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The Geometry of Aggregative Games

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Abstract

We study aggregative games in which players' strategy sets are convex intervals of the real line and (not necessarily differentiable) payoffs depend only on a player's own strategy and the sum of all players' strategies. We give sufficient conditions on each player's payoff function to ensure the existence of a unique Nash equilibrium in pure strategies, emphasizing the geometric nature of these conditions. These conditions are almost best possible in the sense that the requirements on one player can be slightly weakened, but any further weakening may lead to multiple equilibria. The same conditions also permit the analysis of comparative statics and the competitive limit. We discuss the application of these conditions in a range of examples, chosen to illustrate various aspects their use. We also show that all restrictions on payoffs in aggregative games that guarantee the existence of a unique equilibrium of which we are aware are covered by these conditions. When payoffs are sufficiently smooth, these conditions can be tested using derivatives of the marginal payoff and we illustrate these tests in the applications introduced earlier. We also investigate conditions under which the unique equilibrium is locally stable. These hold in particular in a symmetric game under the same conditions required to ensure the existence of a unique equilibrium.

Keywords: Noncooperative game theory, aggregative games, equiibrium existence and unbiqueness.

JEL Classification: C62, C72

1 Introduction

Many commonly studied simultaneous-move games have a similar structure in which each player's payoff is a function of her own strategy and the sum of the strategies of all players. Selten [44] called such games 'aggregative'. Applications include Cournot oligopoly, private provision of public goods, cost and surplus sharing games – of which open access resource games are special cases – and Tullock rent-seeking contests with linear technology. Further applications can be found, via a transformation of the strategy space, in models of competition with differentiated products (Spence, [46], [47], Dixit and Stiglitz [24], Blanchard and Kiyotaki [4]) and in rent-seeking contests with nonlinear technology (Tullock [50], Szidarovzsky and Yakowitz [49]).

Referring to such games, Shubik [45] said: "Games with the above property clearly have much more structure than a game selected at random. How this structure influences the equilibrium points has not yet been explored in depth." A number of authors have studied existence of pure strategic equilibria in aggregative games in the context of specific applications such as Cournot oligopoly (for example McManus [35], [36] and Novshek [43]). Such authors sometimes use methods applicable to a wider range of aggregative games. Indeed, Kukushkin's proof of the existence of an equilibrium of an aggregative game when best replies are non-increasing [33] uses a modification of Novshek's approach to Cournot oligopoly. Dubey et al [26] also establish existence under assumptions of strategic complementarity or substitution, although they use a somewhat different approach (pseudo-potential functions).

In this paper, we focus on uniqueness as well as existence. A unique equilibrium may increase the predictive power (and thus the falsifiability) of the predictions of a model. It also avoids equilibrium selection issues and relieves the modeller of the task of explaining how players overcome coordination problems. Conditions for existence and uniqueness of several aggregative games may be found in the literature. Most intensively studied are the Cournot oligopoly game (Szidarovszky and Okuguchi [48], Kolstad and Mathiesen [32]) and the public goods contribution games (Andreoni [2], Cornes, Hartley and Sandler [9] and Bergstrom, Blume and Varian [3]). More recently, Watts [53] (see also Cornes and Hartley [10]) has established such conditions for cost and surplus sharing game and Szidarovszky and Yakowitz [49] have proved existence and uniqueness in risk-neutral rent-seeking contests. Most of these authors use distinct approaches to establish their results, and yet the fact that all these games are aggregative, together with general results on existence, prompts the question of whether there is a common technique for investigating those situations under which such games are well-behaved. Indeed, our aim in this paper is to develop such techniques and apply them to the games mentioned as well as several others. We also examine when such games have predictable comparative statics and the properties of the large-game (competitive) limit, if it exists. More specifically, we introduce assumptions on the payoffs of a player such that, if the payoffs of all players satisfy these conditions, the game will have a unique equilibrium. Ideally, these conditions will be best possible on individual payoffs, in the sense that, if they are not satisfied, a game can be constructed with such a player and all rivals satisfying the conditions and which exhibits multiple equilibria.

The approach adopted by Novshek and generalized by Kukushkin identifies equilibria as fixed points of the sum of correspondences from the aggregate to the strategy space ("backwards reaction correspondence"), one for each player. If each player's correspondence is single-valued, continuous, decreasing where positive and has large enough supremum, the game will have a unique equilibrium. Conditions under which this holds have been derived for several applications and more generally by Corchon [7], who showed that sufficient conditions for existence of a unique equilibrium in an aggregative game are payoffs that are concave in own strategy and satisfy a condition close to and implied by strategic substitutes, together with compact, convex strategy sets. Such Nash equilibria also have many other desirable properties. However, such conditions may be overly restrictive in applications. For example, in Cournot oligopoly, they rule out iso-elastic demand functions and are they not satisfied in open access resource games with standard assumptions on preferences. Nor do they apply to rent-seeking contests. In all these games, best responses as a function of the aggregate strategy of a player's rivals initially rise and subsequently fall as the aggregate increases from zero. In Section 3, we describe a weaker set of conditions which may be applied to all the above games. These conditions include or generalize all the existence and uniqueness results described above[1]. Although our conditions are less restrictive than Corchon, we are nevertheless able to obtain comparative statics on the behavior of the aggregate and payoffs. For example, we can unambiguously sign the effect on payoffs of adding new players. All these authors use[2] the "backward reaction function" of Novshek [43] and Selten. However, uniqueness requires that the aggregate backward reaction function be decreasing or at least has slope less than unity. Our modification is to divide players' reaction functions by the aggregate strategy to obtain a

[1]Except Kolstad and Mathiesen, who give necessary and sufficient conditions on best response mappings, rather than payoffs, for a unique equilibrium.

[2]Sometimes under different nomenclature.

"share function". Consistency requires the aggregate share function to equal one in equilibrium and, if such functions are decreasing, the equilibrium will be unique.

The layout of the paper is as follows. In Section 2, we formally define aggregative games and describe our notation. In Section 3, we describe our geometrical conditions (regularity) for ensuring existence and uniqueness of Nash equilibria. We also introduce share functions and prove that regularity implies the existence of a continuous share function that is decreasing where positive. Section 4 extends the analysis to comparative statics of payoffs and, in Section 5, we study the (competitive) limit as the number of players becomes large. Throughout these sections, we illustrate our results by discussing their application to Cournot oligopoly games. In Section 6, we consider existence and uniqueness (and comparative statics and competitive limits, where appropriate) for five further applications. The sufficient conditions in Section 3 are applied to the payoffs of individual players and, in Section 7, we investigate their necessity. Firstly, we show how regularity can be slightly weakened for one player in an aggregative game without losing existence, uniqueness and comparative statics results of equilibria. However, no further weakening of these conditions is possible, when applied to individual payoffs. However, when there is a relationship between players' payoffs, a further weakening of these condition may be possible and this is discussed in Section 8. In particular, we investigate problems in which payoffs are identical or, more generally, fall into a finite number of types. In all our analyses, the only smoothness condition we have imposed is continuity. However, regularity can often be tested more conveniently when payoffs are twice differentiable in the interior of the payoff space. Sufficient conditions for regularity are established in Section 9, together with applications to the five examples introduced in Section 6. In Section 10, we discuss local asymptotic stability under a continuous version of best-response dynamics with smooth payoffs. In particular, we show that equilibria of symmetric aggregative games played by regular players are stable. Finally, Section 11 offers conclusions and discusses several extensions of our methodology.

2 Aggregative games

We consider the simultaneous-move game $\mathcal{G} = \left(I, \{S_i\}_{i\in I}, \{\pi_i\}_{i\in I}\right)$, in which each of the finite set of players I has a strategy set $S_i = [0, w_i]$ for some $w_i > 0$. (In some applications, the natural strategy set may be $\mathbb{R}_+$. However, if strategies $x_i > w_i$ are dominated[3], the theory to be described is still applicable.) Denote $\prod_{j \in I} S_j$ by $\mathcal{S}$ and $\prod_{j \in I \setminus \{i\}} S_j$ by $\mathcal{S}_{-i}$. We write $x_i \in S_i$ for Player i's strategy and X for $\sum_{i \in I} x_i$. If $\mathbf{x} \in \mathcal{S}$ is a strategy profile, $\pi_i : \mathcal{S} \longrightarrow \mathbb{R}$ denotes the payoff function of Player i. Henceforth, we assume, without explicit statement, that π_i is continuous except possibly at $\mathbf{x} = \mathbf{0}$. (The exceptional treatment of the origin is useful in some applications[4].)

We call such a game *aggregative*[5] if, for each $i \in I$, there is a function $v_i : \widetilde{S}_i \longrightarrow \mathbb{R}$, where

$$\widetilde{S}_i = \{(x_i, X) : 0 \leq x_i \leq \max\{w_i, X\}\},$$

such that

$$\pi_i(\mathbf{x}) = v_i(x_i, X) \text{ for all } \mathbf{x} \in \mathcal{S} \text{ satisfying } \sum_{i \in I} x_i = X. \tag{1}$$

Since feasibility dictates that $X \leq \sum_{i \in I} w_i$, we could have imposed (1) only for such X. However, we do not restrict attention to such X, since our focus is on conditions on v_i ensuring a unique Nash equilibrium and well-behaved comparative statics for *any* set of competitors with payoffs also satisfying these conditions. Not restricting X also permits the study of limiting equilibria as the number of players becomes large. With slight notational abuse, we shall write the aggregative game as $\mathcal{G} = \left(I, \mathbf{w}, \{v_i\}_{i \in I}\right)$, where $\mathbf{w} = \{w_i\}_{i \in I}$.

To simplify the exposition, it is convenient to focus on non-null ($\mathbf{x} \neq \mathbf{0}$) equilibria. Note that there cannot be a null equilibrium if, for any $i \in I$, there is $x \in (0, w_i]$, for which $v_i(x, x) > v_i(0, 0)$. (In a Cournot oligopoly, the condition says that at least one firm can make positive monopoly profits.) Any equilibrium must satisfy $X > 0$.

[3] An example is a Cournot oligopoly in which average cost is positive and non-decreasing and price approaches or is equal to zero for large output. In such a game, levels of output at which cost exceeds the corresponding price are dominated by null output.

[4] For example, in a rent-seeking game, the sum of payoffs of all players is equal to the rent minus the agregate expenditure on rent-seeking, provided at least one player's expenditure is positive. If all expenditures are zero, so are all payoffs. Hence, the sum of payoffs must be discontinuous at the origin and therefore the payoff of at least one player must also have this property.

[5] Note that aggregative games need not be potential games (and vice versa). For example, Theorem 4.5 of Monderer and Shapley [38] shows that for a Cournot oligopoly game to be a potential game entails linear demand, whereas such a game is aggregative for any demand function.

3 Existence and Uniqueness

In this section, we investigate existence and uniqueness of non-null equilibria in pure strategies[6]. We introduce two assumptions, which we call the *aggregate crossing condition* [ACC] and *radial crossing condition* [RCC]. To describe and exploit these, a little notation and a preliminary lemma are needed.

When $\arg\max_{x_i \in S_i} \pi_i(\mathbf{x})$ is a convex set for all $\mathbf{x}_{-i} \in \mathcal{S}_{-i}$, we shall say that Player i has *convex best responses.* In an aggregative game, best responses depend only on $X_{-i} = \sum_{j \in I \setminus \{i\}} x_j$ and it is convenient to write $B_i(X_{-i})$ for the set of best responses.

Condition 3.1 (Convex best responses) *$B_i(X_{-i})$ is a convex set.*

The continuity properties of v_i imply that B_i has closed graph except possibly at the origin[7]. It is also useful to observe that the graph of B_i satisfies the connectedness property set out in the following lemma[8].

