Distributional Comparative Statics

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Abstract

Distributional comparative statics is the study of how individual decisions and equilibrium outcomes vary with changes in the distribution of economic parameters (income, wealth,productivity, information, etc.). This paper develops new tools to address such issues and illustrates their usefulness in applications. The central development is a condition called *quasi-concave differences* which implies concavity of the policy function in optimization problems without imposing differentiability or quasi-concavity conditions. The general take-away is that many distributional questions in economics which are frustratingly complex and cannot be solved by direct calculations or the implicit function theorem, can be addressed easily with this paper's methods. Several applications demonstrate this: the paper studies increased uncertainty in Bayesian games; it shows how increased dispersion of productivities affects output in the model of Melitz (2003); and it generalizes Carroll and Kimball (1996)'s result on concave consumption functions to the Aiyagari (1994) setting with borrowing constraints.

Keywords: Distributional comparative statics, concave policy functions, income distribution, inequality, uncertainty, heterogenous firms, Bayesian games, dynamic stochastic general equilibrium models, arg max correspondence.

JEL Classification Codes: C61, D80, D90, E20, I30.

1 Introduction

Imagine that the players in a Bayesian game receive less precise private signals and therefore become more uncertain about their environment. In an arms race with incomplete information, for example, countries may be uncertain about arms' effectiveness and opponents' intentions — and the degree of uncertainty is likely to change over time. How will such changes in exogenous distributions affect the Bayesian equilibria, including the mean actions and the actions' variances? In the arms race, will increased uncertainty lead to disarmament or to escalation? The objective of this paper is to develop the tools needed to address such questions and illustrate the methods' use in applications.

Distributional comparative statics (henceforth *DCS*) studies how changes in exogenous distributions affect endogenous distributions in models with optimizing agents. Apart from the effect on equilibrium quantities following increased uncertainty in Bayesian games; the methods developed here are able to address a number of economic problems.

- A monetary policy committee (MPC) sets the interest rate. The public knows that the MPC's objective is to minimize a standard loss function, as in Kydland and Prescott (1977). But how the interest rate affects output and inflation is private information which the MPC may disclose with any desired degree of accuracy. A monetary economist would then want to know how the public's interest rate expectations are affected if the MPC reveals more or less information.

- The incomplete markets model of Aiyagari (1994) features a population of consumers with heterogenous incomes who make consumption and savings decisions subject to borrowing constraints. In this setting, a macroeconomist might wish to know under what conditions on consumers' preferences a Lorenz dominated decrease in inequality will reduce the inequality of outcomes (the variance of savings across the population), or increase per-capita savings.

- In the international trade model of Melitz (2003) a continuum of firms have different productivities. A trade theorist might then want to know if there is "increasing or decreasing returns to diversity", *i.e.*, if increased dispersion of productivities increases or reduces total output, or if increased dispersion increases the variance of output across the firms.

Note that these are all DCS questions since we change an exogenous distribution and ask how an endogenous distribution changes in response. This paper's approach can be used to derive general and economically meaningful answers to such questions. One might instead attempt to proceed by "brute force", *i.e.*, by means of first-order/Euler conditions and repeated use of the implicit function theorem. A general take-away from this paper is that in important situations where such brute force does not work, this paper's methods do; and whether brute force works or not, the tools developed here offer both a simpler and more enlightening way to attack many problems.

Recall from Atkinson (1970) that in standard models of savings, dominating shifts in Lorenz curves reduce or increase aggregate savings according to whether the savings function is concave or convex. As explained in Section 2 and illustrated repeatedly throughout the paper, Atkinson's observation is useful much more generally: concavity or convexity of the function which maps exogenous variables into endogenous ones (the policy function) is the key to answering DCS questions about mean-preserving spreads, second-order stochastic dominance, Lorenz or generalized Lorenz shifts. To address the previous questions, we may therefore focus attention on concavity/convexity of suitable policy functions *provided* we know under what conditions on the primitives of the decision problems, these functions will be concave or convex. The main contribution of this paper is a theorem that offers precisely that, and isolates the critical condition which implies that a decision problem's policy function is concave (or convex). Specifically, it is shown that if the payoff function satisfies a condition called *quasi-concave differences*, then the policy function — and more generally, the policy correspondence — will be concave. Quasi-concave differences is easy to verify in applications, and ensures concavity of the policy function whether or not payoff functions are differentiable, concave, or even quasi-concave. This advances the literature in several ways.

Firstly, it enables us to deal with distributional issues in a number of models which we previously could not handle. Thus in the model of Aiyagari (1994) mentioned above, any attempt at using the implicit function theorem fails because the value function is not differentiable (Section 2.2). Models with ambiguity averse agents — an ambiguity averse MPC in the example above, say — confounds existing methods for similar reasons (Section 3.3). In the trade setting of Melitz (2003), existing methods fail when production sets are not convex (Section 3.3), and so on.

Secondly, the results in this paper allow us to disentangle the fundamental economic conditions that drive our conclusions from unnecessary technical conditions. As this paper's applications illustrate again and again, this can improve our economic understanding substantially. Readers familiar with monotone methods (*e.g.* Topkis (1978), Milgrom and Shannon (1994), Quah (2007)) and with so-called robust comparative statics more generally (*e.g.* Milgrom and Roberts (1994), Acemoglu and Jensen (2015)) will immediately spot the parallel: when one obtains a result under certain sufficient conditions and those conditions are a mixture of critical economic conditions and entirely unnecessary technical conditions, economic intuition is lost because one is unable separate the two (Milgrom and Roberts (1994), p.442-443).[1] In particular, one can usually not predict whether even minor changes in model specifications — replacing a specific functional form with a slightly different one, say — will overturn the results. This problem is particularly severe in models that are computationally complicated, and to be sure, DCS questions are difficult to handle by direct calculations even in the simplest of models.

A setting where the previous advances turn out to be of critical importance is Bayesian games (Section 4). In simple models and under sufficiently strong conditions, one could in principle derive results by repeated use of the implicit function theorem. But in practice, this would require

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[1]Monotone methods have an important role to play in DCS (see Section 2.1), but they are rarely sufficient on their own. In particular, one cannot simply parameterize the exogenous distribution and then apply monotone methods (or the implicit function theorem). This is because DCS questions ask how endogenous distributions change, not how deterministic decision variables change.

a monumental effort as well as a host of unnecessary simplifications, and in the end it would be virtually impossible to identify the economic condition that accounts for the results. Any interpretation would therefore, at best, be a qualified guess. In contrast, this paper's results allow us to deal with changing distributions in Bayesian games in full generality; and if we allow functional forms to be differentiable, the conditions one must check can be characterized explicitly via derivatives and are very easy to work with.[2]

The paper begins in Section 2 by further motivating and exemplifying the DCS agenda. Second 2 also previews the paper's main results without going into too much technical detail. The paper then turns to quasi-concave differences, discusses the intuitive content of the definition, and shows — first in the simplest possible setting (Section 3.1), then under more general conditions (Section 3.2)— that quasi-concave differences implies concavity of the policy function in an optimization problem. An appendix treats the issue under yet more general conditions where the decision vector is allowed to live in an arbitrary topological vector lattice (Appendix III). Section 3.3 contains a practitioner's guide to the results and several fully worked-through examples. Section 4 then tackles DCS in Bayesian games, and Section 5 derives general conditions for concavity of policy functions in stochastic dynamic programming problems. As a concrete application, Section 5 extends Carroll and Kimball (1996) to the setting with borrowing constraints (Aiyagari (1994)). That result plays an important role for various distributional comparative statics questions in macroeconomics (Huggett (2004), Acemoglu and Jensen (2015)) and is also essential for the analysis of inequality in settings where consumers may be credit constrained (Section 2.2).

2 Preview and Motivation

This section previews the paper's results and explains the role of convex and concave policy functions for distributional comparative statics (DCS). The section also discusses several set-ups in which existing methods are unable to address DCS questions.

2.1 Decisions under Uncertainty

A monetary policy committee (MPC) meets to set the rate of interest $x \in X \subseteq \mathbb{R}$. As in Kydland and Prescott (1977), the MPC has a loss function $L(y - y^*, \pi - \pi^*)$ where y denotes output, π denotes inflation, and stars denote natural/target levels. The central bank controls output and inflation via the interest rate, $y = y(x, z)$ and $\pi = \pi(x, z)$ where $z \in Z \subseteq \mathbb{R}$ is a parameter that represents the MPC's assessment of the Lucas supply/Philips curve and the interest rate pass through.[3] The MPC's objective is thus to maximize $u(x, z) = -L(y(x, z) - y^*, \pi(x, z) - \pi^*)$ with respect to x.

A forecaster must predict the MPC's decision. She knows its objective u but only holds certain beliefs about z as represented by a probability measure μ on Z. If everyone is rational, the

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[2]Operationally, the conditions for quasi-concave differences in the differentiable case are on an equal footing with, say, concavity, or supermodularity/increasing differences which can be established, respectively, via the Hessian criterion and the cross-partial derivatives test of Topkis (1978).

[3]Note that we could have equally assumed that the central bank directly chooses inflation and output as in Kydland and Prescott (1977). The reason for the focus on interest rates will become clear when the forecaster is introduced next.

forecaster will therefore arrive at a forecast with distribution

(1) $$\mu_x(A) = \mu\{z \in Z : g(z) \in A\},$$

where A is any Borel set in X and $g : Z \to X$ is the MPC's policy function,

(2) $$g(z) = \arg\max_{x \in X} u(x, z).^4$$

So if the forecaster is asked how likely the MPC is to set the interest rate in the interval between 0.5 and 0.6 %, she will answer "with probability $\mu_x([0.5, 0.6])$" where $\mu_x([0.5, 0.6]) \in [0, 1]$. Her "headline" forecast will be the mean of μ_x. And so on.

Consider now a shift in the forecaster's beliefs μ. For example, she might become more uncertain about the MPC's private signal (a mean-preserving spread to μ), or her beliefs could be subjected to first- or second-order stochastic dominance shifts. Relevant economic examples abound: increased uncertainty could be because the MPC transmits less information to the public, or because it signals decreased ability to control inflation and output. A second-order stochastic dominance increase could be due to an external event such as a more favorable public forecast of output and inflation. For the reader's convenience, the formal definitions follow (see *e.g.* Shaked and Shanthikumar (2007) for an in-depth treatment of stochastic orders).

Definition 1 (Stochastic Orders) Let μ and $\tilde{\mu}$ be two distributions on the same measurable space $(Z, \mathscr{B}(Z))$.[5] Then:

- $\tilde{\mu}$ *first-order stochastically dominates* μ if $\int f(z)\tilde{\mu}(dz) \geq \int f(z)\mu(dz)$ for any increasing function $f : Z \to \mathbb{R}$ such that the integrals are well-defined.

- $\tilde{\mu}$ is a *mean-preserving spread of* μ if $\int f(z)\tilde{\mu}(dz) \geq \int f(z)\mu(dz)$ for any convex function $f : Z \to \mathbb{R}$ such that the integrals are well-defined.

- $\tilde{\mu}$ is a *mean-preserving contraction of* μ if $\int f(z)\tilde{\mu}(dz) \geq \int f(z)\mu(dz)$ for any concave function $f : Z \to \mathbb{R}$ such that the integrals are well-defined.[6]

- $\tilde{\mu}$ *second-order stochastically dominates* μ if $\int f(z)\tilde{\mu}(dz) \geq \int f(z)\mu(dz)$ for any concave and increasing function $f : Z \to \mathbb{R}$ such that the integrals are well-defined.

- $\tilde{\mu}$ *dominates* μ *in the convex-increasing order* if $\int f(z)\tilde{\mu}(dz) \geq \int f(z)\mu(dz)$ for any convex and increasing function $f : Z \to \mathbb{R}$ such that the integrals are well-defined.

When μ shifts (to $\tilde{\mu}$) in accordance with one of these stochastic orders, the natural question is how the forecast's distribution μ_x changes. The following observations provide the answers. Note that by an "increase in μ", we mean that μ is replaced with a distribution $\tilde{\mu}$ that dominates μ in the given stochastic order. Similarly for a "decrease in μ" and a "mean preserving spread to μ" where $\tilde{\mu}$ is dominated by μ and $\tilde{\mu}$ is a mean preserving spread of μ, respectively.[7]

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[4]We assume here that the MPC is able to agree on a single decision (existence and uniqueness).

[5]Here $\mathscr{B}(Z)$ denotes the Borel algebra of Z.

[6]Note that $\tilde{\mu}$ is a mean-preserving contraction of μ if and only if μ is a mean-preserving spread of $\tilde{\mu}$.

[7]Note that there is nothing deep or difficult about the following observations — in fact, they are basically restatements of the definitions. For detailed proofs, please see Appendix I.

1. If g is increasing, any first-order stochastic dominance increase in μ will lead to a first-order stochastic dominance increase in μ_x.

2. If g is concave, any mean-preserving spread to μ will lead to a second-order stochastic dominance decrease in μ_x.

3. If g concave and increasing, any second-order stochastic dominance increase in μ will lead to a second-order stochastic dominance increase in μ_x.

4. If g is convex, any mean-preserving spread to μ will lead to a convex-increasing order increase in μ_x.

5. If g is convex and increasing, any convex-increasing order increase in μ will lead to a convex-increasing order increase in μ_x.

So, going back to the question posed a moment ago: if the policy function in (2) is concave and the forecaster becomes more uncertain about the MPC's private signal (a mean-preserving spread to μ), her forecast's distribution μ_x decreases in the second-order stochastic dominance order (Observation 2). In particular, the headline forecast (the mean of μ_x) decreases and the forecast's variance increases. We return to this case momentarily. But first, consider Observation 1 which concerns a first-order stochastic dominance shift in μ. In this situation, the implicit function theorem (IFT) or monotone methods can be used to establish that g is increasing. Observation 1 then implies that the forecaster's distribution μ_x increases when μ increases (both with respect to the first-order stochastic dominance order). The IFT tells us that g is increasing if $g'(z) \geq 0$ in equation (5) below. Using monotone methods, we know that g will be increasing if u exhibits increasing differences (Topkis (1978)) or satisfies the single-crossing property (Milgrom and Shannon (1994)). So existing results fully enable us to deal with first-order stochastic dominance shifts in the forecaster's beliefs. There are many instances of such reasoning in the literature. For example, the property that first-order stochastic dominance of beliefs implies first-order stochastic dominance of (predicted) actions is the basic criteria for a Bayesian game to exhibit strategic complementarities, and Van Zandt and Vives (2007) provide multiple examples where they use monotone methods to verify that policy functions are increasing.

