The Afternoon Effect with Risk-Averse Bidders

Claudio Mezzetti

University of Leicester

Received Theoretical Wisdom, Part A

The standard symmetric model, with independent types and risk neutral bidders

- Revenue Equivalence

- The price sequence is a martingale (the expected price in auction $k+1$ conditional on the realized price in auction k is equal to the price in auction k)

Received Theoretical Wisdom, Part B

Milgrom and Weber's (1982a,b) model with affiliated types (strong form of correlation)

- Ascending price auction yields more revenue than uniform-price auction, which yields more revenue than discriminatory auction.

- The price sequence is a submartingale (the expected price in auction $k+1$ conditional on the realized price in auction k is *higher* to the price in auction k).

- Disclosing information prior to the auction raises seller's revenue

Stylized Facts

In many sequential auctions prices decline over time (the declining price anomaly, or afternoon effect).

Milgrom and Weber (1982b) evidence: November 1982 sale at Sotheby's (New York) of leases on RCA-owned satellite-based telecommunications transporters (sequential English).

Round	1	2	3	4	5	6	7
Price	14.4	14.1	13.7	13.5	12.5	10.7	11.2

Ashenfelter (1989) evidence: Sotheby's Auction of Chateau Palmer 1961, December 11, 1985

Round	1	2	3
Price	920	800	700

Several explanations of the *puzzle* have been offered. In their original paper Milgrom and Weber (1982b) explained the evidence contradicting their theory with non-equilibrium behaviour by bidders!!!???

They also mentioned risk aversion.

Ashenfelter also appealed to risk aversion.

Does Risk Aversion Solve the Puzzle? The literature has shown:

- Under risk aversion the first price auction yields more revenue than the second-price and the ascending auction.

- Risk aversion does not solve puzzle (1). McAfee and Vincent (1993) studied a two-round second price auction with independent private values. They wrote the utility function of a bidder type x_i paying p_i as

$$U(x_i - p_i)$$

They showed that if U exhibits *nondecreasing absolute risk aversion*, then the expected price in the second round is lower than the expected price in the first round; BUT under the more realistic assumption that U satisfies *decreasing absolute risk aversion*, then an equilibrium in pure strategies doesn't exists (i.e., the sequential auction is not even efficient).

I will model risk aversion differently from the way it has been modeled before and show that risk aversion explains the puzzle.

The most general model form of the utility function would be

$$U((V(x_i, x_{-i}), -p_i)$$

(More general than Maskin Riley (1984)).

Two natural questions are:

Under which conditions on the functions U, V, and ℓ does a monotone equilibrium of a sequential auction exist?

Under which conditions does the equilibrium involve declining prices?

- K identical objects; I bidders, $I > K$; each bidder has unit demand

- Utility is separable in the object and money

$$U_i = V(x_i, x_{-i}) - \ell(p_i)$$

- x_i player i's realized type; types are i.i.d. random variables with common density f and distribution F with support $[\underline{x}, \overline{x}]$

- x_{-i} realized type profile of all players but i

- $\ell(p_i)$ disutility from a payment p_i; strictly increasing and convex

- $V(x_i, x_{-i})$ i's payoff from one object; symmetric in x_j, $j \neq i$, and increasing in all its arguments

- $\frac{\partial V_i}{\partial x_i} \geq \frac{\partial V_i}{\partial x_j}$

- under private values $V(x_i, x_{-i}) = x_i$

- under additive common values: $V(x_i, x_{-i}) = \sum_{j=1}^{I} x_j$

- $Y_j^{(n)}$ is the j-th highest type of bidder out of n; its distribution and density are $F_j^{(n)}$and $f_j^{(n)}$

- $< \pi, p >$ a direct mechanism

- $\pi_i(x_i, x_{-i})$ probability that i wins an object

- $p_i(x_i, x_{-i})$ i's payment

- Bidder i' s expected payoff when his type is x_i , but he reports z_i and all other bidders report truthfully is

$$U_i(z_i; x_i) = \int_{\underline{x}}^{\overline{x}} \dots \int_{\underline{x}}^{\overline{x}} \left[V(x_i, x_{-i})\pi_i(z_i, x_{-i}) - \ell(p_i(z_i, x_{-i}))\right] g(x_{-i}) dx_{-i}$$

where $g(x_{-i}) = \prod_{j \neq i} f(x_j)$

- Letting $U_i(x_i)$ be type x_i's expected payoff in the truthful equilibrium of $\langle \pi, p \rangle$, and using a standard envelope argument yields

$$U_i{}'(x_i) = \int_{\underline{x}}^{\overline{x}} \dots \int_{\underline{x}}^{\overline{x}} \frac{\partial V(x_i, x_{-i})}{\partial x_i} \pi_i(x_i, x_{-i}) g(x_{-i}) dx_{-i}$$