Lemma 3.1 *Suppose that $x_i^0 \in B_i\left(X_{-i}^0\right)$ and $X_{-i}^0 \leq \alpha + \beta x_i^0$, where α and β are real numbers. Then there exists $X'_{-i} \geq X_{-i}^0$ and $x'_i \in B_i\left(X'_{-i}\right)$ such that $X'_{-i} = \alpha + \beta x'_i$.*

Proof. Since the set of (x_i, X_{-i}) satisfying $x_i \in [0, w_i]$ and $X_{-i} \leq \alpha + \beta x_i$ is bounded we can define X_{-i}^{U} to be the least upper bound of X_{-i} subject to $x_i \in B_i(X_{-i})$ and $X_{-i} \leq \alpha + \beta x_i$. Since $x_i \in B_i(X_{-i})$ implies that $0 \leq x_i \leq w_i$, there is a sequence $\left\{\left(x_i^n, X_{-i}^n\right)\right\}$ such that $X_{-i}^n \longrightarrow X'_{-i}$, as $n \longrightarrow \infty$ and $\{x_i^n\}$ is convergent, to x_i^{U}, say. By continuity, $x_i^{\mathrm{U}} \in B_i\left(X'_{-i}\right)$ and $X'_{-i} \leq \alpha + \beta x_i^{\mathrm{U}}$. For any $X_{-i} > X'_{-i}$, there is $x_i \in [0, w_i]$ such that $x_i \in B_i(X_{-i})$ and, by definition of X'_{-i}, we have $X_{-i} > \alpha + \beta x_i$. It follows by a similar continuity and compactness argument that there is an x_i^{L} such that $x_i^{\mathrm{L}} \in B_i\left(X'_{-i}\right)$ and $X'_{-i} \geq \alpha + \beta x_i^{\mathrm{L}}$. If x'_i is chosen to satisfy $X'_{-i} = \alpha + \beta x'_i$, then x'_i is a convex combination of x_i^{L} and x_i^{U} and, by convexity of best responses, $x'_i \in B_i\left(X'_{-i}\right)$. The inequality $X'_{-i} \geq X_{-i}^0$ is immediate from the construction of X'_{-i}. ■

[6] In many applications, preferences over outcomes are naturally assumed to be a continuous weak ordering. To order distributions over outcomes entails a significant strengthening of these assumptions.

[7] Recall that payoffs need not be continuous at the origin.

[8] In fact B_i is connected in the conventional sense but this is more complicated to prove and not needed in the sequel.

For Player i and any $X > 0$, we study the set of strategies x_i that the player can choose in a Nash equilibrium in which the value of the aggregate is X. Each such x_i must be a best response to $X_{-i} = X - x_i$. Hence, the graph of the correspondence that maps X into the set of strategies consistent with equilibrium $X > 0$ is

$$L_i = \left\{ (x_i, X) \in \widetilde{S}_i' : x_i \in B_i (X - x_i) \right\}, \tag{2}$$

where $\widetilde{S}_i' = \widetilde{S}_i \backslash \{\mathbf{0}\}$. Note that L_i is the image of graph of B_i under the linear mapping $(x_i, X_{-i}) \mapsto (x_i, x_i + X_{-i})$ which leads to the following corollary.

Corollary 3.1 *Suppose that $(x_i^0, X^0) \in L_i$ and $X^0 \leq \alpha + \beta x_i^0$, where α and β are real numbers. Then there exists $(x_i', X') \in L_i$ such that $X' = \alpha + \beta x_i'$ and $X' - x_i' \geq X - x_i$.*

Our conditions may now be stated as follows.

Condition 3.2 (ACC) *Player i's best responses satisfy the aggregate crossing condition at X if there is at most one x_i satisfying $(x_i, X) \in L_i$.*

Condition 3.3 (RCC) *Player i's best responses satisfy the radial crossing condition at σ if there is at most one value of X satisfying $(\sigma X, X) \in L_i$.*

Geometrically, these conditions can be visualized graphically with X on the horizontal and x_i on the vertical axis. Then Conditions ACC and RCC state that L_i meets a vertical line at X and a ray through the origin with slope σ at most once. Figure 1, Panel (a), shows a situation in which all three conditions are satisfied. In panel (b), best responses are not everywhere convex. Panels (c) and (d) depict violations of Conditions ACC and RCC respectively. In both of these panels, there is also a value of X_{-i} for which the set of responses is an interval. Our next lemma demonstrates that the appearance of this feature alongside violations of one or the other of the crossing conditions is no coincidence.

Definition 3.4 *Player i is regular if*

1. *$B_i(X_{-i})$ is convex for all $X_{-i} \geq 0$,*
2. *best responses satisfy ACC at all $X > 0$,*
3. *best responses satisfy RCC at all $\sigma \in (0, 1]$.*

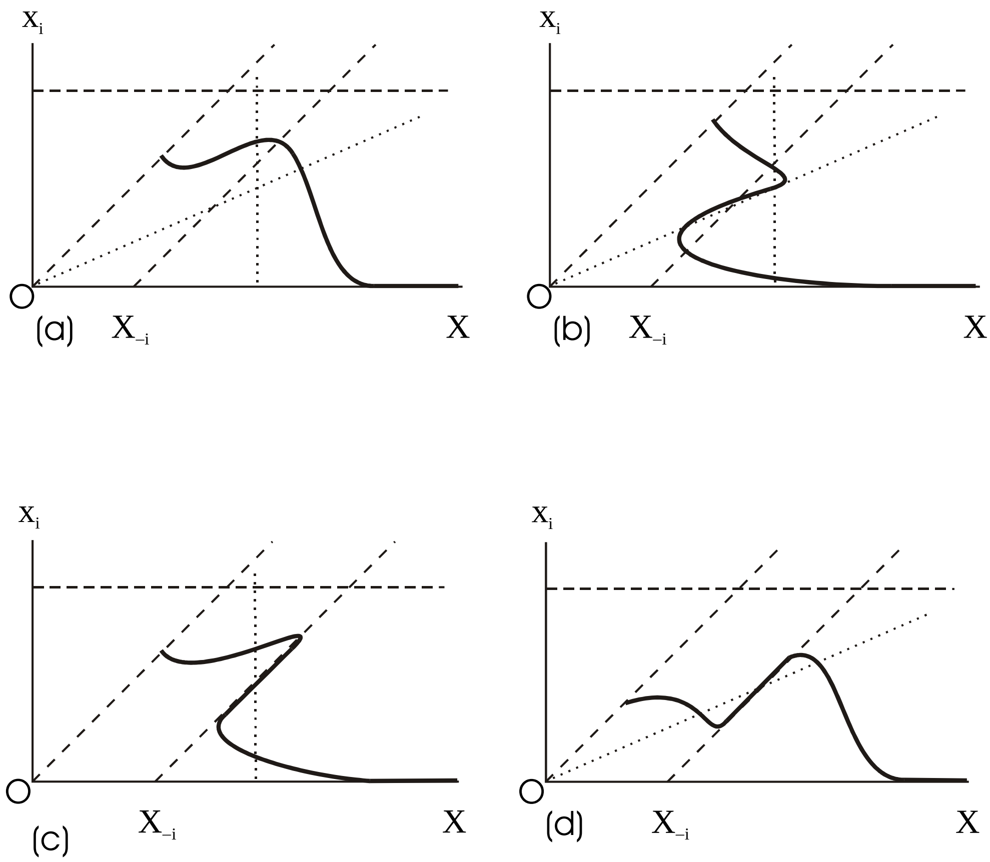

Figure 1:

Our aim in this section is to show that aggregative games played by regular players have unique pure strategy equilibria. We start by showing that, for individual players, best responses are singletons.

Lemma 3.2 *If Player i is regular, B_i is single valued.*

Proof. It is useful to view $B_i(X_{-i})$ as the set of maximizers of v_i on a line of unit slope through $(0, X_{-i})$; formally,

$$B_i(X_{-i}) = \arg\max_{x \in S_i} v_i(x, x + X_{-i}).$$

The lemma is proved by fixing $X_{-i} > 0$ and deriving a contradiction from the supposition that

$$B_i(X_{-i}) = [x^*, x^{**}],$$

where $0 \leq x^* < x^{**} \leq w_i$.

To achieve this, it proves convenient to define $X^* = x^* + X_{-i}$, $X^{**} = x^{**} + X_{-i}$, $\sigma^* = x^*/X^*$, $\sigma^{**} = x^{**}/X^{**}$ and note that $\sigma^* < \sigma^{**}$. We now consider the line through $(\sigma^* X^{**}, X^{**})$ with unit slope: $x = \phi(X) = X - (1 - \sigma^*) X^{**}$. (The construction is illustrated in Figure 2.) Note that maximizers of v_i on this line take the form

$$B_\phi = \{(\phi(X), X) : \phi(X) \in B_i((1 - \sigma^*) X^{**})\}$$

and observe that $X \in [X^*, X^{**}]$ implies $(\phi(X), X) \notin B_\phi$ because of ACC. Similarly, if

$$X^{**} \leq X \leq \frac{1 - \sigma^*}{1 - \sigma^{**}} X^{**},$$

then $\phi(X) \in [\sigma^*, \sigma^{**}]$, which implies $(\phi(X), X) \notin B_\phi$ because of RCC. It follows that there is $(\phi(X), X) \in B_\phi \subset L_i$ which satisfies either (a) $X < X^*$, or (b) $X > (1 - \sigma^*) X^{**}/(1 - \sigma^{**})$. In case (a), we can apply Corollary 3.1 to deduce the existence of $(x', X^*) \in L_i$ such that

$$X^* - x' \geq X - \phi(X) = (1 - \sigma^*) X^{**} > (1 - \sigma^*) X^*.$$

We conclude that $x' < \sigma^* X^* = x^*$ and thus that there are two distinct points of L_i satisfying $X = X^*$, contradicting aggregate crossing. In case (b),

$$\begin{aligned} \phi(X) &= X - (1 - \sigma^*) X^{**} \\ &> \frac{1 - \sigma^*}{1 - \sigma^{**}} X^{**} - (1 - \sigma^*) X^{**} \\ &= \frac{\sigma^{**}}{1 - \sigma^{**}} [X - \phi(X)], \end{aligned}$$

which implies that $\phi(X) > \sigma^{**} X$. We can apply Corollary 3.1 again to deduce the existence of $(x', X') \in L_i$ such that $x' = \sigma^{**} X'$ and

$$(1 - \sigma^{**}) X' = X' - x' \geq X - \phi(X) = (1 - \sigma^*) X^{**},$$

implying $X' > X^{**}$. We conclude that there are two distinct points satisfying $x = \sigma^{**} X$, giving another contradiction, this time with the radial crossing condition. ■

Panels (a) and (b) of Figure 2 illustrate cases (a) and (b) in the proof. We now take the reader through the reasoning involved in case (a) with the help of the figure. Construct the point C, at the intersection of the lines $X = X^{**}$ and $x = \frac{x^*}{X^*} X$. Corollary tells us that there is a point $(x, X) \in L_i$ that lies on the line of slope 1 through C. No such point can lie in the segment BC, since this would violate **ACC**. Also, no such point can lie in the segment CD, since this would violate **RCC**. In case (a), depicted in panel (a), we suppose the point lies in the segment AB. Such a point is marked in the figure. Corollary 3.1 tells us that there must exist a point in L_i that is on the line $X = X^*$ and below the point B. But the existence of such a point violates **ACC**. Thus we cannot have a point that is both in L_i and in the segment AB. Turning to case (b), a similar kind of argument rules out the existence of a point in the segment DE.

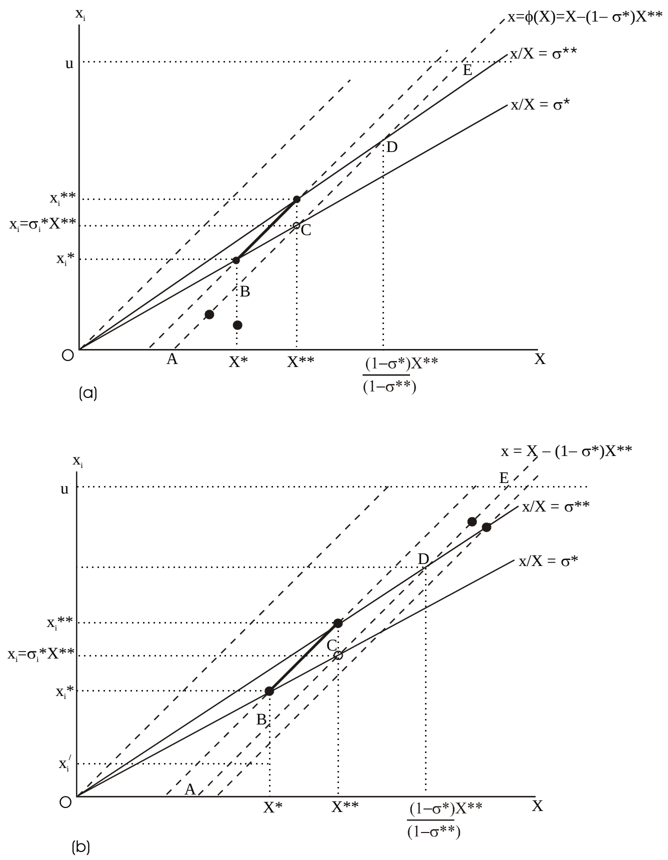

Figure 2:

The case $X_{-i} = 0$ corresponds to the line $x = X$ and is therefore complicated by the possibility of discontinuity at the origin. The radial crossing condition with $\sigma = 1$ implies that $\arg\max v_i(X, X)$ is either a singleton or empty. Hence, there are two possible cases: (i) $v_i(X, X)$ is maximized at

$\underline{X}_i > 0$, or (ii) $v_i(X, X)$ has no maximum in $X > 0$. Since we have assumed that $v_i(x, x) > v_i(0, 0)$ for some $x > 0$, case (ii) can only occur if v is discontinuous at the origin. Note that in case (i), $(\underline{X}_i, \underline{X}_i) \in L_i$. We shall refer to $\underline{X}_i$ as the *participation value* of Player i. In a Cournot oligopoly, $\underline{X}_i$ is the monopoly output of firm i. In case (ii), it is convenient to set $\underline{X}_i = 0$.