Imagine, however, that μ is *not* subjected to a first-order stochastic dominance shift but to a mean-preserving spread as discussed a moment ago, or to a second-order stochastic dominance increase (Observation 3 above). As is clear, we must then (in addition) know whether g is concave to derive the effect on μ_x. Moreover, for the cases covered by Observations 4-5 we must know whether g is convex. For concreteness and to set the stage for Bayesian games (Section 4), assume that the MPC's objective u takes an expected utility form

$$(3) \qquad u(x,z) = \int U(x,\bar{z},z)\eta(\bar{z}).$$

In (3), $\bar{z}$ is public with distribution η and z as before is private to the MPC. Again, economic examples abound. For example, $\bar{z}$ could be the expected price of oil and η its distribution. Or

if $\bar{z}$ takes only a finite number of values (in which case η is a counting measure and the integral actually a summation), $\eta(\bar{z})$ could be the "weight" of a member $\bar{z}$ of the MPC and $U(\cdot, \bar{z}, z)$ that member's individual objective.

This paper's main result (Theorem 1) will immediately allow us to conclude that if u satisfies a condition called *quasi-concave differences* (Definition 2 in Section 3), then the MPC's policy function g will be concave whether or not the objective is concave or even quasi-concave. It also does not matter whether the objective takes the specific form (3), but when it does and U is differentiable, a sufficient condition for quasi-concave differences is that $D_x U(x, \bar{z}, z)$ is concave in (x, z) for almost every $\bar{z} \in Z$.[8] By way of Observation 2 above we thus get a particularly simple condition on the fundamentals of the MPC's objective which implies, for example, that if the MPC chooses to reveal less of its private information to the public, then the forecast's distribution decreases in the sense of second-order stochastic dominance. In particular, the condition has a simple economic interpretation: concavity of $D_x U$ is equivalent to convexity of the MPC's marginal loss function $D_x L$. A convex marginal loss function obtains if the marginal loss is relatively constant or rises slowly when output and inflation are close to their target levels, and rises more rapidly when output and inflation are farther away from the targets.[9] So if the MPC's adversity to an additional rate hike increases at an ever stronger rate the farther the MPC is from its targets, we should expect less information transmission to reduce mean forecasts.[10]

Without this paper's results, repeated use of the implicit function theorem (IFT) provides the only way to address the concavity of g. It is instructive to follow this line of reasoning for a moment. If U is sufficiently smooth, concavity of u is assumed, differentiation under the integral sign is allowed, and the solution is interior for all $z \in Z$, the following first-order condition is necessary and sufficient for an optimum

$$(4) \qquad (D_x u(x, z) =) \int_{\bar{z} \in \bar{Z}} D_x U(x, \bar{z}, z) \eta(\bar{z}) = 0.$$

If the second derivative never equals zero (strict concavity of $u(\cdot, z)$), the IFT determines x as a function of z, $x = g(z)$ where

$$(5) \qquad g'(z) = -\left[\int_{\bar{z} \in \bar{Z}} D_{xx}^2 U(g(z), \bar{z}, z) \eta(\bar{z}) \right]^{-1} \int_{\bar{z} \in \bar{Z}} D_{xz}^2 U(g(z), \bar{z}, z) \eta(\bar{z}).$$

Note that monotone comparative statics is about the sign of g', and as Milgrom and Shannon (1994) convincingly argue, the IFT approach is not ideal for many applications. When the question is concavity of g, the situation is worse since we must determine g'' and so need to apply the IFT one more time. Specifically, we differentiate the right-hand-side of (5) with respect to z and substitute in for $g'(z)$. The resulting expression is rather daunting and of no particular importance to us. It contains a mixture of integrals of second and third derivatives and in contrast to the

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[8]The details of everything being postulated here can be found in Section 3.1.

[9]Note that, strictly speaking, this interpretation requires that the Lucas supply curve is linear in x and z (*i.e.*, the functions $y(x, z)$ and $\pi(x, z)$ are linear). With non-linear relationships, the interest rate pass-through enters the picture and complicates matters. The topic of interpretation will occupy a large part of Section 3.1.

[10]For the related literature on central bank communication see *e.g.* Myatt and Wallace (2014) and references therein.

condition we arrived at using this paper's results above, it may or may not be possible to establish any useful and intuitively transparent condition for $g'' \leq 0$ (concavity) from such an expression.[11] More substantially, in order to apply the IFT twice, a host of unnecessary technical assumptions must be imposed — so even when the IFT provides sufficient conditions for concavity of g, these will not be the most general conditions. As Milgrom and Roberts (1994) and Acemoglu and Jensen (2013, 2015) discuss in detail, this lack of "robustness" generally makes it impossible to disentangle the fundamental economic conditions that drive one's results from superfluous technical assumptions (again see also Milgrom and Shannon (1994), keeping in mind that the situation is worse here because we need to apply the IFT twice). In particular, the IFT requires the MPC's objective to be strictly concave which imposes spurious cross-restrictions on the loss function and the Lucas supply curve. If u is *not* strictly concave, or if it is *not* at least thrice differentiable the IFT is never applicable. In the next subsection we shall encounter a particularly egregious instance of this but even in the current example interesting cases cannot be handled via the IFT. Thus if the MPC displays ambiguity aversion (see Section 3.3), u will not even be once differentiable. And if constraint sets vary — say, if the MPC's maximum acceptable interest rate change depends on economic fundamentals — the IFT's usefulness is similarly confounded.

2.2 Income Allocation and Inequality

In the monetary policy committee example of the previous subsection, the implicit function theorem (IFT) does at least provide a conclusion under suitable technical assumptions. We now turn to an application from macroeconomics where the differentiability requirements of the IFT confounds *any* attempt to use it to establish concavity of the policy function. So here this paper's results provide the only known way to deal with the economic issues raised.

Consider the stochastic income allocation model with Bellman equation

(6) $$v(rx + wz) \quad = \max_{y \in \Gamma(x,z)} \quad u(rx + wz - y) + \beta \int v(ry + wz')\eta(dz').$$

v is the value function and $\Gamma(x,z) = \{y \in \mathbb{R} : -b \leq y \leq rx + wz\}$ is admissible savings given past savings x and labor productivity z which follows an *i.i.d.* process with distribution η. As usual r denotes the interest factor (one plus the interest rate), and w the wage rate. When $b < +\infty$, we have a borrowing constraint and potential market incompleteness which Aiyagari (1994) in an influential paper shows accounts for a number of empirical regularities (see also Acemoglu and Jensen (2015)).

Let $g(rx + wz) = \arg\max_{y \in \Gamma(x,z)} u(rx + wz - y) + \beta \int v(y, z')\eta(dz')$ denote the *savings function*, and $c(rx + wz) = rx + wz - g(rx + wz)$ the *consumption function*. These are, without further elaboration, assumed to be well-defined. Clearly, the savings function is convex if and only if the consumption function is concave.

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[11]Note that the problem in part is that concavity of u simultaneously imposes conditions on second partial derivatives which leads to "entanglement" as discussed in the Introduction and further discussed momentarily. With multidimensional decision variables as explored in Appendix III, the IFT becomes excessively complicated and is rarely useful.

Following Carroll and Kimball (1996), say that u belongs to the Hyperbolic Absolute Risk Aversion (HARA) class if $\frac{u'''u'}{(u'')^2} = k$ for a constant $k \in \mathbb{R}$. Carroll and Kimball (1996) prove that if u belongs to the HARA class, then the consumption function is concave if there is no borrowing constraint ($b = +\infty$) and if the period utility function has a positive third derivative (precautionary savings).

Technically, the proof of Carroll and Kimball (1996) relies on Euler equations and repeated application of the IFT. In particular, this approach requires that the value function v is at least thrice differentiable. This is unproblematic if the borrowing constraint is inactive, but if $b < +\infty$, then the value function will not be thrice differentiable at any point where the borrowing constraint binds.[12] As we shall see in Section 5, this paper's results allow for a simple and direct proof of the concavity of the consumption function which does not rely on Euler equations and does not require differentiability of the value function. In particular, it is shown that the consumption function will be concave for the general HARA class with or without borrowing constraints. Furthermore, the HARA class "pops out" endogenously from an application of our general results — there is no guesswork involved, and no ingenuity is required (contrast with the mathematical ingenuity of Carroll and Kimball (1996)). Note that this added simplicity when it comes to finding suitable sufficient conditions parallels the discussion of sufficient conditions for $g''(z) \leq 0$ that followed equation (5) in the previous example. In fact, the additional ease-of-use makes this paper's results so effective in the stochastic dynamic programming setting that little effort is required to prove a result on the convexity/concavity of policy functions for stochastic dynamic programming problems at the level of generality of the text book treatment of Stokey and Lucas (1989). Thus we are able in Section 5 to address not just the previous income allocation problem but nearly any stochastic dynamic model one can think of applying in macroeconomics and other fields.

Let us finish this section with a brief look at the importance of this paper's results for the study of income inequality. Let $W_i = r x_i + w z_i$ denote income of agent i at a given moment in time and consider a continuum of agents $i \in [0, 1]$ with identical preferences but possibly different incomes. Each individual thus has consumption $c = c(W_i)$ where c was defined a moment ago. Letting η_W denote the frequency distribution of income, per-capita/mean consumption is then

$$\int c(W_i)\eta_W(d W_i).$$

From Atkinson (1970) we know that mean-preserving spreads to η_W are equivalent to increases in inequality in the sense of Lorenz dominance (Atkinson (1970), p. 246-247). Thus by the result of Section 5 previewed a moment ago, a Lorenz increase in inequality implies *lower* mean consumption if u is in the HARA class, whether or not consumers are limited by borrowing constraints. Notice, however, that Observations 2-4 on page 4 allow us to go considerably further. Since the

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[12]Due to the importance of borrowing/liquidity constraints for much applied work, the same authors (Carroll and Kimball (2001)) as well as Huggett (2004) address the concavity question in a framework with borrowing constraints and establish concavity of the consumption function for three special cases of the general HARA class (CRRA, CARA, and quadratic utility, respectively). See also Suen (2015) for more on this topic (note that Suen (2015) and the current paper were written independently of each other).

consumption function c is also increasing under standard conditions, the previous statement extends to generalized Lorenz dominance by Observation 3 on page 4.[13] Of more novelty, we can go beyond considerations of the mean. For example, by Observation 2, a Lorenz increase in inequality will lead to a second-order stochastic dominance decrease in the distribution of consumption when c is concave and increasing. So we can conclude not only that the mean will decrease, but also that the variance will increase. Since the variance measures *inequality of outcomes* as opposed to the inequality of opportunities embodied in the distribution of income, such conclusions are obviously interesting. More generally, the approach developed in this paper opens up a simple and effective way to study inequality in a variety of situations — including situations such as income allocation under borrowing constraints where other approaches fail.

3 Concave Policy Functions

Motivated by the previous section, this section presents the paper's main results on the concavity and convexity of policy functions. The first subsection considers the simplest case of an objective with a one-dimensional decision variable, a fixed constraint set, and a unique optimizer. This simplicity allows us to focus on the new concepts' economic interpretation. In the second subsection, all of these restrictions are relaxed. The last subsection contains a user's guide to the results as well as examples.

3.1 A Simple Case

Let $u : X \times Z \to \mathbb{R}$ be a payoff function where $x \in X \subseteq \mathbb{R}$ is a decision variable and $z \in Z$ a vector of parameters. It is assumed that X is convex and that Z is a convex subset of a vector space. When the associated decision problem $\max_{x \in X} u(x, z)$ has a unique solution for all $z \in Z$, define the *policy function* $g : Z \to X$ by

(7) $$g(z) = \arg\max_{x \in X} u(x, z).$$

The example from Section 2.1 fits into this framework with $g(z)$ being the MPC's interest rate decision given state of the economy z. In that Section, the significance of g being increasing was discussed and it was mentioned that g is increasing if u exhibits increasing differences in x and z. Precisely, this requires that $u(x + \delta, z) - u(x, z)$ is (coordinatewise) increasing in z for all $x \in X$ and $\delta > 0$ with $x + \delta \in X$ (Topkis (1978)). The purpose of this section is to show that concavity of g is ensured by a related condition.

Definition 2 (Quasi-Concave Differences) A function $u : X \times Z \to \mathbb{R}$ exhibits *quasi-concave differences* if for all $\delta > 0$ in a neighborhood of 0, $u(x, z) - u(x - \delta, z)$ is quasi-concave in $(x, z) \in \{x \in X : x - \delta \in X\} \times Z$.

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[13]The generalized Lorenz curve is constructed by scaling up the Lorenz curve by the distribution's mean and is equivalent to second-order stochastic dominance shifts, see *e.g.* Dorfman (1979).

If in this definition $u(x+\delta,z)-u(x,z)$ is instead required to be quasi-convex, u exhibits *quasi-convex differences*. Quasi-convex differences will be shown to imply that g is convex. Conveniently, u exhibits quasi-convex differences if and only if $-u$ exhibits quasi-concave differences, hence there is no reason to distinguish between the two in the following discussion. The first thing to note is that quasi-concave differences is easy to verify for differentiable objectives.

Lemma 1 (Differentiability Criterion) Assume that $u : X \times T \to \mathbb{R}$ is differentiable in $x \in X \subseteq \mathbb{R}$. Then u exhibits quasi-concave differences if and only if the partial derivative $D_x u(x,z)$ is quasi-concave in $(x,z) \in X \times Z$.

Proof. Appendix II. ∎

As an illustration, consider the MPC's expected utility objective from Section 2.1 with u continuously differentiable. We then have,

$$D_x u(x,z) = \int_{\bar{z} \in \bar{Z}} D_x U(x,\bar{z},z) \eta(\bar{z}).$$

Since integration preserves concavity, it immediately follows from Lemma 1 that u exhibits quasi-concave differences if $D_x U(x,\bar{z},z)$ is concave in (x,z) for *a.e.* $\bar{z}$.[14]

More generally, quasi-concavity is a fully tractable condition and so is therefore quasi-concave differences. This makes verification and computation easy as returned to in Section 3.3.

