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Public debt sustainability in a globally integrated market

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Abstract:

Public debt is considered in a multi-country Diamond model, initially, where the deficit is exogenous, giving *bifurcation* maxima, and, subsequently, where it is determined by a policy trade-off between its deficit financing benefit and its crowding-out cost. In the latter case, two effects arise: first, the Chang (1990) financing externality; and second, negative feedback from the debt to the deficit, initially discovered in the data by Bohn (1998), occurs as an endogenous outcome – and one which is strengthened by the degree of international financial integration. The Chang effect implies countries will over-issue public debt under financial globalization, which will also threaten the sustainability of equilibrium in a nonlinear model. The endogenous Bohn feedback effect, although inherently stabilizing, is not decisively so, causing bifurcations to remain, if policy-makers regard deficits and capital as imperfect substitutes. However, for the perfect substitutes case, it works very powerfully even to eradicate the adjustment dynamics and to deliver higher-valued, *corner-point* maxima.

[1] I am grateful for opportunities to present earlier versions of this paper at the *Halle Institute for Economic Research*, Germany in January 2016, and at the *Centre for Growth and Business Cycle Research Conference*, Manchester in July 2016, and for comments from the participants.

1. Introduction

Whenever public debt rises above accustomed levels, questions usually arise about its sustainability. Traditionally, this concern has related to its *dynamic* properties either from a Ricardian or *forward-looking* perspective of from a *backward-looking* one, defined where debt is determined as the accumulation of past deficits. These amount, respectively, to projecting whether a time-path of budget deficits may converge to a steady state of surpluses or whether the rate of economic growth will overtake the interest rate to stabilize public debt as a proportion of GDP. More recently a third view has arisen which considers sustainability with reference to the *existence* properties of nonlinear models, for which key references are Chalk (2000), Rankin and Roffia (2003), Braueninger (2005) and Farmer and Zotti (2010). Even so, the existence and stability properties of a nonlinear model are generally inseparable.

A source of the nonlinearity is found in the Diamond (1958) overlapping generations model due to a combination of non-Ricardian households with finite-lives, and a concave production function, which translates into a concavity for the capital accumulation equation. The first feature means that public debt crowds-out capital, while the second that an excess of it may remove the possibility of a steady state equilibrium altogether. Rankin and Roffia (2003) use the terminology of *bifurcation maximum* to define this particular where there are *two* possible steady-states, one trivial, another economic, which cojoin within an interior range at a unique, knife-edge point where public debt is at its highest sustainable level. They contrast this with the alternative possibility of a *degenerate maximum*, at which point the capital stock is driven to a corner-point of zero. In a later paper, Rankin (2007) finds this pertains to the alternative – but still overlapping generations – model of Blanchard-Yaari.[2]

Braueninger (2005) shows the bifurcation property also arises in a model of endogenous economic growth, where dynamics of the capital stock and the public budget constraint are integrated.[3] An increase in the primary deficit, naturally, raises public debt as a ratio of GDP

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[2] Necessary conditions for a *degenerate* maximum in the Diamond model, where the capital accumulation process is $k_{t+1} = f(k_t)$, are that $f(0) > 0$ and $f'(0) < 1$. This is ruled out in cases of a CES production function with a *finite* elasticity of factor substitution, $\varpi < \infty$. If $0 < \varpi \leq 1$, $f(0) = 0$; while if $1 \leq \varpi < \infty$, $f'(0) \to \infty$.

[3] This depends on the (i) assumption of a Cobb Douglas production function, in giving constant factor shares, $w_t = (1-\mu)y_t$ and $R_t K_t = (1-\mu)y_t$, (ii) the specification of public debt, B_t, as a ratio β, of output, giving $B_t = \beta y_t$, and (iii) the asset accumulation equation, $K_{t+1} + B_t = \rho w_t$, where ρ is the saving rate of the young. Putting these relationships together and defining $G_{t+1} \equiv k_{t+1}/k_t$ gives $R_t / G_{t+1} = \mu / \left(\rho(1-\mu) - \beta\right)$, which is

ratio, but then there is a further and non-linear amplifying response as a fall in the growth rate causes a further rise in the debt-GDP ratio. This model serves as a vantage point for viewing the relationship between economic growth and the *stock* of public debt, which has recently received more empirical attention, following Reinhart and Rogoff (2010), (2011)].[4] Finally, Farmer and Zoppi (2010) show that the bifurcation result extends to a two-country model as a maximum of debt jointly issued by both.

The present paper picks up a number of these themes, and also presents some extensions. It uses Braeuninger's (2005) adaption of the endogenous growth model, but incorporates portfolio-balance where risk-averse households hold public debt as a safe asset along with risky capital. This has the major implication that the debt interest rate is increasing in the size of the stock to make the model even more nonlinear and to give another source of possible bifurcation. Since capital is risky because of unforecastable *total factor productivity* shocks, the model supports the notion of an inverse relationship between macroeconomic uncertainty and the equity premium, as attested to by Lettau *et al* (2007).

Another implication is that the existence of a bifurcation maximum – with the attendant possibility of a catastrophe – depends on the supply of public debt *relative* to the household demand for it. As a consequence, potential collapses may occur through exogenous demand falls caused, for example, by perceptions of reduced macroeconomic volatility, as well as through over-issuance by profligate governments.[5] The bifurcation point arises as a point of tangency between two, upward-sloping schedules. The portfolio demand for public debt is, naturally, always an increasing function of its own rate of return, while its steady state supply, derived from the government financing equation, is, here, also an increasing function, as public debt grows in positive response to its interest servicing cost.

The second main feature of the model is the consideration of an international setting with many countries that may enter a single globally integrated market for public debt. There are N countries whose households demand public debt, while M is the number of countries whose

both also its steady state value and the growth factor of the debt-GDP ratio. This depends only on the right-hand-side parameter set and not on whether growth is exogenous with the interest factor is endogenous or *vice versa*.

[4] Roberts (2014) suggests that the more appropriate relationship in a generalized OLG model is between economic growth and the *change* in public debt.

[5] The fragility of demand becomes a predominant issue, where the repayment of debt is an additional uncertainty.

governments issue it. It is assumed that $N \geq M \geq 1$, where the autarky case is tantamount to the restriction, $N = M = 1$. Generally, the metric M/N may be interpreted as the relative global supply of public debt, and, as such, plays a key role in the model. If $\overline{N}$ is the total number of countries in the world with $N \leq \overline{N}$, then $N/\overline{N}$ is another measure representing international financial integration.

The third main feature, appearing in the latter part of the paper, is the determination of domestic policy on the basis of a given objective. Reasonable assumptions are that the policy-maker values the scope for deficit-financing afforded by issuing public debt – otherwise it would never be issued – as well as private capital, since this is the basis for economic activity, a ubiquitous concern in macroeconomic policy considerations. The elasticity of substitution between the deficit and capital in the policy function is also not without relevance. The deficit is determined as a policy-optimal resolution of the trade-off between the financing benefits and the crowding-out costs of issuing public debt. A surplus may also arise with an infinite elasticity of substitution.

A major result is that negative feedback from the previous level of debt to current deficit, found first in the data by Bohn (1998), arises endogenously under these policy preferences. However, although intrinsically stabilizing, it is not, generally, sufficiently so, because the parameter underlying it also represents a relative preference for running deficits, which has a countervailing effect in any nonlinear model such as this, and is dominating if there is a finite elasticity of substitution in the policy function. For the alternative, infinite case, however, the endogenous feedback effect is so powerful as to eradicate any potential for dynamic instability.

The open economy dimension of the model enhances these results, since the Chang (1990) financing externality comes into play. Under global financial integration, countries can issue public debt in the knowledge that its crowd-out effects will be externalized. The degree of this depends on the number N of globally integrated countries or on the size of any single country relative to the world economy.

The Chang externality has two important implications. First, its effect is equivalent to a having a larger relative preference for running deficits. Countries may issue large amounts of public debt either because they are relatively willing to pay the cost in crowding-out their own capital stocks or because it crowds-outs the capital stocks of other economies, or, indeed, because both of these.