Under the assumptions of the lemma, we can define a best response *function*: which we write $b_i(X_{-i})$. Since it has a closed graph, b_i is a continuous function. It follows that, if L_i crosses the line $X = X^0$, it crosses $X = X'$ for all $X' > X^0$. For, if we define

$$\psi_i(X_{-i}) = b_i(X_{-i}) + X_{-i}, \tag{3}$$

there must be some $X^0_{-i} \leq X^0$ for which $\psi_i\left(X^0_{-i}\right) = X^0 < X'$. Since $\psi_i(X') \geq X'$, the intermediate value theorem implies that there is X'_{-i} satisfying $\psi_i\left(X'_{-i}\right) = X'$ as claimed. The aggregate crossing condition implies that L_i crosses $X = X^0$ exactly once for each X^0 in a semi-infinite interval. This allows us to define a function r_i on this interval, where $(r_i(X^0), X^0)$ for the crossing point. We call r_i the *replacement function*[9] *of* Player i. Note that this function has closed graph (L_i) and is therefore continuous. For our purposes, it is more convenient to use the *share function* defined as $s_i(X) = r_i(X)/X$. The radial crossing condition implies that, for any $\sigma \in (0, 1]$, there is at most one value of X satisfying $s_i(X) = \sigma$. Since $L_i \subset \widetilde{S}_i$, we must also have $s_i(X) \leq w_i/X$ and we can conclude that s_i is strictly decreasing where positive. In case (i), and the domain of both r_i and s_i is $[\underline{X}_i, \infty)$. (If s_i were defined for $X < \underline{X}_i$, we would have $s_i(X) > 1$, which is impossible.) In case (ii), the domain of r_i and s_i is $(0, \infty)$ and we write

$$\overline{\sigma}_i = \sup_{X>0} s_i(X) = \lim_{X \longrightarrow 0^+} s_i(X) \tag{4}$$

for the least upper bound of the share function. The following result summarizes and extends these observations.

Proposition 3.1 *Regularity is a necessary and sufficient condition for the existence of a share function for Player i, which is strictly decreasing where positive and has domain* $[\underline{X}_i, \infty)$ *or* $\mathbb{R}_{++}$. *The former case occurs if and only if i has positive participation value* $\underline{X}_i$ *and* $s_i(\underline{X}_i) = 1$ *and* $s_i(X) < 1$ *for all* $X > \underline{X}_i$. *In either case, either (a) there is* $\overline{X}_i > 0$ *such that* $s_i(X) = 0$ *if and only if* $X \geq \overline{X}_i$, *or (b)* $s_i(X) \longrightarrow 0$ *as* $X \longrightarrow \infty$.

[9] This is our name for the "backwards reaction function". It is intended to capture the idea that the value of the replacement function at X is the output level that, if subtracted from X, will be replaced by the player, maintaining the aggregate level X.

Figure 3 shows the four possible shapes of the graph of the share function. The distinction between the cases (a) and (b) rests on whether L_i meets the

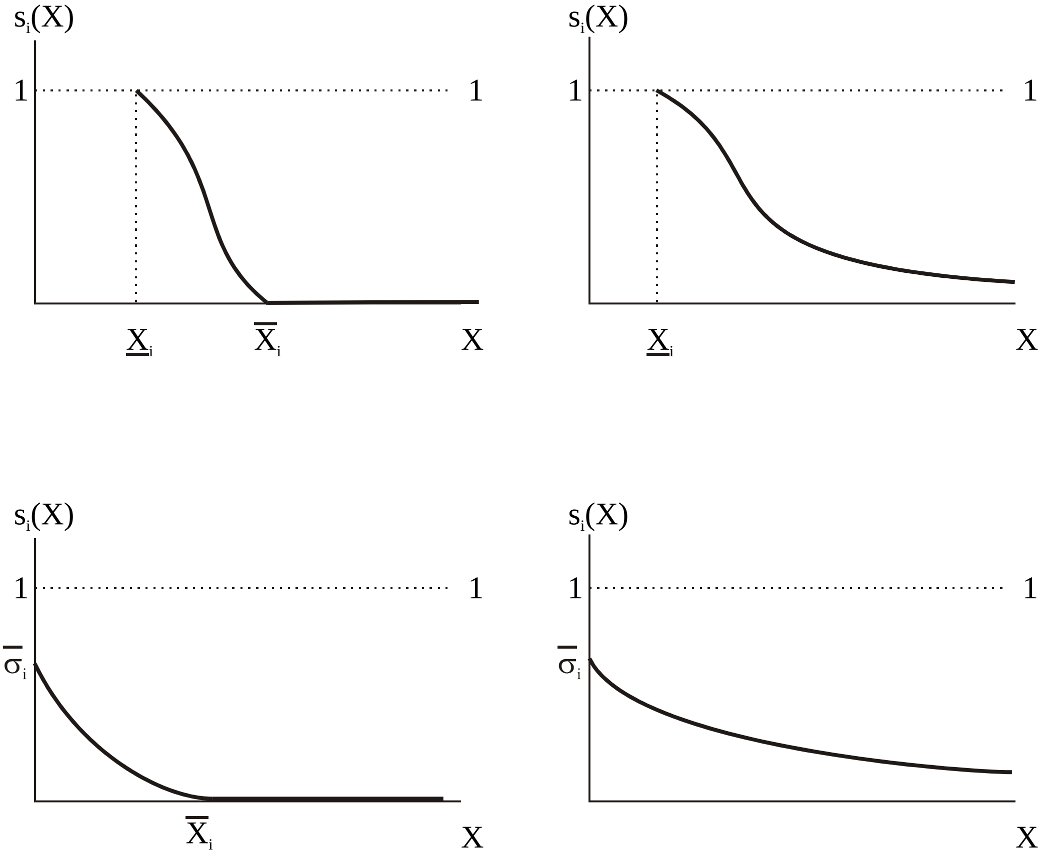

Figure 3:

$x = 0$ axis. If so, $\overline{X}_i$ is the greatest lower bound of the intersection of L_i and this axis. Furthermore, L_i coincides with this axis for $X \geq \overline{X}_i$, otherwise continuity would imply a contradiction of the radial crossing condition (for small enough $\sigma > 0$). We shall refer to $\overline{X}_i$ as the *dropout value* of Player i. In a Cournot oligopoly, $\overline{X}_i$ is the competitive level of output for Player i. That is, the output at which price falls to the marginal cost of Player i at the origin. In case (b), it is convenient to set $\overline{X}_i = +\infty$, so the dropout value is always defined. The assertion that s_i is asymptotic to the axis in this case is a consequence of the inequality $s_i(X) \leq w_i/X$.

Remark 3.5 *A detailed examination of the arguments leading to the proposition shows that the boundedness of the strategy set of Player i is not required* *for all conclusions in the proposition. In particular, if the strategy set is $\mathbb{R}_+$ and best responses are unique (or possibly empty if $X_{-i} = 0$) then all the conclusions in the proposition except (b) remain valid. Indeed, it is straightforward to see that, in case (b), the share function is either strictly increasing or strictly decreasing. In the latter case, a direct argument is needed to establish that the share function vanishes asymptotically.*

Share functions allow us to compute equilibria because of the following result, easily proved by chasing definitions.

Lemma 3.3 *Suppose that all players have share functions. Then $\widehat{\mathbf{x}}$ is a non-null Nash equilibrium if and only if $\widehat{X}$ lies in the domain of s_i and $\widehat{x}_i = \widehat{X} s_i\left(\widehat{X}\right)$ for all $i \in I$, where $\widehat{X} = \sum_{i \in I} \widehat{x}_i$.*

This lemma implies that there is an equilibrium with aggregate value $\widehat{X}$ if and only if the *aggregate share function* $s_I(X) = \sum_{i \in I} s_i(X)$ satisfies $s_I\left(\widehat{X}\right) = 1$. Note that the domain of the aggregate share function is $X \geq \max \underline{X}_i$, where the maximum is over players with finite participation value, if any, and is $\mathbb{R}_{++}$, otherwise. Under the conclusions of Proposition 3.1, the aggregate share function is continuous and approaches zero as $X \longrightarrow \infty$. If at least one player has a positive participation value, the aggregate share function is defined for $X \geq \underline{X} = \max \underline{X}_i$, where the maximum is over all players with positive participation values. If no player has a positive participation value, there is a unique equilibrium if and only if the aggregate share function exceeds 1 for small enough X. This gives the following existence and uniqueness result.

Theorem 3.6 *Suppose that all players in the aggregative game $\mathcal{G} = \left(I, \mathbf{w}, \{v_i\}_{i \in I}\right)$ are regular. If no player has a positive participation value, suppose further that*

$$\sum_{i \in I} \overline{\sigma}_i > 1. \tag{5}$$

Then, $\mathcal{G}$ has a unique non-null Nash equilibrium.

If no player has a positive participation value, and (5) is invalid, $\mathcal{G}$ has no equilibrium.

Figure 4 shows the graphs of share functions in a 3-player game. The thick line is the graph of the aggregate share function, obtained by summing the individual share functions vertically. Note that equilibrium $\widehat{X}$ exceeds $\underline{X}$,

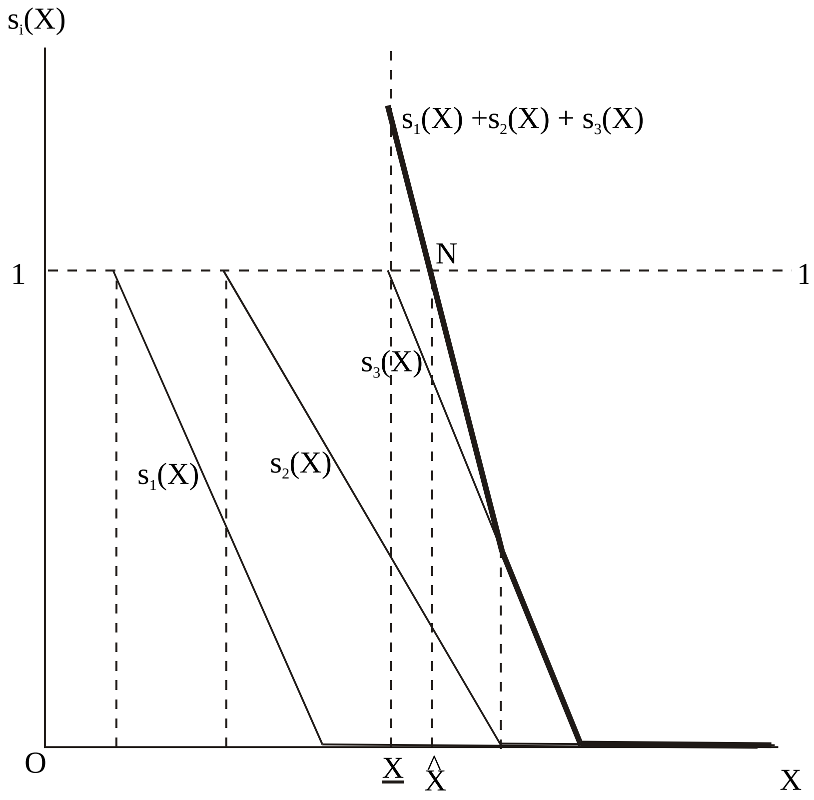

Figure 4:

which explains the terminology 'participation value'. Furthermore, Player i is active ($\widehat{x}_i > 0$) if and only if $\widehat{X} < \overline{X}_i$, which explains the terminology 'dropout value'. It follows from our discussion on participation values that, if the payoff of any player is continuous at the origin, this player has positive participation value and the game has a non-null equilibrium[10].

Theorem 3.6 continues to hold if some or all players have strategy space $\mathbb{R}_+$, provided all share functions which do not meet the X-axis are asymptotic to it. (See Remark 3.6.) In the opposite case, in which all share functions are strictly *increasing*, any equilibrium is still unique and Tarski's theorem can be used to establish existence provided there are values of X for which the aggregate share function is (i) greater than and (ii) less than unity.

In an aggregative submodular game, RCC holds for all players and all

[10]If all payoffs are continuous, existence also follows from standard results [18], [19] when we impose conditions excluding a null equilibrium.