Let us now turn to the economic interpretation of quasi-concave differences. Beginning with a familiar case, let z be income and g a consumption function.[15] When u is differentiable in x, we may plot an *iso-marginal utility diagram*, *i.e.*, a diagram that depicts the iso-marginal utility curves $IMU(c) \equiv \{(z,x) \in Z \times X : D_x u(x,z) = c\}$ for $c \in \mathbb{R}$.
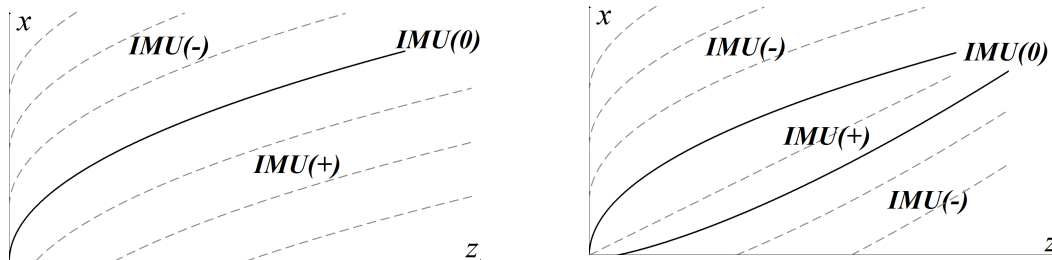
Iso-Marginal Utility diagrams

Figure 1: The zero IMU curve is the Engel curve.

Figure 2: The upper zero IMU curve is the Engel curve.

Informally, an IMU-diagram may be interpreted as follows. Think of marginal utility as "the kick" from consuming one more unit (endorphin release, hunger reduction, etc). Two different

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[14]A rather special case aside, concavity of the integrands $D_x U(\cdot,\bar{z},\cdot)$ is also necessary for u to be quasi-concave. See Debreu and Koopmans (1982).

[15]This will result from the reduced form consumption decision where $u(x,z) = \bar{u}(x,z-px)$ with p the relative price and boundary conditions are imposed so constraints can be ignored.

points on the same iso-marginal utility curve then tell us that a consumer gets the same kick out of consuming an extra unit whether she is poor and consumes little, or rich and consumes more. What "little" and "more" mean exactly tells us whether as income rises, the same kick will follow ever more modest increases in consumption, or conversely, whether to get the same kick the consumer requires ever larger consumption boosts (think chocolate versus cocaine).

Being now more formal, consider the zero IMU curve in Figure 1. Since any IMU curve below $IMU(0)$ is positive ($D_x u(c,z) > 0$) and any IMU curve above $IMU(0)$ is negative ($D_x u(c,z) < 0$), $IMU(0)$ depicts the graph of the consumption function (the Engel curve). As can be seen, concavity of the consumption function obtains because the IMU curves' slopes flatten out as z increases. And this is precisely what quasi-concave differences ensures: quasi-concave differences is equivalent to $D_x u(x,z)$ being quasi-concave (Lemma 1). That $D_x u(x,z)$ is quasi-concave in turn means that the "better marginal utility (MU) sets" $\{(x,z) \in X \times Z : D_x u(x,z) \geq c\}$ are convex. And as we know from standard indifference diagrams, flattening indifference curves are driven by convex better sets ("the diminishing marginal rate of substitution"). Economically, flattening IMU curves means that to keep the additional enjoyment of an extra unit (the kick in the language above) constant, consumption must increase at an ever slower rate with income. This may be because there is another good which the consumer increasingly substitutes towards.[16] But the picture in Figure 1 should more generally be thought of as reflecting an action that as a function of the exogenous variable requires less and less of an increase in order to yield the same efficiency gain, loss reduction, endorphin release, or whatever exact interpretation the application at hand requires. Thus in the MPC example, an increasing and concave policy function means that although increases in z lead to higher interest rates, a further one-unit increase in the interest rate becomes less and less attractive as interest rates increase. For example, we get this when the MPC's marginal loss function is convex, and $y = y(x,z)$ and $\pi = \pi(x)$ are linear functions with y increasing in z and both y and π decreasing in x (so an interest rate increase moves the economy towards the origin of the Lucas supply curve while an increase in z is expansive without being inflationary). In response to an increase in z, the central bank will increase the interest rate to "substitute" some of the additional output for lower inflation. But as the economy keeps expanding, interest rate increases become less and less effective in securing the MPC's targets: at first, increasing x will both shift y and π towards targets (assuming that we begin above those targets). But increasingly, the MPC becomes wary of increasing x to force output back towards target because it will simultaneously be forcing inflation further and further *below* target and this is increasingly upsetting due to the convex marginal loss function.

Now, in each of the previous cases the policy function is increasing (in addition to being concave). But convexity of the better MU sets also captures concavity of g when g is not increasing. If we instead consider an inferior good (a decreasing zero IMU-curve), the interpretation would change — now IMUs become *steeper* as z is increased corresponding to increasing unwillingness

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[16]A necessity good is one whose income elasticity of demand $c'(z)\frac{z}{c(z)}$ is between 0 and 1 (Varian (1992), p.117). When $c(0) = 0$, a concave consumption function implies that the good is a necessity good as seen by taking $y = 0$ in concavity's (differentiable) definition $c(y) \leq c(z) + c'(z)(y - z)$. But as this definition of concavity also shows, there is in general no firm relationship between elasticities and concavity of a policy function. It is thus deliberate that any interpretation in terms of elasticities is avoided.

to consume the good as income goes up — but convexity of the better MU sets remains the characterization of a concave consumption function. Similarly for non-monotonic functions where the two previous interpretations would only apply locally, but convexity of the better MU sets once again drives concavity.[17]

All of the above is straight-forward once we see it in an IMU diagram. What is perhaps not as obvious is that quasi-concavity (or concavity) of u has nothing to do with the story. In Figure 1 the zero IMU's better set (denoted $IMU(+)$) is the entire set below the zero IMU. To be sure, this means that the utility function is quasi-concave in x since for fixed z it tells us that $u(x, z)$ is first increasing and then decreasing in x. But consider now Figure 2 where for fixed z, $u(x, z)$ is first decreasing, then increasing, and then again decreasing in x; and so u is *not* quasi-concave in x. Since the better MU sets are convex, u exhibits quasi-concave differences. The zero IMU "curve" is now a correspondence consisting of two zero IMU curves. Since $u(x, z)$ is decreasing in x below the lower curve and increasing above it, any point on the lower zero-IMU curve is a local minimum. The upper zero IMU curve thus depicts the maxima so, just as in Figure 1, concavity of the consumption function is seen to obtain. Again, the reason is that the better MU sets are convex (quasi-concave differences) although now $IMU(+)$ is the lens between the $IMU(0)$ curves and not everything below $IMU(0)$ as in the quasi-concave setting of Figure 1.

To sum up, quasi-concave differences implies concavity of policy functions whether or not the policy function is monotone and regardless of any concavity or quasi-concavity assumptions. In this light, this Section's main result (Theorem 1 below) will come as no surprise to the reader. In fact, the proof below is just a formalization of the previous graphical argument that avoids using differentiability. There is only one complication related to solutions $g(z)$ touching the lower boundary of X, *i.e.*, solutions such that $g(z') = \inf X$ for some $z' \in Z$. In fact, such solutions will ruin any hope of obtaining a concave policy function for reasons that are easily seen graphically.
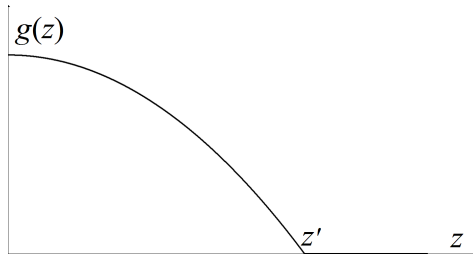

Figure 3: Concavity is destroyed when the policy function touches the lower boundary $\inf X = 0$.

In Figure 3, we see a policy function which at z' touches the lower boundary $\inf X = 0$ of the constraint set $X = \mathbb{R}_+$, and stays at this lower boundary point as z is further increased. It is evident that the resulting policy function will *not* be concave, even though it is concave for $z \leq$ z'. As

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[17]With quasi-convex differences the interpretation is in each instances reversed. Think of giving part of income to charity. Concave better IMU sets (quasi-convex differences) then means that as your income increases, you need to give progressively more of your current income to experience the same "warm glow" (utility gain). Presumably this would be because you need to feel you are making a sufficient sacrifice to get the same marginal utility effect, and you therefore have to progressively give more as a proportion of income for the sacrifice to keep its bite.

discussed at length in the working paper version of this paper (Jensen (2012)), this observation is robust: concave policy functions and lower boundary optimizers cannot coexist save for some *very* pathological cases. Of course, there is no problem if the optimization problem is unconstrained below, *i.e.*, if $\inf X = -\infty$. Nor is there a problem if attention is restricted to interior optimizers (witness Figure 3 where we do have concavity when z is below z').

Theorem 1 (Concavity of the Policy Function) Let Z be a convex subset of a vector space, and $X \subseteq \mathbb{R}$ a convex subset of the reals. Assume that the decision problem $\max_{x \in X} u(x, z)$ has a unique solution $g(z) = \operatorname{argmax}_{x \in X} u(x, z) > \inf X$ for all $z \in Z$. Then if $u : X \times Z \to \mathbb{R}$ exhibits quasi-concave differences, $g : Z \to X$ is concave.

Proof. Pick $z_1, z_2 \in Z$ and let $x_1 = g(z_1)$ and $x_2 = g(z_2)$ be the optimal decisions. Since $x_1, x_2 > \inf X$, there exists $\delta > 0$ such that $u(x_1 - \delta, z_1) - u(x_1, z_1) \leq 0$ and $u(x_2 - \delta, z_2) - u(x_2, z_2) \leq 0$. Letting $\tilde{x}_q = x_q - \delta$ where $q = 1, 2$, this can also be written $u(\tilde{x}_1, z_1) - u(\tilde{x}_1 + \delta, z_1) \leq 0$ and $u(\tilde{x}_2, z_2) - u(\tilde{x}_2 + \delta, z_2) \leq 0$. For $\lambda \in [0, 1]$ set $\tilde{x}_\lambda = \lambda \tilde{x}_1 + (1 - \lambda)\tilde{x}_2$ and $z_\lambda = \lambda z_1 + (1 - \lambda)z_2$. By quasi-concave differences, $u(\tilde{x}_\lambda + \delta, z_\lambda) - u(\tilde{x}_\lambda, z_\lambda) \geq \min\{u(\tilde{x}_1 + \delta, z_1) - u(\tilde{x}_1, z_1), u(\tilde{x}_2 + \delta, z_2) - u(\tilde{x}_2, z_2)\} \geq 0$. Since $u(\tilde{x}_\lambda + \delta, z_\lambda) - u(\tilde{x}_\lambda, z_\lambda) = u(x_\lambda, z_\lambda) - u(x_\lambda - \delta, z_\lambda) \geq 0$ where $x_\lambda = \lambda x_1 + (1 - \lambda)x_2$, this implies that

$$(8) \qquad\qquad u(x_\lambda, z_\lambda) - u(x_\lambda - \delta, z_\lambda) \geq 0 \, .$$

Since u exhibits quasi-concave differences, $u(x, z_\lambda) - u(x - \delta, z_\lambda)$ is quasi-concave in x for any $\delta > 0$ close to zero. To simplify notation, let $\inf X = 0$ from now on. By quasi-concavity in x, $u(x, z_\lambda) - u(x - \delta, z_\lambda)$ is non-decreasing in x on $[\delta, \hat{x}]$ and non-increasing in x on $[\hat{x}, x_\lambda]$ where $\hat{x} \in [\delta, x_\lambda]$. There are three cases:

1. $\hat{x} = \delta$. In this case, $u(x, z_\lambda) - u(x - \delta, z_\lambda)$ is non-increasing in x on $[\delta, x_\lambda]$. Imagine that there is a maximum x^* in $[\delta, x_\lambda)$. Then for all $x \in [x^*, x_\lambda]$: $0 \leq u(x_\lambda, z_\lambda) - u(x_\lambda - \delta, z_\lambda) \leq u(x, z_\lambda) - u(x - \delta, z_\lambda) \leq u(x^*, z_\lambda) - u(x^* - \delta, z_\lambda) \leq 0$, where the first inequality is (8). It follows that $u(x_\lambda, z_\lambda) = u(x^*, z_\lambda)$ so $x_\lambda \in \operatorname{argmax}_{[\delta, x_\lambda]} u(x, z_\lambda)$.

2. $\hat{x} = x_\lambda$. In this case, $u(x, z_\lambda) - u(x - \delta, z_\lambda)$ is non-decreasing in x on $[\delta, x_\lambda]$. Imagine that there is a maximum x^* in $[\delta, x_\lambda)$. Then for any $\epsilon > 0$ with $x^* - \epsilon \geq 0$: $0 \leq u(x^*, z_\lambda) - u(x^* - \epsilon, z_\lambda) \leq u(x^* + \epsilon, z_\lambda) - u(x^*, z_\lambda) \leq u(x^* + 2\epsilon, z_\lambda) - u(x^* + \epsilon, z_\lambda) \leq \dots$ This implies that $u(x^*, z_\lambda) \leq u(x^* + \epsilon, z_\lambda) \leq u(x^* + 2\epsilon, z_\lambda) \leq \dots$. Now pick a (small) $\epsilon > 0$ for which $x_\lambda = x^* + m\epsilon$, some $m \in \mathbb{N}$. We then get that $u(x^*, z_\lambda) \leq u(x^* + m\epsilon, z_\lambda) = u(x_\lambda, z_\lambda)$. So $x_\lambda \in \operatorname{argmax}_{[\delta, x_\lambda]} u(x, z_\lambda)$.

3. $\delta < \hat{x} < x_\lambda$: Repeat the argument of 2 for the initial interval, $[\delta, \hat{x}]$, in order to conclude that $\hat{x} \in \operatorname{argmax}_{[\delta, \hat{x}]} u(x, z_\lambda)$. It follows that $\max_{x \in [\delta, \hat{x}]} u(x, z_\lambda) \leq \max_{x \in [\hat{x}, x_\lambda]} u(x, z_\lambda)$. Next repeat the argument from 1 for the second interval $[\hat{x}, x_\lambda]$ to show that $x_\lambda \in \operatorname{argmax}_{[\hat{x}, x_\lambda]} u(x, z_\lambda)$. Combine to see that $x_\lambda \in \operatorname{argmax}_{[\delta, x_\lambda]} u(x, z_\lambda)$.

We get in all three cases that $x_\lambda \in \operatorname{argmax}_{[\delta, x_\lambda]} u(x, z_\lambda)$. This implies that there exists some $\hat{x} \in \operatorname{argmax}_{[\delta, \sup X]} u(x, z_\lambda)$ with $\hat{x} \geq x_\lambda$. Because $\inf X$ is never optimal and δ can be picked arbitrarily small, $\hat{x} \in \operatorname{argmax}_{[\inf X, \sup X]} u(x, z_\lambda)$. So g is concave: $\lambda g(z_1) + (1 - \lambda)g(z_2) = x_\lambda \leq \hat{x} = g(\lambda z_1 + (1 - \lambda)z_2)$. ∎

Corollary 1 (Convexity of the Policy Function) Let Z be a convex subset of a vector space, and $X \subseteq \mathbb{R}$ a convex subset of the reals. Assume that the decision problem $\max_{x \in X} u(x,z)$ has a unique solution $g(z) = \arg\max_{x \in X} u(x,z) < \sup X$ for all $z \in Z$. Then if $u : X \times Z \to \mathbb{R}$ exhibits quasi-convex differences, $g : Z \to X$ is convex.

