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# All-Pay Auctions with Extra Prize: A Partial Exclusion Principle*

Matthias Dahm†

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**Abstract**

This paper studies the effects of a specific affirmative action policy in complete information all-pay auctions when players differ in ability. We call this policy an extra prize. The contest organiser splits the prize of the competition into a main prize and an extra prize. Extra prizes differ from second prizes, because they are targeted towards disadvantaged (low-ability) agents. We consider a setting with one high-ability and two low-ability contestants and fully characterise equilibrium. Assuming that the contest organiser aims to maximise expected total effort, we show that (i) almost any extra prize is preferable to a standard all-pay auction without extra prize; (ii) the exclusion principle (Baye, Kovenock and de Vries, 1993) can be implemented by a wide range of sufficiently large extra prizes; and (iii) partial exclusion by means of an appropriately chosen extra prize benefits the organiser more than complete exclusion.

*Keywords:* Asymmetric contests, multi-prize contests, equality of opportunity, affirmative action, discrimination, prize structure, exclusion principle

*JEL:* C72, D72, J78

## 1 Introduction

*I don't agree with opening up the Booker for the Americans, I think that's straightforwardly daft. The Americans have got enough prizes of their own. The idea of [the Booker] being Britain, Ireland, the old Commonwealth countries and*

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*new voices in English from around the world gave it a particular character and meant it could bring on writers. If you also include Americans - and get a couple of heavy hitters - then the unknown Canadian novelist hasn't got a chance.*

- Julian Barnes, Man Booker winner in 2011[1]

The above quote refers to a change in the rules for participation of the Man Booker Prize for Fiction. Initially, only novelists from the UK, Commonwealth, Ireland and Zimbabwe were eligible to receive the prize. With the new rules, all fiction in English published worldwide is eligible. This, of course, increases competition. Barnes is concerned that, as a result of the rule change, new novelists will lose against established writers and find it more difficult to win recognition. In this sense we can think of the prize with the initial restricted participation rule as an affirmative action instrument: the Booker prize is a targeted prize for some competitors, in addition to the main competition in which all novelists compete with each other for recognition. The purpose of the present paper is to investigate the incentive effects of such a policy. We do so using an all-pay auction, which is a well established tool for modeling competition. Our main result is that such a prize enhances competition. Consequently, even a contest designer who is not interested in affirmative action per se might decide to establish it.

The competition of novelists described above is a special case of a situation in which players compete by investing a costly and sunk resource in order to increase their probability of winning. In addition, some contestants have more options to receive a reward for their investment than others. This is not unusual.[2] Consider funding for research projects. All researchers in a given country might have access to funding from a central Government agency but only some regions might offer funding through a regional funding agency. Or consider World Chess Championships. The main event is open to all players but in addition some groups (women, junior or seniors) have their own competitions. Another example is Government funding for entrepreneurs. Young entrepreneurs might have access to funding programmes that are only open to them, in addition to funding competitions that are open to all entrepreneurs.[3] A last example is a prize for the best academic paper by a young scientist.

[1]See Mark Brown, 'Julian Barnes: letting US authors compete for Booker prize is 'daft' ', The Guardian, 27, November 2016.

[2]Of course, there are also other international awards complemented by a prize for national competitors. For example, the City Council of Tarragona organises a fireworks competition that has a main (international) prize and a prize for Catalan competitors. In 2009 both prizes were won by the same competitor. Another example is the category of Best Film. In 2011, the Catalan film 'Black Bread' won both the (Spanish) Goya Award and the (Catalan) Gaudí Award. Other film festivals establish additional prizes for example for youth, students or female contestants.

[3]Currently, the Spanish Ministry of Industry has such a programme for entrepreneurs younger than 40 years.

What distinguishes the examples above from the standard contest model of competition is that a player's prospects of receiving a reward for sunk investment depends not only on the magnitude of the investment but also on the player's identity. Players that belong to a disadvantaged group compete for a prize that others do not have access to. This creates a very specific prize structure. All agents compete for the main prize, but only disadvantaged players contest the extra prize. In this sense the competition is characterized by *targeted rewards for sunk investments*. This implies that even though there are two prizes, extra prizes are different from second prizes. The reason is that second prizes, or consolation prizes, are not targeted. They can be won by all agents and change the prize structure for all agents in the same way. In contrast, extra prizes reinforce the incentives of disadvantaged contestants to invest resources in the competition without affecting the prize structure for other agents.

We investigate the effects of extra prizes in an all-pay auction under complete information and with heterogeneous players. Our contribution is twofold. First, we fully characterise equilibrium in a setting with one high-ability and two low-ability contestants, when only the low-ability agents contest the extra prize.[4] We show that the introduction of the extra prize weakens the high-ability agent and strengthens low-ability contestants. In the extreme, when the size of the extra prize is large, the ranking of players in terms of their strengths is reversed; to measure strengths Siegel's (2009) notion of the reach of a player can be appropriately extended. In our model, the extra prize stimulates participation of disadvantaged contestants, because it will always be contested. Moreover, since disadvantaged contestants are symmetric, in equilibrium they use the same strategy. We show that the equilibrium is unique, unless the size of the extra prize is equal to the relative difference in abilities, because then all agents are equally strong in terms of their reach.[5]

Our second contribution is to provide a deeper understanding of the exclusion principle (Baye et al., 1993). This principle says that under some conditions the contest organiser benefits from excluding the most competitive contestant. The reason is that this can level the playing field and strengthen competition among the remaining agents. As a result total expected effort might increase. Complete exclusion of the advantaged contestant from the competition is an extreme case of an extra prize; it is obtained when the extra prize is as large as possible and the main prize is set to zero. We show that complete exclusion

[4] It is well known that the analysis of asymmetric multiple prize all-pay auctions under complete information is very complex (see Cohen and Sela, 2008). Our restriction to three contestants allows us to show uniqueness of equilibrium for almost all sizes of the extra prize. This enables us to compare the contest with extra prize unambiguously to a standard contest without extra price, even though in the latter there is a multiplicity of equilibrium. Moreover, one can show that in a model with further disadvantaged agents of lower abilities, the equilibria studied here remain an equilibrium.

[5] When all contestants have the same reach there is a continuum of equilibria that arises from the possibility that the advantaged player might abstain with different probabilities from the contest. Equilibria with very related properties exist in the standard all-pay auction when agents have the same reach (Baye et al., 1996).

is not necessary. Large extra prizes weaken the advantaged contestant enough so that in equilibrium he abstains with certainty from the contest.[6] Moreover, we show that complete exclusion is not optimal. Choosing the size of the extra prize roughly equal to the relative difference in abilities, the contest organiser can induce considerably higher expected total effort. Competition is strengthened, because this extra prize levels the playing field completely (in terms of the reach of players). Since the advantaged contestant is only excluded from a part of the overall prize, a partial exclusion principle holds. Our analysis also implies that almost any extra prize is preferable to a standard all-pay auction.[7]

Our results complement the analysis in Dahm and Esteve-González (2016). Dahm and Esteve-González investigate extra prizes in a model with an imperfectly discriminating Tullock contest success function. In both models an extra prize levels the playing field and increases the strength of competition. But, contrary to the present paper, in Dahm and Esteve-González an extra prize is only beneficial for intermediate levels of heterogeneity. In this sense the effect of an extra prize is stronger when the contest success function is perfectly discriminatory. This helps to understand why the exclusion principle discovered by Baye et al. (1993) holds in the all-pay auction but, as shown by Fang (2002), does not hold in Tullock contests with economics of scale parameter equal to one.[8]

Since in our model there is a main prize and an extra prize, our paper contributes to the question under which conditions a contest organiser finds it optimal to establish more than one prize in all-pay auctions under complete information.[9] Glazer and Hassin (1988) and Cohen and Sela (2008) provide conditions under which more than one prize should be

[6]Complete exclusion might not be feasible, for instance, for legal reasons.

[7]The reason for why we do not claim that *any* extra prize is preferable to a standard all-pay auction is as follows. As the extra prize goes to zero, the unique equilibrium becomes the equilibrium of the standard all-pay auction in which symmetric players use symmetric strategies. In the standard all-pay auction there is, however, another equilibrium in which only one of the symmetric players is active that has higher expected total effort. Thus depending on how agents coordinate in the standard all-pay auction, in equilibrium total expected effort might be higher in a standard all-pay auction than with a very small extra prize.

[8]The exclusion principle does hold when the economics of scale parameter in a Tullock contest is large, so that the contest approaches the all-pay auction (Alcalde and Dahm, 2010). There are also other not perfectly discriminating contest success functions, different from Tullock's proportional form, for which the principle holds (Alcalde and Dahm, 2007). A variation of the exclusion principle has appeared in Konrad (2006). When firms have silent ownership shares in rivals, then a firm may be able to commit to abstain from the contest. This might be profitable, because it reduces the level of competition. Ownership shares, however, can also increase the level of competition, see Fu and Lu (2013). Menicucci (2006) and Bertoletti (2008) investigate the exclusion principle when the contest organiser is not fully informed about the contestants.