$\sigma \in (0,1)$. Indeed, suppose that

$$x \in B_i(X_{-i}), x' \in B_i(X'_{-i}), X'_{-i} > X_{-i} \Longrightarrow x' \leq x,$$

with strict inequality if $x > 0$. Then, it is immediate that, for any σ satisfying $0 < \sigma < 1$,

$$\sigma X \in B_i((1-\sigma)X)$$

can be satisfied by at most one value of X, which is just RCC. Suppose that, in addition, ACC is satisfied for all $X > 0$ and, if there is a best response to $X_{-i} = 0$, it is unique. Then player i is regular.

However, RCC is a weaker condition than submodularity. Indeed, in the sequel, we shall discuss a class of supermodular search games in which all players are regular. More generally, regularity does not imply monotonic best responses. For example, in a Cournot oligopoly with isoelastic demand and constant (positive) marginal costs, all players are regular, yet best responses are initially increasing but eventually decreasing. Nevertheless, we shall show that submodularity and supermodularity *in addition to regularity* can sometimes yield stronger comparative statics than regularity alone as it allows us to sign the slope of the replacement function.

Proposition 3.2 *Suppose that all players in the aggregative game* $\mathcal{G} = \left(I, \mathbf{w}, \{v_i\}_{i\in I}\right)$ *are regular and the game is submodular [supermodular]. Then the replacement function* r_i *is strictly decreasing [increasing], where positive.*

Proof. We use the fact that $r_i = Xs_i$ satisfies

$$r_i(X) = b_i[\{1 - s_i(X)\}X],$$

where b_i is the best response function. The fact that s_i is nonincreasing implies that $\{1 - s_i(X)\}X$ is strictly increasing in X and therefore that r_i is strictly decreasing where positive if the game is submodular and strictly increasing if it is supermodular. ■

The previous proposition applies to individual players; if some players had increasing and others decreasing best responses, individual replacement functions would inherit these properties. However, such mixed games appear to be uncommon in practice. We shall exploit this proposition in Section 4, which deals with comparative statics.

We illustrate the theorem by applying it to the case of Cournot oligopoly[11]. Suppose that the set of firms is I and firm $i \in I$ chooses its output x_i from

[11] The selection of articles in Daughety [21] covers many aspects of this model.

the set $[0, w_i]$ at cost $c_i(x_i)$. If p denotes the inverse demand function, we assume (without loss of generality) that $p(X) > 0$ for $0 < X < w_i$. The payoff of Player i is

$$\pi_i(\mathbf{x}) = x_i p(X) - c_i(x_i)$$

for $\mathbf{x} \neq \mathbf{0}$ and[12] $\pi_i(\mathbf{0}) = 0$. We impose assumptions on demand and costs. Firstly **C1**: p is twice continuously differentiable and satisfies

$$p'(X) < 0 \text{ and } 2p'(X) + Xp''(X) < 0, \tag{6}$$

for $X > 0$. Note that the latter inequality implies that revenue Xp is strictly concave. The second assumption we shall make is **C2**: c_i is a continuous convex function[13] and satisfies $c_i(0) = 0$.

Firstly, we establish convex best responses by noting that

$$\frac{\partial^2}{\partial x_i^2}[x_i p(X)] = 2\left(1 - \frac{x_i}{X}\right) p'(X) + \frac{x_i}{X}[Xp''(X) + 2p'(X)] < 0.$$

This shows that π_i is a concave function of x_i, so best responses are convex, indeed unique. Furthermore, first order conditions hold in the form $(x_i, X) \in L_i$ if and only if

$$p(X) + x_i p'(X) \in \Delta c_i(x_i), \tag{7}$$

where the right hand side denotes the set (an interval) of slopes of supporting lines to c_i at x_i. It is straightforward to verify ACC. Fix X. The left hand side is strictly decreasing in x_i by **C1** and the right hand side is an increasing correspondence[14]. It follows that (7) can hold for at most one x_i. To check RCC, we can rewrite (7) as

$$p(X) + \sigma_i X p'(X) \in \Delta c_i(\sigma_i X) \tag{8}$$

and note that the derivative of the left hand side with respect to X is

$$(1 - \sigma_i) p'(X) + \sigma_i [Xp''(X) + 2p'(X)] < 0.$$

This verifies the radial crossing condition at $\sigma_i > 0$ and shows that Player i is regular.

[12]The special treatment of the origin allows for the possibility that inverse demand is unbounded for small X.

[13]Continuity restricts the cost function only at the origin.

[14]We say a correspondence F is non-decreasing if $\delta \in F(x_i)$, $\delta' \in F(x_i')$, $x_i' > x_i \implies \delta' \geq \delta$.

If

$$\sup_{X>0} [p(X) + Xp'(X)] > \max \Delta c_i(0), \tag{9}$$

for some player i, the best response to $\mathbf{x}_{-i} = \mathbf{0}$ is positive; i.e. the player has a positive participation value. By Theorem 3.6, there is a unique Cournot equilibrium. When (9) fails for all i, it is convenient to assume that $p(X)$ exceeds $\max \Delta c_i(0)$ for some $X > 0$ (otherwise firm i is inactive against all competition) and the existence of a limiting (absolute) price elasticity η for small X, that is $\eta = -\lim_{X \longrightarrow 0} [p(X)/Xp'(X)]$. In this case, if p has a finite limit as $X \longrightarrow 0$ so does Xp'. If p is unbounded, so is Xp' and we take the limit as $X \longrightarrow 0$ to be $+\infty$. With this interpretation,

$$\overline{\sigma}_i = \eta - \frac{\max \Delta c_i(0)}{\lim_{X \longrightarrow 0} Xp'(X)}$$

and (5) yields a necessary and sufficient condition for a unique equilibrium. In particular, if $p(X)$ is unbounded or $\Delta c_i(0) = \{0\}$, we have $\overline{\sigma}_i = \eta$ for all i and a unique equilibrium exists if and only if $\eta > 1/n$.

Differentiability of the demand cannot be dispensed with (unlike differentiability of cost functions). To illustrate, consider the demand function

$$p(X) = \begin{array}{ll} 2 - X & \text{if } 0 \le X \le 1, \\ 3 - 2X & \text{if } 1 < X \le 3/2, \\ 0 & \text{if } X > 3/2, \end{array}$$

which is concave and strictly decreasing for $X \in [0, 3/2]$. It follows that $Xp(X)$ is strictly concave on the same interval[15]. Suppose that there are two firms, each with $w_i = 1$ and $c_i(x) = x/3$. Then $\pi_i(x_i, \mathbf{x}_{-i})$ is a strictly concave function of x_i. Furthermore, the set of slopes of supporting lines of π_i (with respect to x_i) at $X = 1$ is

$$\Delta_{x_i} \pi_i = \left[\frac{2}{3} - 2x_i, \frac{2}{3} - x_i \right].$$

[15]If

$$Z = \lambda X + (1 - \lambda) X',$$

where $\lambda \in (0, 1)$, concavity of p implies

$$Zp(Z) \ge \lambda X p(X) + (1 - \lambda) X' p(X') + A,$$

where

$$A = \lambda (1 - \lambda)(X' - X)[p(X) - p(X')] > 0,$$

since p is strictly decreasing.

We conclude that (x_1, x_2) is a Nash equilibrium if[16] $x_1 + x_2 = 1$ and $1/3 \leq x_1 \leq 2/3$. Thus, decreasing demand, strictly concave revenue and convex costs permit multiple equilibria (though not multiple values of X).

4 Comparative Statics

In this section, we discuss comparative statics, noting that such analyses are much more intricate in a strategic environment. For example, in a Cournot game in which an idiosyncratic change in its the economic environment causes one firm to reduce its output, other firms may respond by increasing theirs. Consequently, it may not be easy to disentangle these effects to deduce, say, the change in total output. However, if share functions are well-behaved and a mild extra assumption holds, definite results on the aggregate and payoffs follow. The key result on the latter is the following.

Lemma 4.1 *Suppose Player i is regular, has share function s_i and $v_i(x_i, X)$ is strictly increasing in X for all $x_i > 0$. If $X^2 > X^1 > 0$ and $X^1 \geq \underline{X}_i$ (participation value), then*

$$v_i\left(X^1 s_i\left(X^1\right), X^1\right) \leq v_i\left(X^2 s_i\left(X^2\right), X^2\right)$$

and the inequality is strict if $X^1 < \overline{X}_i$.

If $v_i(x_i, X)$ is strictly decreasing in X for all $x_i > 0$, the same results hold with the inequality reversed.

Proof. Since s_i is non-increasing and $s_i(X^1) = 1$ implies $s_i(X^2) < 1$,

$$X^1\left[1 - s_i\left(X^1\right)\right] < X^2\left[1 - s_i\left(X^2\right)\right].$$

From the definition of share functions we have

$$\begin{aligned} v_i\left(X^1 s_i\left(X^1\right), X^1\right) &= \max_{x \geq 0} v_i\left(x, X^1 - X^1 s_i\left(X^1\right) + x\right) \\ &\leq \max_{x \geq 0} v_i\left(x, X^2 - X^2 s_i\left(X^2\right) + x\right) \\ &= v_i\left(X^2 s_i\left(X^2\right), X^2\right). \end{aligned}$$

Note that the continuity of v_i implies that $v_i(0, X)$ is non-decreasing in X. Indeed, equality can occur only if both maximands are 0 and, in particular, only if $s_i(X^1) = 0$.

[16] And 'only if': a more complete analysis shows that all Nash equilibria satisfy these conditions.

The last assertion follows similarly. ■

This lemma can be applied to show that adding extra players to a game increases aggregate output and makes existing players worse or better off according as v_i is decreasing or increasing in X. If one of the additional players is *active* (chooses a positive strategy in equilibrium), currently active players are strictly worse (or better) off.

Theorem 4.1 *Let $\mathcal{G}^k = \left(I^k, \mathbf{w}^k, \{v_i^k\}_{i\in I^k}\right)$ for $k = 1, 2$ and suppose that $I^1 \subset I^2$ and $w_i^1 = w_i^2, v_i^1 = v_i^2$ for $i \in I^1$. Suppose all players in I^2 are regular and $v_i(x_i, X)$ is strictly increasing [decreasing] in X for all $x_i > 0$. If $\mathcal{G}^1$ has a (unique) non-null Nash equilibrium $\widehat{\mathbf{x}}^1$, there is an equilibrium $\widehat{\mathbf{x}}^2$ of $\mathcal{G}^2$. Supposing $\widehat{x}_i^2 > 0$ for some $i \in I^2 \setminus I^1$ and writing $\widehat{X}^k = \sum_{j\in I^k} \widehat{x}_j^k$,*

1. *$\widehat{X}^2 > \widehat{X}^1$,*
2. *inactive players in $\mathcal{G}^1$ are inactive in $\mathcal{G}^2$,*
3. *active players in $\mathcal{G}^1$ are better [worse] off in $\mathcal{G}^2$ than in $\mathcal{G}^1$,*
4. *if the game has decreasing $\{$increasing$\}$ best responses, $\widehat{x}_i^2 \le \{>\}\widehat{x}_i^1$. In the former case, the inequality is strict if $\widehat{x}_i^1 > 0$.*

The requirement that at least one of the additional players be active is not really restrictive. If all additional players are inactive, equilibrium strategies for players in I^1 are unchanged.

Note that regularity alone is not sufficient to allow us to sign individual responses. However, as Part 3 shows, this does not prevent us signing changes in payoffs.

Proof of Theorem 4.1. The existence of an equilibrium of $\mathcal{G}^2$ is an immediate consequence of Theorem 3.6. Then,

$$\sum_{j\in I_2} s_j\left(\widehat{X}^1\right) \ge \sum_{j\in I_1} s_j\left(\widehat{X}^1\right) = 1 = \sum_{j\in I_2} s_j\left(\widehat{X}^2\right).$$

Since each s_i is non-decreasing, we deduce that $\widehat{X}^2 \ge \widehat{X}^1$. Equality could only occur if we had $s_j\left(\widehat{X}^2\right) = 0$ for all $j \in I_2 \setminus I_1$ but this would violate our assumptions and proves Part 1.

If, for some Player i, we have $s_i\left(\widehat{X}^1\right) = 0$, then $s_i\left(\widehat{X}^2\right) = 0$ by Lemma 3.1, which gives Part 2.

Part 2 follows immediately on application of Lemma 4.1 using the result of Part 3.

Part 4 is an immediate consequence of Proposition 3.2. ■

Suppose that there are initially two players. Figure 5 shows the graphs of their share functions, $s_1(X)$ and $s_2(X)$. The associated aggregate share function, graphed by the thick continuous line, takes the value 1 at the Nash equilibrium, $X = \widehat{X}^1$. Now a third player, whose share function is $s_3(X)$, enters the new game. The aggregate share function of the new game is graphed by the thick dashed line, and equilibrium now occurs at $X = \widehat{X}^2$.