Proof. Let $-X \equiv \{-x \in \mathbb{R} : x \in X\}$. Apply Theorem 1 to the optimization problem $\max_{\tilde{x} \in -X} u(-\tilde{x}, z)$ and use that the policy function of this problem is concave if and only if g is convex. ∎

3.2 The General Case

In situations such as the income allocation problem of Section 2.2, it is too restrictive to assume that the constraint set X is fixed. Further, one may face decision problems with multiple solutions unless strict quasi-concavity in x or some similar condition holds.[18] We then face the general decision problem

$$(9) \qquad G(z) = \arg \max_{x \in \Gamma(z)} u(x,z).$$

Here $\Gamma : Z \to 2^X$ is the *constraint correspondence* and $G : Z \to 2^X$ is the *policy correspondence*. A policy function is now a selection from G, *i.e.*, a function $g : Z \to X$ with $g(z) \in G(z)$ for all $z \in Z$. The assumption of a one-dimensional decision variable $X \subseteq \mathbb{R}$ is maintained to keep things simple. Appendix III deals with the general case where X is a subset of a topological vector lattice.

For (9) the result of Topkis (1978) tells us that if u exhibits increasing differences and Γ is an ascending correspondence, then G is ascending. The precise definition of an ascending correspondence is not important for us here; it suffices to say that it naturally extends the notion that a function is increasing to a correspondence. As it turns out, the conclusion of Theorem 1 generalizes in a very similar manner. The only question is how to extend concavity/convexity from a function to a correspondence in a suitable way for our results.[19]

Definition 3 (Concave Correspondences) A correspondence $\Gamma : Z \to 2^X$ is *concave* if for all $z_1, z_2 \in Z$, $x_1 \in \Gamma(z_1)$, $x_2 \in \Gamma(z_2)$, and $\lambda \in [0,1]$, there exists $x \in \Gamma(\lambda z_1 + (1-\lambda)z_2)$ with $x \geq \lambda x_1 + (1-\lambda)x_2$.

For illustrations, see Figures 1-2 where the sets $IMU(+)$ depict graphs of concave correspondences. In parallel with concave/convex functions, $\Gamma : Z \to 2^X$ is said to be *convex* if $-\Gamma : Z \to 2^{-X}$ is concave where $-\Gamma(z) \equiv \{-x \in \mathbb{R} : x \in \Gamma(z)\}$. As mentioned, Definition 3 naturally generalizes concavity of a function to a correspondence. In particular, one immediately sees that if Γ is single-valued, then it is concave if and only if the function it defines is concave (and obviously, Γ is convex if and only if it is convex as a function). Recall that a correspondence $\Gamma : Z \to 2^X$ has a *convex graph* if $\{(x,z) \in X \times Z : x \in \Gamma(z)\}$ is a convex subset of $X \times Z$. Convexity of a correspondence's graph is a much stronger requirement than concavity and convexity of Γ. In fact, a correspondence with a convex graph is both concave and convex.[20] Furthermore, if Γ has a convex graph, it also has

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[18]Note that in the previous subsection, uniqueness of the optimizer was imposed directly to avoid assuming quasi-concavity in the decision variable.

[19]The following definition can be found in Kuroiwa (1996) who also offers an extensive discussion of set-valued convexity. Lemma 2 below appears to be new, however.

[20]To see this, simply pick $x = \lambda x_1 + (1-\lambda)x_2 \in \Gamma(\lambda z_1 + (1-\lambda)z_2)$ in Definition 3.

convex values, *i.e.*, $\Gamma(z)$ must be a convex subset of X for all $z \in Z$. In contrast, convex values is not implied by either concavity or convexity and so will have to be assumed directly when needed (as it will be below).

The following result sheds further light on the definition. It tells us that concavity and convexity of a correspondence is intimately tied to concavity and convexity of extremum selections (when they exist). For most applications, this result is also enough to establish that a given constraint correspondence is concave or convex since it covers inequality constraints where $\Gamma(z) = \{x \in X : \underline{\gamma}(z) \le x \le \overline{\gamma}(z)\}$.

Lemma 2 (Extremum Selection Criteria) If $\Gamma : Z \to 2^X$ admits a greatest selection, $\overline{\gamma}(z) \equiv \sup \Gamma(z) \in \Gamma(z)$ for all $z \in Z$, then Γ is concave if and only if $\overline{\gamma} : Z \to X$ is a concave function. Likewise, if Γ admits a least selection $\underline{\gamma}(z) \equiv \inf \Gamma(z) \in \Gamma(z)$ all $z \in Z$, Γ is convex if and only if $\underline{\gamma}$ is a convex function.

Proof. Only the concave case is proved. Since Γ is concave, we will for any $z_1, z_2 \in Z$, and $\lambda \in [0,1]$ have an $x \in \Gamma(\lambda z_1 + (1-\lambda)z_2)$ with $x \ge \lambda \underline{\gamma}(z_1) + (1-\lambda)\underline{\gamma}(z_2)$. Since $\overline{\gamma}(\lambda z_1 + (1-\lambda)z_2) \ge x$, $\overline{\gamma}$ is concave. To prove the converse, pick $z_1, z_2 \in Z$ and $x_1 \in \Gamma(z_1)$, $x_2 \in \Gamma(z_2)$. Since the greatest selection is concave, $x = \overline{\gamma}(\lambda z_1 + (1-\lambda)z_2) \ge \lambda \overline{\gamma}(z_1) + (1-\lambda)\overline{\gamma}(z_2) \ge \lambda x_1 + (1-\lambda)x_2$. Since $x \in \Gamma(\lambda z_1 + (1-\lambda)z_2)$, Γ is concave. ∎

From this Lemma, one can often immediately spot a concave correspondence. For example, we see that the union of the zero IMU curves in Figure 2 is the graph of a concave correspondence where the lower zero IMU is the least selection and the upper zero IMO the greatest selection. Lemma 2 also shows exactly how concavity and convexity relates to other known convexity concepts for correspondences. In a diagram with z on the first axis and x on the second axis, draw the graph of a concave function $\overline{\gamma}$. Now extend this graph to the graph of a correspondence Γ by drawing freely anything at or below the graph of $\overline{\gamma}$. Then the resulting correspondence is concave by Lemma 2. So in Figure 2 we could draw anything below the upper zero IMU and would still have a concave correspondence. As an aside, it is evident from these Figures that convex values as well as a convex graph are not implied — and it is equally evident that a convex graph implies that the correspondence is both concave and convex (these facts were discussed in a more technical manner a moment ago).

Theorem 2 (Concavity of the Policy Correspondence) Let Z be a convex subset of a vector space, and $X \subseteq \mathbb{R}$ a convex subset of the reals. Assume that the decision problem $\max_{x \in \Gamma(z)} u(x,z)$ has a solution $G(z) = \arg\max_{x \in \Gamma(z)} u(x,z) \ne \emptyset$ for all $z \in Z$ and that the infimum of $\Gamma(z)$ is never optimal, $x \in G(z) \Rightarrow x > \inf \Gamma(z)$. Then if $u : X \times Z \to \mathbb{R}$ exhibits quasi-concave differences and Γ is concave and has convex values, $G : Z \to 2^X$ is concave.

Proof. Pick any $z_1, z_2 \in Z$, $x_1 \in G(z_1)$, and $x_2 \in G(z_2)$. Setting $x_\lambda = \lambda x_1 + (1-\lambda)x_2$, and $z_\lambda = \lambda z_1 + (1-\lambda)z_2$, we must show that there exists $\hat{x} \in G(z_\lambda)$ with $\hat{x} \ge x_\lambda$. As in the proof of Theorem 1, we use quasi-concave differences to conclude that $u(x_\lambda, z_\lambda) - u(x_\lambda - \delta, z_\lambda) \ge 0$ for any sufficiently small $\delta > 0$. We are clearly done if there does not exist $x \in \Gamma(z_\lambda)$ with $x < x_\lambda$. So assume that such

an x exists. By concavity of Γ, there also exists $\tilde{x} \in \Gamma(z_\lambda)$ with $\tilde{x} \geq x_\lambda$. Since Γ has convex values therefore $[x, \tilde{x}] \subseteq \Gamma(z_\lambda)$. Since $x_\lambda \in [x, \tilde{x}]$ and $\inf \Gamma(z_\lambda)$ cannot be optimal, we may now proceed precisely as in the proof of Theorem 1 and use quasi-concavity of $u(x, z_\lambda) - u(x - \delta, z_\lambda)$ in x to show that there must exist a $\hat{x} \in \Gamma(z_\lambda)$ with $\hat{x} \geq x_\lambda$. ∎

Corollary 2 (Convexity of the Policy Correspondence) Let Z be a convex subset of a vector space, and $X \subseteq \mathbb{R}$ a convex subset of the reals. Assume that the decision problem $\max_{x \in \Gamma(z)} u(x, z)$ has a solution $G(z) = \arg\max_{x \in \Gamma(z)} u(x, z) \neq \emptyset$ for all $z \in Z$ and that the supremum of $\Gamma(z)$ is never optimal, $x \in G(z) \Rightarrow x < \sup \Gamma(z)$. Then if $u : X \times Z \to \mathbb{R}$ exhibits quasi-convex differences and Γ is convex and has convex values, $G : Z \to 2^X$ is convex.

Proof. Let $-\Gamma(z) \equiv \{-x \in \mathbb{R} : x \in \Gamma(z)\}$. Apply Theorem 2 to the optimization problem $\max_{\tilde{x} \in -\Gamma(z)} u(-\tilde{x}, z)$ and use that the policy correspondence of this problem is concave if and only if G is convex. ∎

If the conditions of Theorem 2 hold and the policy correspondence is single-valued $G = \{g\}$, then g must be a concave function by Lemma 2. Hence Theorem 1 is a special case of Theorem 2. From Lemma 2 also follows that when u is upper semi-continuous and Γ has compact values so that G has a greatest selection, this greatest selection must be concave.[21] Finally, note that just like in the theory of monotone comparative statics (Topkis (1978), Milgrom and Shannon (1994)), these observations are valid without assuming that the objective function is quasi-concave in the decision variable.

3.3 A User's Guide and Some Examples

This subsection provides a practitioners' guide to Theorems 1 and 2. Focus will be on simple applications. For more involved applications, the reader is referred to the next two sections. We begin with a straight-forward consequence of Lemma 1.

Lemma 3 (Quasi-Concave Differences for Thrice Differentiable Functions) *A thrice differentiable function* $u : X \times Z \to \mathbb{R}$ *where* $X, Z \subseteq \mathbb{R}$ *exhibits quasi-concave differences if and only if*

$$(10)\quad 2D_{xx}^2 u(x,z) D_{xz}^2 u(x,z) D_{xxz}^3 u(x,z) \geq [D_{xx}^2 u(x,z)]^2 D_{xzz}^3 u(x,z) + [D_{xz}^2 u(x,z)]^2 D_{xxx}^3 u(x,z).$$

Proof. (10) is the non-negative bordered Hessian criterion for quasi-concavity of $D_x u(x,z)$ (see *e.g.* Mas-Colell et al (1995), pp.938-939). By Lemma 1, this is equivalent to $u(x,z)$ exhibiting quasi-concave differences. ∎

For applications, one usually only requires either this Lemma, or one of two facts about concavity: that concavity is preserved by integration (used in the MPC example immediately following

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[21] Note that it is unreasonable to expect the *least* selection to be concave also. In fact, this would not characterize any reasonable concavity-type condition for a correspondence (in the case of a correspondence with a convex graph, for example, the greatest selection is concave and the least selection is convex).

Lemma 1) or that the minimum of a family of concave functions is concave (used in the second example below).

Heterogenous Firms in International Trade Models. Consider the model of Melitz (2003). Each firm in a continuum $[0,1]$ chooses output $x \geq 0$ in order to maximize profits. A firm with cost parameter $z > 0$, can produce x units of the output by employing $l = zx + f$ workers where $f > 0$ is a fixed overhead (Melitz (2003), p.1699).[22] The frequency distribution of the cost parameter z across the firms is η_z. With revenue function R, a firm with cost parameter z chooses $x \geq 0$ in order to maximize

(11) $$u(x,z) = R(x) - zx - f.$$

Let $G(z) = \arg\max_{x \geq 0}[R(x) - zx - f]$ denote the optimal output(s) given z. To show that G is concave or convex, we may apply Theorem 1 or the more general Theorem 2. For Theorem 1, $R(x) - zx - f$ must be strictly quasi-concave or satisfy some other condition that guarantees that firms have a unique optimal output level, $G(z) = \{g(z)\}$. As one easily verifies, (10) holds if and only if

(12) $$R''' \leq 0.$$

So by Theorem 1, if the revenue function has a non-positive third derivative and optimizers are unique, the policy function g is concave on $z \in \{z \in Z : g(z) > 0\}$, *i.e.*, when attention is restricted to the set of active firms. It follows from Observation 2 on page 2.1 that aggregate output $\int_{[0,1]} g(z)\, \eta_z(dz)$ decreases when firms become more diverse (a mean preserving spread to the distribution η_z). Such "decreasing returns to diversity" is easily understood in light of (12) which says that the marginal revenue function is concave. A concave marginal revenue function tells us that if we consider two firms that produce x_1 and x_2, respectively, and the firms have different productivities $z_1 \neq z_2$, then the average of their marginal revenues $(R'(x_1) + R'(x_2))/2$ will be *lower* than the marginal revenue of a firm which produces the average output $(x_1 + x_2)/2$ and has the average productivity $(z_1 + z_2)/2$. In particular, when x_1 and x_2 are optimal for the respective firms and the marginal revenues therefore equal zero, the marginal revenue of the average productivity firm will be positive, and it is therefore optimal for it to produce more than the average $(x_1 + x_2)/2$. Putting these observations together, we see that if production is spread across diverse firms, then total output is lower than if production takes place among the same number of more similar firms. For an extensive discussion of concave marginal revenue functions in the theory of production, the reader is referred to Leahy and Whited (1996).