Theorem 1 (*Bidder-payoff Equivalence*) *Bidders' payoffs are the same in every mechanism having the same outcome function* (*for example in any efficient auction*) *and yielding the same payoff to the lowest type. Bidder* i*' s expected payoff is given by*

$$U_i(x_i) = \underline{u} + \int_{\underline{x}}^{x_i} \int_{\underline{x}}^{\overline{x}} \dots \int_{\underline{x}}^{\overline{x}} \frac{\partial V(x, x_{-i})}{\partial x} \pi_i(x, x_{-i}) g(x_{-i}) dx_{-i} dx$$

- In the model of risk averse bidders most commonly analyzed in the literature (e.g., Matthews (1983), McAfee and Vincent (1993)) bidders have a monetary value for the object:

$$U_i = V(x_i - p_i)$$

- With this utility function bidder-payoff equivalence fails. There is interplay between screening and insurance:

$$U_i(z_i; x_i) \;=\; \int_{\underline{x}}^{\overline{x}} \dots \int_{\underline{x}}^{\overline{x}} V\left(x_i - p_i(z_i, x_{-i})\right) \pi_i(z_i, x_{-i}) g(x_{-i}) dx_{-i}$$

and

$$U_i{}'(x_i) = \int_{\underline{x}}^{\overline{x}} \dots \int_{\underline{x}}^{\overline{x}} V'(x_i - p_i(x_i, x_{-i}))\pi_i(x_i, x_{-i}) g(x_{-i}) dx_{-i}$$

- the slope of a bidder's payoff function depends on the outcome *and* payment functions.

The uniform-price auction: all bidders submit a bid, the K highest bidders win an object at a price equal to the $K + \mathbf{1}$'st highest bid.

The ascending (English) auction: there is a price clock raising continuously; bidders decide when to drop. Once a bidder has dropped, it cannot re-enter. The price clock stops and the auction ends when there are only K bidders left, who are declared the winners and pay the price showing on the clock. Under private values (i.e., $U(x_i, x_{-i}) = x_i$) the English auction is strategically equivalent to the uniform-price auction. This is no longer so if values are not purely private.

The discriminatory auction: all bidders submit a bid. The bidders with the K highest bids win an object and pay a price equal to the bid they submitted.

The sequential discriminatory auction: one object is sold in each of K rounds to the highest bidder at a price equal to the highest bid.

The sequential uniform-price auction: one object is sold in each round to the highest bidder at a price equal to the second highest bid in that round.

With sequential auctions there are two cases. The case in which the winning bid is announced, and the case in which it is only announced that a good has been sold.

Theorem 2 *In each round of the sequential discriminatory and the sequential uniform-price auction (with or without winning bids announcements), the expected liability and the expected payoff of a bidder of type* x *are the same*

The Afternoon Effect 1

When bidders are risk averse, prices decline along the equilibrium path of a sequential auction without bid announcements.

Theorem 3 *The price sequences in a sequential discriminatory and in a sequential uniform auction without bid announcements are a super-martingale.*

Proof: Consider the case of a sequential uniform auction.

Suppose we are in round k of the auction and type x of bidder 1 has lost all preceding auctions. Suppose also that in round k bidder x considers raising his bid by a small amount above $\beta_k^{SUn}(x)$. This will only make a difference if after the deviation he wins in round k, while he would have otherwise lost and won in round $k+1$. For this event to happen, it must be that $Y_k^{(I-1)} \simeq x$. Conditional on this event, the *marginal cost* of the deviation is

$$\ell(\beta_k^{SUn}(x)),$$

while the *marginal benefit* is

$$E\left[\ell(\beta_{k+1}^{SUn}(Y_{k+1}^{(I-1)}))|Y_k^{(I-1)} = X_1 = x \leq Y_{k-1}^{(I-1)})\right].$$