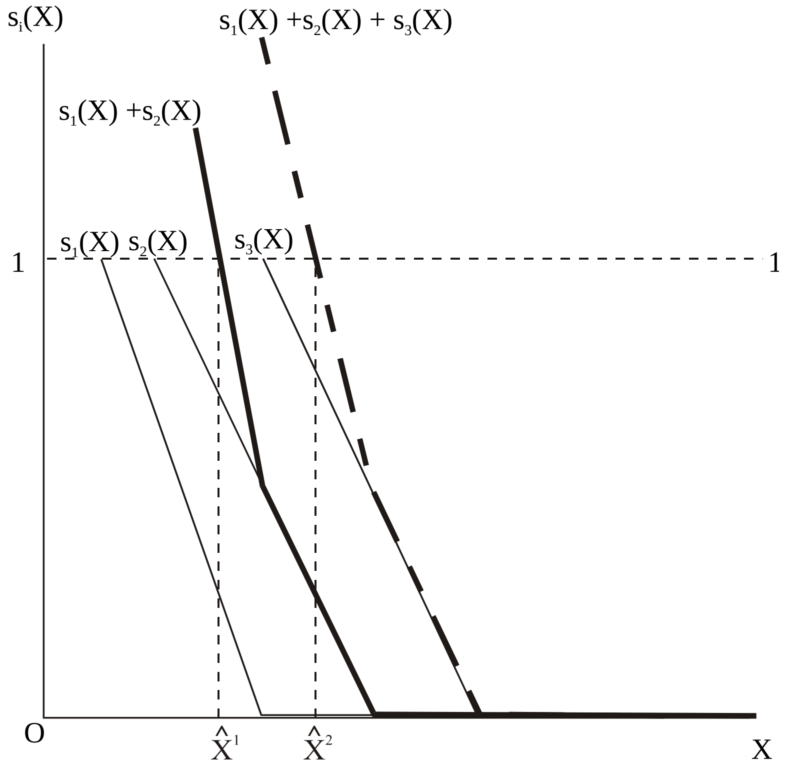

Figure 5:

In the Cournot case, decreasing demand implies that profits strictly decrease with aggregate output, for a given level of firm output. The theorem shows that entry increases output and has an adverse effect on incumbent firms. That a condition such as regularity is needed for such a conclusion was shown by McManus [35], [36].

As a second application of the lemma, we consider the effect of an idiosyncratic change in payoffs to a single player $\hat{\imath} \in I$. This yields two aggregative games, $\mathcal{G}^1$ and $\mathcal{G}^2$, where $\mathcal{G}^k = \left(I^k, \mathbf{w}^k, \{v_i^k\}_{i \in I^k}\right)$ and $w_i^1 = w_i^2, v_i^1 = v_i^2$ for all $i \in I \setminus \{\hat{\imath}\}$. It is convenient to write L_i^k for the graph, in the (x_i, X)-plane, of best responses in $\mathcal{G}^k$, generalizing the notation of Section 2. The next result gives conditions on the change of payoffs for player $\hat{\imath}$ entailing an increase in equilibrium aggregate.

Theorem 4.2 *Suppose (i) that all players in I are regular; (ii) $v_i(x_i, X)$ is strictly increasing [decreasing] in X for all $x_i > 0$ and all $i \in I \setminus \{\hat{\imath}\}$; (iii) $(x^k, X) \in L_{\hat{\imath}}^k$ for $k = 1, 2$ implies that $x^1 \leq x^2$ where this inequality is strict if $x^2 > 0$.*

If $\mathcal{G}^1$ has a (unique) Nash equilibrium non-null $\widehat{\mathbf{x}}^1$, there is an equilibrium $\widehat{\mathbf{x}}^2$ of $\mathcal{G}^2$. Supposing $\widehat{x}_{\hat{\imath}}^2 > 0$ and writing $\widehat{X}^k = \sum_{j \in I^k} \widehat{x}_j^k$,

1. *$\widehat{X}^2 > \widehat{X}^1$,*
2. *players inactive in $\mathcal{G}^1$ are inactive in $\mathcal{G}^2$,*
3. *players other than $\hat{\imath}$, active in $\mathcal{G}^1$, are better [worse] off in $\mathcal{G}^2$ than in $\mathcal{G}^1$.*

   If the game has decreasing {increasing} best responses,

4. *$\widehat{x}_i^2 \leq \{\geq\} \widehat{x}_i^1$ with strict inequality if $\widehat{x}_i^1 > 0$, for $i \in I \setminus \{\hat{\imath}\}$,*
5. *$\widehat{x}_{\hat{\imath}}^2 > \widehat{x}_{\hat{\imath}}^1$.*

Proof. Regularity implies that all players have share functions, which are the same in both games for all players in $I \setminus \{\hat{\imath}\}$. By (iii), $\overline{X}_{\hat{\imath}}^2 \geq \overline{X}_{\hat{\imath}}^1$ and, if $\underline{X}_{\hat{\imath}}^2 \leq X < \overline{X}_{\hat{\imath}}^2$, then $s_{\hat{\imath}}^1(X) < s_{\hat{\imath}}^2(X)$, implying

$$\sum_{j \in I \setminus \{\hat{\imath}\}} s_j^1(X) + s_{\hat{\imath}}^1(X) < \sum_{j \in I \setminus \{\hat{\imath}\}} s_j^2(X) + s_{\hat{\imath}}^2(X)$$

so that $\widehat{X}^2 > \widehat{X}^1$. Parts 2, 3 and 4 are proved as in Theorem 4.1. Part 1 implies that the strategy of at least one player must increase in $\mathcal{G}^2$. In a submodular game, it follows from Part 4 that this player must be $\hat{\imath}$, proving Part 5. When the game is supermodular (has increasing replacement functions), Part 5 follows from:

$$\widehat{x}_{\hat{\imath}}^2 = r_{\hat{\imath}}^2\left(\widehat{X}^2\right) > r_{\hat{\imath}}^1\left(\widehat{X}^2\right) > r_{\hat{\imath}}^1\left(\widehat{X}^1\right) = \widehat{x}_{\hat{\imath}}^1,$$

where the first inequality follows from $s_{\hat{\imath}}^1 < s_{\hat{\imath}}^2$ and the second from Proposition 3.2. ∎

We leave the reader to confirm how the shift in an individual's share function leads to a shift in the aggregate share function and henc a change in the equiibrium value of X. The geometric condition that $L_{\hat{\imath}}^2$ lies above $L_{\hat{\imath}}^1$ is equivalent to the requirement that the best responses of Player $\hat{\imath}$ are higher in $\mathcal{G}^2$ than $\mathcal{G}^1$. In the Cournot game of Section 3, this is equivalent to a reduction in marginal costs, in the sense that, for any $x > 0$,

$$\delta^k \in \Delta c_{\hat{\imath}}^k (x) \Longrightarrow \delta^2 < \delta^1, \tag{10}$$

where $c_{\hat{\imath}}^k$ is the cost function in $\mathcal{G}^k$ for $k = 1, 2$. Then,

$$\left(x^k, X\right) \in L_{\hat{\imath}}^k \Longleftrightarrow p(X) + x^k p'(X) \in \Delta c_{\hat{\imath}}^k \left(x^k\right)$$

and we must have $x^2 \geq x^1$, where this inequality is strict if $x^2 > 0$. Suppose, to the contrary, we had $x^1 > x^2 \geq 0$. Let $\delta^0 \in \Delta c_{\hat{\imath}}^2 (x^1)$, then we would have

$$p(X) + x^2 p'(X) < \delta^0 < p(X) + x^1 p'(X),$$

The first inequality follows from the fact that $\Delta c_{\hat{\imath}}^2$ is strictly increasing and the second from (10). These inequalities would contradict $p'(X) < 0$. Further, $x^1 = x^2 > 0$ also leads to contradiction, in this case of (10). This justifies (iii).

Theorem 4.2 applies only to a change in payoffs of a single player. Obviously, the theorem may be applied cumulatively to changes in the payoffs of a proper subset of players. In some applications, we may wish to analyze a change in all payoffs. For example, an increase in costs in an input market or imposition of a tax may lead to an increase in average and marginal costs for all firms. In general, consider a change in all payoffs in a game in which all players are regular (in both games) and, for all $i \in I$, we have $\left(x^k, X\right) \in L_i^k$ for $k = 1, 2$ implies that $x^1 \leq x^2$ and that this inequality is strict if $x^2 > 0$. Repeated application of Part 1 of the theorem shows us that equilibrium X increases[17]. In general, we are unable to sign changes in individual strategies except in the case of increasing best responses, where we can conclude that all strategies increase.

5 The many-player limit

A well known feature of Cournot oligopoly is the competitive limit. When there are n identical firms and marginal costs are positive, the industry output of the n-player game increases in n and approaches the competitive level

[17]Condition (ii) is only needed for signing the change in payoffs of players whose payoffs do not change. There are no such players in this example and so we do not need to include this condition.

as $n \longrightarrow \infty$. Such a result extends to aggregative games, provided the (common) dropout value is finite. As n increases, the share function moves clockwise about the dropout point, approaching a vertical line as $n \longrightarrow \infty$. It follows that the dropout value is the limiting equilibrium value of the aggregate X. Figure 6 indicates how the aggregate share function rotates, and the equilibrium value of X approaches the common dropout value, as the number of identical players increases.

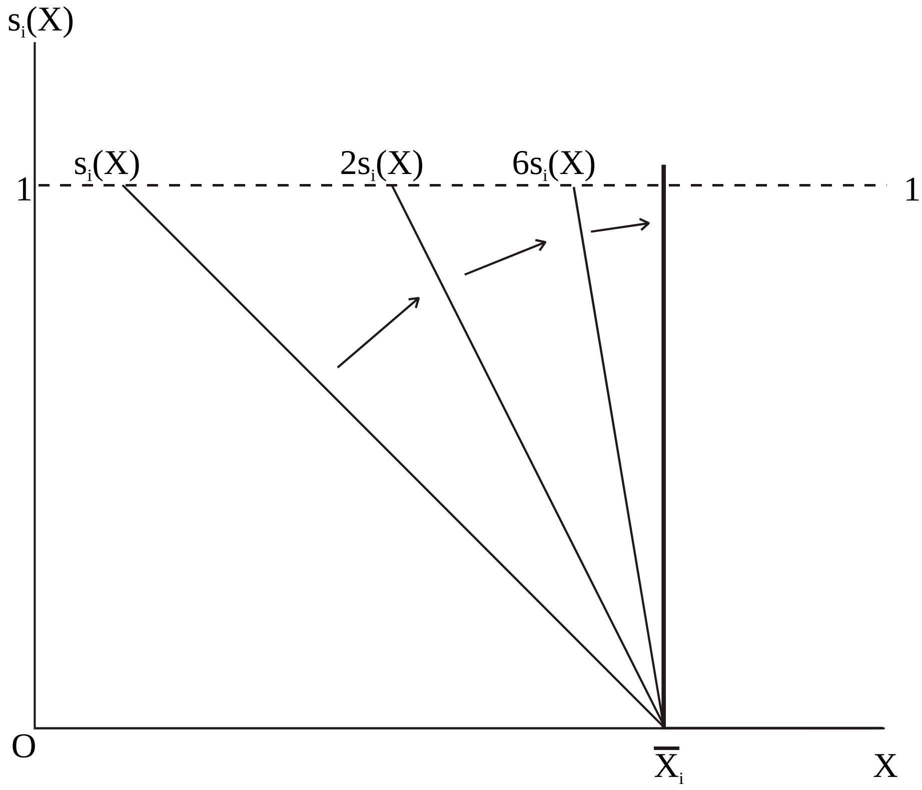

Figure 6:

We can extend this conclusion to non-identical players by considering an infinite sequence $\mathcal{S} = (i_1, i_{2,\ldots})$ of regular players drawn from a finite set T of types of regular player with finite dropout value. All players of type $t \in T$ have the same strategy set and payoff function, which we write $\left[0, w_{(t)}\right]$ and $v_{(t)}(x, X)$, respectively. Suppose that there are $n_t(n)$ players of type t in the first n members of $\mathcal{S}$, where $\sum_{t \in T} n_t(n) = n$, and write $\mathcal{G}^n$ for the game played by the first n members of this sequence: $\left(\{i_k\}_{k=1}^n, (w_1, \ldots, w_n), \{v_k\}_{k=1}^n\right)$ and $\widehat{X}^n$ for its equilibrium aggregate value. Note that Theorem 4.1 implies that $\widehat{X}^{n+1} \geq \widehat{X}^n$ and the inequality is strict if the dropout value for Player $n+1$ exceeds $\widehat{X}^n$. Our aim is to study the sequence $\left\{\widehat{X}^n\right\}$ as $n \longrightarrow \infty$.

