (Morelli and Montero, 2003). Those cooperative solution concepts also require the game to be homogeneous and constant-sum in order for payoffs to be proportional.[4]

[4]Proportionality results can also be obtained in the context of the Baron-Ferejohn

The coalition that forms is the minimal winning coalition with the smallest number of parties. If parties are asymmetric, the smaller parties are never part of the government. One may ask whether proportional payoffs can be achieved for an arbitrary minimal winning coalition by choosing the order of moves appropriately. The answer is negative: for the game $[5; 3, 2, 2, 1, 1]$, there is no order of moves for which coalition $\{1, 4, 5\}$ forms with a proportional payoff division. There are three types of possible orders for which the parties in this coalition move first: [31122], [13122] and [11322]. It can be shown that the first mover gets the whole budget in order [31122], whereas in the other two orders the first mover gets half of the budget.

If the game is not constant-sum and homogeneous, proportionality may break down. In some cases, this is due to the presence of a party that can be "held hostage" by others, as pointed by Morelli (1999).

Example 2 *There are four parties, with* $3, 2, 2$ *and* 1 *votes respectively. The quota is* 5*. If the parties play a demand commitment game in decreasing order the unique subgame perfect equilibrium results in coalition* $\{1, 2\}$ *with* $d_1 = \frac{1}{2}$ *and* $d_2 = \frac{1}{2}$*.*

Party 4 is helpless because there is only one minimal winning coalition it can form. Knowing this, party 3 will either form a coalition with 1 and get $1-d_1$, or set $d_3 = 1-d_2$. Party 2 can then form a coalition with 1 (obtaining $1-d_1$) or set $d_2 = d_1$ and induce coalition $\{2, 3, 4\}$. Anticipating this, party 1 sets $d_1 = \frac{1}{2}$. The game $[5; 3, 2, 2, 1]$ has many homogeneous representations, but in none of them do parties 1 and 2 have the same number of votes.

Proportionality can break down even if no party can be held hostage by others, as the following example illustrates.

Example 3 *Consider the game* $[7; 4, 3, 2, 2, 1, 1]$*. If the parties play a demand commitment game in decreasing order, the unique subgame perfect*

model (see Montero, 2006). However, this proportionality is *ex ante* (ex post the proposer obtains more than half of the total payoff) and in order to hold generally it requires the recognition probabilities to be themselves proportional.

equilibrium results in coalition $\{1,2\}$ *with* $d_1 = d_2 = \frac{1}{2}$.

The game above is constant-sum but not homogeneous. None of the parties can be held hostage by others: given any two parties, each of them can form a minimal winning coalition that does not include the other. Moreover, coalition $\{1,2\}$ has exactly 7 votes. Nevertheless, proportionality fails because $\{1,3,4\}$ and $\{2,3,4\}$ are both minimal winning coalitions. From the point of view of parties 3 and 4, parties 1 and 2 are equally valuable even though they have a different number of votes. If the turn reaches party 3, which of the two coalitions forms will depend on whether d_1 is higher or lower than d_2. Anticipating this, party 2 has two options: it can form a coalition with 1 and get $1 - d_1$, or set $d_2 = d_1$ and induce coalition $\{2,3,4\}$. Party 2 will form a coalition if $1 - d_1 \geq d_1$, or $d_1 \geq \frac{1}{2}$.

4 Appendix: Proof of Theorem 1

The result trivially follows if there is a veto player. In constant-sum games, a veto player must be a dictator, thus $\omega_1 = q$, and $d_1 = 1$ would be the equilibrium outcome. We will assume from now on that $\omega_i < q$ for all i.

We denote as $\mathbb{B}(d,i)$ with $i \in N$ and $d \in \mathbb{R}^{P_i}$ the subgame which begins when it is party i's turn, facing a vector d of demands. At subgame $\mathbb{B}(d,i)$, party i will determine the optimal winning coalition $S \ni i$ to be formed, and will formulate a demand d_i that will lead to S being formed. We will show how party i determines which coalition is optimal as well as how the optimal coalition can be induced by the choice of d_i.

Suppose we are in $\mathbb{B}(d,i)$, and party i plans to make a demand in the belief that a coalition $S \in \Omega(\omega)$ with $i \in S$ will be formed. This coalition should include some parties from $N \backslash P_i$ (party i and possibly parties that move after it) and may also include some predecessors from P_i. Let α be the number of votes controlled by parties in $S \cap (N \backslash P_i)$. Then, the parties in $S \cap P_i$ should control at least $q - \alpha$ votes. We denote as $b(i,\alpha)$ the maximum benefit that can be achieved by buying these $q-\alpha$ votes from parties in P_i.

$$b\,(i,\alpha) := \max\left\{1-d_T : T \subset P_i, \omega_T \geq q-\alpha\right\}.$$

Party i can calculate $b(i,\alpha)$ for every feasible value of α. Notice that not all integers between 0 and q are feasible for every player. First, α cannot be so small that even the votes of all the parties in P_i would not suffice. Let

$$\gamma_0^i := q - \omega_{P_i}.$$

In order for $b(i,\alpha)$ to exist we need $\alpha \geq \gamma_0^i$.

Since party i must be in S, it seems reasonable to require $\alpha \geq \omega_i$ as well. The next lemma shows that this is unnecessary: there is no positive benefit from buying more than $q-\omega_i$ votes.

Lemma 3 *Let $\gamma_0^i \leq \alpha < \omega_i$ and assume no party $j < i$ has made a strictly dominated choice of d_j. Then, $b\,(i,\alpha) \leq 0$. Moreover, $b\,(i,\alpha) = 0$ implies $b\,(i,\omega_i) \geq 0$.*

Proof. Let $T \subset P_i$ such that $\omega_T \geq q-\alpha$. Since $\alpha < \omega_i$, we have

$$\omega_{T\cup\{i\}} = \omega_T + \omega_i > \omega_T + \alpha \geq q.$$

Hence, since the game is homogeneous, $T \cup \{i\}$ cannot be a minimal winning coalition. Moreover, party i is the party with less votes in $T \cup \{i\}$, thus coalition T should be winning. This means that either $d_T \geq 1$ (implying $b(i,\alpha) \leq 0$) or $d_T < 1$, in which case the smallest party in T (party j) would have been strictly better-off by setting a higher demand and forming a coalition, regardless of the actions of the parties moving after j.

Moreover, when $b\,(i,\alpha) = 0$, $b\,(i,\omega_i) \geq 0$ follows from the fact that $b\,(i,\cdot)$ is nondecreasing in the second variable. ∎

We will eliminate strictly dominated strategies, thus in all the subgames we study it will be the case that $b(i,\alpha) \leq 0$ for $\gamma_0^i \leq \alpha < \omega_i$. Otherwise the turn would never have reached party i.

Since there is no positive benefit from buying more than $q - \omega_i$ votes, and (given that there is no benefit left to be divided) the particular value of α is irrelevant if $b(i, \alpha) = 0$, any lower bound between 0 and ω_i can be equivalently used by party i. We take α to be greater or equal to:[5]

$$\gamma^i := \max\left\{1, \gamma_0^i\right\}.$$

Moreover, party i is constrained by the number of votes owned by parties in $N \backslash P_i$. Thus, α must be smaller or equal to

$$\delta^i := \omega_{N \backslash P_i}.$$

Notice that $\delta^{i+1} = \delta^i - \omega_i$ for all $i < n$. Also, $\omega_i < q$ implies $\gamma_0^i \leq \delta^{i+1}$. It follows from lemma 4.9 in Ostmann (1987) that $\omega_i \leq \delta^{i+1}$ for all $i < n$, thus $\gamma^i \leq \delta^{i+1}$ for all $i < n$.

For party 1 only $\alpha \geq q$ is feasible and $b(1, \alpha) = 1$ for all $\alpha \geq q$. For party n, only $\alpha = 1$ is feasible and $b(n, 1)$ is simply n's payoff from buying the votes of one of the cheapest coalitions controlling at least $q - 1$ votes.

The following lemma shows how $b\left(i+1, \alpha\right)$ is determined from $b(i, \cdot)$ and d_i. It may be the case that, having α votes in its pocket, party $i+1$ cannot form a winning coalition without party i. Then $b\left(i+1, \alpha\right) = b\left(i, \alpha + \omega_i\right) - d_i$ irrespective of d_i. Otherwise party $i+1$ will compare the best coalition that includes i with the best coalition that does not include i. Given that i is included in the coalition, $i+1$ needs to buy the remaining votes $(q - (\alpha + \omega_i))$ from P_i, and the best way to do this leads to a benefit of $b(i, \alpha + \omega_i)$; after paying d_i, there would be $b\left(i, \alpha + \omega_i\right) - d_i$ left. Without party i, the maximum benefit from buying $q - \alpha$ votes without buying i's votes is precisely $b\left(i, \alpha\right)$. Party i will then be included if d_i is sufficiently low.