Equating marginal cost and marginal benefit gives the *arbitrage condition*

$$\ell(\beta_k^{SUn}(x)) = E\left[\ell(\beta_{k+1}^{SUn}(Y_{k+1}^{(I-1)}))|Y_k^{(I-1)} = X_1 = x)\right].$$

Recalling that ϕ is the inverse of ℓ,

$$\beta_k^{SUn}(x) = \phi\left(E\left[\ell(\beta_{k+1}^{SUn}(Y_{k+1}^{(I-1)}))|Y_k^{(I-1)} = X_1 = x)\right]\right).$$

Conditional on $Y_k^{(I-1)} \geq X_1 = x \geq Y_{k+1}^{(I-1)}$, the price in round k is

$$
\begin{array}{rcl}
P_k^{SUn} & = & \beta_k^{SUn}(x) \\
& = & \phi\left(E\left[\ell(\beta_{k+1}^{SUn}(Y_{k+1}^{(I-1)}))|Y_k^{(I-1)} = X_1 = x)\right]\right) \\
& > & E\left[\phi\left(\ell(\beta_{k+1}^{SUn}(Y_{k+1}^{(I-1)}))|Y_k^{(I-1)} = X_1 = x)\right)\right] \\
(& = & \text{with risk neutral bidders}) \\
& = & E\left[\beta_{k+1}^{SUn}(Y_{k+1}^{(I-1)}))|Y_k^{(I-1)} = X_1 = x\right] \\
& = & E\left[\beta_{k+1}^{SUn}(Y_{k+1}^{(I-1)}))|Y_k^{(I-1)} \geq X_1 = x \geq Y_{k+1}^{(I-1)}\right] \\
(& \leq & \text{with affiliated types}) \\
& = & E\left[P_{k+1}^{SUn}|Y_k^{(I-1)} \geq X_1 = x \geq Y_{k+1}^{(I-1)}\right] \\
& = & E\left[P_{k+1}^{SUn}|P_k^{SUn} = \beta_k^{SUn}(x)\right]. \qquad \blacksquare
\end{array}
$$

The Afternoon Effect 2

When bidders are risk averse, and values are purely private, prices decline along the equilibrium path of a sequential auction with bid announcements.

Theorem 4 *When values are purely private, the price sequences in a sequential discriminatory and in a sequential uniform auction with bid announcements are a supermartingale.*

Proof: Consider a sequential uniform auction with bid announcements and private values. The equilibrium bidding function does not depend on the announced bids

$$\begin{aligned}
\beta_k^{SUa}(x; y_{k-1}, \ldots) &= \phi\left(E\left[Y_K^{(I-1)} | Y_k^{(I-1)} = X_1 = x < y_{k-1}, \ldots\right]\right) \\
&= \phi\left(E\left[Y_K^{(I-1)} | Y_k^{(I-1)} = X_1 = x\right]\right) \\
&= \beta_k^{SUa}(x).
\end{aligned}$$

With bid announcements, the *arbitrage condition* becomes

$$\begin{aligned}
&\ell\left(\beta_k^{SUa}\left(x; y_{k-1}, \ldots\right)\right) \\
&\qquad = E\left[\ell\left(\beta_{k+1}^{SUa}\left(Y_{k+1}^{(I-1)}; x, y_{k-1}, \ldots\right)\right) | Y_k^{(I-1)} = X_1 = x\right].
\end{aligned}$$

If, in addition, values are private:

$$\ell\left(\beta_k^{SUa}(x)\right) = E\left[\ell\left(\beta_{k+1}^{SUa}\left(Y_{k+1}^{(I-1)}\right)\right) | Y_k^{(I-1)} = X_1 = x\right].$$

Suppose $Y_{k+1}^{(I-1)} < x = X_1 < Y_k^{(I-1)}$. Then, in round k the winner's signal is $Y_k^{(I-1)}$, and bidder 1 of type x is the price setter: $P_k^{SUa} = \beta_k^{SUa}(x)$. In round $k+1$, the winner is bidder 1 of type x, and the price setter's signal is $Y_{k+1}^{(I-1)}$.