Write $\overline{X}_{(t)}$ for the dropout value of players of type $t \in T$ and

$$X^* = \max_{t \in T} \overline{X}_{(t)}. \tag{11}$$

Suppose that $n_{\widehat{t}}(n) \longrightarrow \infty$ as $n \longrightarrow \infty$ for some type $\widehat{t}$ achieving the maximum in (11). We shall show that $\widehat{X}^n \longrightarrow X^*$. To see this, write $s_{(t)}(X)$ for the share function corresponding to $v_{(t)}(x, X)$ and note that, if type $t' \neq \widehat{t}$ has smaller dropout value: $\overline{X}_{(t')} < X^*$, then $s_{(\widehat{t})}\left(\overline{X}_{(t')}\right)$ is positive by Proposition 3.1. It follows that, if $n_{\widehat{t}}(n) > 1/s_{(\widehat{t})}\left(\overline{X}_{(t')}\right)$, then $\widehat{X}^n > \overline{X}_{(t')}$ by Theorem 3.6 and another application of the proposition. We may conclude that for all large enough $n_{\widehat{t}}(n)$, the only active types are those achieving the maximum in (11) and the argument of the first paragraph allows us to conclude that $\widehat{X}^n \longrightarrow X^*$.

It follows from Proposition 3.1 that $s_{(t)}\left(\widehat{X}^n\right)$ is either equal to or approaches zero for all types $t \in T$ and hence the same is true for equilibrium individual strategies. If the payoff to a zero strategy is itself zero, continuity allows us to deduce that payoffs go to zero in the many-player limit. In the Cournot case, more is true: total profit made by all firms goes to zero. The next theorem gives conditions under which this holds for a general aggregative game as well as summarizing the previous discussion.

Theorem 5.1 *Define $\mathcal{S}$ and $\left\{\widehat{X}^n\right\}_{n=1}^{\infty}$ as above and suppose that $n_{\widehat{t}}(n) \longrightarrow \infty$ as $n \longrightarrow \infty$ for some type $\widehat{t}$ achieving the maximum in (11). Then,*

1. $\widehat{X}^n \longrightarrow X^*$,
2. *if $v_{(t)}(0, X) = 0$ for all $X > 0$ and all $t \in T$, then $v_i\left(\widehat{x}_i^n, \widehat{X}^n\right) \longrightarrow 0$, where $\widehat{x}_i^n$ is the equilibrium strategy of Player i in $\mathcal{G}^n$,*
3. *if, in addition, $v_{(t)}(x, X)/x$ has a finite limit as $x \longrightarrow 0^+$ for all $X > 0$ and all $t \in T$, then*

$$\sum_{i=1}^{n} v_i\left(\widehat{x}_i^n, \widehat{X}^n\right) \longrightarrow 0.$$

Proof. We have established the first two parts in the preamble and it only remains to prove Part 3. First observe that by Part 1. we have

$X^*/2 < \widehat{X}^n < 3X^*/2$ for all large enough n. Furthermore, for any $t \in T$, Dini's theorem asserts that the convergence of $v_{(t)}(x, X)/x$ is uniform in $X \in [X^*/2, 3X^*/2]$ and therefore the limit is continuous. Furthermore, if $X > X^*$ and $x < X - X^*$, we have $s_{(t)}(X - x) = 0$ which says that $0 \in B_i(X - x)$. In particular,

$$v_{(t)}(x, X) \leq v_{(t)}(0, X - x) = 0,$$

which implies, by continuity of payoffs,

$$\lim_{x \longrightarrow 0^+} \frac{v_{(t)}(x, X^*)}{x} \leq 0.$$

It follows readily from these observations that, given any $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\|(x, X) - (0, X^*)\| < \delta \Longrightarrow \frac{v_{(t)}(x, X)}{x} < \frac{\varepsilon}{X^*} \text{ for } t = 1, \ldots, T,$$

where $\|\cdot\|$ denotes Euclidean norm.

Since $s_{(t)}(X)X \longrightarrow 0$ as X approaches X^* from below, there is an X' such that $s_{(t)}(X)X < \delta/\sqrt{2}$ if $X' < X < X^*$. Let $X'' = \max\left\{X', X^* - \delta/\sqrt{2}\right\}$. By construction, if $X'' < X < X^*$, then $\left\|\left(s_{(t)}(X)X, X\right) - (0, X^*)\right\| < \delta$. Choose N so that $\widehat{X}^N \geq X''$. Then $n \geq N$ implies $X'' < \widehat{X}^n < X^*$ and therefore,

$$\begin{aligned}
\sum_{m=1}^{n} v_m\left(s_m\left(\widehat{X}^n\right)\widehat{X}^n, \widehat{X}^n\right) &= \sum_{t=1}^{T} n_t(n)\, v_{(t)}\left(s_{(t)}\left(\widehat{X}^n\right)\widehat{X}^n, \widehat{X}^n\right) \\
&< \sum_{t=1}^{T} n_t(n) \frac{\varepsilon}{X^*} s_{(t)}\left(\widehat{X}^n\right)\widehat{X}^n \\
&= \sum_{m=1}^{n} s_m\left(\widehat{X}^n\right) \frac{\widehat{X}^n}{X^*}\varepsilon \leq \varepsilon.
\end{aligned}$$

This establishes the claimed limit. ■

Summing over several types, as in Part 3, is only meaningful when payoffs are cardinal and comparable, such as profits in oligopoly or expenditures in risk-neutral rent seeking. In the latter case, with linear production functions, the sum of payoffs is equal to the value of the rent net of expenditure on rent seeking. Part 3 allows us to deduce that rent-seeking dissipates almost all the rent in a large, asymmetric contest. Even if payoffs are ordinal, we can conclude that, in an obvious extension of notation, $n_t(n)\, v_{(t)}\left(\widehat{x}^n_{(t)}, \widehat{X}^n\right) \longrightarrow 0$. Thus, under the assumptions of Part 3, payoffs of players of type t approach zero faster than $1/n_t(n)$.

It is not necessary to have $n_t(n) \longrightarrow \infty$ for *all* $t \in T$ to draw the conclusions in Theorem 5.1. Indeed, the argument preceding the theorem establishes the stated limits provided that $n_t(n) \longrightarrow \infty$ for at least one type t achieving the maximum in (11). However, if even this fails, it is straightforward to construct examples where none of the conclusions holds. For example, if, say, the first player in $\mathcal{S}$ has dropout point X^* and all other players have dropout point $X^\dagger < X^*$, then it is straightforward to see that $\widehat{X}^n \longrightarrow X^\dagger$ as $n \longrightarrow \infty$.

One way of generating the sequence $\mathcal{S}$ is by making independent random choices from T according to some arbitrary distribution. Then the conditions of Theorem 5.1 hold with probability one. Indeed, Part 1. of the theorem continues to hold even if T is infinite, provided the distribution of dropout points is almost surely bounded above. Indeed, let

$$X^* = \operatorname*{ess\,sup}_{t \in T} \overline{X}_{(t)} < \infty.$$

That is, X^* is the least value of X which satisfies $\Pr\left\{t : \overline{X}_{(t)} \le X\right\} = 1$. In this case, also, X^* can be viewed as the competitive limit.

Proposition 5.1 *Let $\left\{\widehat{X}^n\right\}_{n=1}^{\infty}$ denote the sequence of equilibrium aggregate expenditures of the sequence of games $\{\mathcal{G}^n\}_{n=1}^{\infty}$, described above. With probability one, $\widehat{X}^n \longrightarrow X^*$ as $n \longrightarrow \infty$.*

Proof. We can apply Theorem 4.1 to deduce that $\left\{\widehat{X}^n\right\}_{n=1}^{\infty}$ is a non-decreasing sequence and Lemma 3.1 to deduce that, with probability one, the aggregate share function of $\mathcal{G}^n$ is zero for $X \ge X^*$. Hence, $\widehat{X}^n \longrightarrow X^U$, for some $X^U > 0$, as $n \longrightarrow \infty$ and the probability that $X^U \le X^*$ is one. The proof is completed by showing that, with probability one, the aggregate share function at any $X < X^*$ exceeds unity for all large enough n.

In conjunction with the probability distribution over T, the function which maps the type $t \in T$ into the value of the share function at X, that is $t \longmapsto s_{(t)}(X)$ defines a random variable $S(X)$. Furthermore, the value of the aggregate share function of $\mathcal{G}^n$ at X is $S_1 + \cdots + S_n$, where each S_m is an independent copy of $S(X)$. The definition of X^* implies that $X < \overline{X}_{(t)} < X^*$ with positive probability and for all such t Lemma 3.1 implies that $s_{(t)}(X) > 0$. Hence, $S(X)$ is a non-negative random variable which satisfies $\Pr\{S(X) > 0\} > 0$. It follows that, with probability one, there is n' such that $S_1 + \cdots + S_n \ge 1$ for all $n \ge n'$, as required. ■

6 Applications

In this section, we introduce five further applications chosen to illustrate the application of the aggregate and radial crossing conditions. In each case, we give conditions for regularity and thus for the existence of a unique equilibrium. We also briefly discuss comparative statics and the competitive limit, where appropriate.

6.1 Search games

We consider a version of the "coconut economy" search game introduced by Diamond [22] which omits production and is also discussed by Milgrom and Roberts [37] and Dixon and Somma [25]. Each player i in the set of players I exerts effort x_i in searching for trading partners. Search incurs a benefit which is proportional both to own effort and to the aggregate effort exerted by the other players as well as a cost described by a cost function c_i. The payoff function take the form:

$$\pi_i(\mathbf{x}) = \theta x_i \left(\lambda_i + \sum_{j \neq i} x_j \right) - c_i(x_i),$$

where $\theta > 0$ is a parameter scaling the overall return to search and $\lambda_i \geq 0$ represents a payoff from search effort without meeting a trading partner[18].

Example 6.1 (Search) *Player i's strategy set*[19] *is* $[0, w_i]$ *and payoff is:*

$$v_i(x_i, X) = \theta x_i (\lambda_i + X - x_i) - c_i(x_i).$$

Much of the interest in such games lies in their multiple equilibria and the consequent coordination problems. If $\lambda_i = 0$ for all i, $\widehat{\mathbf{x}} = \mathbf{0}$ is an equilibrium. Here, we focus on unique non-null equilibria, which, if at least one λ_i is positive, will be the unique equilibrium. It is readily checked that this game is supermodular which guarantees existence of an equilibrium as well as monotone comparative statics. We include the game here to illustrate the fact that supermodularity is not inconsistent with regularity and to show that multiple non-null equilibria require marginal costs not to increase too fast. Indeed, we show that if the marginal cost at x increases faster than x there will be a unique non-null equilibrium. Specifically, we impose the following condition on Player i.

[18] Perhaps from finding coconuts lying on the ground.

[19] In the original game, strategy sets were unbounded.

EA The cost function c_i is continuous, differentiable for positive argument and $c_i'(x)/x$ is a positive, strictly increasing function of $x > 0$.

If, for example, $c_i = kx^\alpha$, where $k > 0$, then **EA** is satisfied if and only if $\alpha > 2$.

Proposition 6.1 *If **EA** holds for Player i in the Search game, Example 6.1, then i is regular. The dropout point is positive if and only if $\lambda_i > 0$. If $\lambda_i = 0$,*

$$\overline{\sigma}_i = \left[1 + \lim_{x \longrightarrow 0^+} \frac{c_i'(x)}{\theta x}\right]^{-1}. \tag{12}$$

Proof. Assumption **EA** implies that $c_i'(x)$ is strictly increasing which implies that w_i is the best response to X_{-i} for $X_{-i} \geq X_i^w$, where

$$X_i^w = \max\left\{\frac{c_i'(w_i) - \lambda_i}{\theta}, 0\right\}.$$

The best response to $X_{-i} = 0$ is positive[20] (and equal to the dropout point) if $\lambda_i > 0$.and is $x_i = 0$ if $\lambda_i = 0$. Since Assumption **EA** also implies that $c_i'(x) \longrightarrow 0$ as $x \longrightarrow 0^+$, the (interior) best response x_i to any X_{-i} in the interval $(0, X_i^w)$ satisfies $c_i'(x_i) = \theta(\lambda_i + X_{-i})$, which we can rewrite:

$$X = x_i \left[1 + \frac{c_i'(x_i) - \lambda_i}{\theta x_i}\right]. \tag{13}$$

Since c_i' is strictly increasing, we may conclude that best responses are unique. Furthermore, the right hand side of (13) is increasing in x_i which shows that (13) has at most one solution for any $X > 0$. Note also that, if (13) holds for some $x_i \in (0, w_i)$, then $X < w_i + X_i^w$, so $(w_i, X) \notin L_i$. Similarly, if $X \geq w_i + X_i^w$,

$$X \geq w_i + \frac{c_i'(w_i) - \lambda_i}{\theta} > x_i \left[1 + \frac{c_i'(x_i) - \lambda_i}{\theta x_i}\right]$$

for any $x_i < w_i$, so (13) cannot hold. These observations establish ACC. Similarly, for $x_i = \sigma X$ and $\sigma \in (0, 1)$,

$$\frac{1}{\sigma} = 1 + \frac{c_i'(\sigma X)}{\theta \sigma X} \tag{14}$$

[20]**EA** implies that $c_i'(x) \longrightarrow 0$ as $x \longrightarrow 0$, so marginal payoff approaches λ_i.

and, by Assumption **EA** this equation can have at most one solution in $X > 0$. This is RCC and completes the proof of regularity. ■

Together with Theorem 3.6, the following proposition shows that, if **EA** holds for all players, then there is a unique non-null equilibrium, except possibly if $\lambda_i = 0$ for all i. In the latter case, we also require $\sum_{i \in I} \overline{\sigma}_i > 1$, where $\overline{\sigma}_i$ satisfies (12).