[9]The single-prize all-pay auction under complete information has been studied by Hillman and Samet (1987) and Hillman and Riley (1989), and its equilibrium has been completely characterised by Baye et al. (1996). Glazer and Hassin (1988), Barut and Kovenock (1998), Clark and Riis (1998) and Cohen and Sela (2008) analyse multiple-prize all-pay auctions under different assumptions concerning the contestants' valuations for prizes. Recent work by Siegel (2009, 2010 and 2014) and Xiao (2016) extends this model in many ways, including head starts. Sisak (2009) provides a review of the literature on multiple-prize contests.

established. Barut and Kovenock (1998), however, show that in their model the organiser is indifferent between establishing one or several prizes.[10] Our model differs from these multiple-prize all-pay auctions, because in these papers prizes are not targeted to specific groups of contestants.

Although our model is very stylized, it allows us to draw policy implications. Fu (2006) and Pastine and Pastine (2012) have used a closely related all-pay auction model in order to understand affirmative action in college admissions. Two conclusions that emerge from these papers are the following. First, even if the college only values the academic quality of its students, it may implement affirmative action, because it strengthens competition. Second, it matters how the affirmative action policy is implemented. In both papers the admission test score of the disadvantaged applicants is increased, which creates a biased version of the all-pay auction. But while in Fu (2006) this bias is multiplicative, in Pastine and Pastine (2012) it is additive. As a result, effort incentives are not the same and the resulting equilibrium student-body diversity is different. Our paper complements these studies in important ways, because some real affirmative action policies are based on quotas. These targeted rewards might be better described by our model. Moreover, the details of affirmative action policies vary widely, and so it is important to know that targeted rewards also have the potential to level the playing field and strengthen competition.[11]

The paper is organised as follows. The next section presents our model of extra prizes. We provide a full characterisation of equilibrium in Section 3. Section 4 investigates expected total effort in equilibrium and establishes the partial exclusion principle. The last section contains concluding remarks. All proofs are relegated to an Appendix.

[10] Moldovanu and Sela (2001) analyse a different all-pay auction model with incomplete information and show that when cost functions are convex several prices might be optimal; see also Liu and Lu (2017) for a related model. Clark and Riis (1998) investigate whether in a multiple-prize all-pay auction prizes should be awarded simultaneously or sequentially. Clark and Riis also show that the exclusion principle does not necessarily hold in multiple-prize all-pay auctions. Arbatskaya (2003), however, builds on the results by Barut and Kovenock (1998) and shows that a contest organiser might even in a symmetric setting benefit from excluding contestants, provided valuations depend on the number of contestants.

[11] A different question is whether a contest organiser prefers to establish an extra prize or to bias the selection rule (multiplicatively). Using the analysis in Franke et al. (2014) and the proof of Proposition 3 below one can show that the optimal bias generates higher total expected effort. The examples in Section 4, however, suggest that this difference is very small when the ability of agents is relatively similar. Thus, when—perhaps because of fairness concerns—it is not possible to bias the selection rule, an extra prize might be an interesting alternative policy. Notice also that Pastine and Pastine (2012) argue that many affirmative action policies imply additive bias, which Li and Yu (2015) and Franke et al. (2016) show generates higher total expected effort than multiplicative bias.

# 2 Model

There are three risk-neutral contestants with abilities $\alpha_i$. Agent 1 is advantaged and has high ability, while agents 2 and 3 are homogeneous and have low ability, so that $\alpha_1 \geq \alpha_d > 0$ with $d = 2,3$. Our equilibrium characterisation in Section 3 allows for $\alpha_1 = \alpha_d$ but for simplicity of the exposition in Section 4 we exclude the case in which all agents have the same ability. Contestants compete exerting effort $e_i \in \mathbb{R}_+$ and different abilities are reflected in heterogeneous effort costs $c_i(e_i) = e_i/\alpha_i$. Effort is not recovered.

Contestants compete simultaneously for a budget $B$, which without loss of generality is normalized to one. The budget is split into two prizes $(1-\beta)$ and $\beta$ with $\beta \in [0,1]$. Contestant 1 competes only for the main prize $(1-\beta)$, while the disadvantaged contestants 2 and 3 compete for both prizes. In other words, the set of contestants $N = \{1,2,3\}$ compete for the main prize (MP) of size $(1-\beta)$ and the set of agents $D = \{2,3\}$ contest the extra prize (EP) of size $\beta$. Notice that this implies that, although contestants 2 and 3 exert effort only once, they might win both prizes. This structure of targeted rewards distinguishes our model from other contests with multiple prizes, including second prizes. The contest designer chooses $\beta$ in order to maximise total effort.

We consider an all-pay auction setting in which prizes are assigned as follows. Given a set of agents $A \in \{N,D\}$ competing for prize $k \in \{MP, EP\}$, and given effort choices by the contestants, the win probability of agent $i \in A$ follows

$$p_i^k(e_1,e_2,e_3) = \begin{cases} 1 & \text{if } e_i > e_j, \forall j \in A \text{ with } j \neq i \\ \frac{1}{h} & \text{if } i \text{ ties with } h-1 \text{ others and } \nexists j \in A \text{ such that } e_i < e_j \\ 0 & \text{if } \exists j \in A \text{ such that } e_i < e_j \end{cases}. \tag{1}$$

Thus, given a vector of efforts $(e_1,e_2,e_3)$, player $i$'s expected payoff is

$$U_i(e) = p_i^{MP}(1-\beta) + p_i^{EP}\beta z_i - \frac{e_i}{\alpha_i}, \tag{2}$$

where $z_i \in \{0,1\}$ takes value 1 if $i \in D$, and value 0 otherwise. Note that this model includes two special cases. When $\beta = 0$ or $\beta = 1$ we obtain a standard all-pay auction without extra prize in which the set of contestants is $N$ or $D$, respectively.[12]

[12] In these situations the value of $\beta$ is such that there is only one prize. Consequently, the contestants eligible for this prize compete as in a standard all-pay auction. A similar situation arises if one disadvantaged agent does not contest the extra prize. For instance, if agent 2 exerts zero effort with probability one, then players 1 and 3 compete (for the main prize) as in a standard all-pay auction. The extra prize is not contested and by exerting some positive effort contestant 3 receives it with certainty. We will also use the term standard all-pay auction to refer to such a situation.

# 3 Equilibrium characterization

## 3.1 Preliminaries

It is useful to start by extending Siegel's (2009) notion of the "reach" to our model. In Siegel's model a contestant can win at most one prize and the reach of a contestant is the effort level at which the valuation for winning is zero. Since in our model a contestant can win more than one prize, we define contestant $i$'s reach $r_i$ to be the effort level for which his valuation for winning all prizes that he contests is zero. More precisely,

$$r_i = \max\left\{ e_i \in \mathbb{R} \middle| (1-\beta) + \beta z_i - \frac{e_i}{\alpha_i} = 0 \right\}, \tag{3}$$

with $z_i \in \{0,1\}$ taking value 1 if $i \in D$, and value 0 otherwise. Since rewards are targeted and the advantaged player does not compete for the extra prize, (3) becomes

$$r_1 = \alpha_1(1-\beta) \quad \text{and} \quad r_d = \alpha_d, \tag{4}$$

for $d = 2, 3$. It follows that the order of contestants by their reach depends on $\beta$. We have that $r_1 \geq r_d$ if and only if the size of the extra prize is at most equal to the relative difference in abilities, that is,

$$\beta \leq \frac{\alpha_1 - \alpha_d}{\alpha_1} \equiv \hat{\beta}. \tag{5}$$

We will see that, as in other all-pay auctions, the second highest reach plays an important role in our analysis. In our simple three contestant model the second highest reach is $r_d$ for any size of the extra prize $\beta$.

It is well known that in the standard all-pay auction without extra prize there is no Nash equilibrium in pure strategies. This remains true when an extra prize is introduced (formally this follows from Lemma 2 below). Consequently, we consider Nash equilibria in mixed-strategies. We represent the equilibrium mixed-strategy of contestant $i$ by the cumulative distribution function (cdf) $G_i(e_i)$. Agent $i$ randomises continuously on an interval $S$ if his mixed-strategy contains no mass points and has a strictly increasing cdf almost everywhere on $S$. We denote by $\gamma_i(e_i)$ the mass placed at $e_i$ by contestant $i$'s mixed strategy. We say that a contestant $i$ is *active* if $\gamma_i(0) < 1$. When $\gamma_i(0) = 1$ we say that contestant $i$ *abstains* from the contest. Lastly, when $\gamma_i(0) = 0$ we say that contestant $i$ *never abstains* from the contest.

We are now in a position to describe some basic properties of the equilibrium.

**Lemma 1** *For any $\beta > 0$, in any equilibrium both disadvantaged contestants are active.*

The previous lemma shows that an extra prize is a powerful tool to make sure disadvantaged agents compete. The extra prize will always be contested. The next lemma shows that disadvantaged are not only active but never abstain. In addition, the next lemma establishes further important equilibrium properties.