$$
\begin{aligned}
& E\left[P_{k+1}^{SUa} | P_k^{SUa}\right] \\
& = E\left[P_{k+1}^{SUa} | \beta_k^{SUa}(x)\right] \\
& = E\left[\beta_{k+1}^{SUa}(Y_{k+1}^{(I-1)}) | Y_{k+1}^{(I-1)} < X_1 = x < Y_k^{(I-1)}\right] \\
& = E\left[\phi\left(\ell\left(\beta_{k+1}^{SUa}(Y_{k+1}^{(I-1)})\right)\right) | Y_{k+1}^{(I-1)} < X_1 = x < Y_k^{(I-1)} \ldots\right] \\
& < \phi\left(E\left[\ell\left(\beta_{k+1}^{SUa}(Y_{k+1}^{(I-1)})\right) | Y_{k+1}^{(I-1)} < X_1 = x < Y_k^{(I-1)} \ldots\right]\right) \\
& = \phi\left(E\left[\ell\left(\beta_{k+1}^{SUa}(Y_{k+1}^{(I-1)})\right) | Y_{k+1}^{(I-1)} < X_1 = x = Y_k^{(I-1)} \ldots\right]\right) \\
& = \beta_k^{SUa}(x) \\
& = P_k^{SUa}. \qquad \blacksquare
\end{aligned}
$$

The Afternoon Effect v. The Information Effect

The Information Effect: When bidders are risk neutral, and values are not purely private, prices *increase* along the equilibrium path of a sequential auction with bid announcements.

If bidders are risk averse, then there is a trade-off between the Afternoon and the Information Effect.

Theorem 5 *When bidders are risk neutral, and values are not purely private, the price sequences in a sequential discriminatory and in a sequential uniform auction with bid announcements are a submartingale.*

Proof: Consider a sequential uniform auction. With bid announcements, the *arbitrage condition* becomes

$$\ell\left(\beta_k^{SUa}\left(x; y_{k-1}, \ldots\right)\right)$$
$$= E\left[\ell\left(\beta_{k+1}^{SUa}\left(Y_{k+1}^{(I-1)}; x, y_{k-1}, \ldots\right)\right) | Y_k^{(I-1)} = X_1 = x\right],$$

which in the case of risk-neutral bidders reduces to

$$\begin{aligned} & \beta_k^{SUa}\left(x; y_{k-1}, \ldots\right) \\ = \ & E\left[\beta_{k+1}^{SUa}\left(Y_{k+1}^{(I-1)}; x, y_{k-1}, \ldots\right) | Y_k^{(I-1)} = X_1 = x\right]. \end{aligned}$$

Suppose $Y_{k+1}^{(I-1)} < x = X_1 < Y_k^{(I-1)} < y_{k-1} < \ldots < y_1$. Then, in round k the winner is the bidder with the signal $Y_k^{(I-1)}$, and bidder 1 of type x is the price setter; that is, $P_k^{SUa} = \beta_k^{SUa}(x; y_{k-1}, \ldots)$. In round $k+1$, bidder 1 of type x wins the auction, and the price setter is the bidder with the signal $Y_{k+1}^{(I-1)}$. It follows that

$$E\left[P_{k+1}^{SUa}|P_k^{SUa}\right]$$

$$= E\left[P_{k+1}^{SUa}|\beta_k^{SUa}(x;y_{k-1},...)\right]$$

$$= E\left[\beta_{k+1}^{SUa}(Y_{k+1}^{(I-1)};Y_k^{(I-1)},...)|Y_{k+1}^{(I-1)} < x = X_1 < Y_k^{(I-1)} < y_{k-1},...\right]$$

$$> E\left[\beta_{k+1}^{SUa}(Y_{k+1}^{(I-1)};Y_k^{(I-1)},...)|Y_{k+1}^{(I-1)} < x = X_1 = Y_k^{(I-1)} < y_{k-1},...\right]$$

$$= \beta_k^{SUa}(x;y_{k-1},...)$$

$$= P_k^{SUa}. \qquad \blacksquare$$

Revenue Rankings

Theorem 6 *Suppose ℓ is strictly convex (bidders are risk averse).*

(1) *All bidder types x pay a higher expected price in the discriminatory than in the sequential discriminatory auction with no price announcemenst; hence $R^D > R^{SDn}$.*

(2) *In each round, all bidder types x pay a higher expected price in the sequential discriminatory auction than in the sequential uniform auction; hence $R^{SDn} > R^{SUn}$ and $R^{SDa} > R^{SUa}$.*

(3) *Revenue is higher in the sequential uniform auction without winning bid announcements than in the simultaneous uniform auction, $R^{SUn} > R^U$.*

(4) *If values are not purely private, then all bidder types x pay a higher expected price in the uniform auction than in the English auction; hence $R^U > R^E$. If values are private, then $R^E = R^U$.*

The seller's revenues can be ranked as follows: $R^E < R^U < R^{SUn} < R^{SDn} < R^D$.