The existence of a unique equilibrium remains valid if the strategy space is changed to $\mathbb{R}_+$, provided that we strengthen **EA** by requiring that $c_i'(x)/x$ be unbounded above and the share function vanishes asymptotically. But, if $s_i(X) \nrightarrow 0$ as $X \longrightarrow \infty$, we would have $X^n s_i(X^n) \longrightarrow \infty$ on some sequence $(X^1, X^2, \dots)$. Since (14) must hold with $\sigma = s_i(X)$, taking the limit on the sequence in (14) would lead to a contradiction with the unboundedness of $c_i'(x)/x$. Note that, if any player has unbounded strategy space, the existence of equilibria of a supermodular game is no longer a direct application of Tarski's theorem. Nevertheless, existence can be deduced from Theorem 3.6 (provided there are enough players), for (14) implies that $\overline{\sigma}_i = \lim_{x \longrightarrow 0^+} [\theta x / c_i'(x)]$ and (5) gives a sufficient condition for existence. In particular, if $c_i'(x)/x$ approaches zero for any i, the game will have a unique equilibrium.

We also note that v_i is strictly increasing in X for all $x_i > 0$ so, by Theorem 4.1, additional searchers lead to increased search effort by existing searchers and an improvement in their payoffs.

6.2 Smash-and-Grab games

Recently Bliss [5] has discussed Smash-and-Grab problems in which expected-utility maximizing players undertake an activity (such as burglary) which receives a payoff with probability less than one. Increasing the intensity of the activity increases the potential payoff, but reduces the probability of receiving that payoff.

In the strategic version, Player i selects intensity $x_i (\leq w_i)$ which results in a payoff $u_i(x_i)$ with a probability that depends on the full strategy profile, $\mathbf{x}$. Otherwise, the player receives reservation payoff 0 (gets caught). We focus on the case in which the probability of receiving a payoff depends only on the sum of the intensities chosen by all players and write this probability $h_i(X)$. (The case where h_i is an additively separable can also be handled by a simple transformation.) As with the Cournot application, kinks where h_i becomes zero can be handled by permitting negative values since this does not affect the set of Nash equilibria provided u_i is increasing and h_i is decreasing, as we shall subsequently assume. All payoffs are measured in units of expected utility.

Example 6.2 (Smash and Grab) *Player i's strategy set is $[0, w_i]$ and payoff is expected utility:*

$$\pi_i(\mathbf{x}) = v_i(x_i, X) = u_i(x_i)\, h_i(X).$$

We will apply the following conditions.

SA For $i \in I$,

(i) u_i is continuous, satisfies $u_i(0) > 0$ and is strictly increasing and strictly concave in $x_i \geq 0$;

(ii) h_i is continuous and satisfies $h_i(0) = 1$ and $h_i\left(\sum_{j \neq i} m_j\right) > 0$; for $X > 0$, h_i is continuously differentiable, log concave and satisfies $h_i'(X) < 0$.

Condition (i) specifies that players prefer no reward to getting caught, find the activity profitable and are risk averse. Condition (ii) is satisfied if $h_i(X) = A_i \exp\{-B_i X\}$ or h_i is linear. In the latter case, the graph of h_i must reach the axis beyond $\sum_{j \neq i} m_j$. This condition is necessary, for, if it failed for all players, any strategy profile with $X > \sum_{j \neq i} m_j$ for all i would be an equilibrium.

Proposition 6.2 *If **SA** holds for a player in the Smash and Grab game, Example 6.2, then that player is regular.*

Proof. Since we are examining pure strategy Nash equilibria, we can apply a strictly increasing transformation to payoffs without changing the set of equilibria. Using a logarithmic transformation, we can take the payoff as

$$\ln[u_i h] = \ln u_i(x_i) + \ln h_i(X)$$

and note that **SA** implies that $\ln u_i$ is strictly concave and $\ln h$ is concave. We conclude that best response sets are convex. Furthermore, $(x_i, X) \in L_i$ if and only if

$$-\frac{h_i'(X)}{h_i(X)} \in \Delta \ln u_i(x_i) \tag{15}$$

where $\Delta f(x)$ denotes the set of slopes of supporting lines to f at x. Strict concavity of $\ln u_i$ implies that $\Delta \ln u_i(x_i)$ are disjoint for distinct x_i. This establishes both ACC and RCC. ■

Since payoffs are continuous, existence is assured, so Theorem 3.6 asserts the existence of a unique equilibrium. Further, **SA** implies that v_i is a strictly decreasing function of X, for given x_i and (log-concavity of h_i, in particular) that the game is submodular. We may conclude from Theorem 4.1, that extra players increase equilibrium aggregate intensity, whilst reducing the individual intensities and payoffs of existing players.

6.3 Public good contribution games

Our next application is the classic problem of voluntary subscription to the provision of a public good. Cornes et al. [9] provide a recent discussion of this model. A set I of consumers has to decide non-cooperatively what quantity of a public good to provide. Consumer $i \in I$ chooses how much, x_i, of her income m_i to devote to a public good. Preferences are represented by an ordinal utility function $u_i(y_i, X)$ where y_i is expenditure on private consumption and X is total expenditure on the public good.

Example 6.3 (Pure Public Goods) *Player i's strategy set is* $[0, m_i]$ *and payoff is utility:*

$$\pi_i(\mathbf{x}) = v_i(x_i, X) = u_i(m_i - x_i, X) \text{ \textit{when} } X > 0$$

and $\pi_i(\mathbf{0}) = v_i(0,0) = 0$.

The following is a generalization of a well-known condition.

PA Player $i \in I$ has continuous, strictly increasing preferences and the equal-price income expansion paths is upwards sloping.

PA is most readily exploited in terms of the set of (absolute values of) the marginal rate of substitution which we denote by $MRS_i(y, X)$. [That is, the set of slopes of supporting lines (with X on the horizontal axis) to the upper preference set at (y, X).] In particular, if $1 \in MRS_i(y, X)$ and $\delta' \in MRS_i(y', X')$, where $y, X > 0$, we require $\delta' \leq 1$ if $X' \geq X, y' \leq y$ and $\delta' \geq 1$ if $X' \leq X, y' \geq y$, with strict inequality in both cases if $(y', X') \neq (y, X)$. This requirement is implied by, but weaker than both goods being normal.

Proposition 6.3 *If **PA** holds for a player in the game Public Good Contribution game, Example 6.3, then that player is regular.*

Proof. We have $(x_i, X) \in L_i$ if and only if

$$1 \in MRS_i\,(m_i - x_i, X) \tag{16}$$

and x_i is a best response to X_{-i} if and only if (16) holds with $X = x_i + X_{-i}$. The discussion above shows that multiple best responses are not possible and also verifies ACC. Note also that $(\sigma X, X) \in L_i$ if and only if $1 \in MRS_i\,(m_i - \sigma X, X)$, which, if $\sigma > 0$, can hold for at most one value of X, verifying RCC. ■

Since payoffs are continuous, existence of an equilibrium is assured, which, therefore, is unique. Further, **PA** implies that v_i is a strictly increasing function of X, for given x_i and it follows from (16) that best responses are decreasing. By Theorem 4.1, additional contributions are offset by a reduction in current contributions, but not enough to reduce total public good provision. Consequently, current players, even non-contributors, are made better off. These results reflect the standard notions of free and easy riding discussed in Cornes and Sandler [17] for example.

6.4 Sharing Games

Next, we consider an example of production and cost sharing which generalizes a number of situations such as joint exploitation of a resource with common access and jointly cleaning up pollution. For definiteness, we suppose that costs and output are divided in proportion to input. Other sharing rules are possible many of which also lead to aggregative games. Watts [53] and Cornes and Hartley [10] analyze examples in which one of the functions F or C is an identity. Moulin [40] includes a wide-ranging discussion of models of this type.

A good is jointly produced by a group: I. Player $i \in I$ contributes $x_i \geq 0$ to a productive activity and receives a quantity f_i of the output and faces a personal cost c_i. Preferences are reflected in a utility function $u_i\,(f_i, c_i)$. Total output and cost depend on total contribution, X. Specifically,

$$\sum_{j\in I} f_j = F\,(X) \text{ and } \sum_{j\in I} c_j = C\,(X)\,,$$

where F and C are joint production and cost functions. We suppose nonparticipants obtain no benefit: $\pi_i\,(\mathbf{0}) = u_i\,(0,0)$. We shall assume that output and cost are shared in proportion to contributions, though other sharing rules can be analyzed by the same methods.

Example 6.4 (Joint Production with proportional shares) *Player i's strategy set is $\mathbb{R}_+$ and payoff:*

$$\pi_i(\mathbf{x}) = v_i(x_i, X) = u_i\left(\frac{x_i}{X}F(X), \frac{x_i}{X}C(X)\right) \text{ when } X > 0$$

and $\pi_i(\mathbf{0}) = v_i(0,0) = u_i(0,0)$, where $u_i(f,c)$ is the utility function.

We will apply the following conditions, generalizing Watts [53].

JA (i) The utility function of Player i is strictly quasi-concave, strictly increasing in f_i, strictly decreasing in c_i and represents binormal preferences. There is $w_i > 0$ for which i is indifferent between $(0,0)$ and $(F(w_i), C(w_i))$.

(ii) The production function is continuous, satisfies: $F(0) = 0$, twice differentiable and satisfies $F'(X) > 0$, $F''(X) \leq 0$ in $X > 0$.

(iii) The cost function is continuous, satisfies: $C(0) = 0$, twice differentiable and satisfies $C'(X) > 0$, $C''(X) \geq 0$ in $X > 0$.

Our interpretation of binormality is best explained letting $MRS_i(f,c)$ denote the set of marginal rates of substitution (slopes of supporting lines to the upper preference set, with f on the horizontal axis). Binormality states that, if $\delta \in MRS_i(f,c)$ and $\delta' \in MRS_i(f',c')$, where $f, c > 0$, then[21] $(f',c') > (f,c)$ implies $\delta' < \delta$. This is equivalent to the assumption that income expansion paths are downward sloping.

Under **JA**, w_i can be taken as the upper bound on the strategy space for Player i. To see this, first note that, if $x_i > w_i$, then

$$u_i(F(x_i), C(x)) < u_i(0,0). \tag{17}$$

For, (ii) and (iii) imply $F(w_i) \geq w_i F(x_i)/x_i$ and $C(w_i) \leq w_i C(x_i)/x_i$, so that, if (5.1) were untrue, (i) and the equation

$$\left(\frac{w_i}{x_i}F(x_i), \frac{w_i}{x_i}C(x_i)\right) = \frac{w_i}{x_i}(F(x_i), C(x_i)) + \left(1 - \frac{w}{x_i}\right)(0,0)$$

would imply $u_i(F(w_i), C(w_i)) > u_i(0,0)$, contradicting the definition of w_i. It follows from (5.1) that any $x_i > w_i$ is dominated by $x_i = 0$. To see this, observe that, for any $X \geq x_i$, we have $x_i F(X)/X \leq F(x_i)$ and $x_i C(X)/X \geq C(x_i)$ and therefore

$$u_i\left(\frac{x_i}{X}F(X), \frac{x_i}{X}C(X)\right) \leq u_i(F(x_i), C(x)) < u_i(0,0).$$

[21] Weak vector inequalities are interpreted component-wise and $\mathbf{x} > \mathbf{y}$ means $\mathbf{x} \geq \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$.