By appealing to Observations 1-5 on page 4 one can further go on to predict how the distribution of the firms' outputs changes when η_z is subjected to mean-preserving spreads or other stochastic order changes. For example, the distribution of the outputs of a more diverse set of firms will be second-order stochastically dominated by the distribution of outputs of a less diverse set of firms when (12) holds (this is again by Observation 2). If u exhibits quasi-convex differences (reverse the inequality (10) yielding the condition $R''' \geq 0$), we instead get a convex policy func-

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[22] In terms of Melitz' notation, z is the inverse of the firm's productivity level φ.

tion by Corollary 1; there is "increasing returns to diversity", and all of the previous conclusions are reversed.

The limitations of Theorem 1 are evident in the current situation since monopolistically competitive firms' objectives will often not be strictly quasi-concave or even quasi-concave and so uniqueness of the optimizer is difficult to ensure in general. One can then instead use Theorem 2. Assuming only that solutions exist (so that G is well-defined), we can conclude that G is a concave correspondence when the revenue function R has a non-positive third derivative. Hence the greatest selection from the policy correspondence will be a concave function (Lemma 2), and the maximum aggregate output decreases with a mean-preserving spread to η_z.

If R is assumed to be strictly concave so that $R'' < 0$, we might alternatively have applied the implicit function theorem (IFT) to the first-order condition $R'(x) - z = 0$. This yields $x = g(z)$ where $D_z g = R''$. The IFT is particularly easy to use in this case, and we immediately see that (12) once again ensures concavity of g (this is because $D_z g$ decreasing precisely means that g is concave). Note, however, that Theorem 2 (and also Theorem 1) applies to many situations which the IFT is unable to address.

Ambiguity Aversion. Consider an agent who makes an investment $x \geq 0$ in a project whose expected return depends on a known signal z as well as a draw by nature among the possible states $\bar{z}_1, \ldots, \bar{z}_l$. The expected return is evaluated according to a Choquet/non-additive expected utility criterion with state payoff $U(\bar{z}_k, z)$ and capacities ν. The cost $c(x, z)$ is assumed to be strictly convex and differentiable in x. The policy function is thus,

$$g(z) = \arg\max_x \left[x \min_{\bar{\mu} \in C(\nu)} \sum_{k=1}^{l} \bar{\mu}_k U(\bar{z}_k, z) - c(x, z) \right],$$

where $C(\nu)$ denotes the core of ν.

If U is concave in z and $D_x c(x, z)$ is convex, then $\min_{\bar{\mu} \in C(\nu)} \sum_{k=1}^{l} \bar{\mu}_k U(\bar{z}_k, z) - D_x c(x, z)$ is concave in (x, z) (the first term is concave in z because the minimum of a family of concave functions is concave). By Lemma 1, it follows that the objective exhibits quasi-concave differences and by Theorem 1, $g : Z \to X$ is consequently concave. Since the first-order condition of this problem is not differentiable in z, the implicit function theorem cannot be used to reach this conclusion. Note that the previous conclusions also apply to other settings with ambiguity averse agents. An example here is when the monetary policy committee of Section 2.1 is ambiguity averse with respect to the realization of the oil price $\bar{z}$. The expected utility objective (3) is then replaced with a non-additive utility objective and the previous argument applied.

4   Bayesian Games

The purpose of this section is to use Theorem 1 to deal with increased uncertainty in Bayesian games. In Section 2.1, the central bank (the MPC) was the only agent who made a decision, and that decision was not influenced by the other agent's action (the forecast). The increase in uncertainty on the other hand, affected only the forecaster (by assumption the MPC knew its private signal). In reality, central banks take other agents' responses into account when setting interest

rates — and those agents will be aware of this and take *that* into account. So, any increase in uncertainty spills over between the agents. Such considerations naturally lead us to study DCS in Bayesian games.

A Bayesian game consists of a set of players $\mathscr{I} = \{1, \dots, I\}$, taken here to be finite, where player $i \in \mathscr{I}$ receives a private signal $z_i \in Z_i \subseteq \mathbb{R}$ drawn from a distribution μ_{z_i} on $(Z_i, \mathscr{B}(Z_i))$.[23] Agents maximize their objectives and a Bayesian equilibrium is just a Nash equilibrium of the resulting game (defined precisely in a moment). The question we ask is this: How will the set of equilibria be affected if one or more signal distributions μ_{z_i} are subjected to mean-preserving spreads or second-order stochastic dominance shifts? So if $i = 1$ is the MPC and z_1 represents the MPC's assessment of the Philips/Lucas supply curve and the interest rate pass through, we are back in the setting of Section 2.1 except that we now allow for the game-theoretic interaction between the MPC and the other agents in the economy.

Assuming that private signals are independently distributed, an *optimal strategy* is a measurable mapping $g_i : Z_i \to X_i$ such that for almost every $z_i \in Z_i$,

$$
(13) \qquad g_i(z_i) \in \arg\max_{x_i \in X_i} \int_{z_{-i} \in Z_{-i}} U_i(x_i, g_{-i}(z_{-i}), z_i) \mu_{z_{-i}}(dz_{-i}).
$$

Here $X_i \subseteq \mathbb{R}$ is agent i's action set and $g_{-i} = (g_j)_{j \neq i}$ are the strategies of the opponents. A *Bayesian equilibrium* is a strategy profile $g^* = (g_1^*, \dots, g_I^*)$ such that for each player i, $g_i^* : Z_i \to X_i$ is an optimal strategy given the opponents' strategies $g_{-i}^* : Z_{-i} \to X_{-i}$. Obviously, $g_i : Z_i \to X_i$ is a policy function when it satisfies (13) for all $z_i \in Z_i$. The *optimal distribution* of an agent i is the measure on $(X_i, \mathscr{B}(X_i))$ given by:

$$
(14) \qquad \mu_{x_i}(A) = \mu_{z_i}\{z_i \in Z_i : g_i(z_i) \in A\}, \ A \in \mathscr{B}(X_i)
$$

Note that for the previous description to be consistent we need uniqueness of optimal strategies.

Assumption 1 *For every i, the optimal strategy g_i exists and is unique.*

This assumption is satisfied if X_i is compact and $U_i(x_i, x_{-i}, z_i)$ is strictly concave in x_i (risk aversion), and continuous in (x_i, x_{-i}, z_i). Note that if uniqueness is not assumed, we could instead use Theorem 2 but we favor here simplicity over generality.

For given opponents' strategies the decision problem in (13) coincides with the MPC's objective in Section 2.1. In particular, we know how changes in μ_{z_i} affect the optimal distribution of the player μ_{x_i} when g_i is concave or convex (see 1-5 on page 4). Combining with Theorem 1 we immediately get:

Lemma 4 Consider a player $i \in \mathscr{I}$ and let Assumption 1 be satisfied.

1. If $\int_{z_{-i} \in Z_{-i}} U_i(x_i, g_{-i}(z_{-i}), z_i) \mu_{z_{-i}}(dz_{-i})$ exhibits quasi-concave differences in x_i and z_i, and no element on the lower boundary of X_i ($\inf X_i$) is optimal, then a mean-preserving spread

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[23] $\mathscr{B}(\cdot)$ denotes the Borel subsets of a given set.

to μ_{z_i} will lead to a second-order stochastic dominance decrease in the optimal distribution μ_{x_i}.

2. If $\int_{z_{-i} \in Z_{-i}} U_i(x_i, g_{-i}(z_{-i}), z_i)\mu_{z_{-i}}(dz_{-i})$ exhibits quasi-convex differences in x_i and z_i, and no element on the upper boundary of X_i (sup X_i) is optimal, then a mean-preserving spread to μ_{z_i} will lead to a convex-increasing order increase in the optimal distribution μ_{x_i}.

As we saw in Section 3.1, the conditions of Lemma 4 are easy to verify if U_i is differentiable in x_i (see also the main Theorem below which provides sufficient conditions in the differentiable case). The Lemma tells us that less precise private signals (increased uncertainty) leads to higher variance of any affected player's optimal distribution. Whether the mean actions increase or decrease depends, however, on whether the payoff function exhibits quasi-convex or quasi-concave differences. The story clearly does not end there though. The increase in uncertainty will spill over to the other players and make *everybody's* game environments more uncertain. To deal with this, we need the following straightforward generalization of a result found in Rothschild and Stiglitz (1971).[24]

Lemma 5 Let Assumption 1 be satisfied and let $g_i(z_i, \mu_{x_{-i}}) = \arg\max_{x_i \in X_i} \int U_i(x_i, x_{-i}, z_i)\mu_{x_{-i}}(dx_{-i})$. Then for $j \neq i$:

1. *If $U_i(\tilde{x}_i, x_{-i}, z_i) - U_i(x_i, x_{-i}, z_i)$ is concave in x_j for all $\tilde{x}_i \geq x_i$, then $g_i(z_i, \mu_{x_{-i,-j}}, \tilde{\mu}_{x_j}) \leq g_i(z_i, \mu_{x_{-i}})$ whenever $\tilde{\mu}_{x_j}$ is a mean-preserving spread of μ_{x_j}.*

2. *If $U_i(\tilde{x}_i, x_{-i}, z_i) - U_i(x_i, x_{-i}, z_i)$ is concave and increasing in x_j for all $\tilde{x}_i \geq x_i$, then $g_i(z_i, \mu_{x_{-i}}) \leq g_i(z_i, \mu_{x_{-i,-j}}, \tilde{\mu}_{x_j})$ whenever $\tilde{\mu}_{x_j}$ second-order stochastically dominates μ_{x_j}.*

If in these statements concavity in x_j is replaced with convexity, the first conclusion changes to: $g_i(z_i, \mu_{x_{-i,-j}}, \tilde{\mu}_{x_j}) \geq g_i(z_i, \mu_{x_{-i}})$ whenever $\tilde{\mu}_j$ is a mean-preserving spread of μ_{x_j}; and the second conclusion changes to $g_i(z_i, \mu_{x_{-i}}) \leq g_i(z_i, \mu_{x_{-i,-j}}, \tilde{\mu}_{x_j})$ whenever $\tilde{\mu}_{x_j}$ dominates μ_{x_j} in the convex-increasing order.

Proof. Statement 1 is a direct application of Topkis' theorem (Topkis (1978)) which in the situation with a one-dimensional decision variable and unique optimizers says that the optimal decision will be non-decreasing [non-increasing] in parameters if the objective exhibits increasing differences [decreasing differences]. The conclusion thus follows from the fact that $\int U_i(x_i, x_{-i}, z_i)\mu_{x_{-i}}(dx_{-i})$ exhibits decreasing differences in x_i (with the usual order) and μ_{x_j} (with the mean-preserving spread order $\succeq_{cx}$) if and only if the assumption of the statement holds. Also by Topkis' theorem, if $U_i(\tilde{x}_i, x_{-i}, z_i) - U_i(x_i, x_{-i}, z_i)$ is increasing in x_j for $j \neq i$ and for all $\tilde{x}_i \geq x_i$, then $\tilde{\mu}_{x_j} \succeq_{st} \mu_{x_j} \Rightarrow g_i(z_i, \mu_{x_{-i,-j}}, \tilde{\mu}_{x_j}) \geq g_i(z_i, \mu_{x_{-i}})$ (here $\succeq_{st}$ denotes the first-order stochastic dominance order). From this and Statement 1 follows Statement 2 because it is always possible to split

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[24]Rothschild and Stiglitz (1971) consider mean-preserving spreads in the differentiable case. If U_i is differentiable in x_i, the main condition of Lemma 5 is equivalent to the concavity of $D_{x_i}U_i(x_i, x_{-i}, \cdot)$ which exactly is the assumption of Rothschild and Stiglitz (1971). See also Athey (2002) for related results. It is by combining such (known) results with this paper's new results that we are able to make progress on DCS in Bayesian games.

a second order stochastic dominance increase $\succeq_{cvi}$ into a mean preserving contraction $\succeq_{cv}$ followed by a first order stochastic dominance increase (Formally, if $\tilde{\mu}_{x_j} \succeq_{cvi} \mu_{x_j}$, then there exists a distribution $\hat{\mu}_{x_j}$ such that $\tilde{\mu}_{x_j} \succeq_{st} \hat{\mu}_{x_j} \succeq_{cv} \mu_{x_j}$). The convex case is proved by a similar argument and is omitted. ∎

For given distributions of private signals $\mu_z = (\mu_{z_1}, \ldots, \mu_{z_I})$ let $\Phi(\mu_z)$ denote the *set of equilibrium distributions, i.e.,* the set of optimal distributions (14) where $(g_1^*, \ldots, g_I^*)$ is one of the (possibly many) Bayesian equilibria. Fix a given stochastic order $\succeq$ on the probability space of optimal distributions and consider a shift in the distribution of private signals from μ_z to $\tilde{\mu}_z$. In the Theorem below, $\succeq$ will be either the second-order stochastic dominance order or the convex-increasing order; and the shift from μ_z til $\tilde{\mu}_z$ will be a mean-preserving spread. The set of equilibrium distributions then *increases* in the order $\succeq$ if

(15) $\forall \mu_x \in \Phi(\mu_z) \, \exists \tilde{\mu}_x \in \Phi(\tilde{\mu}_z)$ with $\tilde{\mu}_x \succeq \mu_x$ and $\forall \tilde{\mu}_x \in \Phi(\tilde{\mu}_z) \, \exists \mu_x \in \Phi(\mu_z)$ with $\tilde{\mu}_x \succeq \mu_x$

If the order $\succeq$ is reversed in (15), the set of equilibrium distributions *decreases*. If $\Phi(\mu_z)$ and $\Phi(\tilde{\mu}_z)$ have least and greatest elements, then (15) implies that the least element of $\Phi(\mu_z)$ will be smaller than the least element of $\Phi(\tilde{\mu}_z)$ and the greatest element of $\Phi(\mu_z)$ will be smaller than the least element of $\Phi(\tilde{\mu}_z)$ (Smithson (1971), Theorem 1.7). In particular, if the equilibria are unique and we therefore have a function ϕ such that $\Phi(\mu_z) = \{\phi(\mu_x)\}$ and $\Phi(\tilde{\mu}_z) = \{\phi(\tilde{\mu}_x)\}$, we get $\phi(\mu_z) \preceq \phi(\tilde{\mu}_z)$ which simply means that the function ϕ is increasing.