It is also the case that $R^{SUa} < R^{SDa}$.

Hence, the discriminatory auction raises the highest revenue among all the allocation efficient auctions in which only the winners pay.

Proof. (1) $R^D > R^{SDn}$ By the bidder-payoff equivalence theorem, since the discriminatory and the sequential discriminatory auction are both efficient, the expected payoff and the expected payment disutility of bidder i with type x in the two auctions is the same:

$$\ell(\beta^D(x)) = E\left[\ell\left(\widetilde{P}_i^{SDn}\right)|X_i = x > Y_K^{(I-1)}\right].$$

Using the fact that ϕ is concave, we have

$$\begin{aligned}
E\left[\widetilde{P}_i^D|X_i = x > Y_K^{(I-1)}\right] &= \beta^D(x) \\
&= \phi\left(E\left[\ell\left(\widetilde{P}_i^{SDn}\right)|X_i = x > Y_K^{(I-1)}\right]\right) \\
&> E\left[\phi\left(\ell\left(\widetilde{P}_i^{SDn}\right)\right)|X_i = x > Y_K^{(I-1)}\right] \\
&= E\left[\widetilde{P}_i^{SDn}|X_i = x > Y_K^{(I-1)}\right]
\end{aligned}$$

■

Conditional on x and on winning, in a discriminatory auction the price a bidder pays is a constant, while in a sequential discriminatory auction it varies depending on which auction the bidder wins. Risk averse bidders are willing to pay a premium so as to fully insure against price fluctuations.

Similarly, in both a discriminatory and a uniform auction bidders do not pay when they lose. They only pay when they win. Since bidders are risk averse, they are willing to pay a premium so as to make sure that they face a constant payment when winning, as in a discriminatory auction, rather than a random payment, as in a uniform auction.

Proof. (3) $R^{SUn} > R^U$. Since in a sequential uniform auction the price sequence is a supermartingale, it is

$$E\left[P_{k+1}^{SUn}|P_k^{SUn} = p\right] < p;$$

hence, taking expectations on both sides yields

$$E[P_{k+1}^{SUn}] < E[P_k^{SUn}].$$

It follows that

$$\begin{aligned}
R^{SUn} &= \sum_{k=1}^{K} E\left[P_k^{SUn}\right] \\
&> KE\left[P_K^{SUn}\right] \\
&= KE\left[P^U\right] \\
&= R^U
\end{aligned}$$

where $E\left[P_K^{SUn}\right] = E\left[P_K^U\right]$ follows from $\beta_K^{SUn}(x) = \phi(v(x,x)) = \beta^U(x)$. ■

In the last round of a sequential uniform-price auction with no announcements, all remaining bidders bid as much as they bid in a uniform-price auction. It follows that the price in the last round of the sequential auction is the same as the price in the uniform-price auction. Since the price sequence declines over time, it must be the case that prices are higher in early rounds of the sequential uniform-price auction with no announcements, than in a uniform-price auction. Hence $R^{SUn} > R^{U}$.

Proof. (4) $R^U > R^E$. Suppose first that values are not entirely private. Conditional on $Y_{K+1}^{(I)} = x$:

$$\begin{aligned} E\left[R^U | Y_{K+1}^{(I)} = x\right] &= K\phi\left(v\left(x,x\right)\right) \\ &= K\phi\left(E\left[v^\alpha\left(x,x,Y_{K+2}^{(I)},...,Y_I^{(I)}\right)\right]\right) \\ &> KE\left[\phi\left(v^\alpha\left(x,x,Y_{K+2}^{(I)},...,Y_I^{(I)}\right)\right)\right] \\ &= E\left[R^E | Y_{K+1}^{(I)} = x\right], \end{aligned}$$

values not being entirely private implies that $v^\alpha\left(x,x,Y_{K+2}^{(I)},...,Y_I^{(I)}\right)$ is a non degenerate function of the random variables $Y_{K+2}^{(I)},...,Y_I^{(I)}$.