Whether d_i is sufficiently low depends on the demands of the parties in P_i. Because parties may be complements, in some cases no positive demand by i would be low enough, as the following example illustrates.

[5] A lower bound of 1 has the advantage of being independent of i and allowing division by all values of α, but the proof can be adapted to any other choice.

Consider the game $[10; 7, 3, 3, 3, 1, 1, 1]$. Let $i = 3$, $i + 1 = 4$. We have $b(3,7) = \max(1 - d_1, 1 - d_2)$ and $b(3,4) = 1 - d_1$. Having 7 votes in its pocket, party 3 may buy the votes of either party 1 (with a benefit of $1 - d_1$) or party 2 (with a benefit of $1 - d_2$). On the other hand, having only 4 votes, party 3 must buy the votes of party 1, with a benefit of $1 - d_1$.

If party 4 wants to compute $b(4,4)$ it compares $1 - d_1$ and $1 - d_2 - d_3$. Thus in this particular case parties 2 and 3 are complements. If d_3 is high, then $b(4,4) = 1 - d_1$, which is precisely $b(3,4)$. If *both* d_3 and d_2 are sufficiently low, then $b(4,4) = 1 - d_2 - d_3$ and $b(3,7) = 1 - d_2$, hence $b(4,4) = b(3, 4 + \omega_3) - d_3$. If $d_2 > d_1$, no positive value of d_3 is sufficiently low.

Lemma 4 *Assume we are in* $\mathbb{B}(d, i+1)$. *Let* α *such that* $\gamma_0^{i+1} \le \alpha \le \delta^{i+1}$. *Then* $\gamma_0^i \le \alpha + \omega_i \le \delta^i$ *and furthermore:*

a) if $\alpha < \gamma_0^i$, *then* $b(i+1, \alpha) = b(i, \alpha + \omega_i) - d_i$*;*

b) if $\alpha \ge \gamma_0^i$, *then* $b(i, \alpha)$ *exists and*

$$b(i+1, \alpha) = \max\{b(i, \alpha), b(i, \alpha + \omega_i) - d_i\}.$$

Proof. We have to prove that $\gamma_0^i \le \alpha + \omega_i \le \delta^i$. It is straightforward:

$$\alpha \le \delta^{i+1} \Longrightarrow \alpha + \omega_i \le \delta^{i+1} + \omega_i = \delta^i.$$

$$\alpha \ge \gamma_0^{i+1} = q - \omega_{P_{i+1}} \Longrightarrow \alpha + \omega_i \ge q - \omega_{P_{i+1}} + \omega_i = q - \omega_{P_i} = \gamma_0^i.$$

a) If $\alpha < \gamma_0^i$, every $T \subset P_{i+1}$ with $\omega_T \ge q - \alpha$ satisfies $i \in T$. Then:

$$\begin{aligned} b(i+1, \alpha) &= \max_{T \subset P_{i+1} : \omega_T \ge q - \alpha} (1 - d_T) = \max_{T \subset P_{i+1} : i \in T, \omega_T \ge q - \alpha} (1 - d_T) \\ &= \max_{T \subset P_i : \omega_T \ge q - \alpha - \omega_i} (1 - d_T) - d_i = b(i, \alpha + \omega_i) - d_i. \end{aligned}$$

b) If $\alpha \geq \gamma_0^i$, $b(i,\alpha)$ is well defined and

$$\begin{aligned} b(i+1,\alpha) &= \max_{T\subset P_{i+1}:\omega_T\geq q-\alpha}(1-d_T) \\ &= \max\left\{\max_{T\subset P_{i+1}:i\notin T,\omega_T\geq q-\alpha}(1-d_T), \max_{T\subset P_{i+1}:i\in T,\omega_T\geq q-\alpha}(1-d_T)\right\} \\ &= \max\left\{\max_{T\subset P_i:\omega_T\geq q-\alpha}(1-d_T), \max_{T\subset P_i:\omega_T\geq q-\alpha-\omega_i}(1-d_T)-d_i\right\} \\ &= \max\{b(i,\alpha), b(i,\alpha+\omega_i)-d_i\}. \end{aligned}$$

■

We have defined the best way to form a coalition that contains α votes from $N\backslash P_i$ and at least $q-\alpha$ votes from P_i. It remains to choose the optimal value of α, and the optimal demand d_i.

We denote as Σ^i the set of values between γ^i and δ^i that maximize $b(i,\alpha)/\alpha$. Thus:

$$\Sigma^i := \arg\max_{\gamma^i\leq\alpha\leq\delta^i}\frac{b(i,\alpha)}{\alpha}$$

The next lemma shows that the only interesting bargaining occurs when $b(i,\sigma^i)\geq 0$ for some/all[6] $\sigma^i\in\Sigma^i$.

Lemma 5 *Assume we are in a subgame perfect equilibrium (SPE) of* $\mathbb{B}(d,i)$*. If* $b(i,\sigma^i)<0$ *for some/all* $\sigma^i\in\Sigma^i$*, then every party gets zero.*

Proof. Since $b(i,\sigma^i)/\sigma^i$ is maximum, we deduce that $b(i,\alpha)<0$ for every $\alpha\geq\gamma^i$. The same occurs for $\alpha=0$ since $b(i,\alpha)$ is nondecreasing in α. This means that no winning coalition can be formed. ■

Thus, if $b(i,\sigma^i)<0$ for some/all $\sigma^i\in\Sigma^i$, party i formulates an arbitrary demand and the game eventually ends with no coalition being formed.

From now on, we will assume that $b(i,\sigma^i)\geq 0$ for all $\sigma^i\in\Sigma^i$. We will show that in equilibrium party i always chooses some $\alpha\in\Sigma^i$.

[6]Of course, $b(i,\sigma)\geq 0$ for some $\sigma\in\Sigma^i$ implies $b(i,\sigma)\geq 0$ for all $\sigma\in\Sigma^i$.

The following lemma shows that all values of α between $\delta^{i+1}+1$ and $\delta^{i+1}+\omega_i=\delta^i$ lead to the same $b(i,\alpha)$. The extra votes are not valuable because they are not enough to replace any party from P_i.

For example, in the game $[10;7,3,3,3,1,1,1]$, $\delta^4=6$ and $\delta^5=3$. Consider the situation of party 4. If it takes $\alpha=4$, there are two ways to form a winning coalition: buying the votes of party 1, or buying the votes of parties 2 and 3. Thus, $b(4,4)=\max(1-d_1,1-d_2-d_3)$. If instead it takes $\alpha=5$ or $\alpha=6$, exactly the same parties are needed: none of party 4's predecessors can be dispensed with despite the extra votes.

Lemma 6 *Assume we are in the subgame $\mathbb{B}(d,i)$. Then*

$$\left\{T\subset P_i:\omega_T\geq q-\left(\delta^{i+1}+\alpha\right)\right\}=\left\{T\subset P_i:\omega_T\geq q-\delta^i\right\}$$

for all $\alpha=1,2,...,\omega_i$.

Proof. "$\subset$" Let $T\subset P_i$ such that $\omega_T\geq q-\left(\omega_{N\backslash P_{i+1}}+\alpha\right)$. Then

$$\omega_T\geq q-\left(\omega_{N\backslash P_i}-\omega_i+\alpha\right)=q-\omega_{N\backslash P_i}+(\omega_i-\alpha)\geq q-\omega_{N\backslash P_i}.$$

"$\supset$" Let $T\subset P_i$ such that $\omega_T\geq q-\omega_{N\backslash P_i}$. Then, $T\cup(N\backslash P_i)$ is winning and contains party i. We study two cases:

- $T\cup(N\backslash P_i)\setminus\{i\}=T\cup(N\backslash P_{i+1})$ is also winning. Then, $\omega_T\geq q-\omega_{N\backslash P_{i+1}}$ and the result is proved.
- $T\cup(N\backslash P_{i+1})$ is losing. Then, since the game is constant-sum, we conclude that its complement, $(N\backslash T)\cap P_{i+1}$, is winning and contains party i as the weakest member. By taking out party i, we obtain the coalition $(N\backslash T)\cap P_i$ which is losing (since its complementary $T\cup(N\backslash P_i)$ is winning). Thus, $(N\backslash T)\cap P_{i+1}$ is minimal winning and $T\cup(N\backslash P_{i+1})$ is maximal losing. Hence, under Corollary 1:

$$\omega_T=q-\omega_{N\backslash P_{i+1}}-1\geq q-\omega_{N\backslash P_{i+1}}-\alpha.$$