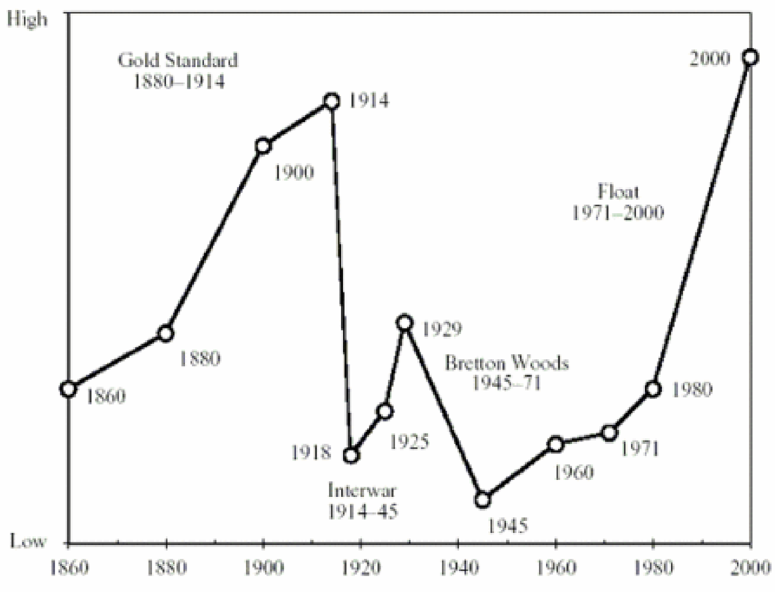
It follows from this, if where countries want to run deficits, they will jointly over issue public debt. It is suggested that this is a likely explanation of the expansion that actually happened in the second era of financial globalization following the breakdown of Bretton Woods in the 1970s. However, another prediction of the Chang externality is that financial integration may even threaten the sustainability of a steady state equilibrium in a nonlinear model such as this. Nash-acting countries, in concert, may issue more public debt than the international economy is willing to hold, so that increasing globalization may lead to its own undoing.

The rest of paper is structured as follow. In *Section 2* some historical data is presented to establish some perspective for the analysis. *Section 3* presents the basic form of the model, which is Diamond (1965) augmented with a portfolio framework and with endogenous growth. In *Section 4*, the more standard case of exogenous public deficits is considered. *Section 5* introduces a policy function, which leads to an endogenous Bohn feedback effect. *Section 6* extends the discussion, and *Section 7* provides a brief summary.

2. An historical overview

It might be useful by first considering some long-run data. The first *figure* from Obstfeld and Taylor (2003) presents their assessment of global capital mobility over the long period 1860-2000. This shows there was a first period of *globalization* from the mid-nineteenth century to outbreak of the First World War. The war itself then led to its demise, which was then followed by short-lived recovery at the end of hostilities, only to be soon terminated by the breakdown in international economic relations and the occurrence of the Second World War. Again, the end of this other war led to a second recovery in international capital mobility, which was relatively mild in being hamstrung by the operational requirements of the new, Bretton Woods system of fixed exchange rates. It was not until the early 1970s, when this system began to breakdown, and at the end of this decade, when this breakdown was practically complete, that capital mobility really shot off, giving rise to the second period of financial *globalization*.

Figure One: International financial mobility, 1860-2000

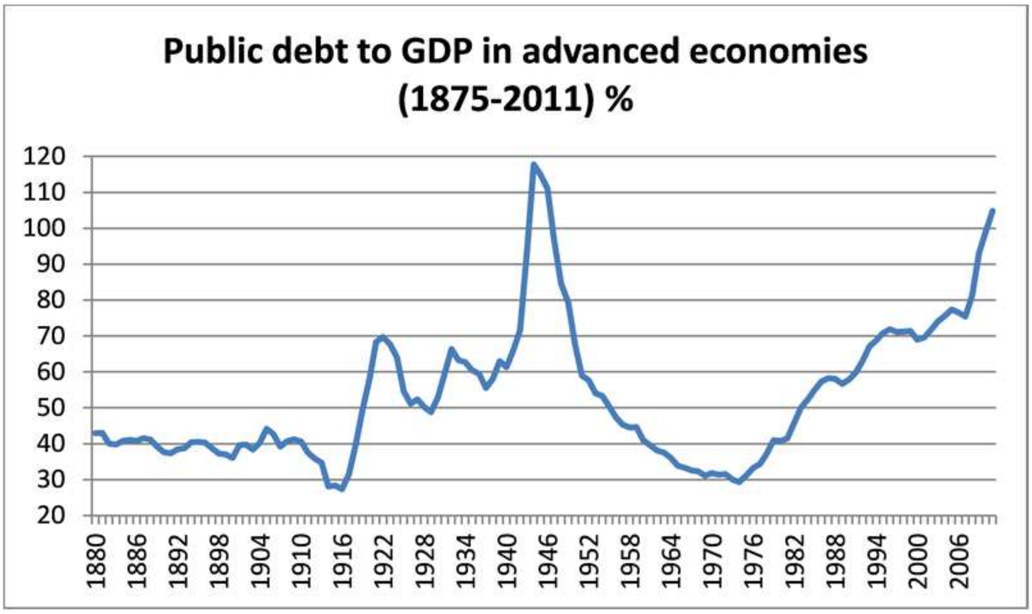


Source: Obstfeld and Taylor (2003).

and at the end of this decade, when this breakdown was practically complete, that capital mobility really shot off, giving the second period of financial *globalization*.

The second *figure* depicts IMF data for a joint measure of public debt for a number of major countries for broadly the same extended history.

Figure Two



Source: IMF World Economic Outlook.

With regard to the nexus between international financial mobility and public debt we suggest that there are at least two main relationships at work in this data. First, it is apparent that during each of the two, twentieth century, world wars, there were sharp rises in public debt accompanied by a loss of international capital mobility, where the latter began to recover each time only after the end of hostilities. Global disintegration of the pathological kind thus has a financial dimension to it, while normal levels of tax revenue are insufficient to cover war-time expenditures, at least, for the protagonist nations. There are, thus, circumstances in which a *negative* association between international financial mobility and public debt arises, because they are joint consequences of a common cause – war.

There is a second relationship, about which the model below is concerned. There was a notable increase in global capital mobility at the incipient breakdown of the Bretton Woods system of fixed exchange rates in 1971, which turned into a surge by the time of was more or less completely over round about 1980 as many countries moved to floating. The previous policy of enforcing capital controls in order to sustain tenuous exchange rate parities was no longer pursued, which freed countries to lend to each other. In the wake of these policy changes, there followed steep rises in public debt, which reversed the steady decline following the end of the Second World War. The rise in the ratio of public debt to GDP for these *advanced countries* was almost of a war-time order of magnitude: rising from 30% to 75% over the period that started in the 1970s and concluded just *before* the onset of the global financial crisis in 2007.

This more recent and cataclysmic event saw sharp rises in public debt, as governments entrusted future tax payers with the bad debts of the banks. Although of indubitable significance, it has nevertheless diverted too much attention from the earlier and more sustained episode, which is comparable importance in its implications for the normal conduct of fiscal policy. This is that financial openness *combined with* a predisposition towards running large fiscal deficits may promote global expansions in public debt through the working of the Chang externality

In light of this hypothesis, one might make a comparison with the earlier experience of financial globalization, prior to the outbreak of the First World War, which saw no accompanying increase in public debt, indeed, even an evident decline over the period. However, far from challenging the basic hypothesis, it serves to endorse it to insofar as the late nineteenth century was characterized as a period of low budget deficits – and even of persistent surpluses for two of the major players, the UK and the US. An appetite for running persistent primary deficits in peacetime is, in long historical terms, quite recent, and part and parcel of commitments to macroeconomic stabilization policy and to associated preferences for big government.

In addition, there has been a third, more recent episode of international financial integration, in the 1999 formation of *European Monetary Union*. Detken *et al* (2004) claims that this was a *stronger* example than the preceding one to the extent that the currency fluctuation risk was eliminated between the participant nations. He made this claim in order to sound an alarm in light of the warning given by Chang (1990). A later, empirical assessment by Debrun *et al* (2008), however, finds, that, prior to the financial crisis, these *Euro* countries were probably constrained from issuing large amounts of public debt in being submitted to an accompanying fiscal framework. This indeed supports the previous point that international financial mobility will not promote a general growth in public debt, only if there is a basic predisposition towards running large deficits.

The two main periods of financial globalization differed for at least one reason: Schularick and Steger (2010) find that whilst capital mobility conferred certain benefits of economic growth on a number of countries in the first period of globalization, this is not so apparent in the second one. The distinction between globalization that enhances and does not enhance economic growth surely hinges on the kind of financial assets that are being traded internationally. The late nineteenth century saw, in particular, capital-rich Britain exporting capital to other, emerging nations, thus promoting an extension of economic development. By contrast, as shown by Schularick (2006), the second era of financial globalization in the late twentieth century has been characterized more by high income countries practicing asset diversification for purposes of risk-management. This would entail assets that do not necessarily facilitate physical investment and, as such, would include public debt instruments. We now turn to a model that picks up on some of these themes.