**Lemma 2** *For any $\beta > 0$, in any equilibrium the following holds:*

1. *Both disadvantaged contestants employ the same mixed-strategy $G_2 = G_3$ and obtain an expected equilibrium payoff of zero.*
2. *For all $i$, there is no contestant $i$ who places mass on $(r_d, \infty)$, $G_i$ contains no atoms in the half open interval $(0, r_d]$ and the disadvantaged contestants do not place an atom at zero.*

As in other all-pay auctions, no contestant exerts more effort than the second highest reach $r_d$. Disadvantaged contestants use the same strategy, randomise continuously on $[0, r_d]$, place no atom anywhere, and dissipate all rents from the competition. The advantaged agent, however, might place an atom at zero. As in the standard all-pay auction without extra prize, equilibrium implies that there are no atoms at points different from zero. Importantly, since Lemma 2 rules out that at equilibrium disadvantaged contestants use different strategies, it must hold that the set of active agents is either $D$ or $N$. It also allows us to use the notation $G_1(e_1)$ and $G_d(e_d)$ for the cdfs of the advantaged and disadvantaged agents' efforts $e_1 \geq 0$ and $e_d \geq 0$, respectively, with $d = 2, 3$.

## 3.2 Large extra prizes

It is reasonable to expect that the size of the extra prize determines the characteristics of the equilibrium. On one hand, if the extra prize is very small, one might expect it not to affect behaviour much, so that the equilibrium is similar to the standard three agent all-pay auction. On the other hand, if the extra prize is very large, the advantaged contestant is excluded from a large part of the prize and one might expect the equilibrium to be similar to the standard two player all-pay auction among disadvantaged agents. In the present and the following subsection we show that this intuition is indeed true. We start with a large extra prize.

**Proposition 1** *For any configuration of abilities $\alpha_1 \geq \alpha_d$ the following holds:*

*(i) If and only if $\beta \geq \hat{\beta}$, there is a mixed-strategy equilibrium in which the advantaged agent abstains and disadvantaged agents play the same strategy characterized by the probability distribution function $G_d(e_d)$ for efforts $e_d \geq 0$ with*

$$G_d(e_d) = \begin{cases} \frac{e_d}{\alpha_d} & \text{if } e_d \in [0, r_d] \\ 1 & \text{if } e_d \geq r_d \end{cases} . \tag{6}$$

*The equilibrium payoff of all contestants is zero.*

*(ii) If $\beta > \hat{\beta}$, then the equilibrium described in part (i) is unique.*

Notice that the previous proposition covers the case in which $\beta = 1$, so that there is only the extra prize and no main prize. In this case the contest reduces to a standard all-pay auction in which only the disadvantaged agents compete, as the advantaged agent is completely excluded. Theorem 1 in Baye et al. (1996) states that in this case the unique equilibrium is as described in the statement.[13] Proposition 1 shows that this unique equilibrium remains unchanged, provided the extra prize is strictly larger than the relative difference in abilities. This implies that if there is not much difference in abilities even a small extra prize can already be sufficient to discourage the advantaged agent from participating. The intuition for this is that, although in such a case the advantaged contestant is formally only excluded from a small part of the overall prize, this weakens him enough to convert the disadvantaged agents in the strong contestants (as measured by the reach of agents). As in Theorem 1 of Baye et al. (1996) a contestant exerts zero effort with probability one, if he competes against two strictly stronger players.

## 3.3 Small extra prizes

We turn now to small extra prizes. Since Lemma 2 implies that disadvantaged agents employ the same strategy, one might expect the equilibrium to be related to the symmetric equilibrium in the standard three agent all-pay auction. This intuition turns out to be correct. Stating this result formally requires to define the following real number $\Phi$.

**Lemma 3** *For any $\alpha_1$, $\alpha_d$ and $\beta > 0$, there exists a unique $\Phi > 0$ such that*

$$\frac{\Phi}{\alpha_d}\sqrt{\frac{\alpha_1}{(1-\beta)(\alpha_1(1-\beta)-\alpha_d+\Phi)}}-\frac{\beta}{1-\beta}=0. \tag{7}$$

We are now in a position to state the following result.

**Proposition 2** *For any configuration of abilities $\alpha_1 \geq \alpha_d$ the following holds:*

*(i) If and only if $\beta \leq \hat{\beta}$, there is a mixed-strategy equilibrium which is characterized by probability distribution functions $G_1(e_1)$ and $G_d(e_d)$ for the advantaged and disadvantaged agents' efforts $e_1 \geq 0$ and $e_d \geq 0$, respectively, with*

$$G_1(e_1) = \begin{cases} 0 & \text{if } e_1 \in [0,\Phi] \\ \frac{e_1}{\alpha_d}\sqrt{\frac{\alpha_1}{(1-\beta)(\alpha_1(1-\beta)-\alpha_d+e_1)}}-\frac{\beta}{1-\beta} & \text{if } e_1 \in [\Phi, r_d] \\ 1 & \text{if } e_1 \geq r_d \end{cases} \tag{8}$$

[13] For completeness we mention that in Baye et al. (1996) contestants differ in their valuations for the prize, while in our model agents differ in ability. Starting with an equilibrium in Baye et al.'s model, it is, however, straightforward to modify the cdfs and obtain an equilibrium in the standard all-pay auction with heterogeneity in abilities.

*and*

$$G_d(e_d) = \begin{cases} \frac{e_d}{\alpha_d \beta} & \text{if } e_d \in [0,\Phi] \\ \sqrt{\frac{\alpha_1-\alpha_d}{\alpha_1(1-\beta)} + \frac{e_d}{\alpha_1(1-\beta)} - \frac{\beta}{1-\beta}} & \text{if } e_d \in [\Phi, r_d] \\ 1 & \text{if } e_d \geq r_d \end{cases} . \tag{9}$$

*The equilibrium payoff of contestant 1 is $\hat{\beta} - \beta$, while the disadvantaged agents earn zero.*

*(ii) If $0 < \beta < \hat{\beta}$, then the equilibrium described in part (i) is unique.*

In the equilibrium described in the previous proposition, the advantaged contestant is aggressive, in the sense that he exerts a minimum effort of $\Phi$.[14] This allows him to obtain a positive equilibrium payoff. Disadvantaged agents compete with the advantaged contestant for high effort levels and compete with each other for the extra prize for low effort efforts (lower than $\Phi$). In fact, for low effort levels the disadvantaged agents compete as in (6) with the distributions rescaled by $1/\beta$, the size of the extra prize. As a result, the rent of disadvantaged agents is completely dissipated.

Notice that Proposition 2 covers the case in which $\beta = 0$. In this case the contest reduces to a standard all-pay auction for the main prize and there is no extra prize. Baye et al. (1996) have shown that there is a continuum of equilibria, including one in which the two disadvantaged agents play the same strategy. Indeed, for $\beta = 0$ we have that $\Phi = 0$ and (8) and (9) coincide with cdfs described in Theorem 2 in Baye et al. (1996) when the two disadvantaged agents play the same strategy. Proposition 2 indicates how this equilibrium changes as the extra prize is introduced. In particular, it is straightforward to show that as $\beta$ increases, the disadvantaged contestants become more aggressive (in the sense of first-order stochastic dominance). In addition, when $\beta$ is small enough compared to $\hat{\beta}$, the advantaged contestant also seems to become more aggressive.[15] We will discuss these issues further in Section 4.

## 3.4 Intermediate extra prizes

We consider now the special case of an intermediate extra prize, which is equal to the relative difference in abilities. The analysis so far has already established that there exist at least two equilibria – one in which the advantaged contestant abstains ($\gamma_1(0) = 1$) and one in which he never abstains ($\gamma_1(0) = 0$). The next result shows that there is a continuum of equilibria and that the two aforementioned equilibria are extreme cases of this continuum.

[14] A similar feature appears in a standard all-pay auction with additive bias in the function assigning the win probabilities, see Li and Yu (2012).

[15] When $\beta$ is close to $\hat{\beta}$, one can find configurations of abilities for which an increase in $\beta$ makes the advantaged contestant less aggressive.

To state this result formally, it is convenient to introduce the following notation. Given a parameter $\gamma_1(0) \in [0,1]$, we define the real number

$$\lambda \equiv ((1-\beta)\gamma_1(0)+\beta)^2 \,\alpha_d. \tag{10}$$

Notice that, since $\lambda$ is strictly increasing in $\gamma_1(0)$, we have that $\lambda \in [\beta^2\alpha_d, \alpha_d]$.[16]

**Proposition 3** *Let $\beta = \hat{\beta}$. For any configuration of abilities $\alpha_1 > \alpha_d$ there is a continuum of mixed-strategy equilibria in which disadvantaged agents play the same strategy and the advantaged contestant abstains with probability $\gamma_1(0) \in [0,1]$, where $\gamma_1(0)$ is a free parameter, and randomises continuously over the interval $[\lambda, r_d]$, where $G_1(\lambda) = \gamma_1(0)$. More precisely, the equilibrium is characterized by probability distribution functions $G_1(e_1)$ and $G_d(e_d)$ for the advantaged and disadvantaged agents' efforts $e_1 \geq 0$ and $e_d \geq 0$, respectively, with*

$$G_1(e_1) = \begin{cases} \gamma_1(0) & \text{if } e_1 \in [0,\lambda] \\ \frac{\alpha_1}{\alpha_d}\sqrt{\frac{e_1}{\alpha_d}} - \frac{\alpha_1-\alpha_d}{\alpha_d} & \text{if } e_1 \in [\lambda, r_d] \\ 1 & \text{if } e_1 \geq r_d \end{cases} \tag{11}$$