$\blacksquare$

Corollary 2 *In* $\mathbb{B}(d,i)$, *we have* $b\left(i,\delta^{i+1}+\alpha\right)=b\left(i,\delta^{i}\right)$ *for all* $\alpha=1,2,...,\omega_i$. *Moreover, for all* $\sigma^i\in\Sigma^i$, *if* $b\left(i,\sigma^i\right)>0$,

$$\sigma^i>\delta^{i+1}\Longrightarrow\sigma^i=\delta^{i+1}+1.$$

Proof. Under Lemma 6, it is clear that $b\left(i,\delta^{i+1}+\alpha\right)=b\left(i,\delta^{i}\right)$ for all $\alpha=1,2,...,\omega_i$, since they minimize d_T on the same coalitions T. Hence, if $b\left(i,\delta^{i}\right)>0$,

$$\frac{b\left(i,\delta^{i+1}+\alpha\right)}{\delta^{i+1}+\alpha}<\frac{b\left(i,\delta^{i+1}+1\right)}{\delta^{i+1}+1}$$

for all $\alpha=2,...,\omega_i$ and thus the maximum is $b\left(i,\delta^{i+1}+1\right)/\left(\delta^{i+1}+1\right)$. Let $\sigma^i\in\Sigma^i$ such that $\sigma^i>\delta^{i+1}$. Since $\sigma^i=\delta^{i+1}+\alpha$ for some $\alpha=1,2,...,\omega_i$, we conclude the result. $\blacksquare$

Now we define the maximum demand party i can make at $\mathbb{B}(d,i)$. This depends on what party $i+1$ can achieve without party i. If party $i+1$ decides to exclude party i, it is in a similar situation to party i except that it has less feasible values for α. It will be choosing an α between γ^i and δ^{i+1}, and the maximum benefit from buying $q-\alpha$ votes without party i is precisely $b(i,\alpha)$. We define T^i as the set of values between γ^i and δ^{i+1} that maximize $b\left(i,\alpha\right)/\alpha$ (recall that $\gamma^i\leq\delta^{i+1}$, so the interval is nonempty).

$$T^i:=\underset{\gamma^i\leq\alpha\leq\delta^{i+1}}{\arg\max}\,\frac{b\left(i,\alpha\right)}{\alpha}.$$

Let $\tau^i\in T^i$. Because $\gamma^i\leq\tau^i$ and $\tau^i\leq\delta^{i+1}<\delta^i$, $\frac{b\left(i,\tau^i\right)}{\tau^i}\leq\frac{b\left(i,\sigma^i\right)}{\sigma^i}$ for all $\sigma^i\in\Sigma^i$.

For any values of $\sigma^i\in\Sigma^i$ and $\tau^i\in T^i$, we define

$$d_i^*:=\left\{\begin{array}{ll}\frac{\omega_i b\left(i,\sigma^i\right)}{\sigma^i} & \text{if } \sigma^i\leq\delta^{i+1}\\ b\left(i,\sigma^i\right)-\frac{\sigma^i-\omega_i}{\tau^i}b\left(i,\tau^i\right) & \text{if } \sigma^i>\delta^{i+1} \text{ and } b\left(i,\tau^i\right)\geq 0\\ b\left(i,\sigma^i\right) & \text{if } \sigma^i>\delta^{i+1} \text{ and } b\left(i,\tau^i\right)<0.\end{array}\right\}\quad(1)$$

It is easy to prove that d_i^* is independent of the particular choice of σ^i and τ^i. By definition, $\frac{b\left(i,\tau^i\right)}{\tau^i}$ and $\frac{b\left(i,\sigma^i\right)}{\sigma^i}$ are independent of the τ^i and σ^i chosen. Also, $b\left(i,\tau^i\right) \geq 0$ for some $\tau^i \in T^i$ if and only if $b\left(i,\tau^i\right) \geq 0$ for all $\tau^i \in T^i$. If $b\left(i,\sigma^i\right) = 0$ for some σ^i, then $b\left(i,\sigma^i\right) = 0$ for all σ^i and $b\left(i,\tau^i\right) \leq 0$ for all $\tau^i \in T^i$. Thus, $d_i^* = 0$ regardless of the choice of σ^i and τ^i. If $b\left(i,\sigma^i\right) > 0$, d_i^* is the same for all $\sigma^i \leq \delta^{i+1}$. If $\sigma^i > \delta^{i+1}$, $\sigma^i = \delta^{i+1} + 1$. If Σ^i contains some $\sigma^i \leq \delta^{i+1}$ as well as $\sigma^i = \delta^{i+1} + 1$, d_i^* will still be independent of the choice of σ^i because in this case $T^i = \Sigma^i \backslash \{\delta^{i+1} + 1\}$, thus $\frac{b\left(i,\tau^i\right)}{\tau^i} = \frac{b\left(i,\sigma^i\right)}{\sigma^i}$.

In order to prove that d_i^* is the equilibrium demand of party i, the following lemmas will be useful. Notice that $\sigma^i > \omega_i$ for some $\sigma^i \in \Sigma^i$ implies $i < n$, because $\omega_n = 1$ and $\gamma^n = \delta^n = 1$.

Lemma 7 *Assume we are in $\mathbb{B}(d,i)$. If $\sigma^i > \omega_i$ for some $\sigma^i \in \Sigma^i$ and party i demands $d_i \leq d_i^*$, then*

$$\gamma^{i+1} \leq \sigma^i - \omega_i \leq \delta^{i+1} \tag{2}$$

$$b\left(i+1, \sigma^i - \omega_i\right) = b\left(i,\sigma^i\right) - d_i \tag{3}$$

and, given any $\tau^i \in T^i$,

$$\frac{b\left(i+1,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i} \geq \left\{ \begin{array}{cl} \frac{b\left(i,\sigma^i\right)}{\sigma^i} & \text{if } \sigma^i \leq \delta^{i+1} \\ \frac{b\left(i,\tau^i\right)}{\tau^i} & \text{if } \sigma^i > \delta^{i+1} \text{ and } b\left(i,\tau^i\right) \geq 0 \\ 0 & \text{if } \sigma^i > \delta^{i+1} \text{ and } b\left(i,\tau^i\right) < 0. \end{array} \right\} \tag{4}$$

Furthermore, inequality in (4) is strict iff $d_i < d_i^*$.

Proof. Let $\sigma^i \in \Sigma^i$ such that $\sigma^i > \omega_i$. We first prove (2):

$$\begin{array}{rcl} \sigma^i & \leq & \delta^i \Longrightarrow \sigma^i - \omega_i \leq \delta^i - \omega_i = \delta^{i+1}. \\ \sigma^i & \geq & \gamma^i \geq \gamma_0^i = q - \omega_{P_i} \Longrightarrow \sigma^i - \omega_i \geq q - \omega_{P_i} - \omega_i = q - \omega_{P_{i+1}} = \gamma_0^{i+1}. \\ \sigma^i & > & \omega_i \Longrightarrow \sigma^i - \omega_i > 0 \Longrightarrow \sigma^i - \omega_i \geq 1. \end{array}$$

We have just proven that $\sigma^i - \omega_i$ is a feasible value of α for party $i+1$. Notice that for $b\left(i,\sigma^i\right) > 0$, homogeneity implies $\sigma^i - \omega_i \geq \omega_{i+1}$.

We prove now (3) and (4). Under Lemma $4a$), (3) is true when $\sigma^i - \omega_i < \gamma_0^i$. Then (4) follows immediately by replacing $\frac{b\left(i+1,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i}$ by $\frac{b\left(i,\sigma^i\right)-d_i}{\sigma^i-\omega_i}$ and then using $d_i \le d_i^*$.[7] Assume then $\sigma^i - \omega_i \ge \gamma_0^i$. We have two cases:

1. If $\sigma^i \le \delta^{i+1}$, then $d_i^* = \frac{\omega_i b\left(i,\sigma^i\right)}{\sigma^i}$. Since $\sigma^i \in \Sigma^i$, re-arranging terms,

$$\begin{aligned}\frac{b\left(i,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i} &\le \frac{b\left(i,\sigma^i\right)}{\sigma^i} \Longrightarrow b\left(i,\sigma^i-\omega_i\right) \le b\left(i,\sigma^i\right) - \frac{\omega_i b\left(i,\sigma^i\right)}{\sigma^i} \\ &\Longrightarrow b\left(i,\sigma^i-\omega_i\right) \le b\left(i,\sigma^i\right) - d_i.\end{aligned}$$

   Hence, (3) follows under lemma $4b$). Moreover

$$\begin{aligned}\frac{b\left(i+1,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i} &= \frac{b\left(i,\sigma^i\right)-d_i}{\sigma^i-\omega_i} \ge \frac{b\left(i,\sigma^i\right)-\frac{\omega_i b\left(i,\sigma^i\right)}{\sigma^i}}{\sigma^i-\omega_i} \\ &= \frac{\left(\sigma^i-\omega_i\right)b\left(i,\sigma^i\right)}{\sigma^i\left(\sigma^i-\omega_i\right)} = \frac{b\left(i,\sigma^i\right)}{\sigma^i}\end{aligned}$$

   with strict inequality iff $d_i < d_i^*$.