3. The model

3.1 Public debt dynamics

The public budget constraint is given by

$$b_t = x_t + R_t^B b_{t-1}, \tag{1}$$

where debt in period t , b_t , is issued to finance the current primary deficit, x_t , plus the cost, $R_t^B b_{t-1}$, of servicing the outstanding debt, b_{t-1}, where R_t^B is the interest factor on that debt. To focus on the basics, we make no distinction between spending and taxes, which are considered, respectively, to be waste and non-distortionary.

Using the following definitions for the debt-GDP ratio, the deficit-GDP ratio and the growth factor,

$$\beta_t \equiv b_t / E(y_t), \qquad \gamma_t \equiv x_t / E(y_t), \qquad G_t \equiv E(y_t) / E(y_{t-1}), \tag{2}$$

allows the budget constraint in (1) may be written in ratio form,

$$\beta_t = \gamma_t + (R_t^B / G_t)\beta_{t-1} \tag{3}$$

The dynamics are defined as being *backward-looking* in the sense that public debt is determined as the accumulation of past deficits. An alternative, forward-looking or Ricardian view treats public debt as the present value of future net surpluses.

The inequality condition $R_t^B < G_t$ is both a *necessary* and *sufficient* condition for the stability and for the existence of a steady state equilibrium in the purely backward-looking case, [6] [7] but only if the long-run values of R_t^B and G_t are either exogenous or independent of the public debt to give a linear adjustment process. The present model generates a nonlinear process or, more specifically, one where the debt-GDP ratio β_t is a convex, as well as a positive, function of its past value, β_{t-1}, because of two features. One already established by Braeuninger (2007) is that public debt crowd-outs productive capital to reduce economic growth and thus the *denominator* term of R_t^B / G_t . The second, introduced here, is that a rise in public debt also causes a *numerator* rise, as risk-averse investors require greater compensation for holding more on asset with defined asset demands. Risk-aversion also has a stabilizing effect in inducing

[6] The condition $R_t^B < G_t$ does not imply *dynamically inefficiency* under endogenous growth, as Saint-Paul (1992) shows.

[7] The alternative, *forward-looking* or Ricardian case requires $R_t^B > G_t$ for stability. Chalk (2000) uses a different terminology in describing this this as the condition required for “intertemporal budget balance” as opposed $R_t^B < G_t$, which is one for “flow budget constraint”.

households to accept lower rates of return on this safer asset, which slows down its rate of accumulation according to equation (3). Nevertheless, the inequality condition $R_t^B < G_t$ may. at most, only be a necessary one in a nonlinear model.

If an alternative Ricardian view is taken, the reverse condition $R_t^B > G_t$ is required to ensure for the condition of forward-stability that future primary surpluses do not grow faster than the interest rate at which they are discounted. In choosing between these two views, it may be useful to consider some evidence, although these possibilities may only be reflections of periodic and country-specific policy choices and constraints rather than ubiquitous economic features. Historical data, collected and summarized by Dutta *et al* (2000) for five major economies over the period 1900-1989, is presented in the following *table*.

Table One: Rates of return and of economic growth for five major economies for 1900-1989 from Dutta et al (2000)			
	Public debt return % pa	*Economic growth % pa*	*Equity return % pa*
USA	*0.92*	*3.16*	*6.33*
UK	*-0.02*	*1.87*	*4.42*
France	*-1.90*	*2.42*	*8.40*
Germany	*3.30*	*2.88*	*8.93*
Japan	*-2.00*	*4.42*	*7.71*
Average	***0.60***	***2.95***	***7.16***

Besides the not surprising result that the average return on equity exceeds average economic growth rate for all countries, this latter measure, in turn, is found to exceed the average return on public debt for four of the five. Thus, a majority support a backward-looking – as opposed to a Ricardian – view of public debt, one on which this present paper intends to focus. However, the fifth case, which is consistent with long-run budget surpluses, cannot be dismissed, and does emerge as another, albeit, polar possibility also found to arise in model.

Also of relevance to any discussion of dynamic stability is the empirical finding of Bohn (1998) in US data of *stabilizing*, negative (positive) feedback from the outstanding debt to the primary deficit (surplus),

$$\gamma_t = \gamma_{0,t} - \gamma_1 \beta_{t-1} \tag{4}$$

Greiner et al (2007) discovered the same response for a number of EU countries, while Mauro (2013) *et al* investigated this response to confirm it for an extensive number of both countries and time periods.

Three broad questions may be asked in this regard: how might this mechanism work; how might it arise; and how effective might it be? First, on *prima facie* grounds it might seem that the feedback response of equation (4) would be unnecessary for the stability of public debt in a country where the inequality $R_t^B < G_t$ holds on average for very long periods of time, which would include the US of Bohn's study. This would categorically true, if each of these two terms is either exogenous or determined independently of public debt. However, if at least one of these two terms is a function of public debt, as both are in the model presented below, then the feedback mechanism might work by limiting the growth of public debt in order to ensure that inequality $R_t^B < G_t$ indeed does hold, at least, in the long-run. Alternatively, if the reverse case of $R_t^B > G_t$ persists for long periods of time, as for Germany in the *table* above, then this feedback mechanism might work by generating the long-run fiscal surpluses, which alone are consistent with any steady-state with this particular inequality condition.

The second question is how might it arise? A main finding of the model, which comes from assuming that policy-makers have preference weightings over both the primary deficit and the accumulation of private capital accumulation, is that the feedback response may occur quite endogenously. Thus, its presence may solely be due as an outcome of another policy rather than as an implementation of an independent rule in its own right.

Thirdly, we find this mechanism is weak for the specified range of policy preferences. The fact that deficit spending is persistently chosen implies bifurcation maxima for public debt, and rules-out any steady state other than one with the particular inequality condition of $R_t^B < G_t$. The endogenous feedback response is generally too weak to prevent this kind of maximum, and cannot remove the associated possibility of a collapse, should the primary deficit persistently exceed a threshold. This, in essence, is because the parameter that supports the endogenous and stabilizing response is the same one that defines a policy preference for running deficits, which is inherently destabilizing in a nonlinear model such as this. Of these countervailing effects, the preference one dominates the feedback one, for the general case of an imperfect elasticity of substitution between the two arguments of the policy function.

But, there is also a notable exception, if the elasticity of substitution is infinite, namely, where policy-makers regard the primary deficit and private capital as perfect substitutes. This, despite some weighting on deficits, allows for the possibility of budget surpluses, which leads to a higher potential maximum value for public debt at a corner-point, raising the return on public debt, so that the alternative condition $R_t^B > G_t$ holds, which alone is consistent with a steady state of surpluses. In this case, the feedback response transpires to be very powerful.

3.2 Production

Each economy consists of a continuum of firms of measure one, each indexed z and producing an output $y_t(z)$. Production in each firm is a function of its own inputs of labour and capital, $l_t(z)$ and $k_t(z)$, under constant returns to scale,

$$y_t(z) = (1+\varepsilon_t) D_t l_t(z)^{1-\mu} k_t(z)^{\mu}, \qquad E(\varepsilon_t) = 0, \qquad D_t \equiv A k_t^{1-\mu} \tag{5}$$

Output also depends on total factor productivity, $(1+\varepsilon_t)D_t$, which moves stochastically around a trend variable, D_t, due to a common shock, ε_t, which has a zero mean. This variable is common to all firms as an aggregate technology, which is reflected in the aggregate (unindexed) capital stock, k_t, as considered in Romer (1989).