*and*

$$G_d(e_d) = \begin{cases} \frac{e_d}{((1-\beta)\gamma_1(0)+\beta)\alpha_d} & \text{if } e_d \in [0,\lambda] \\ \sqrt{\frac{e_d}{\alpha_d}} & \text{if } e_d \in [\lambda, r_d] \\ 1 & \text{if } e_d \geq r_d \end{cases} . \tag{12}$$

*In any equilibrium the equilibrium payoffs of all contestants are zero.*

The previous proposition bridges the equilibria in Propositions 1 and 2. When $\gamma_1(0) = 1$, then $\lambda$ equals $r_d$ and (11) prescribes that the advantaged contestant abstains, while (12) becomes (6). As $\gamma_1(0)$ decreases, $\lambda$ decreases and the equilibrium has a similar structure to the one in Proposition 2: there is an interval of high effort levels on which all contestants are active and an interval of low effort levels on which only the disadvantaged agents are active contesting only the extra prize. Again, for low effort levels the disadvantaged agents compete as in (6) with the distributions rescaled by $1/((1-\beta)\gamma_1(0)+\beta)$, which represents the part of the overall prize that is uncontested by the advantaged agent. In the extreme, when $\gamma_1(0) = 0$, then $\lambda$ is largest and Proposition 3 becomes the special case of Proposition 2 in which $\beta = \hat{\beta}$.

Moreover, since in Proposition 3 the size of the extra prize $\hat{\beta}$ is equal to the relative difference in abilities, all contestants have the same reach. This has two implications that parallel the standard all-pay auction. First, all rents are completely dissipated and all equilibrium payoffs are zero. Second, the possibility that one contestant places mass at zero creates the

[16] For simplicity we write $\lambda$ instead of $\lambda(\gamma_1(0))$.

possibility of equilibrium multiplicity. But while in Baye et al. (1996) the identity of the agent placing mass at zero is arbitrary, in our model the existence of the extra prize implies that disadvantaged contestants never abstain from the contest.[17]

# 4 A partial exclusion principle

In this subsection we use the equilibrium characterizations of the previous section to compare partial exclusion of the advantaged agent by means of an extra prize to two benchmarks. These benchmarks are, on one hand, a standard all-pay auction without extra prize and, on the other hand, complete exclusion of the advantaged agent. As mentioned before, our model reduces to these benchmarks by setting $\beta = 0$ and $\beta = 1$, respectively. Since complete exclusion is only an interesting option for a contest organiser when abilities differ, we assume in this section that $\alpha_1 > \alpha_d$.

Consider first the benchmark of $\beta = 0$, which has been analysed in Baye et al. (1996). There is a continuum of equilibria that are not revenue equivalent. Revenue is maximized in the equilibria in which one disadvantaged agent abstains. For later reference we call this "best case" equilibrium the "asymmetric standard all-pay auction" equilibrium and state that it generates an expected sum of effort of $\alpha_d (\alpha_d + \alpha_1) / (2\alpha_1)$. Revenue is minimized in the equilibrium in which the disadvantaged contestants use the same strategy. As we have seen the symmetric equilibrium is closely related to the equilibrium in Proposition 2.

Consider now the case $\beta = 1$, which also has been analysed in Baye et al. (1996). Complete exclusion of the advantaged contestant from the competition is a special case of our Proposition 1 and yields an expected sum of effort equal to $\alpha_d$. Our setting is a special case of Baye et al.'s (1993) exclusion principle. Consequently, our assumption $\alpha_1 > \alpha_d$ implies that the organiser of the competition benefits from complete exclusion, as total expected effort in the "asymmetric standard all-pay auction" equilibrium is lower than $\alpha_d$. Since we know from Proposition 1 that the unique equilibrium under complete exclusion does not change for smaller but sufficiently large extra prizes, we immediately have the following variation of the exclusion principle.

**Corollary 1** *Compared to a standard all-pay auction, the contest organiser benefits strictly from any extra prize of size $\beta > \hat{\beta}$, because it implements the exclusion principle.*

[17]The case of three contestants with equal valuations (and therefore the same reach) is a special case of Theorem 1 in Baye et al. (1996). Proposition 3 covers this case by setting $\alpha_1 = \alpha_d$, implying that the size of the extra prize $\hat{\beta}$ is equal to zero. In this case (11) and (12) coincide with the mixed-strategies in Baye et al.'s symmetric three player example. Baye et al. have shown that there is a unique symmetric equilibrium and that there is a continuum of asymmetric equilibria in which one contestant places mass at 0. The only difference between (11) and (12) for $\alpha_1 = \alpha_d$ and Baye et al. is that in our setting the identity of the contestant placing mass at 0 is not arbitrary.

This result is interesting, because it implies that complete exclusion of the advantaged contestant is straightforward to implement. It is not necessary to formally exclude the advantaged agent from the competition, which might not be feasible for legal or ethical reasons. Such an explicit entry barrier can be avoided, because it suffices to establish an extra prize that is larger than the relative difference in abilities. Interestingly, if this relative difference is small, say 1/6, then the extra prize can be quite small, for example 1/5.

Given Corollary 1, an important question is whether the contest organiser can do better than implementing the exclusion principle. Our main result reveals that this is the case.

**Proposition 4** *Compared to complete exclusion of the advantaged agent, the contest organiser benefits strictly from partial exclusion by means of a sufficiently large extra prize of size* $\beta < \hat{\beta}$.

The intuition for this result is simple. When the size of extra prize is equal to the relative difference in abilities, there is a continuum of equilibria (Proposition 3). Broadly speaking these equilibria differ in the size of the atom that the advantaged contestant places at the origin. The smaller the size of the atom, the more aggressive the advantaged contestant becomes (in the sense of first-order stochastic dominance) and even though the disadvantaged agents become less aggressive, the first effect is stronger than the second. This implies that an extra prize equal to the relative difference in abilities improves almost always strictly over complete exclusion. The only exception is the equilibrium in which the advantaged contestant places an atom of mass one at the origin, in which case the same result as under complete exclusion is obtained.[18] Consider now an extra prize of a size a little smaller than the relative difference in abilities. Because of the continuity of the cdfs and because for smaller extra prizes the equilibrium is unique (Proposition 2), it follows that the contest organiser can be certain that partial exclusion by means of an appropriately chosen extra prize improves strictly upon complete exclusion.[19]

We summarise the discussion of this section with the help of Figure 1. The figure displays five examples. In each example we fix $\alpha_1 = 1$, while $\alpha_d$ takes values 1/10, 1/4, 1/2, 3/4 and 9/10. To distinguish these cases we denote the threshold $\hat{\beta}$ by $\hat{\beta}^{\alpha_d}$. Given the values for $\alpha_d$, the thresholds $\hat{\beta}^{\alpha_d}$ are equal to 9/10, 3/4, 1/2, 1/4 and 1/10. The horizontal axis

[18] In the Appendix in the proof of Proposition 4 we provide expressions for individual and total expected effort as a function of the atom that the advantaged contestant places at the origin.

[19] Because of the mathematical complexity of the expression for total expected effort when $\beta < \hat{\beta}$, we have not been successful to prove that total expected effort is strictly increasing in $\beta$ for all $\beta < \hat{\beta}$. We conjecture, however, that this is true. It is straightforward to show that as $\beta$ increases the disadvantaged contestants become more aggressive (in the sense of first-order stochastic dominance). The examples that we have calculated suggest that the same is true for the advantaged contestant unless $\beta$ is very close to $\hat{\beta}$. But even when $\beta$ is very close to $\hat{\beta}$ and the aggressiveness of the advantaged contestant is reduced, our examples suggest that this is overwhelmed by the increased aggressiveness of the disadvantaged contestants so that total expected effort continues to increase.