2. If $\sigma^i > \delta^{i+1}$, recall that $\gamma^i \le \sigma^i - \omega_i \le \delta^{i+1}$. Then for any $\tau^i \in T^i$

$$\frac{b\left(i,\tau^i\right)}{\tau^i} \ge \frac{b\left(i,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i}.$$

   Re-arranging terms,

$$\begin{aligned}\frac{\sigma^i-\omega_i}{\tau^i} b\left(i,\tau^i\right) &\ge b\left(i,\sigma^i-\omega_i\right) \\ &\Longrightarrow b\left(i,\sigma^i-\omega_i\right) + b\left(i,\sigma^i\right) - \frac{\sigma^i-\omega_i}{\tau^i} b\left(i,\tau^i\right) \le b\left(i,\sigma^i\right) \\ &\Longrightarrow b\left(i,\sigma^i-\omega_i\right) + d_i \le b\left(i,\sigma^i\right).\end{aligned}$$

   Hence, (3) follows under lemma $4b$).

   To show (4), we distinguish two subcases:

[7] Actually, $\sigma^i - \omega_i < \gamma_0^i$ implies $\sigma^i \le \delta^{i+1}$, so two of the three cases are void.

(a) If $b\left(i,\tau^{i}\right)\geq 0$, then $d_{i}^{*}=b\left(i,\sigma^{i}\right)-\frac{\sigma^{i}-\omega_{i}}{\tau^{i}}b\left(i,\tau^{i}\right)$

$$\begin{aligned}\frac{b\left(i+1,\sigma^{i}-\omega_{i}\right)}{\sigma^{i}-\omega_{i}} &= \frac{b\left(i,\sigma^{i}\right)-d_{i}}{\sigma^{i}-\omega_{i}}\\ &\geq \frac{b\left(i,\sigma^{i}\right)-b\left(i,\sigma^{i}\right)+\frac{\sigma^{i}-\omega_{i}}{\tau^{i}}b\left(i,\tau^{i}\right)}{\sigma^{i}-\omega_{i}}\\ &= \frac{b\left(i,\tau^{i}\right)}{\tau^{i}}\end{aligned}$$

with strict inequality iff $d_{i}<d_{i}^{*}$.

(b) If $b\left(i,\tau^{i}\right)<0$, then $d_{i}^{*}=b\left(i,\sigma^{i}\right)$ and thus

$$\frac{b\left(i+1,\sigma^{i}-\omega_{i}\right)}{\sigma^{i}-\omega_{i}}=\frac{b\left(i,\sigma^{i}\right)-d_{i}}{\sigma^{i}-\omega_{i}}\geq 0$$

with strict inequality iff $d_{i}<d_{i}^{*}$.

■

Lemma 8 *Assume we are in $\mathbb{B}\left(d,i+1\right)$ and $\sigma^{i}>\omega_{i}$ for some $\sigma^{i}\in\Sigma^{i}$.*

a) If $d_{i}<d_{i}^{*}$, then $i\in S$ for all $S\in\underset{T\subset P_{i+1}:\omega_{T}\geq q-\sigma^{i+1}}{\arg\max}\left(1-d_{T}\right)$ and all $\sigma^{i+1}\in\Sigma^{i+1}$.

b) If $d_{i}=d_{i}^{*}$, then $\sigma^{i}-\omega_{i}\in\Sigma^{i+1}$. Moreover, $S\in\underset{T\subset P_{i}:\omega_{T}\geq q-\sigma^{i}}{\arg\max}\left(1-d_{T}\right)$ implies $S\cup\{i\}\in\underset{T\subset P_{i+1}:\omega_{T}\geq q-\left(\sigma^{i}-\omega_{i}\right)}{\arg\max}\left(1-d_{T}\right)$.

c) If $d_{i}=d_{i}^{*}$, given $\sigma^{i+1}\in\Sigma^{i+1}$ and $S\in\underset{T\subset P_{i+1}:\omega_{T}\geq q-\sigma^{i+1}}{\arg\max}\left(1-d_{T}\right)$, $i\in S$ implies $S\cap P_{i}\in\underset{T\subset P_{i}:\omega_{T}\geq q-\sigma^{i}}{\arg\max}\left(1-d_{T}\right)$ for some $\sigma^{i}\in\Sigma^{i}$.

Proof. a) Let $\sigma^{i+1}\in\Sigma^{i+1}$. Suppose there exists $S\in\underset{T\subset P_{i+1}:\omega_{T}\geq q-\sigma^{i+1}}{\arg\max}\left(1-d_{T}\right)$ such that $i\notin S$. Then, $b\left(i+1,\sigma^{i+1}\right)=b\left(i,\sigma^{i+1}\right)$. We see three cases:

1. If $\sigma^{i}\leq\delta^{i+1}$,

$$\frac{b\left(i+1,\sigma^{i+1}\right)}{\sigma^{i+1}}=\frac{b\left(i,\sigma^{i+1}\right)}{\sigma^{i+1}}\leq\frac{b\left(i,\sigma^{i}\right)}{\sigma^{i}}\overset{\text{(Lemma 7)}}{<}\frac{b\left(i+1,\sigma^{i}-\omega_{i}\right)}{\sigma^{i}-\omega_{i}}$$

which contradicts that $\sigma^{i+1}\in\Sigma^{i+1}$.

2. If $\sigma^i > \delta^{i+1}$,

$$\frac{b\left(i+1,\sigma^{i+1}\right)}{\sigma^{i+1}} = \frac{b\left(i,\sigma^{i+1}\right)}{\sigma^{i+1}} \leq \frac{b\left(i,\tau^i\right)}{\tau^i} \overset{\text{(Lemma 7)}}{<} \frac{b\left(i+1,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i}$$

which contradicts that $\sigma^{i+1} \in \Sigma^{i+1}$.

b) Let α such that $\gamma^{i+1} \leq \alpha \leq \delta^{i+1}$. Under Lemma 4, either $b\left(i+1,\alpha\right) = b\left(i,\alpha+\omega_i\right) - d_i$ or $b\left(i+1,\alpha\right) = b\left(i,\alpha\right)$. We have to prove that

$$\frac{b\left(i+1,\alpha\right)}{\alpha} \leq \frac{b\left(i+1,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i}.$$

If $b\left(i+1,\alpha\right) = b\left(i,\alpha\right)$, we proceed like in case a).

If $b\left(i+1,\alpha\right) = b\left(i,\alpha+\omega_i\right) - d_i$, we have three cases:

1. If $\sigma^i \leq \delta^{i+1}$, then

$$\begin{aligned}\frac{b\left(i+1,\alpha\right)}{\alpha} &= \frac{b\left(i,\alpha+\omega_i\right)-d_i}{\alpha} = \frac{b\left(i,\alpha+\omega_i\right)-\frac{\omega_i b\left(i,\sigma^i\right)}{\sigma^i}}{\alpha} \\ &\leq \frac{\frac{b\left(i,\sigma^i\right)}{\sigma^i}\left(\alpha+\omega_i\right)-\frac{\omega_i b\left(i,\sigma^i\right)}{\sigma^i}}{\alpha} \\ &= \frac{b\left(i,\sigma^i\right)}{\sigma^i} \overset{\text{(Lemma 7)}}{\leq} \frac{b\left(i+1,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i}.\end{aligned}$$

2. If $\sigma^i > \delta^{i+1}$ and $b\left(i,\tau^i\right) \geq 0$ for some/all $\tau^i \in T^i$, then either $\frac{b(i,\alpha+\omega_i)}{\alpha+\omega_i} \leq \frac{b\left(i,\tau^i\right)}{\tau^i}$ (if $\alpha+\omega_i \leq \delta^{i+1}$) or $b\left(i,\alpha+\omega_i\right) = b\left(i,\sigma^i\right)$ (if $\alpha+\omega_i > \delta^{i+1}$, by Corollary 2).