Workers are deemed to be immune from the effects of production uncertainty by being paid their *expected* marginal product,

$$w_t(z) = (1-\mu) D_t l_t(z)^{-\mu} k_t(z)^{\mu}, \qquad D_t \equiv A k_t^{1-\mu}$$

It then follows that the remaining factor of capital alone bears all the uncertainty in paying out the residual from the *stochastic* output under the assumption of internally constant returns to scale. As both factor inputs, as well as the wage, are determined in advance of the output realization, the owners of capital stand either to gain or lose from the stochastic output shock. Dividing their *ex post* income term by the predetermined capital stock determines the uncertain return on capital,

$$R_t^K = (1+\varepsilon_t) D_t l_t(z)^{1-\mu} k_t(z)^{\mu-1} - (1-\mu) D_t l_t(z)^{1-\mu} k_t(z)^{\mu-1}, \qquad D_t \equiv A k_t^{1-\mu}$$

These assumptions, first, imply that the managers of the firm are acting on the behalf of risk-neutral owners. Since, as we have assumed that worker-households are spared the effects of output uncertainty and, having assumed they are also risk-averse, we might appeal to the theory *implicit labour contracts*.[8] Although the underlying assumptions of this theory are not presently laid out, implicit contracts might allow that in the event of a negative shock, $\varepsilon_t < 0$, firms neither sack workers nor reduce their hours or wages, while the owners of capital bear the full brunt. Owner compensation rests on the result that in the event of a positive shock, $\varepsilon_t > 0$, they receive the entire residual of output gains, while workers are disallowed wage increases. Secondly, the assertion that the owners of capital bear *all* the output risk is suggestive of the equity financing of capital under unlimited liability. Thirdly, the assumption of risky capital provides the basis of the portfolio specification of the model. Finally, the security of wage income, which exactly constitutes the saving base in a two-period Diamond model in the absence of bequests and of second period work allows economic growth to be deterministic within an otherwise stochastic model.

Assuming symmetry across firms and normalizing the labour input in each firm employment level of unity, so that $k_t(z) = k_t$, $l_t(z) = l_t = 1$, $\forall z$, gives the following solutions for output, for the wage and for the realized and the distribution of returns on capital,

$$y_t = (1+\varepsilon_{t+1})Ak_t \tag{6}$$

$$w_t = (1-\mu)Ak_t, \tag{7}$$

$$R_t^K = (\mu + \varepsilon_t)A, \qquad E(R_t^K) = \mu A, \quad \text{var}(R_t^K) = A^2\,\text{var}(\varepsilon_t). \tag{8}$$

3.3 Household saving and portfolio balance

Households conform to the Diamond (1965) framework in that they for two periods and derive utility from consumption in each. They choose a sure level of consumption when young, c_t^Y, and save for a future consumption which is necessarily uncertain when they acquire risky capital assets. The distribution of consumption when old, c_{t+1}^O is characterized by its mean and variance, $E(c_{t+1}^O)$ and $\text{var}(c_{t+1}^O)$, and each of these moments enters the lifetime utility function,

[8] Seminal references are Baily (1974), Gordon (1974) and Azariadis (1975).

$$U_t = (1-\rho)\ln c_t^Y + \rho \ln\left(E(c_{t+1}^O) - \alpha \frac{\text{var}(c_{t+1}^O)}{2s_t} \right), \tag{9}$$

The parameter ρ is the rate of relative time-preference, and α is a degree of risk-aversion. The specification is such that the variance of second period consumption is deflated by first period saving. This may be interpreted as a form of precautionary saving, since saving here also directly yields utility in the presence of uncertainty (where $\text{var}(c_{t+1}^O) > 0$) quite apart from its role in smoothing (expected) consumption. Furthermore, with regard more practical modelling considerations, this specification preserves the logarithmic form of the utility function to generate a linear saving function, since the term $\text{var}(c_{t+1}^O)$ is proportional to the *square* of s_t, so that $\text{var}(c_{t+1}^O)/s_t$, as well as c_t^Y and $E(c_{t+1}^O)$, is *linear* in s_t, thus facilitating a separability between the household's saving and portfolio decisions.

Young households supply an inelastic unit of work in return for the wage, w_t, and save by acquiring both capital, k_{t+1}, which has the return factor of R_{t+1}^K, and public debt, b_t, which yields R_{t+1}^B. The old consume their accumulated asset income, do not work, nor leave a bequest to the young. The household budget constraints, respectively, when young and old, are

$$c_t^Y = w_t - k_{t+1} - b_t, \qquad c_{t+1}^O = R_{t+1}^K k_{t+1} + R_{t+1}^B b_t. \tag{10}$$

It is useful to give definitions for the portfolio shares,

$$b_t = \omega_t s_t, \qquad k_{t+1} = (1-\omega_t)s_t, \tag{11}$$

so that in applying the budget constraints in (9), the utility function in (8) becomes

$$\ln U_t^Y = (1-\rho)\ln(w_t - s_t) + \rho \ln s_t$$
$$+ \rho \ln\left((1-\omega_t)E(R_{t+1}^K) + \varpi_t R_{t+1}^B - \frac{\alpha}{2}(1-\varpi_t)^2 \text{var}(R_{t+1}^K) \right)$$

This is maximized by a choice of saving,

$$s_t = \rho w_t \tag{12}$$

and by a portfolio choice,

$$1-\varpi_t = \frac{E(R_{t+1}^K) - R_{t+1}^B}{\sigma}, \quad \varpi_t = \frac{\sigma + R_{t+1}^B - E(R_{t+1}^K)}{\sigma}, \quad \sigma \equiv \alpha \operatorname{var}(R_{t+1}^K), \tag{13}$$

where σ is a composite parameter that measures the amount of risk multiples by the degree of aversion to it.

3.4 The effect of public debt on its return and on economic growth

3.4.1 Autarky

As point of reference, the closed-economy may first be considered. Equations (11), (12) and (13) imply the following demand for public debt,

$$b_t = \left(1 - \sigma^{-1}\left(E(R_{t+1}^K) - R_{t+1}^B\right)\right)\rho w_t \ . \tag{14}$$

It is also convenient to define

$$G^* \equiv \rho(1-\mu)A, \tag{15}$$

as the economic growth factor that occurs in the absence of debts and deficit, and which, as such, is also its maximum value. General economic growth in the autarky case is

$$G = G^* - A\beta \tag{16}$$

The return on public debt is deemed to be endogenous, so that equation (13) is inverted. Along with equations (2), (7) and (16) this gives

$$R_{t+1}^B = E(R_{t+1}^K) - \sigma + \left(\sigma A/G^*\right)\beta_t \ , \ \ E(R_{t+1}^K) > \sigma \ , \ \ \partial R_{t+1}^B \big/ \partial \beta_t = \sigma\, A/G^* > 0 \,. \tag{17}$$

Given the constant mean return on capital, where $E(R_{t+1}^K) > R_{t+1}^B$, the positive response $\partial R_{t+1}^B / \partial \beta_t > 0$ positive reflects a falling risk-premium. Equations (16) and (17) are the autarkic special cases of the general model to follow.

3.4.2 A globally integrated public debt market with globally immobile capital

The essential assumption of this analysis is that there exists a globally integrated market for public debt, so that there is a single factor of return, R_{t+1}^B, facing all countries. There are N of them, indexed i, out of a possibly higher number $\overline{N}$, (ie. $N \le \overline{N}$), whose residents enter a globally integrated market for public debt. The ratio $N/\overline{N}$ thus may thus be interpreted as measure of *global market integration*, although $\overline{N}$ plays no role in the analysis.

However, of these N public debt demanding countries, there is a possibly smaller number M, $1 \le M \le N \le \overline{N}$, with governments that issue it. Since it would raise questions if countries with the financial standing to offload debt onto the global market could not induce their own citizens to hold it, we assume that hat enables them to globally acceptable public debt would have residents that would not themselves accept it, we discount the possibility that $M > N$. Finally, if countries are identical in all other respects, the ratio M/N may then be interpreted as the global supply of public debt relative to its global demand, which is defined as,

$$\lambda \equiv M/N, \tag{18}$$

and which plays an important role in the model.

The second main assumption is that, although public debt markets are fully integrated internationally, there is no global market for corporate finance. This means that domestic economic growth will depend only on domestic preferences and conditions, implying a form of the Feldstein-Horioka (1990) result. A possible justification for this autarkic assumption is that while the financial instruments through which government borrowing is conducted are assumed to be transparent, homogeneous and safe, the borrowing of firms may be dogged by the standard problems of asymmetric information, which may be resolved but only at the national level.