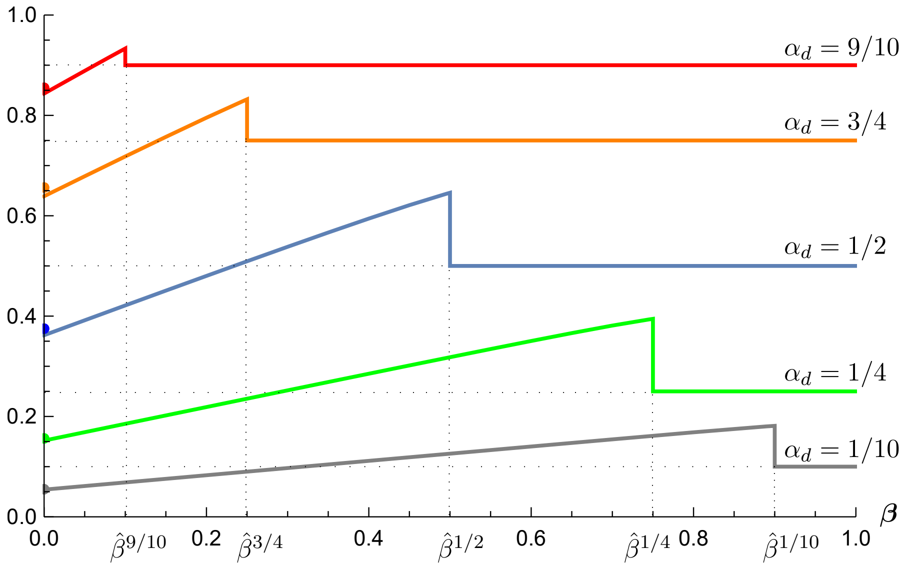


Figure 1: The size of the extra prize and the sum of expected effort

indicates the size of the extra prize $\beta$, while the vertical axis measures the expected sum of effort. Each curve represents a different value for $\alpha_d$ and includes for $\beta = 0$ the equilibrium in which the disadvantaged contestants employ the same strategy. The isolated point higher but very close to each curve at $\beta = 0$ indicates for each example total expected effort in the asymmetric standard all-pay auction equilibrium. In line with the exclusion principle, we see that $\beta = 1$ generates strictly higher total expected effort than $\beta = 0$. We also see that complete exclusion can be implemented by a wide range of sufficiently large extra prizes. The vertical parts of the curves correspond to the continuum of equilibria when $\beta = \hat{\beta}^{\alpha_d}$. Choosing an extra prize equal to the relative difference in abilities, the contest organiser cannot be worse-off than under complete exclusion. Total expected effort is maximized when the extra prize is approximately equal to the relative difference in abilities.[20] Notice also that a wide range of sizes for the extra prize allow the contest organiser to strictly improve over complete exclusion and that almost any extra prize improves over a standard all-pay auction.

[20] If the contest organiser is certain that the equilibrium is played in which the advantaged agent does not place mass at zero, then it is optimal to set the extra prize equal to $\hat{\beta}$. If this is not the case, then he can avoid coordination on an unfavourable equilibrium by reducing the extra prize a little bit. A technical issue concerns the existence of the optimal extra prize. That may be solved by making the realistic assumption that a smallest monetary unit exists.

# 5 Concluding remarks

This paper analysed the effects of establishing an extra prize for disadvantaged agents in an all-pay auction under complete information. The overall prize value is split in a main prize and an extra prize. All types of contestants compete for the main prize, but only low-ability agents can win the extra prize. We fully characterise equilibrium in a setting with one high-ability and two low-ability contestants. Assuming that the contest organiser aims to maximize expected total effort, our main result establishes a partial exclusion principle. Partial exclusion of the advantaged agent from part of the overall prize value by means of an appropriately chosen extra prize benefits the organiser more than complete exclusion.

We have also shown that almost any extra prize is preferable to a standard all-pay auction. This result is robust to a variation of our informational assumptions. Following Menicucci (2006) and Bertoletti (2008) assume that the contest organiser is not fully informed about the contestants but that contestants still know each others' abilities. In such a situation, given a size of the extra prize, the equilibria we have characterised will remain unchanged. Without further information, however, the organiser will not be able to choose the size of the extra prize that maximises total expected effort. But as we have seen he will still be very likely to be able to improve upon a standard all-pay auction. And if the organiser has some idea about relative abilities, then he might even do considerably better than with complete exclusion.

An interesting avenue for future research is to generalise the prize structure of our model of extra prizes to a more general model of targeted rewards for sunk investments. For instance, our model is a special case of a multiple-prize all-pay auction in which the set of agents competing for each of the prizes is (potentially) different. Such a model might capture interesting features of real contests and allow to ask novel questions regarding contest design. Concerning design, one could ask which agents should compete for which prizes in order, say, to maximise total effort. Concerning the task to build more realistic contest models, consider lobbying. Interest groups are usually affected by many different policies, but not all groups care about all issues. Our analysis of extra prizes suggests that interest groups affected by multiple issues compete harder. Lastly, consider the contests for funding of research projects mentioned in the Introduction.[21] Say there are two regions. Each of the regions offers a funding competition, in addition to the competition organised by the central Government agency. What is the optimal degree of decentralization of research funds?

[21] Beviá and Corchón (2015) compare centralized and decentralized contests for example for research funds but do not allow that a contestant competes at the same time in both contests.

# A Appendix

In this Appendix we provide a proof for the results stated in the main text. We use the notation $u_i(e_i, G_{-i})$ to indicate contestant $i$'s payoff from bidding $e_i$ when the other two agents employ strategies $G_{-i}$. We indicate by $\underline{s}_i$ and $\bar{s}_i$ the lower and upper bounds of the supports of player $i$'s mixed-strategy, respectively. Contestant $i$'s expected equilibrium payoff is indicated by $u_i^*$.

## A.1 Proof of Lemma 1

Without loss of generality suppose that contestant 2 abstains. In such a case agent 3 receives the extra prize and competes with contestant 1 in a standard all-pay auction for the main prize of size $(1-\beta)$. It is well known that in the unique equilibrium of this all-pay auction the upper bound of the mixed-strategies of contestants 1 and 3 is $(1-\beta)\alpha_3$. Suppose now contestant 2 deviates and bids $(1-\beta)\alpha_3$. This yields $1-(1-\beta)\alpha_3/\alpha_2 = \beta > 0$, which is better than abstaining. Q.E.D.

## A.2 Proof of Lemma 2

We prove Lemma 2 through a series of claims.

**Claim 1** $\bar{s}_i \leq r_d$ *for all* $i$.

**Proof:** Notice that there is no contestant $i$ who employs a strategy that places mass on $(r_i, \infty)$. The reason is that it implies strictly negative payoffs, while setting $e_i = 0$ avoids losses. Thus, we have $\bar{s}_i \leq r_i$ for all $i$. Suppose $\beta \geq \hat{\beta}$. In this case we have that $r_1 \leq r_d$, implying the statement. Suppose now $\beta < \hat{\beta}$, in which case we have that $r_1 > r_d$. But since disadvantaged agents do not put mass on $(r_d, r_1]$, contestant 1 has no incentive to use a strategy with $\bar{s}_1 > r_d$. Q.E.D.

**Claim 2** $\underline{s}_d = 0$, $\gamma_d(0) = 0$ *and* $u_d^* = 0$, *for* $d = 2, 3$.

**Proof:** Suppose the first statement is not true and without loss of generality let $\underline{s}_2 \geq \underline{s}_3$ with $\underline{s}_2 > 0$. There are three cases to consider.

1. $\underline{s}_2 = \underline{s}_3 > 0$. On one hand, if there does not exist $d$ with $d = 2, 3$ such that $\gamma_d(\underline{s}_d) > 0$, then $u_2(\underline{s}_2, G_{-2}) = -\underline{s}_2/\alpha_2 < 0$. On the other, if such a $d$ exists, say $d = 2$, then the other disadvantaged contestant 3 could profitably increase $\underline{s}_3$ slightly (unless $\underline{s}_3 = \alpha_3$ or $\underline{s}_1 \geq \underline{s}_3 = \beta\alpha_3$, in which case contestant 3 could profitably reduce $\underline{s}_3$ to zero).

2. $\underline{s}_2 > \underline{s}_3 > 0$. In this case we have that $u_3(\underline{s}_3, G_{-3}) = -\underline{s}_3/\alpha_3 < 0$ and contestant 3 could profitably reduce $\underline{s}_3$ to zero.

3. $\underline{s}_2 > \underline{s}_3 = 0$. There are again three cases.

   (a) $\underline{s}_2 = \underline{s}_1$. On one hand, if $\gamma_2(\underline{s}_2) = 0$, then $u_1(\underline{s}_1, G_{-1}) = -\underline{s}_1/\alpha_1 < 0$. On the other, if $\gamma_2(\underline{s}_2) > 0$, then contestant 1 could profitably increase $\underline{s}_1$ slightly (unless $\underline{s}_1 = \alpha_1(1-\beta)$, in which case contestant 1 could profitably reduce $\underline{s}_1$ to zero).

   (b) $\underline{s}_2 > \underline{s}_1$. It follows that $\underline{s}_1$ must be equal to zero, because otherwise $u_1(\underline{s}_1, G_{-1}) = -\underline{s}_1/\alpha_1 < 0$. Similarly, for any $i = 1, 3$ bidding $e_i$ such that $\underline{s}_2 > e_i > 0$ is unprofitable and bidding $e_i = \underline{s}_2$ can only be profitable if $\gamma_2(\underline{s}_2) > 0$. In the latter case, however, contestant $i$ could, as in cases 1 and 3(a), profitably increase $e_i$ slightly. Thus $G_i(\underline{s}_2) = G_i(0)$ must hold for $i = 1, 3$. This implies that contestant 2 could profitably decrease $\underline{s}_2$.

   (c) $\underline{s}_2 < \underline{s}_1$. Again, any bid $e_3$ with $\underline{s}_2 > e_3 > 0$ is unprofitable and $e_3$ with $e_3 = \underline{s}_2$ can only be profitable if $\gamma_2(\underline{s}_2) > 0$. In the latter case, however, contestant 3 could (similarly to case 1) profitably increase $e_3$ slightly. Thus $G_3(\underline{s}_2) = G_3(0)$ must hold, in which case contestant 2 could profitably decrease $\underline{s}_2$.

This proves that $\underline{s}_d = 0$.