   If $\alpha+\omega_i \leq \delta^{i+1}$,

$$\begin{aligned}\frac{b\left(i+1,\alpha\right)}{\alpha} &= \frac{b\left(i,\alpha+\omega_i\right)-d_i}{\alpha} = \frac{b\left(i,\alpha+\omega_i\right)-b\left(i,\sigma^i\right)+\frac{\sigma^i-\omega_i}{\tau^i}b\left(i,\tau^i\right)}{\alpha} \\ &\leq \frac{\frac{\alpha+\omega_i}{\tau^i}b\left(i,\tau^i\right)-\frac{\sigma^i}{\tau^i}b\left(i,\tau^i\right)+\frac{\sigma^i-\omega_i}{\tau^i}b\left(i,\tau^i\right)}{\alpha} \\ &= \frac{b\left(i,\tau^i\right)}{\tau^i} \overset{\text{(Lemma 7)}}{\leq} \frac{b\left(i+1,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i}.\end{aligned}$$

If $\alpha + \omega_i > \delta^{i+1}$,

$$\begin{aligned}\frac{b(i+1,\alpha)}{\alpha} &= \frac{b(i,\alpha+\omega_i) - d_i}{\alpha} = \frac{b(i,\alpha+\omega_i) - b\left(i,\sigma^i\right) + \frac{\sigma^i-\omega_i}{\tau^i} b\left(i,\tau^i\right)}{\alpha} \\ &= \frac{\sigma^i - \omega_i}{\alpha} \frac{b\left(i,\tau^i\right)}{\tau^i}.\end{aligned}$$

If $b\left(i,\sigma^i\right) > 0$, corollary 2 implies $\sigma^i = \delta^{i+1} + 1$. Then $\alpha - \omega_i > \delta^{i+1}$ implies $\alpha + \omega_i \geq \delta^{i+1} + 1 = \sigma^i$, or $\left(\sigma^i - \omega_i\right)/\alpha \leq 1$. If $b(i,\sigma^i) = 0$, $b(i,\tau^i) = 0$, implying $\frac{\sigma^i-\omega_i}{\alpha}\frac{b(i,\tau^i)}{\tau^i} = \frac{b(i,\tau^i)}{\tau^i}$. In either case,

$$\frac{b(i+1,\alpha)}{\alpha} \leq \frac{b\left(i,\tau^i\right)}{\tau^i} \overset{\text{(Lemma 7)}}{\leq} \frac{b\left(i+1,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i}.$$

3. If $\sigma^i > \delta^{i+1}$ and $b\left(i,\tau^i\right) < 0$ for some/all $\tau^i \in T^i$, then either $b(i,\alpha+\omega_i) < 0$ (if $\alpha+\omega_i \leq \delta^{i+1}$) or $b(i,\alpha+\omega_i) = b\left(i,\sigma^i\right)$ (if $\alpha+\omega_i > \delta^{i+1}$, by Corollary 2).

   If $b(i,\alpha+\omega_i) < 0$,

   $$\frac{b(i+1,\alpha)}{\alpha} = \frac{b(i,\alpha+\omega_i) - d_i}{\alpha} < -\frac{d_i}{\alpha} \leq 0 \leq \frac{b\left(i+1,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i}.$$

   If $b(i,\alpha+\omega_i) = b\left(i,\sigma^i\right)$,

   $$\frac{b(i+1,\alpha)}{\alpha} = \frac{b(i,\alpha+\omega_i) - d_i}{\alpha} = 0 \leq \frac{b\left(i+1,\sigma^i-\omega_i\right)}{\sigma^i-\omega_i}.$$

We now prove the second statement. Let $S \in \underset{T\subset P_i:\omega_T \geq q-\sigma^i}{\arg\max} (1-d_T)$. We have to prove $b\left(i+1,\sigma^i-\omega_i\right) = 1 - d_{S\cup\{i\}}$. Using (3),

$$b\left(i+1,\sigma^i-\omega_i\right) = b\left(i,\sigma^i\right) - d_i = 1 - d_S - d_i = 1 - d_{S\cup\{i\}}.$$

c) Since $i \in S$, $b(i+1,\sigma^{i+1}) = b(i,\sigma^{i+1}+\omega_i) - d_i$, or

$$b(i,\sigma^{i+1}+\omega_i) = b(i+1,\sigma^{i+1}) + d_i. \tag{5}$$

Let $\sigma^i > \omega_i$. We have shown that $\sigma^i - \omega_i \in \Sigma^{i+1}$, thus

$$\frac{b(i+1,\sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i+1,\sigma^i-\omega_i)}{\sigma^i-\omega_i}. \tag{6}$$

1. If $\sigma^i \leq \delta^{i+1}$ for some $\sigma^i \in \Sigma^i$, it follows from (6) and (3) that

$$\frac{b(i+1,\sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i,\sigma^i)}{\sigma^i}. \tag{7}$$

Then

$$\frac{b(i,\sigma^{i+1}+\omega_i)}{\sigma^{i+1}+\omega_i} \overset{(5)}{=} \frac{b(i+1,\sigma^{i+1})+d_i}{\sigma^{i+1}+\omega_i} = \frac{b(i+1,\sigma^{i+1})+\frac{\omega_i}{\sigma^i}b\left(i,\sigma^i\right)}{\sigma^{i+1}+\omega_i}$$
$$\overset{(7)}{=} \frac{\frac{\sigma^{i+1}}{\sigma^i}b\left(i,\sigma^i\right)+\frac{\omega_i}{\sigma^i}b\left(i,\sigma^i\right)}{\sigma^{i+1}+\omega_i} = \frac{b(i,\sigma^i)}{\sigma^i}.$$

Hence $\sigma^{i+1}+\omega_i \in \Sigma^i$ and $b\left(i,\sigma^{i+1}+\omega_i\right) = b\left(i+1,\sigma^{i+1}\right)+d_i = 1-d_{S\cap P_i}$.

2. If $\sigma^i > \delta^{i+1}$ for all $\sigma^i \in \Sigma^i$, $\delta^{i+1}+1$ always belongs to Σ^i.

Suppose $S\cap P_i \notin \underset{T\subset P_i:\omega_T\geq q-\sigma^i}{\arg\max}(1-d_T)$ for all $\sigma^i \in \Sigma^i$. Then it must be the case that for any σ^i either $\omega_{S\cap P_i} < q-\sigma^i$, or $\omega_{S\cap P_i} \geq q-\sigma^i$ but $1-d_{S\cap P_i}$ is not maximal.

Suppose $\omega_{S\cap P_i} < q-\sigma^i$ for all $\sigma^i \in \Sigma^i$. Since $\delta^{i+1}+1 \in \Sigma^i$, it follows from Lemma 6 that $\omega_{S\cap P_i} < q-\delta^i$. But then $\omega_{S\cap P_i}+\omega_{N\setminus P_i} = \omega_{S\cap P_{i+1}}+\omega_{N\setminus P_{i+1}} < q$, contradicting the assumption that $\omega_{S\cap P_{i+1}} \geq q-\sigma^{i+1}$.

Suppose $\omega_{S\cap P_i} \geq q-\sigma^i$ but $1-d_{S\cap P_i} < 1-d_T$ for some $\sigma^i \in \Sigma^i$ and $T \subset P_i$ with $\omega_T \geq q-\sigma^i$.

If $\sigma^{i+1}+\omega_i > \delta^{i+1}$,

$$b(i,\sigma^{i+1}+\omega_i) \overset{(6)}{=} b(i,\sigma^i) > 1-d_{S\cap P_i} = b(i+1,\sigma^{i+1})+d_i$$

contradicting (5).

If $\sigma^{i+1}+\omega_i \leq \delta^{i+1}$, $\frac{b(i,\sigma^{i+1}+\omega_i)}{\sigma^{i+1}+\omega_i} \leq \frac{b(i,\tau^i)}{\tau^i}$. There are two possibilities:

- If $b(i,\tau^i) \geq 0$, it follows from (6) and (3) that $\frac{b(i+1,\sigma^{i+1})}{\sigma^{i+1}} = \frac{b(i,\tau^i)}{\tau^i}$. Then

$$\begin{aligned}\frac{b(i,\tau^i)}{\tau^i}(\sigma^{i+1}+\omega_i) &\geq b(i,\sigma^{i+1}+\omega_i) \overset{(5)}{=} b(i+1,\sigma^{i+1})+d_i = \\ &= \frac{b(i,\tau^i)}{\tau^i}\sigma^{i+1} + b\left(i,\sigma^i\right) - (\sigma^i-\omega_i)\frac{b\left(i,\tau^i\right)}{\tau^i}\end{aligned}$$

implying $\frac{b(i,\tau^i)}{\tau^i} \geq \frac{b(i,\sigma^i)}{\sigma^i}$, thus $\frac{b(i,\tau^i)}{\tau^i} = \frac{b(i,\sigma^i)}{\sigma^i}$. Then $\frac{b(i,\sigma^{i+1}+\omega_i)}{\sigma^{i+1}+\omega_i} = \frac{b(i,\sigma^i)}{\sigma^i}$. Hence $\sigma^{i+1}+\omega_i \in \Sigma^i$ and the result follows.

- If $b(i,\tau^i) < 0$, it follows from (6) and (3) that $\frac{b(i+1,\sigma^{i+1})}{\sigma^{i+1}} = 0$. Then

$$b(i,\sigma^{i+1}+\omega_i) = b(i+1,\sigma^{i+1}) + d_i = b(i,\sigma^i).$$

Hence $b(i,\sigma^i) = 1 - d_{S\cap P_i}$ and the result follows.