Theorem 3 (Mean Preserving Spreads in Bayesian Games) Consider a Bayesian game as described above and let $\mu_z = (\mu_{z_i})_{i \in \mathscr{I}}$ and $\tilde{\mu}_z = (\tilde{\mu}_{z_i})_{i \in \mathscr{I}}$ be two distributions of private signals.[25]

1. Suppose all assumptions of Lemma 4.1 are satisfied and $U_i(\tilde{x}_i, x_{-i}, z_i) - U_i(x_i, x_{-i}, z_i)$ is increasing and concave in x_{-i} for all $\tilde{x}_i \geq x_i$. If $\tilde{\mu}_{z_i}$ is a mean-preserving spread of μ_{z_i} for any subset of the players, then the set of equilibrium distributions decreases in the second-order stochastic dominance order (in particular the agents' mean actions decrease, and the actions' variance increase).

2. Suppose all assumptions of Lemma 4.2 are satisfied and $U_i(\tilde{x}_i, x_{-i}, z_i) - U_i(x_i, x_{-i}, z_i)$ is increasing and convex in x_{-i} for all $\tilde{x}_i \geq x_i$. If $\tilde{\mu}_{z_i}$ is a mean-preserving spread of μ_{z_i} for any subset of the players, then the set of equilibrium distributions increases in the convex-increasing order (in particular the agents' mean actions increase, and again the actions' variance increase).

If U_i is differentiable in x_i, all of these assumptions are satisfied if $D_{x_i} U_i(x_i, x_{-i}, z_i)$ is increasing in x_{-i} and either concave in (x_i, z_i) and x_{-i} [case 1] or convex in (x_i, z_i) and x_{-i} [case 2].

Proof. As in previous proofs, let $\succeq_{cvi}$ denote the concave-increasing (second-order stochastic dominance) order and $\succeq_{st}$ denote the first-order stochastic dominance order. Recast the game in

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[25]In the following statements, it is to be understood that any distribution μ_{z_i} that is *not* replaced with a mean-preserving spread $\tilde{\mu}_{z_i}$ is kept fixed.

terms of optimal distributions: Agent i's problem is to find a measurable function g_i which for *a.e.* $z_i \in Z_i$ maximizes $\int_{x_{-i} \in X_{-i}} U_i(x_i, x_{-i}, z_i) \mu_{x_{-i}}(d x_{-i})$. The policy function $x_i = g_i(z_i, \mu_{x_{-i}})$ determines $\mu_{x_i}(A) = \mu_{z_i}\{z_i \in Z_i : g_i(z_i, \mu_{x_{-i}}) \in A\}$ ($A \in \mathscr{B}(X_i)$). An equilibrium is a vector $\mu_x^* = (\mu_{x_1}^*, \ldots, \mu_{x_I}^*)$ such that for all $i \in \mathscr{I}$: $\mu_{x_i}^*(A) = \mu_{z_i}\{z_i \in Z_i : g_i(z_i, \mu_{x_{-i}}^*) \in A\}$, all $A \in \mathscr{B}(X_i)$. Letting $f_i(\mu_{x_{-i}}, \mu_{z_i})$ denote agent i's optimal distribution given $\mu_{x_{-i}}$ and μ_{z_i}, an equilibrium is a fixed point of $f = (f_1, \ldots, f_I)$. By 2 of Lemma 5 and Observation 1 on page 4, $\tilde{\mu}_{x_j} \succeq_{cvi} \mu_{x_j} \Rightarrow f_i(\mu_{x_{-i,-j}}, \tilde{\mu}_{x_j}, \mu_{z_i}) \succeq_{st} f_i(\mu_{x_{-i}}, \mu_{z_i})$ for all $j \neq i$. Since first-order stochastic dominance implies second-order stochastic dominance, $f_i(\mu_{x_{-i,-j}}, \tilde{\mu}_{x_j}, \mu_{z_i}) \succeq_{st} f_i(\mu_{x_{-i}}, \mu_{z_i}) \Rightarrow f_i(\mu_{x_{-i,-j}}, \tilde{\mu}_{x_j}, \mu_{z_i}) \succeq_{cvi} f_i(\mu_{x_{-i}}, \mu_{z_i})$. It follows that the mapping f is monotone when μ_x's underlying probability space is equipped with the product order $\succeq_{cvi}^I$. Again with the order $\succeq_{cvi}$ on optimal distributions, it follows from Lemma 4 that each f_i is decreasing in μ_{z_i} with the convex (mean-preserving spread) order on μ_{z_i}'s underlying probability space. f will also be continuous (it is a composition of continuous functions) and so the theorem's conclusions follow directly from Theorem 3 in Acemoglu and Jensen (2015) (the conditions of that Theorem are immediately satisfied when f is viewed as a correspondence). For the second statement of the theorem the argument is precisely the same except that one now equips the set of optimal distributions with the convex-increasing order and notes that f_i is monotone when the private distributions μ_{z_i}'s underlying probability spaces are equipped with the mean preserving spread order. The differentiability conditions presented at the end of the theorem follow from Lemma 1. ∎

Note that under the assumptions of Theorem 3, the game is supermodular. Under the additional conditions of the following corollary, the game is monotone (*i.e.*, a Bayesian game of strategic complementarities, see Van Zandt and Vives (2007)). The proof follows along the same lines as the proof of Theorem 3 and is omitted.

Corollary 3 (Second-Order Stochastic Dominance Changes) If in addition to the assumptions of Theorem 3, it is assumed that $U_i(\tilde{x}_i, x_{-i}, z_i) - U_i(x_i, x_{-i}, z_i)$ is increasing in z_i, then if $\tilde{\mu}_{z_i}$ second-order stochastically dominates μ_{z_i} for any subset of the players, the set of equilibria decreases in the second-order stochastic dominance order in case 1. In case 2., the set of equilibria increases in the convex-increasing order when $\tilde{\mu}_{z_i}$ dominates μ_{z_i} in the convex-increasing order for any subset of the players.

There are many interesting applications of Theorem 3, ranging from auction theory to the Diamond search model. Here we will study a Bayesian version of the classical arms race game from the field of conflict resolution (see *e.g.* Milgrom and Roberts (1990), p.1272), and ask whether increased uncertainty about arms' effectiveness and opponents' intentions leads to an intensification of the arms race or not. But before getting to that, let us round off our leading example of a monetary policy committee.

In Section 2.1, $\bar{z}$ in the expected utility formulation (3) was interpreted as an additional exogenous variable (the oil price for example). That formulation was chosen deliberately because it also

covers Bayesian games where $\bar{z}$ is now instead the private signal of a second agent ("the public"):

$$(16) \qquad u_1(x_1, z_1) = \int U_1(x_1, g_2(z_2), z_1) \mu_{z_2}(d z_2).$$

z_1 is the MPC's private signal, and $-U_1$ its loss function. As discussed in Section 2.1, the loss function incorporates the MPC's assessment of the Lucas supply curve which depends on the MPC's private signal z_1 but now also depends on what the public does, $g_2(z_2)$. What the public does is, in turn, uncertain to the MPC since it depends on the public's private signal z_2. We already discussed both the conditions for and interpretation of u_1 exhibiting quasi-concave differences in Sections 2.1 and 3.1. What is new is the condition in Lemma 5. Taking here condition 2. in Lemma 5, the intuition is straight-forward: if we think of x_2 as a measure of economic activity, condition 2 says that the MPC's marginal payoff (so minus the loss) is increasing and concave in x_2, higher activity lowers the marginal loss associated with raising the interest rate (the loss function is decreasing in x_2), but decreasingly so as the level of activity increases (the loss function is convex in x_2). This makes a lot of sense economically. All else equal, higher activity makes interest increases less painful in the eyes of an MPC, but this effect is bound to diminish as the economy progressively overheats.

Leaving a detailed "story" for future work, we may now add the public (agent $i = 2$), or we could of course add more than a single agent and extend (16) accordingly. We then ensure that the public's objective function exhibits quasi-concave differences. For example, the public might be thought of as generating economic activity, and a concave policy function would mean that investments, etc., are decreasing at a steepening rate in the rate of interest. Adding the conditions of Lemma 5, we may then apply Theorem 3. This result tells us that if uncertainty increases (either for the central bank or the public) then in the Bayesian equilibrium both the interest rate and the activity variable decrease in the sense of second-order stochastic dominance. So if uncertainty increases for exogenous reasons, mean activity as well as the mean interest rate decrease, and the economy will fluctuate more (the variances increase).

Finally, consider the arms-race advertised a moment ago. Consider two countries, $i = 1, 2$, with identical state payoff functions $u_i(x_i, x_{-i}, z_i) = B(x_i - x_{-i} - z_i) - c\, x_i$. B is a strictly concave function and $c > 0$ a constant cost parameter. z_i is a random variable that reflects the relative effectiveness of the arms — real or imagined (for example a domestic media frenzy might correspond to a mean-preserving spread to z_i).[26] Assuming that B is sufficiently smooth, we can use the conditions at the end of Theorem 3. By strict concavity, $D_{x_i} u(x_i, x_{-i}, z_i) = B'(x_i - x_{-i} + z_i) - c$ is increasing in x_{-i}, and the question is therefore whether it is also either convex or concave in (x_i, z_i) and in x_{-i}. Obviously, this depends entirely on whether B' is convex or concave, *i.e.*, on whether the third derivative of B is positive or negative. In the convex case (positive third derivative), the countries' policy functions are convex. Hence greater uncertainty will increase the affected country's (or countries') expected stock of arms as well as the variance (Lemma 4, which specifically says that given the other country's strategy, greater uncertainty will lead to a convex-increasing shift in the optimal distribution). The increased variance of the affected country's stock of arms will lead to a

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[26]A myriad of other specifications would of course be possible, for example costs could instead be random. This section's results may be applied for any such specification.

more uncertain environment for the other country and make *it* accumulate more arms (Lemma 5). This escalation continues until an equilibrium is reached with higher mean stocks of arms and greater uncertainty about the exact size of the arsenals (Theorem 3).[27] Note that a positive third derivative means that the countries are "prudent" (Kimball (1990)) — a well-understood behavioral trait that also plays a key role in other settings such as in income allocation problems (Carroll and Kimball (1996)). Of course, prudence, which in the words of Kimball (1990) (p.54) is "the propensity to prepare and forearm oneself in the face of uncertainty", has rather more beneficial consequences in income allocation models than it does in arms races. It is therefore not uniformly good news that experimental evidence seems to suggest that most people are prudent (Nussair et al (2011)). But of course, prudence may be situation-dependent or imprudent politicians may be elected. In this case B will have a negative third derivative, and the countries' policy functions will be concave so that greater uncertainty lowers the mean stock of arms in equilibrium. Note however, that according to Theorem 3, the variance will still increase, so whether decision makers are prudent or not, the risk of exceptionally high stocks of arms and the negative consequences in case of war still increases when the environment becomes more uncertain.

5 Stochastic Dynamic Programming

This section uses Theorem 2 to study the topic introduced in Section 2.2, namely the concavity/convexity of policy functions in stochastic dynamic programming problems. In income allocation problems, the relationship between earning risk and wealth accumulation is guided by whether the consumption function is concave or convex (Huggett (2004)). And in macroeconomics, it determines the effect of increased individual uncertainty on aggregate market outcomes in large dynamic economies (Acemoglu and Jensen (2015)) an example of which is the Aiyagari-model mentioned in Section 2.2 and returned to below.

The treatment and notation follows Chapter 9 of Stokey and Lucas (1989). The dynamic programming problem is,

$$
(17) \quad \begin{aligned} \max \quad & \mathbb{E}_0[\textstyle\sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1}, z_t)] \\ \text{s.t.} \quad & \begin{cases} x_{t+1} \in \Gamma(x_t, z_t), \ t = 0, 1, 2, \dots \\ \text{where } (x_0, z_0) \gg 0 \text{ are given.} \end{cases} \end{aligned}
$$

In comparison with Stokey and Lucas (1989), two structural restrictions are made to simplify the exposition (but the results easily generalize, see Remarks 1-2 at the end of this section): first, the z_t's are assumed to be *i.i.d.* with distribution μ_z; second, only the one-dimensional case is considered, *i.e.*, it is assumed that $x_t \in X \subseteq \mathbb{R}$ and $z_t \in Z \subseteq \mathbb{R}$. We also maintain the standard assumption of a concave objective even though our general results in fact do not even require quasi-concavity (Remark 3 discusses this further).

Both X and Z are assumed to be convex sets equipped with their Borel σ-algebras.[28] The *value*

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[27]Note that since the conditions of Corollary 3 are satisfied, these conclusions are valid not just for mean-preserving spreads but for second-order stochastic dominance decreases more generally.

[28]For our result on the policy function $g(x, z)$'s convexity in x, it may alternatively be assumed that Z is a countable set equipped with the σ-algebra consisting of all subsets of Z (see Stokey and Lucas (1989), Assumption 9.5.a.).

function $v : X \times Z \to \mathbb{R}$ *of* (17) *is*,

$$(18) \qquad v(x,z) = \sup_{y \in \Gamma(x,z)} \left[u(x,y,z) + \beta \int v(y,z') \mu_z(dz') \right].$$

The following assumptions are from Stokey and Lucas (1989), Chapter 9.

Assumption 2 $\Gamma : X \times Z \to 2^X$ *is non-empty, compact-valued, continuous, and has a convex graph, i.e., for all* $x, \tilde{x} \in X$, $z \in Z$, *and all* $\lambda \in [0,1]$: $\lambda y + (1-\lambda)\tilde{y} \in \Gamma(\lambda x + (1-\lambda)\tilde{x}, z)$ *whenever* $y \in \Gamma(x,z)$ *and* $\tilde{y} \in \Gamma(\tilde{x}, z)$.

Assumption 3 $u : X \times X \times Z \to \mathbb{R}$ *is bounded and continuous, and* $\beta \in (0,1)$. *Furthermore,* $u(x,y,z)$ *is concave in* (x,y) *and strictly concave in* y.

Note that Assumption 2 in particular requires Γ to have a *convex graph*. As discussed in the first paragraph after Definition 3, this implies that Γ is a concave as well as a convex correspondence (*cf.* the conditions of Theorem 2). Under Assumptions 2-3, the value function $v = v(x,z)$ is uniquely determined, continuous, and concave in x. Furthermore, the *policy function* $g : X \times Z \to X$ is the well-defined and continuous function

$$(19) \qquad g(x,z) = \arg \sup_{y \in \Gamma(x,z)} \left[u(x,y,z) + \beta \int v(y,z') \mu_z(dz') \right].$$

Theorem 4 (Convex Policy Functions in Dynamic Stochastic Programming Problems) Consider the stochastic dynamic programming problem (17) under Assumptions 2-3 and let $g : X \times Z \to X$ denote the policy function (19). Assume that $u(x,y,z)$ is differentiable and satisfies the following upper boundary condition: $\lim_{y^n \uparrow \sup \Gamma(x,z)} D_y u(x, y^n, z) = -\infty$ (or in some other way ensure that $\sup \Gamma(x,z)$ will never be optimal given (x,z)). Then the policy function g is convex in x if $D_x u(x,y,z)$ is non-decreasing in y and there exists a $k \geq 0$ such that $\frac{1}{1-k}[-D_y u(x,y,z)]^{1-k}$ is concave in (x,y) and $\frac{1}{1-k}[D_x u(x,y,z)]^{1-k}$ is convex in (x,y).[29] If in addition $\Gamma(x, \cdot)$ is a convex correspondence and $\frac{1}{1-k}[-D_y u(x,y,z)]^{1-k}$ is concave in (y,z), then the policy function g will also be convex in z.