Consider now the second statement. Without loss of generality let $\gamma_2(0) > 0$. Contestant 3 has an incentive to raise $\underline{s}_3$ by $\epsilon > 0$ but very small.

Lastly, consider the third statement. Without loss of generality consider contestant 2 and notice that $\gamma_3(0) = 0$ implies $u_2^* = u_2(0, G_{-2}) = 0$. Q.E.D.

**Claim 3** *For all $i$, $G_i$ contains no atoms in the half open interval* $(0, r_d]$.

**Proof:** Suppose the cdf of one of the disadvantaged agents, say $G_2$, has an atom at $\tilde{e}_2 \in (0, \alpha_d]$. Suppose $\bar{s}_1 \geq \tilde{e}_2$. In such a case contestant 3's win probability for the extra prize of size $\beta$ has an upward jump at $\tilde{e}_2$. Suppose $\bar{s}_1 < \tilde{e}_2$. In such a case contestant 1's win probability for the main prize and contestant's 3 win probability for both prizes have an upward jump at $\tilde{e}_2$. In both cases adapting the argument in the proof of Lemma 5 in Baye et al. (1996) allows to conclude that there must be an $\epsilon$-neighborhood below $\tilde{e}_2$ in which neither contestant 1 nor 3 put mass, implying that it is not an equilibrium strategy for agent 2 to put mass at $\tilde{e}_2$. Suppose now $G_1$ has an atom at $\tilde{e}_1 \in (0, \alpha_d]$. The fact that the win probability of contestants 2 and 3 for the main prize has an upward jump at $\tilde{e}_1$ allows to reach a similar contradiction. Q.E.D.

**Claim 4** *Suppose $e \in (0, r_d]$ is a point of increase in $G_i$ for $i \in D$. Then $e$ is also a point of increase in $G_j$ for all $j \in D$.*

**Proof:** We start with an observation. Consider the disadvantaged contestants and denote them $i$ and $j$. If $e \in (0, \alpha_d]$ is a point of increase in $G_i$, then contestant $i$ must receive his equilibrium payoff at $e$.[22] Using claim 2 we see that

$$u_i(e, G_{-i}) = G_j(e)(G_1(e)(1-\beta)+\beta) - \frac{e}{\alpha_d} = 0. \tag{13}$$

Since $e$ might or might not be a point of increase in $G_j$ it follows from claim 2 that

$$u_j(e, G_{-j}) = G_i(e)(G_1(e)(1-\beta)+\beta) - \frac{e}{\alpha_d} \leq 0. \tag{14}$$

Expressions (13) and (14) imply that

$$\big(G_i(e) - G_j(e)\big)(G_1(e)(1-\beta)+\beta) \leq 0. \tag{15}$$

Since $G_1(e)(1-\beta)+\beta > 0$, we conclude that

$$G_i(e) \leq G_j(e). \tag{16}$$

Without loss of generality suppose that $y \in (0, \alpha_d]$ is a point of increase in $G_2$ but not in $G_3$. Since by Claims 2 and 3 there are no mass points in $[0, \alpha_d]$ and using the definition of a cdf, there must exist $z \neq y$ with $z \in (0, \alpha_d]$ such that $G_3(z) = G_2(y)$ and $z$ is a point of increase in $G_3$. There are two cases to consider.

1. $z < y$. Since $z$ is a point of increase in $G_3$, we can apply (16) and establish that $G_3(z) \leq G_2(z)$. By the properties of a cdf we have that $G_2(z) \leq G_2(y)$. Thus we obtain $G_3(z) \leq G_2(z) \leq G_2(y)$. Our initial assumption that $G_3(z) = G_2(y)$ allows to establish that

$$G_3(z) = G_2(z) = G_2(y) \tag{17}$$

   must hold. Therefore each $w \in (z, y)$ cannot be a point of increase in $G_2$ and, since each point must be a point of increase for at least two contestants (Lemma 7 in Baye et al., 1996, applies), $w$ must be a point of increase in $G_3$. Applying (16) allows then to establish that $G_3(w) \leq G_2(w)$. By the properties of a cdf we obtain $G_3(z) \leq G_3(w) \leq G_2(w) \leq G_2(y)$. But from (17) we see that for each $w \in (z, y)$ these weak inequalities must hold with strict equality, contradicting that $z$ is a point of increase in $G_3$.

2. $z > y$. Since $y$ is a point of increase in $G_2$, we can apply (16) and establish that $G_2(y) \leq G_3(y)$. By the properties of a cdf we have that $G_3(y) \leq G_3(z)$. Thus we

[22] We use the following definition of a point of increase. Consider a function $f: A \to \mathbb{R}$, defined on a convex set $A \subset \mathbb{R}$. The point $e_0$ is a point of increase in $f$ if for all $\epsilon > 0$, there exists $e \in (e_0, e_0 + \epsilon)$ such that $f(e) > f(e_0)$.

obtain $G_2(y) \leq G_3(y) \leq G_3(z)$. Our initial assumption that $G_3(z) = G_2(y)$ allows to establish that

$$G_2(y) = G_3(y) = G_3(z) \tag{18}$$

must hold. Therefore each $w \in (y,z)$ cannot be a point of increase in $G_3$ and $w$ must be a point of increase in $G_2$. Applying (16) allows then to establish that $G_2(w) \leq G_3(w)$. By the properties of a cdf we obtain $G_2(y) \leq G_2(w) \leq G_3(w) \leq G_3(z)$. But from (18) we see that for each $w \in (y,z)$ these weak inequalities must hold with strict equality, contradicting that $y$ is a point of increase in $G_2$.

Q.E.D.

**Claim 5** *Suppose $e \in (0, r_d]$ is a point of increase in $G_2$ and $G_3$. Then $G_2 = G_3$ at $e$.*

**Proof:** Claim 2 implies that (13) and (14) must both hold with equality. Thus (16) must hold with equality too. Q.E.D.

**Claim 6** *$\bar{s}_d = r_d$, for $d = 2, 3$.*

**Proof:** Claims 4 and 5 imply $\bar{s}_2 = \bar{s}_3$. Let $\bar{s}_2 = \bar{s}_3 < \alpha_d$. Since $\bar{s}_2 = \bar{s}_3$ holds, contestant 1 has no incentive to use a strategy with $\bar{s}_1 > \bar{s}_2$. Consider the payoff to contestant 2 from bidding $\bar{s}_2$. This yields $u_2(\bar{s}_2, G_{-2}) = 1 - \bar{s}_2/\alpha_d > 0$, contradicting Claim 2. Q.E.D.

**Claim 7** *$G_2(e) = G_3(e)$, for all $e \in [0, r_d]$.*

**Proof:** Since each point $e \in (0, r_d]$ must be a point of increase for at least two contestants (Lemma 7 in Baye et al., 1996, applies) and using Claim 4, we conclude that each point $e \in (0, r_d]$ is a point of increase for contestants 2 and 3. Applying Claim 5 we have that $G_2(e) = G_3(e)$, for all $e \in (0, r_d]$, implying that $G_2(e) = G_3(e)$ must hold at $e = 0$. Q.E.D.

Lemma 2 follows directly from Claims 1–7. Q.E.D.

## A.3 Proof of Proposition 1

We start with part (i). Under the assumption that disadvantaged agents employ (6), the expected payoff of the advantaged contestant from any $e_1$ is

$$u_1(e_1, G_{-1}) = \begin{cases} \left(\frac{e_1}{\alpha_d}\right)^2 (1-\beta) - \frac{e_1}{\alpha_1} & \text{if } e_1 \in [0, \alpha_d] \\ (1-\beta) - \frac{e_1}{\alpha_1} & \text{if } e_1 \geq \alpha_d \end{cases}. \tag{19}$$

Suppose $\beta \geq \hat{\beta}$ and that disadvantaged agents employ (6). Since $\beta \geq \hat{\beta}$ is equivalent to $r_d \geq r_1$ and since agent 1 only considers deviations to $e_1 \in [0, r_1]$, we have that $\alpha_d \geq (1-\beta)\alpha_1 \geq e_1$ must hold. Hence for each such $e_1$ we have that $U_1(e_1) \leq 0$ if and only if

$$e_1 \quad \leq \quad \frac{(\alpha_d)^2}{\alpha_1(1-\beta)}, \tag{20}$$

which under our assumptions is true. Lastly, observe that when the advantaged agent exerts zero effort with probability one, the disadvantaged contestants compete as in a standard two player all-pay auction. From Hillman and Riley (1989) and Baye et al. (1996), we know that in such a situation the (unique) equilibrium is characterized by (6). Thus, the strategies in the statement constitute an equilibrium, because for each agent, given the strategies of the other players, there is no positive effort level that yields a strictly higher payoff.

Now let $\beta < \hat{\beta}$. Suppose that the advantaged agent 1 abstains and that disadvantaged agents employ (6). Agent 1 can profitably deviate to $e_1 = \alpha_d$, as

$$u_1(e_1, G_{-1}) \quad = \quad (1-\beta) - \frac{\alpha_d}{\alpha_1} > 0 \Leftrightarrow \alpha_d < (1-\beta)\alpha_1, \tag{21}$$

which is equivalent to $\beta < \hat{\beta}$.