■

Lemma 9 *Assume we are in* $\mathbb{B}(d,i+1)$ *and* $d_i > d_i^*$.

a) If $b\left(i,\tau^i\right) \geq 0$ *for some/all* $\tau^i \in T^i$, *then* $i \notin S$ *for all* $S \in \underset{T\subset P_{i+1}:\omega_T\geq q-\sigma^{i+1}}{\arg\max}(1-d_T)$ *and all* $\sigma^{i+1} \in \Sigma^{i+1}$.

b) If $b\left(i,\tau^i\right) < 0$ *for some/all* $\tau^i \in T^i$, *then every party obtains zero.*

Proof. *a*) Let $\sigma^{i+1} \in \Sigma^{i+1}$ and $\tau^i \in T^i$. We need to prove that $b\left(i,\sigma^{i+1}\right)$ exists and $b\left(i,\sigma^{i+1}\right) > b\left(i,\sigma^{i+1}+\omega_i\right) - d_i$. This will be due to party $i+1$ having the option of setting $\alpha = \sigma^i$ (if $\sigma^i \leq \delta^{i+1}$) or $\alpha = \tau^i$ (if $\sigma^i > \delta^{i+1}$). We examine each case in turn:

1. If $\sigma^i \leq \delta^{i+1}$, then $d_i > \frac{\omega_i b\left(i,\sigma^i\right)}{\sigma^i}$.

   Since $\sigma^i \leq \delta^{i+1}$, $b(i+1,\sigma^i)$ exists. Moreover, lemma 4*b*) implies

$$b(i+1,\sigma^i) \geq b(i,\sigma^i). \tag{8}$$

In principle, there are three possibilities for σ^{i+1}: either $\sigma^{i+1} < \gamma_0^i$, or $\sigma^{i+1} \geq \gamma_0^i$ and $b\left(i,\sigma^{i+1}\right) \leq b\left(i,\sigma^{i+1}+\omega_i\right) - d_i$, or $\sigma^{i+1} \geq \gamma_0^i$ and

$b\left(i,\sigma^{i+1}\right) > b\left(i,\sigma^{i+1}+\omega_i\right) - d_i$. We will show that the first two possibilities lead to a contradiction. In both cases, Lemma 4 implies

$$b\left(i+1,\sigma^{i+1}\right) = b\left(i,\sigma^{i+1}+\omega_i\right) - d_i. \tag{9}$$

From (9) we can deduce:

$$\begin{aligned}
\frac{b\left(i+1,\sigma^{i+1}\right)}{\sigma^{i+1}} &= \frac{b\left(i,\sigma^{i+1}+\omega_i\right) - d_i}{\sigma^{i+1}} \\
&< \frac{b\left(i,\sigma^{i+1}+\omega_i\right) - \frac{\omega_i b\left(i,\sigma^i\right)}{\sigma^i}}{\sigma^{i+1}} \\
&\leq \frac{\frac{\left(\sigma^{i+1}+\omega_i\right) b\left(i,\sigma^i\right)}{\sigma^i} - \frac{\omega_i b\left(i,\sigma^i\right)}{\sigma^i}}{\sigma^{i+1}} \\
&= \frac{b\left(i,\sigma^i\right)}{\sigma^i} \overset{(8)}{\leq} \frac{b\left(i+1,\sigma^i\right)}{\sigma^i}.
\end{aligned}$$

which contradicts that $\sigma^{i+1} \in \Sigma^{i+1}$. Thus, $\sigma^{i+1} \geq \gamma_0^i$ (*i.e.* $b\left(i,\sigma^{i+1}\right)$ does exist) and $b\left(i,\sigma^{i+1}\right) > b\left(i,\sigma^{i+1}+\omega_i\right) - d_i$. We conclude then that $i \notin S$ for all $S \in \underset{T \subset P_{i+1}:\omega_T \geq q-\sigma^{i+1}}{\arg\max} (1-d_T)$.

2. If $\sigma^i > \delta^{i+1}$, then $d_i > b\left(i,\sigma^i\right) - \frac{\sigma^i-\omega_i}{\tau^i} b\left(i,\tau^i\right)$.

   Under Lemma 4*b)*:

$$b\left(i+1,\tau^i\right) = \max\left\{b\left(i,\tau^i\right), b\left(i,\tau^i+\omega_i\right) - d_i\right\} \geq b\left(i,\tau^i\right). \tag{10a}$$

   Suppose $b\left(i,\sigma^{i+1}\right)$ does not exist (i.e. $\sigma^{i+1} < \gamma_0^i$), or $b\left(i,\sigma^{i+1}\right)$ exists and $b\left(i,\sigma^{i+1}\right) \leq b\left(i,\sigma^{i+1}+\omega_i\right) - d_i$. In both cases, under Lemma 4,

$$b\left(i+1,\sigma^{i+1}\right) = b\left(i,\sigma^{i+1}+\omega_i\right) - d_i. \tag{11}$$

   We will prove that (11) leads to a contradiction, so that $b\left(i,\sigma^{i+1}\right)$ exits and $b\left(i,\sigma^{i+1}\right) > b\left(i,\sigma^{i+1}+\omega_i\right) - d_i$, which implies $i \notin S$ for all $S \in \underset{T \subset P_{i+1}:\omega_T \geq q-\sigma^{i+1}}{\arg\max} (1-d_T)$ as desired.

   We have two cases:

- If $\sigma^{i+1}+\omega_i \leq \delta^{i+1}$. Then $\frac{b(i,\sigma^{i+1}+\omega_i)}{\sigma^{i+1}+\omega_i} \leq \frac{b(i,\tau^i)}{\tau^i}$ and

$$\begin{aligned}
\frac{b\left(i+1,\sigma^{i+1}\right)}{\sigma^{i+1}} &= \frac{b\left(i,\sigma^{i+1}+\omega_i\right)-d_i}{\sigma^{i+1}} \\
&< \frac{b\left(i,\sigma^{i+1}+\omega_i\right)-b\left(i,\sigma^i\right)+\frac{\sigma^i-\omega_i}{\tau^i}b\left(i,\tau^i\right)}{\sigma^{i+1}} \\
&\leq \frac{\frac{\sigma^{i+1}+\omega_i}{\tau^i}b\left(i,\tau^i\right)-\frac{\sigma^i}{\tau^i}b\left(i,\tau^i\right)+\frac{\sigma^i-\omega_i}{\tau^i}b\left(i,\tau^i\right)}{\sigma^{i+1}} \\
&= \frac{b\left(i,\tau^i\right)}{\tau^i} \overset{(10a)}{\leq} \frac{b\left(i+1,\tau^i\right)}{\tau^i}
\end{aligned}$$

  which is a contradiction.

- If $\sigma^{i+1}+\omega_i > \delta^{i+1}$, then under Corollary 2, $b\left(i,\sigma^{i+1}+\omega_i\right) = b\left(i,\sigma^i\right)$. If $b\left(i,\sigma^i\right)>0$, $\sigma^i=\delta^{i+1}+1$ and $\sigma^{i+1}+\omega_i \geq \sigma^i$, which implies $\left(\sigma^i-\omega_i\right)/\sigma^{i+1} \leq 1$. If $b\left(i,\sigma^i\right)=0$, $b\left(i,\tau^i\right)=0$. Hence:

$$\begin{aligned}
\frac{b\left(i+1,\sigma^{i+1}\right)}{\sigma^{i+1}} &= \frac{b\left(i,\sigma^{i+1}+\omega_i\right)-d_i}{\sigma^{i+1}} \\
&< \frac{b\left(i,\sigma^{i+1}+\omega_i\right)-b\left(i,\sigma^i\right)+\frac{\sigma^i-\omega_i}{\tau^i}b\left(i,\tau^i\right)}{\sigma^{i+1}} \\
&= \frac{\sigma^i-\omega_i}{\sigma^{i+1}}\frac{b\left(i,\tau^i\right)}{\tau^i} \leq \frac{b\left(i,\tau^i\right)}{\tau^i} \overset{(10a)}{\leq} \frac{b\left(i+1,\tau^i\right)}{\tau^i}
\end{aligned}$$

  which is a contradiction.

b) Recall that we assumed $b(i,\sigma^i) \geq 0$ for all $\sigma^i \in \Sigma^i$. Thus, $b\left(i,\tau^i\right)<0$ for some $\tau^i \in T^i$ implies $\sigma^i > \delta^{i+1}$. Under Corollary 2, this means $b\left(i,\sigma^i\right) = b\left(i,\delta^i\right)$. Let α be such that $\gamma^{i+1} \leq \alpha \leq \delta^{i+1}$. Under Lemma 4, we have two cases:

1. $b\left(i+1,\alpha\right)=b\left(i,\alpha+\omega_i\right)-d_i$. Then

$$b\left(i+1,\alpha\right) < b\left(i,\alpha+\omega_i\right)-b\left(i,\delta^i\right).$$

   Since $\alpha+\omega_i \leq \delta^i$, $b\left(i,\alpha+\omega_i\right) \leq b\left(i,\delta^i\right)$ and thus $b\left(i+1,\alpha\right)<0$.