Equation (14) generalizes to give a global demand for public debt, $\sum_{i=j}^{N} b_{i,t} = \sum_{j=1}^{N} \left(1 - \sigma_j^{-1}\left(E(R_{j,t+1}^K) - R_{t+1}^B\right)\right)\rho_j w_{j,t}$. As the corresponding supply is $\sum_{i=1}^{M} b_{i,t}$, there is an international market equilibrium where

$$\sum_{i=1}^{M} b_{i,t} = \sum_{i=j}^{N} \left(1 - \sigma_j^{-1}\left(E(R_{j,t+1}^K) - R_{t+1}^B\right)\right)\rho_j w_{j,t} \tag{19}$$

Common cross-country distribution returns on capital and degrees of risk-aversion are assumed: $E(R_{j,t+1}^K) = E(R_{t+1}^K)$ and $\mathrm{var}(R_{j,t+1}^K) = \mathrm{var}(R_{t+1}^K)$, $\forall j$, so that $\sigma_i = \sigma$, $\forall i$. Equation (19) may then be inverted to give a world equilibrium public debt interest factor,

$$R^B_{t+1} = E(R^K_{t+1}) + \sigma\left(\frac{\sum_{i=1}^{M}\beta_{i,t}E(y_{i,t})}{\sum_{j=1}^{N}\rho_j(1-\mu_j)E(y_{j,t})} - 1\right),$$

$$\text{where} \quad \frac{\partial R^B_{t+1}}{\partial \beta_{i,t}} = \sigma\left(\frac{E(y_{i,t})}{\sum_{j=1}^{N}\rho_j(1-\mu_j)E(y_{j,t})}\right) \tag{20}$$

as a generalization of (17).

Similarly, economic growth in each country is given by

$$G_{i,t+1} = \left(1 - \frac{\sum_{i=1}^{M}\beta_{i,t}E(y_{i,t})}{\sum_{j=1}^{N}\rho_j(1-\mu_j)E(y_{j,t})}\right)\rho_i(1-\mu_i)A,$$

$$\text{where} \quad \frac{\partial G_{i,t+1}}{\partial \beta_{i,t}} = -\frac{\rho_i(1-\mu_i)y_{i,t}}{\sum_{i=j}^{N}\rho_j(1-\mu_j)y_{j,t}}A, \tag{21}$$

as a generalization of equation (16). This shows that economic growth in each country is affected by its own public debt only to the extent that this contributes to the global aggregate, and that this contribution is inversely related to the number of financially integrated countries.

This has two important implications, one of theoretical significance, one empirical. First, the crowding-out effect of issuing public debt is externalized, which, as Chang (1990) shows, has implications for the size of public debt, and which, here, also threatens its sustainability. Secondly, cross-country correlation and regression analysis, following the empirical analysis of Reinhart and Rogoff (2010), which treat countries as if they were closed-economies, may lead to misspecifications in precluding the possibility of external crowding-out effects to under-estimate the effect of issuing public debt on economic growth.

To illustrate these points, it is helpful to consider the case of cross-country symmetry, where the derivatives in equations (20) and (21) reduce to $\partial G_{i,t+1} / \partial \beta_{i,t} = -A/N$ and $\partial R^B_{t+1} / \partial \beta_{i,t} = \sigma A / G^* N$, which highlights the importance to the domestic comparative statics of international financial integration, as measured by the parameter N, and which also shows that a single country's perception that its fiscal policy will not affect its borrowing costs may be roughly correct. Finally, to recap, autarky in the capital market ties a country's investment to its saving in line with the Feldstein and Horioka (1990) result, which is reflected in the response $\partial G_{i,t+1} / \partial \rho_i > 0$ of equation (21).

4. Exogenous primary deficit ratios and global integration

This first part of the analysis treats the deficit ratio as exogenous, in line with Braeuninger (2005), while also being comparable with Rankin and Roffia (2003) and Farmer and Zotti (2010), where the debt is exogenous. An exogenous value of the deficit ratio, $\gamma_{i,t}$, for each debt-issuing country, along with the application of equations (2), (20) and (21) for each debt-issuing one, z, gives

$$\beta_{z,t} = \gamma_{z,t} + \left(\frac{E(R_t^K)}{1 - \sum_{i=1}^{M} \beta_{i,t-1} y_{i,t-1} \Big/ \sum_{j=1}^{N} \rho_j (1-\mu) y_{j,t}} - \sigma \right) \frac{\beta_{z,t-1}}{G_z^*} \quad G_z^* \equiv \rho_z (1-\mu) A$$

Applying cross-country symmetry both with regard to saving rates, so that $\rho_z, \rho_j = \rho$, $\forall j$, and to income levels $y_{i,t-1}, y_{j,t-1} = y_{t-1}$, $\forall i, j$, gives

$$\beta_{i,t} = \gamma_{i,t} + \left(\frac{E(R_t^K)}{G^* - \lambda A \beta_{i,t-1}} - \frac{\sigma}{G^*} \right) \beta_{i,t-1}, \quad \text{where} \quad G^* \equiv \sigma(1-\mu) A \;, \tag{22}$$

where, λ, to recap, is the global supply of public debt relative to its demand.

It may also be mentioned that the previous level of the global average debt, defined as $\beta_{t-1} = N^{-1} \sum_{i=1}^{M} \beta_{i,t-1}$, has a positive effect on the current public debt of each country, $\partial \beta_{i,t} / \partial \beta_{t-1} > 0$, $i = 1,..,M$, through effecting both economic growth and the common rate of return. Thus, the model predicts cross-country positive serial correlation and co-movements across countries. The dynamics are first-order, monotonic and convex,

$$\frac{\partial \beta_{i,t}}{\partial \beta_{i,t-1}} = \frac{E(R_t^K) G^*}{\left(G - \lambda A \beta_{i,t-1}\right)^2} - \frac{\sigma}{G^*} > 0, \qquad \forall \beta_{i,t-1} > 0 \quad \text{as} \quad E(R_t^K) > \sigma \,,$$

$$\frac{\partial^2 \beta_{i,t}}{\partial {\beta_{i,t-1}}^2} = \frac{2 \lambda A E(R_t^K) G^*}{\left(G - \lambda A \beta_{i,t-1}\right)^3} > 0 \tag{23}$$

Note that the parameter σ , which is the degree of risk aversion multiplied by the *ex ante* volatility of the returns on capital, to some extent is stabilizing. This is because increases in it raise the portfolio demand for public debt, thereby reducing its return and thus the rate at which it accumulates. Naturally, the relative global supply of public debt, λ , has the opposite effect.

The steady state solution is

$$\beta_i^-, \beta_i^+ = \frac{\left(G^* + \sigma - E(R^K) + \lambda\gamma_i A\right)G^*}{2(G^* + \sigma)\lambda A} \pm \sqrt{(..)^2 - \frac{\lambda\gamma_i G^{*2}}{(G^* + \sigma)\lambda A}}. \tag{24}$$

Its quadratic form has certain implications with regard to the existence, which are given in the following *result*.

Result 1: *An exogenous primary deficit ratio implies there is a bifurcation maximum, where the steady state of the product* $\lambda\gamma_j$ *is given by*

$$(\lambda\gamma_i)^{\max} = A^{-1}\left(G^* + \sigma + E(R^K) - 2\sqrt{(G^* + \sigma)E(R^K)}\right),$$

where there is a corresponding, maximum debt-GDP ratio[9] *for a representative debt-issuing country of*

$$\beta^{\max} = \frac{1}{\lambda A}\left(G^* - \sqrt{\frac{E(R^K)G^{*2}}{G^* + \sigma}} \right) \tag{25}$$

There are three possibilities with regard to the existence of a steady state equilibrium.

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[9] A procedure for determining the bifurcation maximum of a quadratic solution is as follows. If the general solution (for the debt ratio) takes the form $\beta_i = x_0 + x_1\gamma_i \pm \sqrt{(x_0 + x_1\gamma_i)^2 - z_0 - z_1\gamma_i}$, where x_0, x_1, z_0 and z_1 are coefficients composites. A bifurcation occurs where the *discriminant* $\sqrt{\left(x_0 + x_1\gamma_i^{\max}\right)^2 - z_0 - z_1\gamma_i^{\max}}$ is zero and so where $\beta_i^{\max} = x_0 + x_1\gamma_i$. A zero value for the discriminant occurs where $\gamma_i^{\max} = z_1/2x_1^2 - x_0/x_1 - \sqrt{\left(z_1/2x_1^2\right)^2 - x_0 z_1/x_1^3 + z_0/x_1^2}$. [Note there is a second solution where the companion root is positive, but around which all values are degenerate.] Substituting $\gamma_i^{\max}$ back into $\beta_i^{\max} = x_0 + x_1\gamma_i$ gives $\beta_i^{\max} = z_1/2x_1\left(1 - \sqrt{1 - 4x_0x_1/z_1 + 4x_1^2 z_0/z_1^2}\right)$, which is a bifurcation maximum value of the endogenous variable (the debt ratio).