Consider now part (ii). By Lemma 2 the set of active agents is either $D$ or $N$. If it is $D$, then, as already mentioned, the unique equilibrium is described in part (i). So suppose the set of active agents is $N$. In such a case there must exist $e \in (0, r_d]$ which is a point of increase for all three contestants. In the proof of Proposition 2 below we show that the only compatible cdfs are described in (8) and (9) and that this implies that $u_1(e, G_{-1}) = \hat{\beta} - \beta < 0$. Under the assumption that $\beta > \hat{\beta}$, contestant 1 is strictly better-off abstaining and thus the equilibrium is unique. Q.E.D.

## A.4 Proof of Lemma 3

Notice that the left hand side of (7) is continuous. Moreover, since $\beta > 0$, (7) is strictly negative at $e_1 = 0$ and equal to one at $e_1 = \alpha_d$. Applying Bolzano's Theorem we conclude that there exists $\Phi$ such that (7) holds. It can be shown that the left hand side of (7) is strictly increasing in $\Phi$. Thus there is a unique $\Phi$ such that (7) holds. Q.E.D.

## A.5 Proof of Proposition 2

Notice first that $G_1(e_1)$ and $G_d(e_d)$ as defined in (8) and (9) are well defined distribution functions. In particular, since the function in the second branch of (8) is the left hand side of (7), we have already established that its density function is strictly positive. It can also be shown that the first and the second branch of (9) intersect for the effort level $\Phi$.[23]

[23] Details are available upon request.

Consider now part (i). Suppose $\beta \le \hat{\beta}$ and that contestant 1 and one disadvantaged agent, say contestant 2, employ (8) and (9). The expected payoff of contestant 3 for any $e_3$ is then

$$u_3(e_3, G_{-3}) = \begin{cases} \frac{e_3}{\alpha_d \beta}\beta - \frac{e_3}{\alpha_d} & \text{if } e_3 \in [0, \Phi] \\ \sqrt{\frac{\alpha_1 - \alpha_d}{\alpha_1(1-\beta)} + \frac{e_3}{\alpha_1(1-\beta)} - \frac{\beta}{1-\beta}} \frac{e_3}{\alpha_d} \sqrt{\frac{\alpha_1(1-\beta)}{\alpha_1(1-\beta) - \alpha_d + e_3}} - \frac{e_3}{\alpha_d} & \text{if } e_3 \in [\Phi, \alpha_d] \\ 1 - \frac{e_3}{\alpha_d} & \text{if } e_3 \ge \alpha_d \end{cases}. \tag{22}$$

Simplifying allows to conclude that for any $e_3 \ge 0$ we have $u_3(e_3, G_{-3}) \le 0$, with strict equality for $e_3 \in [0, \alpha_d]$.

Suppose now that agents 2 and 3 follow the strategies in the statement. Consider agent 1. We have that

$$u_1(e_1, G_{-1}) = \begin{cases} \left(\frac{e_1}{\alpha_d \beta}\right)^2 (1-\beta) - \frac{e_1}{\alpha_1} & \text{if } e_1 \in [0, \Phi] \\ \left(\frac{\alpha_1 - \alpha_d}{\alpha_1(1-\beta)} + \frac{e_1}{\alpha_1(1-\beta)} - \frac{\beta}{1-\beta}\right)(1-\beta) - \frac{e_1}{\alpha_1} & \text{if } e_1 \in [\Phi, \alpha_d] \\ 1 - \beta - \frac{e_1}{\alpha_1} & \text{if } e_1 \ge \alpha_d \end{cases}. \tag{23}$$

For $e_1 \in [\Phi, \alpha_d]$ we obtain $u_1(e_1, G_{-1}) = \hat{\beta} - \beta$, while $u_1(e_1, G_{-1})$ decreases strictly with $e_1$ for $e_1 > \alpha_d$. Consider $u_1(e_1, G_{-1})$ for $e_1 \in [0, \Phi]$. It is straightforward to show that $u_1(e_1, G_{-1})$ is a strictly convex function that takes value zero at $e_1 = 0$ and at $e_1 = (\alpha_d)^2 \beta^2 / (\alpha_1(1-\beta))$. Moreover, it is strictly decreasing at $e_1 = 0$. Thus the most profitable deviation is either $e_1 = 0$ or $e_1 = \Phi$. From the continuity of $G_d$ and the arguments for $e_1 \in [\Phi, \alpha_d]$, it follows then that the payoff for $e_1 \in [0, \Phi]$ cannot exceed contestant 1's equilibrium payoff. Thus, the strategies in the statement constitute an equilibrium, because for each agent, given the strategies of the other players, there is no positive effort level that yields a strictly higher payoff.

Suppose now $\beta > \hat{\beta}$. Assume that contestant 1 and the disadvantaged agents employ (8) and (9), respectively. By the same arguments as before, the expected payoff of contestant 1 is $u_1(e_1, G_{-1}) = \hat{\beta} - \beta$. Under the assumption that $\beta > \hat{\beta}$, this payoff is strictly negative and contestant 1 can profitably deviate by reducing his bid to zero.

Consider now part (ii). By Lemma 2 the set of active agents is either $D$ or $N$. If it is $D$, then contestant 1 abstains and the unique equilibrium is described in Proposition 1. Consider contestant 1 and assume he bids $e_1 = \alpha_d$. This yields $u_1(e_1, G_{-1}) = 1 - \beta - \alpha_d / \alpha_1 = \hat{\beta} - \beta > 0$. Now suppose the set of active agents is $N$. Then by Claim 3 and the fact that each point must be a point of increase for at least two contestants (Lemma 7 in Baye et al., 1996, applies), there must exist $e \in (0, r_d]$ which is a point of increase for all three contestants. For each such $e$ the only compatible cdfs are described in (8) and (9). Lastly notice that we cannot have that $\underline{s}_1 > \Phi$, as this would require that contestant 1 places an atom at zero and hence implies zero payoffs. Thus, (8) and (9) describe the unique equilibrium. Q.E.D.

## A.6 Proof of Proposition 3

Notice first that $G_1(e_1)$ and $G_d(e_d)$ as defined in (11) and (12) are well defined distribution functions.

Suppose $\beta = \hat{\beta}$ and that contestant 1 and a disadvantaged agent, say contestant 2, employ (11) and (12). The expected payoff of contestant 3 for any $e_3$ is

$$u_3(e_3, G_{-3}) = \begin{cases} \frac{e_3}{((1-\beta)\gamma_1(0)+\beta)\alpha_d}((1-\beta)\gamma_1(0)+\beta) - \frac{e_3}{\alpha_d} & \text{if } e_3 \in [0,\lambda] \\ \sqrt{\frac{e_3}{\alpha_d}}\left(\left(\frac{\alpha_1}{\alpha_d}\sqrt{\frac{e_3}{\alpha_d}} - \frac{\alpha_1-\alpha_d}{\alpha_d}\right)(1-\beta)+\beta\right) - \frac{e_3}{\alpha_d} & \text{if } e_3 \in [\lambda, \alpha_d] \\ 1 - \frac{e_3}{\alpha_d} & \text{if } e_3 \geq \alpha_d \end{cases} . \tag{24}$$

Using that $\beta = \hat{\beta}$ and simplifying, we obtain that $u_3(e_3, G_{-3}) \leq 0$, as

$$u_3(e_3, G_{-3}) = \begin{cases} \frac{e_3}{\alpha_d} - \frac{e_3}{\alpha_d} & \text{if } e_3 \in [0,\lambda] \\ \sqrt{\frac{e_3}{\alpha_d}}\sqrt{\frac{e_3}{\alpha_d}} - \frac{e_3}{\alpha_d} & \text{if } e_3 \in [\lambda, \alpha_d] \\ 1 - \frac{e_3}{\alpha_d} & \text{if } e_3 \geq \alpha_d \end{cases} . \tag{25}$$

Suppose now that agents 2 and 3 follow the strategies in the statement. Consider agent 1. We have that

$$u_1(e_1, G_{-1}) = \begin{cases} \left(\frac{e_1}{((1-\beta)\gamma_1(0)+\beta)\alpha_d}\right)^2 (1-\beta) - \frac{e_1}{\alpha_1} & \text{if } e_1 \in [0,\lambda] \\ \frac{e_1}{\alpha_d}(1-\beta) - \frac{e_1}{\alpha_1} & \text{if } e_1 \in [\lambda, \alpha_d] \\ 1-\beta - \frac{e_1}{\alpha_1} & \text{if } e_1 \geq \alpha_d \end{cases} . \tag{26}$$

Again using that $\beta = \hat{\beta}$ and simplifying, we obtain

$$u_1(e_1, G_{-1}) = \begin{cases} \left(\frac{e_1}{((1-\beta)\gamma_1(0)+\beta)\alpha_d}\right)^2 \frac{\alpha_d}{\alpha_1} - \frac{e_1}{\alpha_1} & \text{if } e_1 \in [0,\lambda] \\ \frac{e_1}{\alpha_1} - \frac{e_1}{\alpha_1} & \text{if } e_1 \in [\lambda, \alpha_d] \\ \frac{\alpha_d}{\alpha_1} - \frac{e_1}{\alpha_1} & \text{if } e_1 \geq \alpha_d \end{cases} . \tag{27}$$

To conclude that $u_1(e_1, G_{-1}) \leq 0$ it remains to show that for $e_1 \in [0,\lambda]$ it holds that

$$\left(\frac{e_1}{((1-\beta)\gamma_1(0)+\beta)\,\alpha_d}\right)^2 \frac{\alpha_d}{\alpha_1} - \frac{e_1}{\alpha_1} \leq 0 \Leftrightarrow e_1 \leq ((1-\beta)\gamma_1(0)+\beta)^2\,\alpha_d = \lambda.$$

This, of course, is true in the first branch of (27).