If values are private, then $v\left(x,x,Y_{K+2}^{(I)},...,Y_I^{(I)}\right) = x$, and $E\left[R^E | Y_{K+1}^{(I)} = x\right] = E\left[R^E | Y_{K+1}^{(I)} = x\right] = \phi(x)$, which also implies $R^U = R^E$. ∎

Information Disclosure

That the ascending auction raises the lowest revenue among the standard auctions is the opposite of what happens in the Milgrom and Weber (1982) model with affiliated values and risk neutral bidders.

That the ascending auction raises a lower revenue than the uniform-price auction suggests that (committing to a policy of) revealing information is bad for the seller. This is also in sharp contrast with Milgrom and Weber (1982).

Theorem 7 *In each round of a sequential auction, all bidder types x pay a higher expected price if the preceding winning bids are not announced than if they are announced; hence $R^{SDa} < R^{SDn}$ and $R^{SUa} < R^{SUn}$.*

Conclusions

- A new, tractable model of risk averse bidders with independent signals.
- Bidders payoff equivalence holds: all bidder types receive the same payoff in all auctions that: (1) induce the same outcome (e.g., all efficient auctions) and (2) yield the same payoff (e.g., zero) to the lowest type.
- Revenue equivalence does not hold: the discriminatory yields higher revenue then the uniform which yields higher revenue than the ascending auction.

- If bidders are *risk averse*, in sequential auctions *without bid announcement* prices decline over time (*afternoon effect*).

- If bidders are *risk averse* and have *private values*, in sequential auctions *with bid announcements* prices decline over time (*afternoon effect*).

- If bidders do *not* have *private values* and are *risk neutral*, in sequential auctions *with bid announcements* prices increase over time (*information effect*).

- If bidders do *not* have *private values* and are *risk averse*, in sequential auctions *with bid announcements* prices may increase or decrease over time (Trade-off between the *information effect* and the *afternoon effect*).

The all-pay auction: all bidders pay the bid they submit. The bidders who submit the K highest bids win an object; often used as a model of political lobbying.

Theorem 8 *All bidder types x pay a higher expected price in the all-pay auction than in the discriminatory auction; hence $R^D < R^A$.*

Conditional on x, in an all-pay auction the price a bidder pays is a constant, while in a discriminatory auction it is zero if the bidder loses and it is positive if he wins the auction. Risk averse bidders are willing to pay a premium so as to fully insure against price differences between winning and losing. It is thus intuitively clear that the all-pay auction maximizes revenue among all efficient auctions.

Theorem 9 *The bidding function in the uniform-price auction satisfies*

$$\begin{aligned}\ell\left(\beta^U(x)\right) &= E\left[V(X_i, X_{-i})|X_i = x, Y_K^{(I-1)} = x\right] \\ &=: v(x,x)\end{aligned}$$

Hence, letting $\phi = \ell^{-1}$ be the inverse of ℓ:

$$\beta^U(x) = \phi\left(v(x,x)\right)$$

The seller's expected revenue is

$$R^U = KE\left[\phi\left(v\left(Y_{K+1}^{(I)}, Y_{K+1}^{(I)}\right)\right)\right]$$

Let

$$v^{\alpha}(x, y_K, y_{K+1}..., y_{I-1}) =$$

$$E\left[V(X_i, X_{-i})|X_i = x, Y_K^{(I-1)} = y_K, Y_{K+1}^{(I-1)} = y_{K+1}, ..., Y_{I-1}^{(I-1)} = y_{I-1}\right]$$

Theorem 10 *In an ascending (English) auction, the equilibrium bidding strategy of bidder i with signal x is as follows.*

(1) *If no bidder has dropped out before, then i drops out at when $\ell(p) = v^{\alpha}(x, x, ..., x)$, or, equivalently, when the price equals $\phi\left(v^{\alpha}(x, x, ..., x)\right)$.*

(2) *If k bidders have dropped out, revealing their signals to be $y_{I-1}, y_{I-2}, y_{I-k}$, then bidder i drops out at price $\phi\left(v^{\alpha}(x, x, ..., x, y_{I-k}, ..., y_{I-1})\right)$.*