(i) If $\lambda\gamma_i < (\lambda\gamma_i)^{\max}$, there are technically two steady state solutions for β_j, where the solution in equation (24) contains only real parts. The lower valued solution β_i^- is economically meaningful both in having regular comparative static properties, $\partial\beta_i^- \big/ \partial\gamma_i > 0$, and in being locally stable, $0 < \partial\beta_{i,t}^- \big/ \partial\beta_{i,t-1}^- < 1$. [By contrast, the higher valued solution β_i^+ is associated with the perverse response, $\partial\beta_i^+ \big/ \partial\gamma_i < 0$, as well as being locally unstable, $\partial\beta_{i,t}^+ \big/ \partial\beta_{i,t-1}^+ > 1$.]

(ii) If $\lambda\gamma_i = (\lambda\gamma_i)^{\max}$, there is a unique steady state maximal solution at a point of bifurcation maximum. This is given by equation (25), which borders locally stability, as $\partial\beta_{i,t}^{\max} \big/ \partial\beta_{i,t-1}^{\max} = 1$.

(iii) If $\lambda\gamma_i > (\lambda\gamma_i)^{\max}$, there is no steady state, and the two "solutions" each have imaginary parts.

A key feature is that the existence of any equilibrium with (positive) public debt requires that $G^* + \sigma > E(R^K)$. Thus, accounting for the possibility that $E(R^K) > G^*$ occurs as an empirical reality requires a sufficient degree of risk aversion and/or capital return volatility in the parameter σ.

The analysis may cast either in terms of a representative, debt-issuing country's deficit ratio, γ_i, or of the relative global supply of public debt, λ, or of the product of the two, as the global average deficit ratio. If countries jointly over-issue public debt through generally creating large primary deficits, there may be no steady state equilibrium at all. This essentially replicates the result of Farmer and Zoppi (2010), who considered a two-country case with some other extensions. Alternatively, the movement from a situation where a small number of countries each issue a jointly sustainable amount of public debt to one where additional countries enter to issue the same amount may also lead to catastrophic loss of equilibrium. That is to say that the effects of γ_i and of λ are as perfect substitutes or, more precisely, that it is the global average deficit-GDP ratio, $\lambda\gamma_i$, that is important. Thus, in principle, the global market may collapse by becoming over-crowded through too much financial globalization, unless this also unleashes compensating increases in the demands for public debt by mobilizing saving.

Also, note that risk aversion raises the size of the steady state maximum, as $\partial\beta^{\max}/\partial\sigma > 0$ in equation (25), which works through raising the degree of model nonlinearity. However, this effect is distinct from the previously mentioned one, pertaining to situations below the maximum, where increasing risk-aversion leads to a lower steady-state level of public debt by reducing the rate at which it accumulates. To conclude this *Section*, we assign some values to the parameters in order to elucidate the results.

Table Two: Steady state solution values in the exogenous deficit ratio case

where $\lambda = 1,\ G^* = 4,\ E(R^K) = 5,\ \sigma = 4.5,\ A = 15,$

$\mu = 1/3, \rho = 2/5$, *35 year periods*

	Deficit ratio γ	Public debt ratio β	Portfolio share of public debt ϖ	Growth rate pa	Return on public debt as rate pa	Risk premium as rate pa
Min	$\rightarrow 0$	0	0	4.04%	-1.96%	6.67%
	0.0077	0.0093	0.035	3.93%	-1.31%.	6.02%
	0.0154	0.0199	0.075	3.81%	-0.51%	5.22%
	0.0231	0.0331	0.124	3.65%	0.16%	4.55%
Max	0.0308	0.0621	0.233	3.26%	1.26%	3.45%

Not only for this choice of values, but generally for any applied to the two-period Diamond model, the calculated maximum value for the public debt value and its associated values will be extremely small in comparison with real word magnitudes. The maximum sustainable public debt to GDP ratio here is 6%, which is dwarfed by the still, possibly, low value of 30% that was stipulated maximum as an entry criterion for joining the Euro. This may partly be credited to the specification of half-life periods instead of the annual ones that pertaining to actual data. Some crude reconciliation might be made multiplying any calculated debt-GDP ratio in a two period model by value like 35, as one half the proverbial “three score years and ten”. However, any procedure that multiplies the denominator of the debt-GDP ratio, while leaving the numerator intact, is surely a violation of the behavioural feature that debt is determined in relation to GDP or that part of it which is saved. It may be more appropriate to look at a portfolio share, but, even so, we still find a seemingly quite low value of 23%.

In answer to this, previous research in Roberts (2014A) shows that the paucity of these values may sufficiently be explained by the fact that *asset stocks* are synonymous with the *asset flows* in a two-period OLG, where household save in only one single period, and if they leave no bequests. In principle, within a model with households saving over two (or more) periods, it is possible to obtain high values for the stocks of public debt, which are commensurate with actual values, along with low values for the flows, which are, more properly, the source of crowding-out.

5. Endogenous policy and global integration

5.1 Policy preferences for current public expenditure and future capital

The remaining part of the analysis considers the effects of policy on the size and the sustainability of public debt. First we specify what we consider to be a plausible objective function that values both current public expenditure, $x_{i,t}$, and the prospective capital stock, $k_{i,t+1}$, with reference to baseline levels, $\bar{x}_{i,t}$ and $\bar{k}_{i,t+1}$. Of the i countries issuing public debt, we consider a representative one, z, with the policy objective function,

$$H_{z,t} = \left(\eta\left(x_{z,t} - \bar{x}_{z,t}\right)^{(\gamma-1)/\gamma} + (1-\eta)\left(k_{z,t+1} - \bar{k}_{z,t+1}\right)^{(\gamma-1)/\gamma}\right)^{\gamma/(\gamma-1)}, \quad \bar{x}_{z,t} \geq 0, \quad \bar{k}_{z,t} > 0,$$

$$0 \leq \eta \leq 1, \quad 0 < \gamma \tag{26}$$

The parameter η represents a policy preference for *current* net expenditure relative to *future* capital accumulation, which, as such, should implicit contain an element of time-preference. The second parameter, γ , is the elasticity of substitution between these two variables, which is also important for the analysis. In particular, the possibility of primary surpluses ($x_{z,t} < 0$) is ruled-out except in the linear case of an infinite degree of substitution ($\gamma \rightarrow \infty$), where public deficits and private capital are regarded by the policy-maker as perfect substitutes.

The government budget constraint of equation (1) and equations (11)-(13) and (19), which together determine the process of capital accumulation, allow equation (26) to be written with respect to the current level of public debt, $b_{z,t}$,

$$H_{z,t} = \left(\eta \left(b_{z,t} - R_t^B b_{z,t-1} - \bar{x}_{z,t} \right)^{(\gamma-1)/\gamma} + (1-\eta)\left(k_{z,t+1} - \sum_{i=1}^{M} b_{i,t} / N - \bar{k}_{z,t+1} \right)^{(\gamma-1)/\gamma} \right)^{\gamma/(\gamma-1)}$$

Choosing a deficit, given a predetermined debt to be serviced, is equivalent to a choice of $b_{z,t}$.

A trade-off arises, because the issue of public debt both facilitates increased public expenditure and/or cuts in taxes, while also crowding-out capital, and this is resolved optimally where

$$\eta \left(b_{z,t} - R_t^B b_{z,t-1} - \bar{x}_{z,t} \right)^{-1/\gamma} - ((1-\eta)/N)\left(k_{z,t+1} - \sum_{i=1}^{M} b_{i,t} / N - \bar{k}_{z,t+1} \right)^{-1/\gamma} = 0 \qquad (27)$$

It is clear that a greater degree of global integration in public debt markets, captured by a higher value of N, is equivalent to the policy-maker being less concerned with economic growth, that is, in having a higher value of η. The parameter N, if larger than unity, represents the degree of global integration and constitutes the *sine qua non* of the Chang externality. It has an equivalent effect as the preference weight η, which is to say that the policy-maker may be relatively unconcerned with the crowding-out effect of issuing public debt, either because of a weak preference for private capital accumulation (η is high) or because this is largely externalized (N is high) – or, indeed, because of both of these. Thus moving from autarky to global integration (to a higher N) is tantamount to switching to policy-makers with more inclination towards running deficits (with higher η s).