Proof. See Section 5.1. ∎

Theorem 4 has a host of applications in macroeconomics. For example, it is applied in Acemoglu and Jensen (2015) to study how increased uncertainty affects the equilibria in large dynamic economies. As a concrete example, consider the decision problem faced by an individual consumer in the model of Aiyagari (1994). Let $r > 0$ and $w > 0$ denote, respectively, the factor of interest and wage rate. Let $\Gamma(x,z) = \{y \in [-\underline{b}, \overline{b}] : y \leq rx + wz\}$, and let $\tilde{u}$ be a strictly concave and strictly increasing period utility function. The income allocation problem can then be written as a dynamic programming problem:

$$(20) \qquad \begin{aligned} \max \quad & \mathbb{E}_0[\sum_{t=0}^{\infty} \beta^t \tilde{u}(rx_t + wz_t - x_{t+1})] \\ \text{s.t.} \quad & \begin{cases} x_{t+1} \in \Gamma(x_t, z_t), \ t = 0, 1, 2, \ldots \\ \text{where } (x_0, z_0) \gg 0 \text{ are given.} \end{cases} \end{aligned}$$

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[29]In the limit case $k = 1$, $\frac{1}{1-k}[f(x)]^{1-k}$ is by convention equal to $\log(f(x))$.

Note that this formulation explicitly incorporates borrowing limits (Aiyagari (1994)). Note also that x_t is savings at date t, so the policy function $g(x,z)$ is the *savings function* and the *consumption function* is[30]

(21) $$c(x,z) = rx + wz - g(x,z).$$

c is concave in x ("concavity of the consumption function") if and only if g is convex in x ("convexity of the savings function"). In terms of this section's general notation we have $u(x,y,z) = \tilde{u}(rx + wz - y)$. It is easy to verify (and well known) that Assumptions 2-3 are satisfied. Under a standard boundary condition on $\tilde{u}$, we will never have $g(x,z) = \sup \Gamma(x,z)$, *i.e.*, the consumer will not choose zero consumption at any date.

We have $D_y u(x,y,z) = -\tilde{u}'(rx+wz-y)$ and $D_x u(x,y,z) = r\tilde{u}'(rx+wz-y)$. $D_x u$ is strictly increasing in y since $\tilde{u}$ is strictly concave. By Theorem 4, the consumption function is consequently concave if $\frac{1}{1-k}[\tilde{u}'(rx+wz-y)]^{1-k}$ is concave and $\frac{1}{1-k}[r\tilde{u}'(rx+wz-y)]^{1-k}$ is convex in (x,y). When the Hessian determinants of $D_x u$ and $D_y u$ exist — *i.e.*, when $\tilde{u}$ is thrice differentiable as in Carroll and Kimball (1996) — they both equal zero, and it is then straightforward to verify that the previous conditions on $\tilde{u}$ will hold if and only if

(22) $$\frac{\tilde{u}'\tilde{u}'''}{(\tilde{u}'')^2} = k \geq 0.$$

Thus we find the condition that $\tilde{u}$ must be of the HARA-form introduced in Section 2.2. That $\tilde{u}$ is of the HARA-form is also assumed by Carroll and Kimball (1996) who establish concavity of the consumption function when there are no borrowing limits. One easily verifies that the condition for convexity in z (last sentence of Theorem 4) is also satisfied when $\tilde{u}$ is of the HARA-form. Thus we have generalized the result of Carroll and Kimball (1996) to the incomplete markets setting with borrowing constraints.

5.1 Proof of Theorem 4

The value and policy functions will equal the pointwise limits of the sequences $(v^n)_{n=0}^\infty$ and $(g^n)_{n=0}^\infty$ determined by:

(23) $$v^{n+1}(x,z) = \sup_{y \in \Gamma(x,z)} \left[u(x,y,z) + \beta \int v^n(y,z')\mu_z(dz') \right], \text{ and}$$

(24) $$g^n(x,z) = \arg \sup_{y \in \Gamma(x,z)} \left[u(x,y,z) + \beta \int v^n(y,z')\mu_z(dz') \right].$$

Under the Theorem's conditions, v^n is concave in x for all n. Since a concave function is absolutely continuous, v^n will be absolutely continuous in x for all n. The following result presents conditions under which an absolutely continuous function is quasi-concave. It generalizes the sufficiency part of Lemma 1.

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[30]Note that if we let $W = rx + wz$ (income), we will always have that g and c are functions of W. See Section 2.2 where this was made explicit.

Lemma 6 Assume that $u : X \times Z \to \mathbb{R}$ is absolutely continuous in $x \in X$, *i.e.*, assume that $u(x,z) = \alpha(a,z) + \int_a^x p(\tau,z) \, d\tau$ for a Lebesgue integrable function $p : X \times Z \to \mathbb{R}$ (here $a = \inf X$ and α is a function that does not depend on x). Then u exhibits quasi-convex differences [quasi-concave differences] if $p(x,z)$ is quasi-convex [quasi-concave].

Proof. The statement can be verified by going through the proof of Lemma 1 and everywhere replace u's derivative with p. ∎

Next we need a result on how to ensure that the sum of two functions exhibits quasi-concave or quasi-convex differences.

Lemma 7 Let u be of the form $u(x,z) = f(x,z) + h(-x,z)$ where $f, h : X \times Z \to \mathbb{R}$ are differentiable with $D_x f(x,z) = f'_x(x,z) \geq 0$ and $D_x h(x,z) = -h'_x(-x,z) \leq 0$. Then u exhibits quasi-convex differences on the subset $\{(x,z) \in X \times Z : D_x u(x,z) \leq 0\}$ if there exists a $k \geq 0$ such that $\frac{1}{1-k}[f'_x]^{1-k}$ is convex and $\frac{1}{1-k}[h'_x]^{1-k}$ is concave.[31] If instead $\frac{1}{1-k}[f'_x]^{1-k}$ is concave and $\frac{1}{1-k}[h'_x]^{1-k}$ is convex, then u exhibits quasi-concave differences on the subset $\{(x,z) \in X \times Z : D_x u(x,z) \geq 0\}$.

Proof. In the online appendix (Jensen (2015), Lemma 3). ∎

From now on, call a function $f : X \to \mathbb{R}$ *k-convex* [*k-concave*] if $\frac{1}{1-k}[f(x)]^{1-k}$ is convex [concave] where, as previously mentioned, the case $k = 1$ is taken to mean log-convex [log-concave] by convention.[32]

Lemma 8 Given z, let a denote the least point at which $v^n(\cdot,z)$ is defined. Assume that $-D_y u(x,y,z)$ is k-concave in (x,y) [k-concave in (y,z)], and that $v^n(y,z) = \int_a^y p^n(\tau,z) \, d\tau$ where $p^n(\cdot,z)$ is k-convex. Then $g^n(x,z)$ is convex in x [convex in z].

Proof. This is a direct application Theorem 2's corollary (Corollary 2) to the optimization problem in (24). We consider here only the convexity of g^n in x (the exact same argument implies convexity in z under the Lemma's square-bracketed assumption). Except for quasi-convex differences in (x,y), all the assumptions of Corollary 2 are clearly satisfied (in particular, Γ is a convex correspondence as mentioned immediately after Assumption 3). To see that quasi-convex differences holds, we use Lemma 6 and must thus verify that,

$$
\text{(25)} \qquad D_y u(x,y,z) + \beta \int p^n(y,z') \mu_z(dz'),
$$

is quasi-convex in (x,y) on the relevant set which, allowing for solutions at lower boundary points is $A \equiv \{(x,y) \in X^2 : D_y u(x,y,z) + \beta \int D_y v^n(y,z') \mu_z(dz') \leq 0\}$. To see that this holds, first use that k-convexity is preserved under integration (Jensen (2015), Lemma 2) to conclude that when $p^n(y,z')$ is k-convex in y, $\beta \int p^n(y,z') \mu_z(dz')$ is k-convex in y. Then use Lemma 7. ∎

To finish the proof we need just one last technical result.

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[31] In the limit case $k = 1$, $\frac{1}{1-k}[f'_x]^{1-k}$ is by convention equal to $\log(f'_x)$ (similarly for h'_x).

[32] Quite a bit can be said about such functions, but since this is mainly a mathematical distraction from the point of view of this paper, further investigation has been relegated to the online appendix.

Lemma 9 Assume that $D_x u(x, y, z)$ is k-convex in (x, y) and non-decreasing in y and that $g^n(x, z)$ is convex in x. Then $v^{n+1}(x, z) = \int_a^x p^{n+1}(\tau, z) \, d\tau$ where $p^{n+1}(\cdot, z)$ is k-convex.

Proof. Since v^{n+1} is absolutely continuous, we can (abusing notation slightly) write it as: $v^{n+1}(x, z)$ $= \int_a^x D_x v^{n+1}(\tau, z) \, d\tau$. In particular, $D_x v^{n+1}(x, z)$ exists almost everywhere and when it exists $D_x v^{n+1}(x, z) = D_x u(x, g^n(x, z), z)$ by the envelope theorem. k-convexity of $p^{n+1}(x, z) \equiv D_x u(x, g^n(x, z), z)$ in x now follows immediately from the fact that k-convexity is preserved under convex, increasing transformations (Jensen (2015), Lemma 1). ∎

To prove that g is convex, consider the value and policy function iterations (23)-(24). Start with any value function v^0 such that $v^0(y, z) = \int_a^y p^0(\tau, z) \, d\tau$ where $p^0(\cdot, z)$ is k-convex. Then by Lemma 8, g^0 is convex. Hence by Lemma 9, $v^1(y, z) = \int_a^y p^1(\tau, z) \, d\tau$ where $p^1(\cdot, z)$ is k-convex. Repeating the argument, g^1 is convex and $v^2(y, z) = \int_a^y p^2(\tau, z) \, d\tau$ where $p^2(\cdot, z)$ is k-convex. And so on ad infinitum. The pointwise limit of a sequence of convex function is convex, hence $g(\cdot, z) = \lim_{n \to \infty} g^n(\cdot, z)$ is convex. The same argument applies for convexity in z, concluding the proof of Theorem 4.

Remark 1 (General Markov Processes) The previous proof goes through without any modifications if z_t is allowed to be a general Markov process, *i.e.*, if the functional equation (18) is replaced with:

$$(26) \qquad v(x, z) = \sup_{y \in \Gamma(x, z)} \left[u(x, y, z) + \beta \int v(y, z') Q(z, dz') \right],$$

where Q is z_t's transition function. Indeed, the previous proof goes through line-by-line if we instead begin with the functional equation (26).

Remark 2 (Multidimensional Strategy Sets) The proof also easily extends to the case where X and Z are multidimensional (a case treated in the Appendix). The only modification needed is in the proof of Lemma 8 where now Theorem 5 in the appendix is needed to conclude that g^n is convex, in place of Theorem 2. Thus Theorem 4 extends to the multidimensional case if we in addition assume that u is supermodular in y, that Γ's values are lower semi-lattices, and that optimizers stay away from the upper boundary.[33]

Remark 3 (Dispensing with Concavity Assumptions) The results of this section can be extended to cases where the objective function is not concave in (x, y) and/or is not strictly concave in y (see Assumption 3). What is critical for the proof is that the value function is absolutely continuous so that Lemma 6 can be applied. Of course, absolute continuity of the value function can be established under weak direct conditions (see *e.g.* Milgrom and Segal (2002)) instead of being derived as a consequence of concavity as done above. Instead of a convex policy function one then gets a convex policy correspondence.

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[33] In particular, the objective function in (24) is supermodular in y when u is supermodular in y because supermodularity/increasing differences is preserved under integration (Topkis (1998), Theorem 2.7.6.) and $v^n(y, z')$ is therefore supermodular in y for all n.

6 Conclusion

This paper contributes to distributional comparative statics (DCS), *i.e.*, to the study of how changes in exogenous distributions affect endogenous distributions in economic models. Most DCS questions can be answered if suitable policy functions are either concave or convex. In the main theoretical contribution of the paper (Theorem 1), it is shown that concavity of the policy function hinges on an intuitive as well as easily verifiable condition on the primitives of a model, namely *quasi-concave differences*. That observation parallels Topkis' theorem (Topkis (1978)) which ensures that the policy function is increasing (strategic complementarity) when the objective function exhibits increasing differences. Theorem 2, as well as Theorem 5 in the Appendix, extends the result to policy correspondences (multiple optimizers) and multi-dimensional action sets.

Several areas of application were discussed including uncertainty comparative statics, international trade models of heterogenous firms (Melitz (2003)), the macroeconomic modeling of inequality, and stochastic dynamic programming. In all of these, the concavity of suitably defined policy functions turns out to drive the conclusions, ultimately owing to Observations 1-5 in Section 2.1 (page 4) and the fact that Lorenz dominance is equivalent to mean-preserving spreads and generalized Lorenz dominance is equivalent to decreases in second-order stochastic dominance (see the discussion at the end of Section 2.2). As a concrete illustration of uncertainty comparative statics, a Bayesian arms race is studied and it is found that "prudence" (Kimball (1990)) determines whether mean stocks of arms increase or decrease when uncertainty goes up — but in all cases, a more uncertain environment also leads to higher equilibrium variance and thus greater uncertainty about the scale of destruction in the event of a war. The stochastic dynamic programming results are illustrated by generalizing a result due to Carroll and Kimball (1996) to allow for borrowing constraints. These results play a key role for distributional comparative statics in dynamic stochastic general equilibrium (DSGE) models — a theme taken up in Acemoglu and Jensen (2015) who study, for example, how increased uncertainty about future earnings prospects affects output per worker in the Aiyagari (1994) model.