2. $b(i+1,\alpha) = b(i,\alpha)$. Then $\gamma_0^i \leq \alpha \leq \delta^{i+1}$ and
$$\frac{b(i+1,\alpha)}{\alpha} \leq \frac{b\left(i,\tau^i\right)}{\tau^i} < 0$$
and thus $b(i+1,\alpha) < 0$.

Since $b(i+1,\alpha) < 0$ for all α, we conclude $b\left(i+1,\sigma^{i+1}\right) < 0$ for all $\sigma^{i+1} \in \Sigma^{i+1}$ and thus by Lemma 5 all the parties get zero. $\blacksquare$

Let us consider the following strategy profile for the parties. In $\mathbb{B}(d,n)$, party n forms a coalition $S \cup \{n\}$ with $S \in \underset{T \subset P_n : \omega_T \geq q - \omega_n}{\arg\max} (1-d_T)$ after demanding $d_n = 1 - d_S$. If there is more than one possible choice of S, party n uses the following **tie-breaking rule**: First, select only the coalitions that contain the party with the highest index (party $n-1$, or, if party $n-1$ is in none of the coalitions, party $n-2$ etc.). If there are several coalitions containing this party, select the ones that contain the party with the second highest index, etc., until only one coalition is left.

Let $i < n$ and assume we have defined the strategies for parties in $\mathbb{B}(d,i+1)$. In $\mathbb{B}(d,i)$, party i proceeds as follows:

1. If $\sigma^i > \omega_i$ for all $\sigma^i \in \Sigma^i$, party i demands $d_i = d_i^*$ given as in (1).

2. If $\Sigma^i = \{\omega_i\}$, party i forms coalition $S \cup \{i\}$ with $S \in \underset{T \subset P_i : \omega_T \geq q - \omega_i}{\arg\min} d_T$. If there is more than one possible choice of S, party i uses the tie-breaking rule: Among all the optimal coalitions $S \in \underset{T \subset P_i : \omega_T \geq q - \omega_i}{\arg\min} d_T$, party i selects the ones that contain the party with the highest index ($i-1$, or, if party $i-1$ is in none of the coalitions, party $i-2$, etc). If there are several coalitions containing this party, select the ones that contain the party with the second highest index, etc., until only one coalition is left.

3. If $\{\omega_i\} \subsetneq \Sigma^i$, party i can anticipate the coalition S^* that will be formed should it demand d_i^* and its followers play the strategies we have defined.

(a) If $i \notin S^*$, party i forms coalition $S \cup \{i\}$ with $S \in \underset{T \subset P_i : \omega_T \geq q - \omega_i}{\arg\min} d_T$. If there is more than one possible S, party i uses the tie-breaking rule.

(b) If $i \in S^*$, party i compares the coalitions $S \in \underset{T \subset P_i : \omega_T \geq q - \omega_i}{\arg\max} (1 - d_T)$ and $S^* \cap P_i$. Among them, party i selects a coalition following the tie-breaking rule. If S^* is chosen, party i demands $d_i = d_i^*$ given as in (1). If $S \neq S^*$ is chosen, then party i demands $1 - d_S = b(i, \omega_i) = d_i^*$.and forms coalition $S \cup \{i\}$.

The role of the tie-breaking rule is to ensure that parties have a best response at all stages (cf. Example **??**).

Proposition 1 *The above strategies constitute a SPE for any* $\mathbb{B}(d, i)$.

Proof. We proceed by backwards induction on i. For $i = n$, its strategy is clearly optimal.

Assume now the result is true for $\mathbb{B}(d, i+1)$ and moreover assume the following two conditions hold:

Condition 1 The formed coalition satisfies

$$S \cap P_{i+1} \in \underset{T \subset P_{i+1} : \omega_T \geq q - \sigma^{i+1}}{\arg\max} (1 - d_T)$$

for some $\sigma^{i+1} \in \Sigma^{i+1}$. (This condition holds trivially for $i + 1 = n$ because $\Sigma^n = \{\omega_n\}$).

Condition 2 The above S and σ^{i+1} are such that $S \cap P_{i+1}$ is one of the most favorable sets for party i (*i.e.* $i \notin S$ implies $i \notin T$ for all $T \in \underset{T \subset P_{i+1} : \omega_T \geq q - \sigma^{i+1}}{\arg\max} (1 - d_T)$ and all $\sigma^{i+1} \in \Sigma^{i+1}$). Among them, it is one of the most favorable to party $i - 1$, etc. (This condition holds for $i + 1 = n$ because $\Sigma^n = \{\omega_n\}$ and n applies the tie-breaking rule).

We check that this remains true for $\mathbb{B}(d, i)$. Let $\tau^i \in T^i$. We have two cases:

1. If $\sigma^i > \omega_i$ for all $\sigma^i \in \Sigma^i$, then it is straightforward to check that party i obtains stricly less than d_i^* by forming coalition. If i demands d_i^*, $S \cup \{i\} \in \underset{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}}{\arg\max} (1 - d_T)$ for $\sigma^{i+1} = \sigma^i - \omega_i \in \Sigma^{i+1}$. The induction hypothesis (Conditions 1 and 2) implies that d_i^* will be accepted. Assume party i deviates by demanding $d_i > d_i^*$. If $b\left(i, \tau^i\right) \geq 0$, under Lemma 9$a$) party i does not belong to any coalition in $\underset{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}}{\arg\max} (1 - d_T)$ for any $\sigma^{i+1} \in \Sigma^{i+1}$ and its final payoff is zero under the induction hypothesis (Condition 1). If $b\left(i, \tau^i\right) < 0$, under Lemma 9$b$), its final payoff is zero.

   Moreover, Conditions 1 and 2 hold for i. Condition 1 follows from Lemma 8b) and the induction hypothesis applied to Conditions 1 and 2. Condition 2 follows from the tie-breaking rule applied by the party $j > i$ that eventually forms coalition.

2. If $\omega_i \in \Sigma^i$, then $1 - d_S = b\left(i, \omega_i\right) = d_i^*$ for all $S \in \underset{T \subset P_i: \omega_T \geq q - \omega_i}{\arg\max} (1 - d_T)$. This means that if party i forms a winning coalition it obtains a final payoff of $b\left(i, \omega_i\right)$. Suppose party i deviates and demands $d_i > b\left(i, \omega_i\right)$. It is enough to check that $i \notin S$ for all $S \in \underset{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}}{\arg\max} (1 - d_T)$ and all $\sigma^{i+1} \in \Sigma^{i+1}$. Under the induction hypothesis applied to Condition 1, this means that party i will not be included in any eventual winning coalition, and its final payoff will be zero, while the original strategy yields a nonnegative payoff.

   For constant-sum homogeneous games it is always the case that $\omega_i \leq \delta^{i+1}$, thus $b(i+1, \omega_i)$ is well defined. Under Lemma 4b),

   $$b\left(i+1, \omega_i\right) = \max\left\{b\left(i, \omega_i\right), b\left(i, 2\omega_i\right) - d_i\right\} \geq b\left(i, \omega_i\right) \tag{12}$$

   Suppose that $i \in S$ for some $S \in \underset{T \subset P_{i+1}: \omega_T \geq q - \sigma^{i+1}}{\arg\min} d_T$ and some $\sigma^{i+1} \in \Sigma^{i+1}$. This means

   $$b\left(i+1, \sigma^{i+1}\right) = b\left(i, \sigma^{i+1} + \omega_i\right) - d_i$$

and hence

$$\begin{aligned}
\frac{b\left(i+1,\sigma^{i+1}\right)}{\sigma^{i+1}} &= \frac{b\left(i,\sigma^{i+1}+\omega_i\right)-d_i}{\sigma^{i+1}} \\
&< \frac{b\left(i,\sigma^{i+1}+\omega_i\right)-b\left(i,\omega_i\right)}{\sigma^{i+1}} \\
&\leq \frac{\frac{\sigma^{i+1}+\omega_i}{\omega_i}b\left(i,\omega_i\right)-b\left(i,\omega_i\right)}{\sigma^{i+1}} \\
&= \frac{\left(\sigma^{i+1}+\omega_i\right)b\left(i,\omega_i\right)-\omega_i b\left(i,\omega_i\right)}{\omega_i\sigma^{i+1}} \\
&= \frac{b\left(i,\omega_i\right)}{\omega_i} \overset{(12)}{\leq} \frac{b\left(i+1,\omega_i\right)}{\omega_i}
\end{aligned}$$

which is a contradiction. This contradiction proves that $i \notin S$ for all $S \in \underset{T\subset P_{i+1}:\omega_T \geq q-\sigma^{i+1}}{\arg\min} d_T$, as desired.