5.2 *The main theoretical results*

A first main result and one that is insightful for some later ones concerns the determination of the primary deficit. This is expressed in terms of growth factors, where $\bar{G}_{z,t} \equiv \bar{k}_{z,t} / k_{z,t-1}$ is defined as the reservation growth factor for capital over the period t for country z, where the steady state equivalent (omitting t indexation) within symmetric equilibrium is $\bar{G}$.

$$\gamma_{z,t} = \frac{\bar{\gamma}_i + (\eta N/(1-\eta))^{\gamma} \left(G^*/A - \bar{G}/A \right)}{1 + \lambda(\eta N/(1-\eta))^{\gamma}} - \frac{\lambda(\eta N/(1-\eta))^{\gamma}}{1 + \lambda(\eta N/(1-\eta))^{\gamma}} \left(\frac{E(R^K)}{G^* - \lambda A \beta_{i,t-1}} - \frac{\sigma}{G^*} \right) \beta_{z,t-1}$$

(28)

Result 2: *If $\eta > 0$, the policy objective function in equation (26) implies negative feedback from the previous level of the public debt to the current primary deficit, as in Bohn (1998), (b) which is nonlinear, if these variables are expressed as ratios of their contemporaneous GDPs, and (c) with a magnitude of response that is increasing in N, the number of globally integrated countries.*

Proof: This follows from the signs of the three derivatives,

(a) $$\frac{\partial \gamma_{i,t}}{\partial \beta_{i,t-1}} = -\frac{\lambda(\eta N/(1-\eta))^{\gamma}}{1+\lambda(\eta N/(1-\eta))^{\gamma}}\left(\frac{E(R^K)G^*}{(G^* - \lambda A\beta_{i,t-1})^2} - \frac{\sigma}{G^*}\right) < 0 \,,$$

(b) $$\frac{\partial^2 \gamma_{i,t}}{\partial {\beta_{i,t-1}}^2} = -\frac{2\lambda^2 A(\eta N/(1-\eta))^{\gamma}}{1+\lambda(\eta N/(1-\eta))^{\gamma}}\left(\frac{E(R^K)G^*}{(G^* - \lambda A\beta_{i,t-1})^2}\right) < 0$$

(c) $$\partial\left|\frac{\partial \gamma_{i,t}}{\partial \beta_{i,t-1}}\right| \Big/ \partial N = \frac{\gamma\lambda(\eta/(1-\eta))^{\gamma})N^{\gamma-1}}{(1+\lambda(\eta N/(1-\eta))^{\gamma})^2}\left(\frac{E(R^K)G^*}{(G^* - \lambda A\beta_{i,t-1})^2} - \frac{\sigma}{G^*}\right) > 0$$

The key point is that preferences for public expenditure lead to feedback from the debt to the primary deficit. First, if $\eta = 0$, where the policy-maker has no concern for increasing the deficit above a reservation level, so that $\gamma_t = \bar{\gamma}$, and there is no material change from the previous analysis of an exogenous deficit. Secondly, if $\eta > 0$, the stabilizing response that Bohn (1988) discovered in the data arises as a consequence of a policy preference for running deficits. Thus, this empirical finding could also be a side-effect of a separate policy rather than the implementation of an independent policy in its own right. Furthermore, this strength of this response when endogenous, as here, is increasing in degree of global integration, reflected in the parameter N.

The dynamic equation for public debt as a ratio of GDP is solved as

$$\beta_{z,t} = \frac{\bar{\gamma} + ((G^* - \bar{G})/A)\phi}{1+\lambda\phi} + \frac{1}{1+\lambda\phi}\left(\frac{E(R^K)}{G^* - \lambda A\beta_{i,t-1}} - \frac{\sigma}{G^*}\right)\beta_{z,t-1}, \quad \phi \equiv \left(\frac{\eta N}{1-\eta}\right)^{\gamma} \qquad (29)$$

The composite parameter ϕ summarizes pressures to raise the public debt both through preferences (in η) and through the externality (in N), where the effect of these factors is magnified by the elasticity of substitution, γ , in the policy objective function, if $\eta N/(1-\eta)>1$, but, otherwise, deflated where $\eta N/(1-\eta)<1$.

There is a steady state solution,

$$\beta_z^-,\beta_z^+=\frac{\psi_0+\psi_1\phi}{2(\varpi_0+\varpi_1\phi))}\pm\sqrt{\frac{1}{4}\left(\frac{\psi_0+\psi_1\phi}{\varpi_0+\varpi_1\phi}\right)^2-\left(\frac{\pi_0+\pi_1\phi}{\omega_0+\varpi_1\phi}\right)},$$

$$\phi\equiv\left(\frac{\eta N}{1-\eta}\right)^{\gamma},$$

$$\psi_0\equiv G^*+\sigma-E(R^K)+\lambda A\bar{\gamma}, \qquad \psi_1\equiv\lambda(2G^*-\overline{G}),$$

$$\pi_0\equiv G^*\bar{\gamma}, \qquad \pi_1\equiv(G^*-\overline{G})G^*/A,$$

$$\omega_0\equiv(1+\sigma/G^*)\lambda A, \qquad \varpi_1\equiv\lambda^2 A. \qquad (30)$$

This allows us to establish the main *result*.

Result 3: *(a) If the primary deficit and economic growth are regarded as imperfect substitutes by policy-makers or $\gamma<\infty$ in equation (26), there is a bifurcation maximum for public debt, at the point* $\tilde{\phi}=\dfrac{2(\pi_0\varpi_1+\pi_1\varpi_0)-\psi_0\psi_1}{\psi_1{}^2-4\pi_1\varpi_1}-\sqrt{(..)^2+\dfrac{4\pi_0\omega_0-\psi_0{}^2}{\psi_1{}^2-4\pi_1\varpi_1}}$ *, such that*

- *(i)* *If $(\eta N/(1-\eta))^{\gamma}<\tilde{\phi}$, there are technically two steady state solutions for β_j , containing only real parts, as given by equation (30), with the same properties as in in Result 1.*
- *(ii)* *If $(\eta N/(1-\eta))^{\gamma}=\tilde{\phi}$, there is a unique steady state solution at the point of bifurcation maximum.*
- *(iii)* *If $(\eta N/(1-\eta))^{\gamma}>\tilde{\phi}$, there is no steady state, and the two "solutions" each have imaginary parts.*

(b) If the primary deficit and economic growth are regarded as perfect substitutes by policy-makers or that $\gamma\to\infty$ in equation (26),

(i) *there is a corner point maximum for public debt,* $\hat{\beta}_i^{\max} \to (G^* - \overline{G}) / \lambda A$, *if* $\eta N/(1-\eta) > 1$,

(ii) *there is an without public debt, if* $\eta N/(1-\eta) < 1$.

The $\gamma < \infty$ case is discussed first. If primary deficits and economic growth are regarded as imperfect substitutes by policy-makers, there is a bifurcation maximum with respect to the parameters set ϕ, since this now determines the primary deficit, instead of it being exogenous. Above a threshold value, $\tilde{\phi}$, either because of strong very preferences for running deficits (in η) or a large financing externality (through N), there is no sustainable steady state equilibrium for public debt.

Some values for the deficit-capital "imperfect substitutes" case are shown in the upper part of *Table Three.*

Table Three: Steady state solution values for the endogenous deficit case

where $\lambda = 1,\ G^* = 4,\ \overline{G} = 1.2,\ E(R^K) = 5,\ \bar{\gamma} = 0,$

$\sigma = 4.5,\ A = 15,\ \mu = 1/3, \rho = 2/5,$ *and 35 year periods*

	Policy prefences/integration $\phi \equiv \eta N/(1-\eta)$	(endog) deficit ratio γ	Public debt ratio β	Portfolio share of public debt ϖ	Growth rate pa	Return on public debt as rate pa	Risk premium as rate pa
$\gamma < \infty$	$\to 0$	0	0	0	4.04%	-1.96%	6.67%
	0.134	0.021	0.029	0.109	3.70%	-0.03%	4.74%
	0.259	0.041	0.074	0.278	3.08%	0.74%	3.97%
	>0.259	*no steady state exists*					
$\gamma \to \infty$	<1	0	0	0	4.04%	-1.96%	6.67%
	>1	-0.383	0.187	0.701	0.52%	3.77%	0.94%

There are again bifurcation values that are of the same, low order of magnitude as for the previous, exogenous deficit case, as presented in the *Table Two*. Thus, the Bohn feedback effect, although present, is not decisive in this case. This is because its presence depends on the same parameters that encourage governments to run up deficits, which is problematic for sustainability in a nonlinear model such as this. Thus, the evidence that primary deficits respond negatively to the debt is not sufficient to allay concerns about debt sustainability.