7 Appendices

Appendix I: Proof of Observations 1-5 in Section 2.1

The distribution μ_x is the image measure of μ under g. Hence $\int f(x)\mu_x(dx) = \int f(g(z))\mu(dz)$ for any function $f : X \to \mathbb{R}$ such that the integrals are well-defined. Each claim thus amounts to saying that for classes of functions $\mathscr{F}$ and $\mathscr{F}_x$, if $\int h(z)\bar{\mu}(dz) \geq \int h(z)\mu(dz)$ for all $h \in \mathscr{F}$, then $\int f(g(z))\bar{\mu}(dz) \geq \int f(g(z))\mu(dz)$ for all $f \in \mathscr{F}_x$. In the case of Observation 1, $\mathscr{F}$ and $\mathscr{F}_x$ both equal the class of increasing functions and the claim follows from the fact that $f \circ g$ is increasing when both f and g are increasing. For Observation 2, $\mathscr{F}$ is the class of convex functions and $\mathscr{F}_x$ the class of decreasing, convex functions, and the claim follows because $f \circ g$ is convex when g is concave and f is convex and decreasing. For Observation 3, both $\mathscr{F}$ and $\mathscr{F}_x$ equal the class of increasing, concave functions and the conclusion follows because $f \circ g$ is increasing and concave when both f and g are increasing and concave. Observations 4-5 are proved by the same arguments as

Observations 2-3 and may be omitted.

Appendix II: Proof of Lemma 1

To facilitate the results on multi-dimensional strategy sets in Appendix III, the general case where $X \subseteq \mathbb{R}^n$ is considered. The statement to be provided is that if u is differentiable in x, it will exhibit quasi-concave differences if and only if $D_{x_j} u(x,z)$ is quasi-concave in (x,z) for all $j = 1,\ldots,n$ (evidently, Lemma 1 is a special case of this statement).

"$\Rightarrow$": Since $D_{x_j} u(x,z) = \lim_{\delta \to 0} \frac{u(x+\delta\epsilon_j, z) - u(x,z)}{\delta}$ where ϵ_j denotes the j'th unit vector, and quasi-convexity is preserved under pointwise limits (Johansen (1972)), each partial derivative $D_{x_j} u(x,z)$ is quasi-convex at (x,z) when u exhibits quasi-convex differences at (x,z). "$\Leftarrow$": This direction is not easy. The idea is to prove the contrapositive by contradiction (note that since quasi-convexity is not preserved under integration, we cannot use the fundamental theorem of calculus). So we assume that u does not exhibit quasi-convex differences, that each partial derivative $D_{x_j} u(x,z)$ is quasi-convex, and then derive a contradiction. For $\alpha \in [0,1]$ set $x_\alpha \equiv \alpha x_0 + (1-\alpha)x_1$ and $z_\alpha = \alpha z_0 + (1-\alpha)z_1$. Say that u exhibits quasi-convex differences in the direction $\eta > 0$ at (x_0, z_0, α) if for all $\delta_n > 0$ in some neighborhood of 0:

$$(27) \quad u(x_\alpha + \delta_n \eta, z_\alpha) - u(x_\alpha, z_\alpha) \leq \max\{u(x_0 + \delta_n \eta, z_0) - u(x_0, z_0), u(x_1 + \delta_n \eta, z_1) - u(x_1, z_1)\}$$

It is easy to see that if u exhibits quasi-convex differences (on all of $X \times Z$), then it exhibits quasi-convex differences in all directions $\eta > 0$ at all $(x,z,\alpha) \in X \times Y \times [0,1]$. Let ϵ_j denote the j'th unit vector (a vector with 1 in the j'th coordinate and zeroes everywhere else). Since a function is quasi-convex in all directions if and only if it is quasi-convex in all unit/coordinate directions ϵ_j, we may (as always) restrict attention to the directions of the coordinates in the previous statement. Hence if u *does not* exhibit quasi-convex differences, there will exist a coordinate direction ϵ_j, $(x_0, z_0), (x_1, z_1) \in X \times Y$, $\hat{\alpha} \in [0,1]$ and a sequence $\delta_n \downarrow 0$ such that for all n:

$$(28) \; u(x_{\hat{\alpha}} + \delta_n \epsilon_j, z_{\hat{\alpha}}) - u(x_{\hat{\alpha}}, z_{\hat{\alpha}}) > \max\{u(x_0 + \delta_n \epsilon_j, z_0) - u(x_0, z_0), u(x_1 + \delta_n \epsilon_j, z_1) - u(x_1, z_1)\}$$

Note that we necessarily have $\hat{\alpha} \in (0,1)$ when the previous inequality holds. Intuitively, the inequality says that there exists a point (x_α, z_α) on the line segment between (x_0, z_0) and (x_1, z_1) at which $u(\cdot + \delta_n \epsilon_j, \cdot) - u(\cdot, \cdot)$ takes a strictly higher value than at any of the endpoints. Now, divide through (28) with δ_n and take limits:

$$D_{x_j} u(x_{\hat{\alpha}}, z_{\hat{\alpha}}) \geq \max\{D_{x_j} u(x_0, z_0), D_{x_j} u(x_1, z_1)\}$$

Since $D_{x_j} u(\cdot, \cdot)$ is quasi-convex, it follows that: $D_{x_j} u(x_{\hat{\alpha}}, z_{\hat{\alpha}}) = \max\{D_{x_j} u(x_0, z_0), D_{x_j} u(x_1, z_1)\}$. Assume without loss of generality that $D_{x_j} u(x_0, z_0) \geq D_{x_j} u(x_1, z_1)$. Since $D_{x_j} u(x_0, z_0)$ is quasi-convex and $0 < \hat{\alpha} < 1$, it follows that either (i) $D_{x_j} u(x_0, z_0) = D_{x_j} u(x_\alpha, z_\alpha)$ for all $\alpha \in [0, \hat{\alpha}]$ or (ii) $D_{x_j} u(x_1, z_1) = D_{x_j} u(x_\alpha, z_\alpha)$ for all $\alpha \in [\hat{\alpha}, 1]$ (or both).[34] Consider case (i) (the proof in case (ii) is similar). When (i) holds, u's restriction to the line segment between (x_0, z_0) and $(x_{\hat{\alpha}}, z_{\hat{\alpha}})$ must

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[34]A quasi-convex function's restriction to a convex segment as the one considered here can always be split into two segments, one which is non-increasing and one which is non-decreasing (and in the present situation, there *must* first

necessarily be of the form: $u(x,z) = c\,x_j + g(x_{-j}, z)$ where $c_n = D_{x_n} u(x_0, z_0)$ (a constant) and x_{-j} denotes all coordinates of x except for the j'th one (remember that j is the positive coordinate of e_j). But then $u(x_{\hat{a}} + \delta_n \epsilon_j, z_{\hat{a}}) - u(x_{\hat{a}}, z_{\hat{a}}) = u(x_0 + \delta_n \epsilon_j, z_0) - u(x_0, z_0) = c\,\delta_n \epsilon_j$ which implies that:

(29) $u(x_{\hat{a}} + \delta_n \epsilon_j, z_{\hat{a}}) - u(x_{\hat{a}}, z_{\hat{a}}) \leq \max\{u(x_0 + \delta_n \epsilon_j, z_0) - u(x_0, z_0), u(x_1 + \delta_n \epsilon_j, z_1) - u(x_1, z_1)\}$

Comparing (29) with (28) we have a contradiction, and the proof is complete.

Appendix III: Multi-dimensional Decision Variables

This appendix treats concavity of the policy correspondence in the case where the decision vector is allowed to live in an arbitrary ordered topological vector lattice V, $x \in X \subseteq V$. Note that in this setting, the order mentioned in Definition 3 is the order inherited from V. So if $V = \mathbb{R}^N$ with the usual Euclidean order, the theorem below implies that each coordinate correspondence $G_n : Z \to 2^{X^n} \subseteq 2^{\mathbb{R}}$ is concave in the sense discussed in detail in Section 3.2.

The multi-dimensional setting forces us to make some additional assumptions. In comparison with Theorem 2, X must be a lattice, u must be supermodular in the decision vector, and Γ must be upper semi-lattice valued. Finally, Γ's values must be order convex rather than merely convex.[35] Finally, the boundary conditions must be suitably generalized as we turn to first. It should be noted that all of these assumptions automatically are satisfied when X is one-dimensional. Hence the result to follow encompasses Theorem 2.

First, the basic definitions. Say that a point $x \in \Gamma(z)$ lies on the *upper* [*lower*] *boundary* of $\Gamma(z)$ if there does not exist an $x' \in \Gamma(z)$ with $x' \gg x$ ($x' \ll x$). The upper boundary is denoted by $\overline{B}(\Gamma(z))$ and the lower boundary is denoted by $\underline{B}(\Gamma(z))$. What we are going to require in the theorem below is precisely as in Theorem 2 except that the infimum is replaced with the lower boundary.

Next, X must be a *lattice*, *i.e.*, if x and x' lie in X so do their infimum $x \wedge x'$ and supremum $x \vee x'$. If $X \subseteq \mathbb{R}^n$ with the usual Euclidean/coordinatewise order, the infimum (supremum) is simply the coordinatewise minimum (maximum). Assuming that X is a lattice is actually a very weak additional requirement in the present framework because it is the constraint correspondence Γ that determines the feasible set. It is the next assumption that really has "bite". A lower semi-lattice [upper semi-lattice] is a subset $A \subset X$ with the property that if $x, x' \in A$ then the infimum $x \wedge x'$ [supremum $x \vee x'$] also lies in A. Either is of course weaker than being a lattice. The lower or upper semi-lattice is *order-convex* if $a, b \in A$, and $a \leq a' \leq b$ imply $a' \in A$ (in words, if the set contains an ordered pair of elements, it contains the entire order interval between these elements). Note that order-convexity is stronger than convexity in general, although the two coincide in the one-dimensional case. A budget set is an order-convex lower semi-lattice (it is not a lattice), and

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be non-increasing segment since the function's value weakly decreases between the endpoints). On the convex line segment between (x_0, z_0) and (x_1, z_1) we have in the present situation that the function begins at $D_{x_j} u(x_0, z_0)$, again takes the value $D_{x_j} u(x_0, z_0)$ at (x_a, z_a) and then moves to a weakly lower value $D_{x_j} u(x_1, z_1)$ at the end-point (x_1, z_1). It follows that if $D_{x_j} u(\cdot, \cdot)$ is not constant on the first interval (corresponding to $\alpha \in [0, a]$) it must strictly decrease and then strictly increase on this interval, which implies that $D_{x_j} u(\cdot, \cdot)$ is constant on the second of the two intervals.

[35]Note that this once again precisely parallels monotone comparative statics. In that setting supermodularity and lattice-type assumptions are also unnecessary/trivially satisfied in the one-dimensional case but must be imposed in multiple dimensions.

31

a firm's input requirement set is an order-convex upper semi-lattice (but again not a lattice). As these examples indicate, the fact that we avoid assuming that Γ's values are sublattices of X greatly expands the scope of the theorem below.

Finally, u must be supermodular in the choice variables. The well-known definition is as follows.

Definition 4 (Topkis (1978)) *The objective function $u : X \times Z \to \mathbb{R}$ is supermodular in x if $u(x \vee x', z) + u(x \wedge x', z) \geq u(x, z) + u(x', z)$ for all $x, x' \in X$ and for all $z \in Z$. If u is twice differentiable in x and $X \subseteq \mathbb{R}^n$, it is supermodular in x if and only if the Hessian matrix $D^2_{xx} u(x, z) \in \mathbb{R}^{n \times n}$ has non-negative off-diagonal elements (for all x and z).*

We are now ready to state and prove the main result with multi-dimensional action sets. Note that as in theorems 1-2, the boundary condition is trivially satisfied if the optimization problem is unrestricted or attention is restricted to interior solutions.

Theorem 5 (Concavity of the Policy Correspondence, Multidimensional Case) Let Z be a convex subset of a vector space and X a convex lattice. Define the policy correspondence $G(z) = \arg\sup_{x \in \Gamma(z)} u(x, z)$ where $u : X \times Z \to \mathbb{R}$ is supermodular in x and $\Gamma : Z \to 2^X$ has order-convex values. Assume that $G(z)$ is non-empty and compact for all $z \in Z$. Then:

1. The policy correspondence G is concave if $u : X \times Z \to \mathbb{R}$ exhibits quasi-concave differences, Γ is concave and upper semi-lattice valued, and $x \in G(z) \Rightarrow x \notin \underline{B}(\Gamma(z))$ for all $z \in Z$.

2. The policy correspondence G is convex if $u : X \times Z \to \mathbb{R}$ exhibits quasi-convex differences, Γ is convex and lower semi-lattice valued, and $x \in G(z) \Rightarrow x \notin \overline{B}(\Gamma(z))$ for all $z \in Z$.

Proof of Theorem 5. The convex case 2. is proved (the proof of the concave case is similar). Pick $z_1, z_2 \in Z$, $x_1 \in G(z_1)$, and $x_2 \in G(z_2)$. Exactly as in the proof of Theorem 1, we can use quasi-convex differences to conclude that for some $\delta \gg 0$, $u(x_\alpha, z_\alpha) \geq u(x_\alpha + \delta, z_\alpha)$ for all $\alpha \in [0, 1]$. Hence by quasi-concavity of u in x, $u(x, z_\alpha)$ is non-increasing for $x \geq x_\alpha$. We wish to show that for all α there exists $\hat{x} \in G(z_\alpha)$ with $\hat{x} \leq x_\alpha$. Pick any $x \in G(z_\alpha)$. I am first going to prove that,

(30)
$$x \wedge x_\alpha \in \Gamma(z_\alpha)$$

Since Γ has convex values, there exists some $\tilde{x} \in \Gamma(z_\alpha)$ with $\tilde{x} \leq x_\alpha$. We have $x \in \Gamma(z_\alpha)$ (since $x \in G(z_\alpha)$) and so since Γ's values are lower semi-lattices, $x \wedge \tilde{x} \in \Gamma(z_\alpha)$. But $x \wedge \tilde{x} \leq x \wedge x_\alpha \leq x$, hence $x \wedge x_\alpha \in \Gamma(z_\alpha)$ because Γ has order-convex values. That was what we wanted to show. Next use supermodularity of $u(\cdot, t)$ and the fact that $u(\cdot, z_\alpha)$ is non-increasing for $x \geq x_\alpha$ (implies that $u(x_\alpha, z_\alpha) \geq u(x \vee x_\alpha, z_\alpha)$) to conclude that:

(31)
$$u(x, z_\alpha) - u(x \wedge x_\alpha, z_\alpha) \leq u(x \vee x_\alpha, z_\alpha) - u(x_\alpha, z_\alpha) \leq 0$$

(30)-(31) imply that $x \wedge x_\alpha \in G(z_\alpha)$. But since clearly $x \wedge x_\alpha \leq x_\alpha$ this completes the proof. ∎

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