The expected revenue in an English auction is

$$R^E = KE\left[\phi\left(v^{\alpha}\left(Y_{K+1}^{(I)}, Y_{K+1}^{(I)}, Y_{K+2}^{(I)}, ..., Y_I^{(I)}\right)\right)\right]$$

Theorem 11 *The symmetric equilibrium bidding function in the discriminatory auction satisfies*

$$\ell\left(\beta^D(x)\right) = E\left[v\left(Y_K^{(I-1)}, Y_K^{(I-1)}\right)|Y_K^{(I-1)} \leq x\right]$$

Hence,

$$\beta^D(x) = \phi\left(E\left[v\left(Y_K^{(I-1)}, Y_K^{(I-1)}\right)|Y_K^{(I-1)} \leq x\right]\right)$$

The seller's expected revenue is

$$R^D = \sum_{k=1}^{K} E\left[\phi\left(E\left[v\left(Y_K^{(I-1)}, Y_K^{(I-1)}\right)|Y_K^{(I-1)} \leq x\right]|x = Y_k^{(I)}\right)\right]$$

Theorem 12 *The symmetric equilibrium bidding function in the all-pay auction is given by*

$$\beta^{AP}(x) = \phi\left(F_K^{(I-1)}(x)E\left[v(Y_K^{(I-1)}, Y_K^{(I-1)})|Y_K^{(I-1)} \leq x\right]\right)$$

$$v_k^{\beta}(x, y_k, ..., y_1) = E\left[V(X_i, X_{-i}) | X_i = x, Y_k^{(I-1)} = y_k, ..., Y_1^{(I-1)} = y_1\right]$$

Theorem 13 *Along the equilibrium path of the symmetric equilibrium of the sequential discriminatory (first-price) auction with price announcements, the bidding functions satisfy:*

$$\ell\left(\beta_K^{SDa}(x; y_{K-1}, ...)\right) =$$

$$E\left[v_K^{\beta}\left(Y_K^{(I-1)}, Y_K^{(I-1)}, y_{K-1}, ...\right) | Y_K^{(I-1)} \le x \le Y_{K-1}^{(I-1)}\right]$$

$$\ell\left(\beta_k^{SDa}(x; y_{k-1}, ...)\right) =$$

$$E\left[\ell\left(\beta_{k+1}^{SDa}(Y_k^{(I-1)}; Y_k^{(I-1)}, y_{k-1}, ...)\right) | Y_k^{(I-1)} \le x \le Y_{k-1}^{(I-1)} = y_{k-1}\right]$$

$$w_k^{\beta}(x, y_k, y_{k-1}, ..., y_1) =$$

$$E\left[V(X_i, X_{-i}) | X_i = x, Y_k^{(I-1)} = y_k, Y_{k-1}^{(I-1)} \geq y_{k-1}, ..., Y_1^{(I-1)} \geq y_1\right]$$

Theorem 14 *Along the equilibrium path of the symmetric equilibrium of the sequential discriminatory (first-price) auction with no price announcements the bidding functions satisfy:*

$$\begin{aligned}
\ell\left(\beta_K^{SDn}(x)\right) &= E\left[w_K^{\beta}(Y_K^{(I-1)}, Y_K^{(I-1)}, x, x, ...) | Y_K^{(I-1)} \leq x \leq Y_{K-1}^{(I-1)}\right] \\
\ell\left(\beta_k^{SDn}(x)\right) &= E\left[\ell\left(\beta_{k+1}^{SDn}(Y_k^{(I-1)})\right) | Y_k^{(I-1)} \leq x \leq Y_{k-1}^{(I-1)}\right]
\end{aligned}$$

Theorem 15 *On the equilibrium path, the bidding functions in a sequential uniform auction without winning bids announcements are*

$$\begin{aligned} \beta_K^{SUn}(x) &= \phi(v(x,x)) \\ \beta_k^{SUn}(x) &= \beta_{k+1}^{SDn}(x) \end{aligned}$$

Theorem 16 *On the equilibrium path, the bidding functions in a sequential uniform auction with winning bids announcements are*

$$\beta_K^{SUa}(x; y_{K-1}, ...) = \phi\left(v_K^{\beta}(x, x, y_{K-1}, ..., y_1)\right)$$

$$\beta_k^{SUa}(x; y_{k-1}, ..., y_1) = \beta_{k+1}^{SDa}\left(x; x, y_{k-1}, ..., y_1\right)$$