More generally, if deficits are valued, $\eta > 0$, financial globalization will lead to a general expansion in public debt, and this we believe is depicted in *Figures One* and *Two* of *Section Two*, following the demise of Bretton Woods and the move to financial globalization. The fact that this did not happen in the first era of globalization may be explained by an entirely different policy environment or, in the present context, by assuming $\eta = 0$, or that running deficits was not then on the earlier agenda. It also follows from the nonlinearity of this model, that too much globalization can actually over-burden the debt market, where in concert Nash-acting countries issue more than the world is willing to hold.

The perfect substitutes case, where $\gamma \to \infty$ is also qualitatively different, since it yields a corner point maximum for public debt at $(G^* - \overline{G})/A$, if $\eta N/(1-\eta) > 1$. This is also at a point where the growth factor is pushed down to its reservation minimum of $\overline{G}$, assumed to be at least unity in value. With respect to the portfolio share of public debt, the *table* gives a figure of 0.701, which seems plausible empirically, despite the strictures of the model. A key point is that corner-points allow higher maximum values than bifurcations.[10]

The argument is now made that, compared with the previous bifurcation case, the endogenous Bohn response where $\gamma \to \infty$ works very powerfully to stabilize the model. First, at a corner point steady state maximum, the public debt is very much higher, such that its factor of return is raised above the growth factor, $R^B > G$. This is inherently destabilizing according to the foundational public debt dynamic process – exclusive of the feedback effect – given by equation (3), where the implication is that $\partial\beta_t / \partial\beta_{t-1} > 1$. Second, but *inclusive* of it, as in equation (29), it is apparent that assuming $\gamma \to \infty$ eliminates the debt dynamics altogether to give $\partial\beta_t / \partial\beta_{t-1} = 0$. It is also note that the inequality $R^B > G$ indicates a steady state of budget surpluses, thus pointing to a conclusion that here the endogenous feedback effect

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[10] This corner-point solution also comes from a quadratic equation. The other solution one is clearly degenerate in giving a public debt ratio that is even higher at G^*/A, but with a growth *factor* of zero.

essentially works not by reducing budget deficits but by generating surpluses, which in turn depends on the extreme perfect substitutes case of policy preferences.

To conclude this analysis, we look at another numerical example for the $\gamma \to \infty$ case. If $1/3 < \eta < 1/2$, a movement from a position of autarky, where for each country, effectively, $N = 1$, to one of two country, financial integration where $N = 2$, will cause the public debt as a ratio of GDP ratio in each symmetric country to rise from zero to 0.187 or as a portfolio share from zero to 0.701. Thus, countries may be induced to issue public debt under globalization but not under aurarky.

6. Further considerations

6.1 Country differences

The main results have been measured against the benchmark of a cross-county symmetry, allowing in a non-essential division between the debt issuing and non-issuing ones. The structure of the model is such that, in the event of country differences, there would be no cross-country relationship between public debt and economic growth, because it is the *global aggregate* of public debt that alone matters for each country. However, there are a number of reasons why countries might issue divergent levels of public debt, but here without any consequences for their own growth rates. [11]

The logic of the Chang externality implies that smaller countries will become more internationally indebted, because their internal crowding-out costs will be relatively lower. The parameter N has been interpreted as the number countries with residents with a demand for public debt, as if all countries were of equal size and as if there were no national differences in saving rates and attitudes towards risk. It is possible, alternatively, to regard $N(j)$ as the inverse of the global weight of country j in the world economy, where $\sum_{i=1}^{\tilde{N}} 1/N(i) = 1$, $\tilde{N}$ being the number of countries. Replacing N with $N(j)$ just implies smaller countries will issue more debt. This offers another explanation of the empirical finding of Alesina and Wacziarg (1998) of an inverse cross-country between government expenditure and country size.

[11] Pursuing to its end the logic of a "steady-state" requires that all countries ultimately grow at the same rate.

This, however, is not always the case, and larger countries like the UK, certainly historically, and the US today have issues amounts of globally held public debt that are relative to their still relatively large shares of global GDP. This phenomenon may be possibly explained by another, more behavioural difference between countries. Although cross-country Nash behaviour has underpinned the analysis, and underlies the Chang financing externality, there are also potential country gains from *moving first* or acting as a *Stackelberg* issuer of public debt. A verification of this point may be found by considering the implies cross-country response of $\partial b_{z,t} / \partial b_{i,t} < 0$ in equation (27), where, symmetrically, $\partial b_{i,t} / \partial b_{z,t} < 0$, which suggests that there is an incentive to move first. Thus, in a globally integrated market, a single country might potentially hog a given global demand for public debt. Or, perhaps, more historically, some countries were the first to institute stock exchanges, to which there was a global access but where their own debts were favoured.

6.2 Default risk

An implicit assumption has been that the returns on public debt are perfectly safe without any no default probability, which is questionable, particularly for the case of a bifurcation maximum. Introducing the possibility of default is a possible extension to the analysis. The well-established issue of self-fulfilling crises then arises, where a default premium raises interest rates to increase this possibility.

There is, however, also a more specific point be made with respect to the present multi-country model. Although a global financial collapse is possible, it probably makes sense to acknowledge inevitable differences between countries with some having a greater tendency to default than others. In this case, it is no longer to tenable to assume that the public debts of all countries may be regarded as perfect substitutes. It then follows that a set of more or less financially *homogeneous* countries that do hold each other's public debts would be credited with similarly insignificant probabilities of default. Prospective defaulters might ultimately be forced to leave the "club", if they had been allowed to join it in the first place.

6.3 Global integration and economic growth

Globalization in conjunction with the Chang externality leads to an over-issue of public debt, and this crowds-out capital in the overlapping generations' model at the cost of reducing economic growth. The present model could be amended by eschewing its present Feldstein-Horika feature to allow for greater portfolio diversification where households hold international capital as well as public debt. Hedging against country-specific shocks would raise the overall portfolio share of productive capital, leading to a compensating rise in economic growth. This would be mitigated, however, by the effect of a lower demand for public debt, leading to higher rates of return and thus debt accumulation.[12]

A more favourable view of financial globalization in having net benefits for economic growth could be accommodated the present model without undercutting the mechanisms on which the main analysis stands. Even so, as already mentioned with reference to Schularick and Steger (2010), it is not clear that second era of globalization, also characterized by a general expansion in public debt, has not delivered the same economic gains noted for the first era, about which this present model also has something say.

7. Summary

A model of public debt has been presented. The first part developed some of the existing literature by incorporating a well-defined or portfolio demand for public debt, which adds to the non-linearity that is known to characterize the Diamond model. This made it possible to incorporate more than one interest rate and to replicate empirical patterns of relative asset returns. A multi-country case also served to slightly generalize the analysis of Farmer and Zotti (2010).

The multi-country case comes into its own when it is considered along with endogenous policy, because of two important implications. One is the Chang (1990) financing externality, which may be of empirical relevance in providing a ready explanation for the global expansion in public debt that followed the financial liberalization after Bretton Woods. Furthermore, this externality also has relevance to any discussion of debt sustainability within nonlinear models of open economies.

[12] It appears to be something of a paradox that under backward-looking dynamics, a fall in the demand for public debt raises its ultimate size through increasing its servicing costs. This is because the steady-state demand and supply curves each positively sloped with the former locally flatter than the latter.

The second is that the Bohn (1990) feedback response of the deficit/surplus to the debt. This is found in the data, while occurring here endogenous feature whenever policy-makers, plausibly, trade-off the financing gains of issuing debt against the crowding-out costs that may occur. Although the model could *explain* this stabilizing effect, it did not generally win the day, because the parameters on which it is based also reflect a countervailing tendency for running deficits. Consequently, bifurcation maxima were found to persist at low values of public debt and with the associated possibilities of fold catastrophes when higher. There was also a case of linear policy preferences, where the Bohn mechanism worked very powerfully to introduce a corner-point and, thus, higher maximum for public debt, which seemed empirically plausible when expressed as a portfolio share.

The model demonstrates that any attempt to answer the topical question "*how large can public debt get?*" ought to consider the demand-side factors that determine the interest rate at which it accumulates, as well as the wider economic environment with regard to the degree international openness and to the fundamental framework of policy. The present analysis made a move in these directions, but there is scope for a further treatment.

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