Thus, the strategies in the statement constitute an equilibrium, because for each agent, given the strategies of the other players, there is no positive effort level that yields a strictly higher payoff. Q.E.D.

## A.7 Proof of Proposition 4

We start with the following claim that characterises the expected sum of effort in the continuum of equilibria for $\beta = \hat{\beta}$.

**Claim 8** *The expected sum of effort in the continuum of equilibria for $\beta = \hat{\beta}$ (described in Proposition 3) is given by*

$$\alpha_d + \frac{\alpha_1 - \alpha_d}{3}\left(1 - \left(\left(1-\hat{\beta}\right)\gamma_1(0) + \hat{\beta}\right)^3\right). \tag{28}$$

*This expression is strictly decreasing in $\gamma_1(0)$ and equals $\alpha_d$ for $\gamma_1(0) = 1$.*

**Proof:** We first derive (28). The expected effort of contestant 1 is

$$\begin{aligned} E(e_1) &= \int_\lambda^{\alpha_d} g_1(e_1)e_1 de_1 = \int_\lambda^{\alpha_d} \frac{\alpha_1}{\alpha_d}\frac{\sqrt{e_1}}{2\sqrt{\alpha_d}} de_1 = \frac{\alpha_1}{\alpha_d}\frac{(e_1)^{3/2}}{3\sqrt{\alpha_d}}\bigg|_\lambda^{\alpha_d} \\ &= \frac{\alpha_1}{3}\left(1 - \left(\left(1-\hat{\beta}\right)\gamma_1(0) + \hat{\beta}\right)^3\right). \end{aligned} \tag{29}$$

The expected effort of a disadvantaged contestant is

$$\begin{aligned} E(e_d) &= \int_0^\lambda g_d(e_d)e_d de_d + \int_\lambda^{\alpha_d} g_d(e_d)e_d de_d \\ &= \int_0^\lambda \frac{e_d}{\alpha_d\left(\left(1-\hat{\beta}\right)\gamma_1(0) + \hat{\beta}\right)} de_d + \int_\lambda^{\alpha_d} \frac{\sqrt{e_d}}{2\sqrt{\alpha_d}} de_d \\ &= \frac{(e_d)^2}{2\alpha_d\left(\left(1-\hat{\beta}\right)\gamma_1(0) + \hat{\beta}\right)}\Bigg|_0^\lambda + \frac{(e_d)^{3/2}}{3\sqrt{\alpha_d}}\bigg|_\lambda^{\alpha_d} \\ &= \frac{\alpha_d}{6}\left(\left(1-\hat{\beta}\right)\gamma_1(0) + \hat{\beta}\right)^3 + \frac{\alpha_d}{3} \end{aligned} \tag{30}$$

The expected sum of effort is hence

$$\frac{\alpha_1}{3}\left(1 - \left(\left(1-\hat{\beta}\right)\gamma_1(0) + \hat{\beta}\right)^3\right) + \frac{\alpha_d}{3}\left(\left(1-\hat{\beta}\right)\gamma_1(0) + \hat{\beta}\right)^3 + \frac{2\alpha_d}{3}, \tag{31}$$

which is the same as (28). It is straightforward to verify that (28) equals $\alpha_d$ for $\gamma_1(0) = 1$ and strictly decreases with $\gamma_1(0)$ under our assumption that $0 < \alpha_d < \alpha_1$. Q.E.D.

To conclude the proof notice that for $\beta < \hat{\beta}$ the unique equilibrium is described in Proposition 2. Moreover, since the associated density functions to the cdfs in (8) and (9) are continuous in $\beta$, it follows that the expected sum of effort is also continuous in $\beta$. In addition, the statement of Proposition 2 includes the case of $\beta = \hat{\beta}$ in which case the expressions in (8) and (9) reduce to the equilibrium in Proposition 3 described by (11) and (12) with $\gamma_1(0) = 0$. This implies that for $\beta < \hat{\beta}$ large enough, the expected sum of effort in the unique equilibrium is strictly larger than $\alpha_d$, the expected sum of effort when the advantaged agent is excluded. Q.E.D.

## References


Alcalde, J. and M. Dahm. "Tullock and Hirshleifer: a Meeting of the Minds." *Review of Economic Design*, 11(2), 2007, 193–208.

Alcalde, J. and M. Dahm. "Rent Seeking and Rent Dissipation: A Neutrality Result." *Journal of Public Economics*, 94(1-2), 2010, 1–7.

Arbatskaya, M. "The exclusion principle for symmetric multi-prize all-pay auctions with endogenous valuations." *Economics Letters*, 80, 2003, 73–80.

Baye M.R., D. Kovenock, and C.G. de Vries. "Rigging the Lobbying Process: An Application of the All-Pay Auction." *American Economic Review*, 83(1), 1993, 289–294.

Baye M.R., D. Kovenock, and C.G. de Vries. "The all-pay auction with complete information." *Economic Theory*, 8, 1996, 291–305.

Barut, Y. and D. Kovenock. "The symmetric multiple prize all-pay auction with complete information." *European Journal of Political Economy*, 14, 1998, 627–644.

Bertoletti, P. "A note on the Exclusion Principle." *Journal of Mathematical Economics*, 44, 2008, 1215–1218.

Beviá, C. and L.C. Corchón. "Centralized vs Decentralized Contests." *Economics Letters*, 137, 2015, 32–35.

Clark, D. and C. Riis. "Competition over more than one prize." *American Economic Review*, 88, 1998, 276–289.

Cohen, C. and A. Sela. "Allocation of prizes in asymmetric all-pay auctions." *European Journal of Political Economy*, 24, 2008, 123–132.

Dahm, M. and P. Esteve-González. "Affirmative Action through Extra Prizes." Working Paper, University of Nottingham, 2016.

Fang, H. "Lottery Versus All-Pay Auction Models of Lobbying." *Public Choice*, 112(3–4), 2002, 351–371.

Franke, J., C. Kanzow, W. Leininger, and A. Schwartz. "Lottery versus All-Pay Auction Contests: A Revenue Dominance Theorem." *Games and Economic Behavior*, 83, 2014, 116–126.

Franke, J., W. Leininger, and C. Wasser. "Optimal Favoritism in All-Pay Auctions and Lottery Contests." Working Paper, University of Dortmund, 2016.

Fu, Q. “A Theory of Affirmative Action in College Admissions.” *Economic Inquiry*, 44(3), 2006, 420–428.

Fu, Q. and J. Lu. “Competitive effect of cross-shareholdings in all-pay auctions with complete information.” *Internation Journal of Industrial Organization*, 31, 2013, 267–277.

Glazer, A. and R. Hassin. “Optimal Contests.” *Economic Inquiry*, 26(1), 1988, 133–143.

Hillman, A.L. and D. Samet. “Dissipation of contestable rents by a small number of contenders.” *Public Choice*, 54, 1987, 63–82.

Hillman, A.L. and J.G. Riley. “Politically contestable rents and transfers.” *Economics and Politics*, 1, 1989, 17–39.

Konrad, K.A. “Silent Interest and All-Pay Auctions.” *Internation Journal of Industrial Organization*, 24, 2006, 701–713.

Li, S. and J. Yu. “Contests with Endogenous Discrimination.” *Economics Letters*, 117, 2012, 834–836.

Liu, X. and J. Lu. “Optimal prize-rationing strategy in all-pay contests with incomplete information.” *Internation Journal of Industrial Organization*, 50, 2017, 57–90.

Menicucci, D. “Banning bidders from all-pay auctions.” *Economic Theory*, 29, 2006, 89–94.

Moldovanu, B. and A. Sela. “The optimal allocation of prizes in contests.” *American Economic Review*, 91, 2001, 542–558.

Pastine, I. and T. Pastine. “Student Incentives and Preferential Treatment in College Admisssions.” *Economics of Education Review*, 31, 2012, 123–130.

Siegel, R. “All-Pay Contests.” *Econometrica*, 77(1), 2009, 71–92.

Siegel, R. “Asymmetric Contests with Conditional Investments.” *American Economic Review*, 100(5), 2010, 2230–60.

Siegel, R. “Asymmetric Contests with Head Starts and Nonmonotonic Costs.” *American Economic Journal: Microeconomics*, 6(3), 2014, 59–105.

Sisak, D. “Multiple-Prize Contests – The Optimal Allocation of Prizes.” *Journal of Economic Surveys*, 23(1), 2009, 82–114.

Xiao, J. “Asymmetric all-pay contests with heterogeneous prizes.” *Journal of Economic Theory*, 163, 2016, 178–221.