We now check that Conditions 1 and 2 hold for i. If party i forms coalition, Condition 1 holds with $\sigma^i = \omega_i$, and Condition 2 holds because of the tie-breaking rule. If party i demands d_i^* so that S^* is induced, it must be the case that $\{\omega_i\} \subsetneq \Sigma^i$. Hence, there exists $\sigma^i \in \Sigma^i$ with $\sigma^i > \omega_i$. Then, Condition 1 follows from Lemma 8b) and the induction hypothesis applied to Conditions 1 and 2. Condition 2 follows from the tie-breaking rule applied by the party that eventually forms coalition.

■

The next proposition shows uniqueness of equilibrium *payoffs*. Equilibrium *strategies* are not unique for some subgames. In subgames $\mathbb{B}(d,i)$ where no coalition can be formed (i.e., $b(i,\sigma^i) < 0$), any demand vector is part of a SPE and equilibrium payoffs are always 0 for all parties. Multiplicity may also arise in subgames where a coalition can be formed but $d_i^* = 0$, as the following example illustrates.

Example 4 *Consider the game* $[5;3,2,2,1,1]$ *and suppose* $d_1 = d_2 = 1$. *Equilibrium strategies at* $\mathbb{B}(d,3)$ *are not unique, but equilibrium payoffs are.*

At $\mathbb{B}(d,3)$ we have $d_3^* = 0$ and $\Sigma^3 = \{2,3\}$. If we look at this subgame in isolation, several equilibrium outcomes are possible: coalition $\{1,3\}$ (associated to $\sigma^3 = 2$), coalition $\{2,3,4\}$ or $\{2,3,5\}$ (associated to $\sigma^3 = 3$), coalition $\{2,3,4,5\}$ (which is not a minimal winning coalition), coalition $\{1,4,5\}$ (which does not include party 3), or even no winning coalition at all. Intuitively, since the parties in $\{3,4,5\}$ cannot get a positive payoff, they are indifferent between all these situations. However, parties that have moved before are *not* indifferent. If we take into account that the strategies must be part of an equilibrium for all the subgames, and in particular for subgame $\mathbb{B}(d,2)$, some of the equilibrium strategies at $\mathbb{B}(d,3)$ are not equilibrium strategies for $\mathbb{B}(d,2)$ and are discarded (cf. example **??**). In particular, a coalition containing party 2 must be formed in order for party 2 to have a best response at $\mathbb{B}(d,2)$. Nevertheless, multiplicity remains: after party 2 sets $d_2 = 1$, there are three possible equilibrium coalitions: $\{2,3,4\}$, $\{2,3,5\}$ and $\{2,3,4,5\}$. Nevertheless, all equilibrium strategies lead to the same payoffs.

Proposition 2 *Assume we are in a SPE in $\mathbb{B}(d,i)$. If $b(i,\sigma^i) \geq 0$ for some/all $\sigma^i \in \Sigma^i$, party i's payoff is d_i^* as defined in (1); otherwise party i's payoff is zero.*

Proof. We proceed by backwards induction on i. We prove the following three hypotheses:

1. If $b(i,\sigma^i) < 0$, all parties get zero in every SPE of $\mathbb{B}(d,i)$.
2. If $b(i,\sigma^i) > 0$, party i receives $d_i^* > 0$ in every SPE of $\mathbb{B}(d,i)$ and the coalition that forms satisfies $S \cap P_i \in \underset{T \subset P_i : \omega_T \geq q - \sigma^i}{\arg\max} (1 - d_T)$ for some $\sigma^i \in \Sigma^i$.
3. If $b(i,\sigma^i) = 0$,

   a) party i gets $d_i^* = 0$ in every SPE of $\mathbb{B}(d,i)$;

b) there is a SPE of $\mathbb{B}(d,i)$ in which a winning coalition forms;

c) if a winning coalition S forms, then $S \cap P_i \in \underset{T \subset P_i : \omega_T \geq q - \sigma^i}{\arg\max} (1 - d_T)$ for some $\sigma^i \in \Sigma^i$.

The induction hypothesis holds for party n. Now suppose it holds for party $i+1$. Does it hold for party i?

1. If $b(i, \sigma^i) < 0$, all parties get zero (Lemma 5).

2. If $b(i, \sigma^i) > 0$, party i cannot get more than d_i^* by forming coalition. If party i demands more than d_i^* and $b(i, \tau^i) \geq 0$, we know from Lemma 9a) that $i \notin \underset{T \subset P_{i+1} : \omega_T \geq q - \sigma^{i+1}}{\arg\max} (1 - d_T)$ for all $\sigma^{i+1} \in \Sigma^{i+1}$. The induction hypothesis implies that party i gets zero. If party i demands more than d_i^* and $b(i, \tau^i) < 0$, we know from Lemma 9b) that party i gets zero.

   Now we show that party i can get at least d_i^*. This is immediate if $\omega_i \in \Sigma^i$. Suppose $\omega_i \notin \Sigma^i$. Since $b(i, \sigma^i) > 0$, we know $d_i^* > 0$. The value of d_{i+1}^* induced by d_i^* may be strictly positive or 0. Suppose party i demands $d_i < d_i^*$. Then the corresponding value of d_{i+1}^* is strictly positive. Under Lemma 8a), party i belongs to all coalitions associated with some element of Σ^{i+1}, and the induction hypothesis for $d_{i+1}^* > 0$ implies that party i gets d_i. Thus, the perfectness of the equilibrium implies that d_i^* is accepted (otherwise, party i would not have a best response).

   Moreover, Lemma 8c), the induction hypothesis and the fact that d_i^* is accepted imply that the coalition that forms satisfies $S \cap P_i \in \underset{T \subset P_i : \omega_T \geq q - \sigma^i}{\arg\max} (1 - d_T)$ for some $\sigma^i \in \Sigma^i$.

3. If $b(i, \sigma^i) = 0$, then $d_i^* = 0$ and, moreover, $\alpha \in \Sigma^i$ if and only if $b(i, \alpha) = 0$.

$a)$ It is trivial that party i gets $d_i^* = 0$. If $d_i > d_i^*$, the induction hypothesis implies that no coalition to which party i belongs will form.

$b)$ There is an equilibrium of the subgame in which a coalition associated with $\sigma^i \in \Sigma^i$ forms. This is clearly the case for $\omega_i \in \Sigma^i$. Otherwise, it is optimal for party i to demand $d_i^* = 0$. Then $b\left(i+1, \sigma^{i+1}\right) = 0$ for all $\sigma^{i+1} \in \Sigma^{i+1}$ and the induction hypothesis implies that there is a SPE of $\mathbb{B}\left(i, d\right)$ in which a winning coalition is formed.

$c)$ Assume a winning coalition S is formed with $S \cap P_i \notin \underset{T \subset P_i : \omega_T \geq q - \sigma^i}{\arg\max} \left(1 - d_T\right)$ for all $\sigma^i \in \Sigma^i$. This means that, for a given $\sigma^i \in \Sigma^i$, either $\omega_{S \cap P_i} \geq q - \sigma^i$ but $1 - d_{S \cap P_i}$ is not maximal, or $\omega_{S \cap P_i} < q - \sigma^i$.

Assume first there exists $\sigma^i \in \Sigma^i$ such that $\omega_{S \cap P_i} \geq q - \sigma^i$ but $1 - d_{S \cap P_i}$ is not maximal. Since $b\left(i, \sigma^i\right) = 0$, this means $d_{S \cap P_i} > 1$ and it cannot be optimal at any subgame to form S.

Assume now $\omega_{S \cap P_i} < q - \sigma^i$ for all $\sigma^i \in \Sigma^i$. Since $b\left(i, \sigma^i\right) = 0$ and $b\left(i, \alpha\right)$ is nondecreasing in α, $\delta^i \in \Sigma^i$; thus $\omega_{S \cap P_i} < q - \delta^i$. This means $\omega_{S \cap P_i} + \omega_{S \cap (N \setminus P_i)} < q$. Thus, S is not a winning coalition.

■

Corollary 3 *In any SPE, the coalition of Lemma 2 forms with each party demanding* $d_i = \frac{\omega_i}{q}$.

Proof. Denote this coalition by S^*. Because of lemma 2, $S^* = P_{l+1}$ for some value of l. We can show $d_i = \frac{\omega_i}{q}$ for $i = 1, ..., l$ by induction on i.

Party 1 finds $\Sigma^1 = \{q\}$ and, since $q \leq \delta^2$ (due to the absence of veto players and the game being constant-sum) sets a demand $d_1^* = \frac{\omega_1}{q}$. Given this demand, $q - \omega_1 \in \Sigma^2$.

Assume now $d_j = \frac{\omega_j}{q}$ for all $j \in P_i$, and $q - \omega_{P_i} \in \Sigma^i$. Then,

$$d_i^* = \frac{\omega_i b\left(i, q - \omega_{P_i}\right)}{q - \omega_{P_i}} = \frac{\omega_i \left(1 - \frac{\omega_{P_i}}{q}\right)}{q - \omega_{P_i}} = \frac{\omega_i}{q}.$$

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