Proposition 6.4 *If **JA** holds for a player in the Joint Production Game, Example 6.4, then that player is regular.*

Proof. We have $(x_i, X) \in L_i$ if and only if

$$\frac{MC\,(x_i/X, X)}{MF\,(x_i/X, X)} \in MRS_i\left(\frac{x_i}{X}F\,(X)\,,\frac{x_i}{X}C\,(X)\right), \tag{18}$$

where MC [MC] are convex combinations of the marginal and average product [cost]:

$$\begin{aligned} MC\,(\theta, X) &= \theta C'\,(X) + (1-\theta)\,AC\,(X)\,, & (19)\\ MF\,(\theta, X) &= \theta F'\,(X) + (1-\theta)\,AF\,(X)\,, & (20) \end{aligned}$$

where $AC\,(X) = C\,(X)\,/X$ and $AF\,(X) = F\,(X)\,/X$. Note that an increase in θ decreases MC, since the marginal cost exceeds the average cost under (ii). Similarly, an increase in X increases MC since both marginal and average costs fall under (ii). For MF, similar arguments show the changes to be in the opposite direction.

To study best responses, we hold X_{-i} fixed and observe that an increase in x_i decreases the first component and increases the second component of both MC and MF and therefore increases MC, decreases MF and decreases their ratio. Furthermore, $x_iF\,(X)\,/X$ the correspondence mapping x_i to the right hand side of (18) is strictly decreasing in x_i. It follows that (18) has at most one solution; in particular, best responses are convex.

To verify ACC, we now hold X fixed and observe that the left hand side of (18) is strictly increasing in x_i. (The numerator is increasing and the denominator decreasing.) As before, the right hand side is a decreasing correspondence and (18) has at most one solution.

Finally, we observe that $(\sigma X, X) \in L_i$ if and only if

$$\frac{MC\,(\sigma, X)}{MF\,(\sigma, X)} \in MRS_i\,(\sigma F\,(X)\,, \sigma C\,(X))$$

and holding σ fixed, note that, once again, the left hand side is increasing in X and the right hand side a decreasing correspondence in X. This verifies RCC, completing the proof. ■

Since payoffs are continuous at the origin, existence is assured, so Theorem 3.6 asserts the existence of a unique equilibrium. Furthermore, under **JA**, increasing X decreases average product (since F is concave) and increases average cost (since C is convex). Hence, utility decreases. These conclusions, summarized in the next corollary, generalize the existence and uniqueness results of Watts [53].

Corollary 6.1 *If* $\boldsymbol{SA}$ *holds for all players in the game 6.2, the game has a unique equilibrium. Adding players increases equilibrium* X *and makes existing active players worse off.*

6.5 Contests

Our final application concerns contests for a biddable rent with risk averse contestants in which each player's probability of winning the rent is proportional to some (production) function of their expenditure on rent-seeking. The corresponding game played by risk neutral contestants is strategically equivalent to a Cournot oligopoly model with unit elastic demand, provided production functions are strictly increasing (Vives [52]). However, if some or all players are risk averse this is no longer true. Nevertheless, the game is still aggregative[22] and can be analyzed using the methods described above.

Formally, suppose Player $i \in I$ spends $y_i \geq 0$ on seeking an indivisible rent R which can be won by only one player. The probability that i wins the rent is given by the contest success function:

$$p_i(\mathbf{y}) = \frac{f_i(y_i)}{\sum_{j \in I} f_j(y_j)},$$

where f_i is a strictly increasing function. We assume that Player i is risk averse or risk neutral and has preferences over lotteries described by a von Neumann-Morgenstern utility function u_i. In this example, it is useful to transform the state space by writing $x_i = f_i(y_i)$. Since f_i is strictly increasing, it has an inverse function which we denote g_i.

Example 6.5 (Rent Seeking) *Player i's strategy set is* $[0, f_i(R)]$ *and payoff is expected utility:*

$$\pi_i(\mathbf{y}) = v_i(x_i, X) = \frac{x_i}{X} u_i(R - g_i(x_i)) + \left[\frac{X - x_i}{X}\right] u_i(-g_i(x_i)) \text{ when } X > 0$$

and $\pi_i(\mathbf{0}) = v_i(0,0) = u_i(0)$.

Consider the following condition.

[22] Nitzan [41] gives a useful survey of rent seeking, Hillman and Katz [31] investigate risk aversion and Cornes and Hartley [11] give an exhaustive discussion of the case of many players, each with constant absolute risk aversion.

RA Player i

(i) is either risk averse with constant absolute risk aversion or risk neutral;

(ii) has a continuous, concave production function satisfying $f_i(0) = 0$ and which is differentiable in $x > 0$ and satisfies $f_i'(x) > 0$.

Note that the second part of the condition implies that $g_i(0) = 0$, $g_i'(x) > 0$ for $x > 0$ and that g_i is convex. The first part of the condition requires that $u_i(z) = 1 - \exp\{-\alpha_i z\}$ with $\alpha_i > 0$ or $u_i(z) = z$. Existence and uniqueness in the case when all players are risk neutral was established by Szidarovzsky and Okuguchi[49]. Since this case is strategically equivalent to a Cournot oligopoly game with unit-elastic demand and cost function g_i for Player i, it is covered by the discussion of that game in Section 3.

Proposition 6.5 *If **RA** holds for Player i in the Rent-seeking Game, Example 6.5, then that player is regular. In this case, Player i has a finite dropout value if and only if* $\underline{g}_i' = \inf_{x>0} g_i'(x) > 0$, *in which case the dropout value satisfies* $\overline{X}_i = \beta_i / \underline{g}_i'$, *where*

$$\beta_i = \frac{1 - \exp\{-\alpha_i R\}}{\alpha_i}$$

if $\alpha_i > 0$ *and* $\beta_i = R$ *if* $\alpha_i = 0$.

Proof. When $0 < x_i < X$, a calculation shows that $(x_i, X) \in L_i$ if and only if it is a zero of the function $\widetilde{\gamma}_i$, where

$$\widetilde{\gamma}_i(x_i, X) = \frac{\beta_i (X - x_i)}{X (X - \alpha_i \beta_i x_i)} - g_i'(x_i).$$

Holding X_{-i} fixed, the derivative of the first term with respect to x_i is

$$\frac{\beta_i (X - x_i)}{X^2 (X - \alpha_i \beta_i x_i)^2} [\alpha_i \beta_i (X + x_i) - 2X] < 0,$$

since $x_i < X$ and $\alpha_i \beta_i \leq 1$. Furthermore, g_i' is a strictly increasing function and we may conclude that $\widetilde{\gamma}_i$ is strictly decreasing in x_i. So Player i has convex best responses.

ACC is a consequence of the fact that $\widetilde{\gamma}_i$ is a strictly decreasing function of x_i for $x_i \in (0, X)$. RCC can be verified by observing that $(\sigma X, X) \in L_i$ if and only if

$$\frac{\beta_i (1 - \sigma)}{X (1 - \alpha_i \beta_i \sigma)} - g_i'(\sigma X) = 0$$

and the left hand side is strictly decreasing in X.

To prove the remaining assertions, note that convexity of g_i implies that

$$\underline{g}_i' = \lim_{x \longrightarrow 0^+} g_i(x).$$

Since $\widetilde{\gamma}_i$ is a strictly decreasing function of x_i for given X_{-i}, then $x_i = 0$ is a best response to X_{-i} if and only if

$$\lim_{x \longrightarrow 0^+} \widetilde{\gamma}_i(x, x + X_{-i}) = \frac{\alpha_i \beta_i}{X_{-i}} - \alpha_i \underline{g}_i' \leq 0.$$

This holds for some X_{-i} if and only if $\underline{g}_i' > 0$ and, in that case, it holds when $X_{-i} \geq \beta_i / \alpha_i \underline{g}_i'$. Note that this also establishes the formula for $\overline{X}_i$. ■

For $X > 0$, the share function s_i satisfies

$$\alpha_i X \widetilde{\gamma}_i = 1 - \frac{1 - \alpha_i \beta_i}{1 - \alpha_i \beta_i s_i(X)} - \alpha_i X g_i' [X s_i(X)] = 0.$$

It follows that $\overline{\sigma}_i = \lim_{X \longrightarrow 0^+} s_i(X) = 1$ and, therefore, from Theorem 3.6 that the game has a unique equilibrium, provided there are two or more players.

Furthermore, v_i can be written in the form

$$u_i(-g_i(x_i)) + \frac{x_i}{X} [u_i(R - g_i(x_i)) - u_i(-g_i(x_i))]. \tag{21}$$

Since u_i is strictly increasing, this shows that $v_i(x_i, X)$ is strictly decreasing in X for $x_i > 0$. It follows from Theorem 4.1 that additional contestants make existing active contestants worse off. Note that we cannot, in general, sign the changes in *individual* expenditure, since we do not have monotonic best responses, Indeed, we cannot even conclude that aggregate *expenditure*: $\sum_{i \in I} y_i$ increases. Whilst aggregate X certainly does increase, there is typically no simple mapping, let alone a monotonic function, from X to aggregate expenditure, except when f_i is linear and identical for all i.

Finally, note that Player i has a finite dropout value if and only if $\underline{g}_i > 0$ and, since g_i is the inverse function of f_i, this holds if and only if

$$\underline{f}_i' = \sup_{y > 0} f_i'(y) < \infty. \tag{22}$$

For example, with the transformation function $f_i(y) = c_i y^r$ introduced by Tullock, the dropout value is finite if $r = 1$ but not if $r < 1$. If we normalize the utility function u_i to satisfy $u_i(0) = 0$, then $v_i(0, X) = 0$ for any $X > 0$

and, noting that $u_i(-g_i(x))$ is a concave function of x, taking the value 0 when $x = 0$, we can deduce from (21) that $v_i(x, X)/x$ has a finite limit as $x \longrightarrow 0$ through positive values. Given any $x > 0$, there is $\xi \in (0, x)$ such that

$$\frac{u_i(-g_i(x))}{x} = -u_i'(-g_i(\xi))\, g_i'(\xi),$$

from which we conclude

$$\frac{v_i(x, X)}{x} \longrightarrow \alpha_i \underline{g}_i' - \frac{u_i(R)}{X} \text{ as } x \longrightarrow 0^+.$$

It follows from Theorem 5.1 that, in large games played by a finite set of types satisfying **RA** and (22), the total payoff approaches zero. Further applications of this approach to contests may be found in [11].

7 Weak regularity

Study of the applications above prompts the question of whether regularity is a necessary, as well as sufficient, condition *on individual players* for a unique equilibrium. Clearly, if a player had a share function that was strictly increasing where positive, multiple equilibria would be possible for certain choices of share functions for the other players. Equilibria in the knife-edge case of a share function that was decreasing but not always strictly (where positive) is less clear. Indeed, if a single player has a share function which is non-increasing where positive, but may have horizontal segments of its graph at share values between 0 and 1, and all other players are regular, equilibrium will still be unique. For the share functions of the regular players are strictly decreasing where positive, so the aggregate share function will be strictly decreasing at share value 1 which implies at most one equilibrium. However, if the graphs of two or more players had such horizontal sections, the graph of the aggregate share function could have a horizontal segment with unit share[23], resulting in multiple equilibrium values of X. In Figure 7, the share function of each of three players has a horizontal stretch at the share value of 1/3. In the example shown, there is a continuum of equilibrium values of X. Similarly, it is possible to have vertical sections in the graph of one player (turning the share function into a correspondence) without losing existence and uniqueness. Once again, were two or more players to have a

[23]This requires the equilibrium aggregate to exceed the dropout values of all regular players.

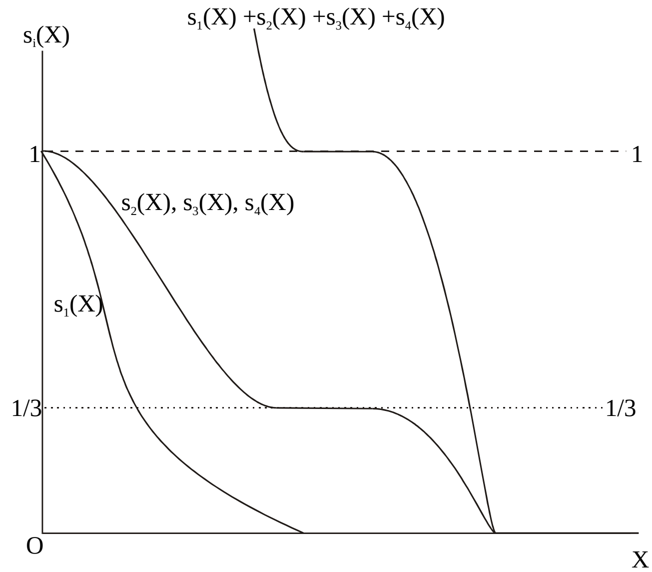

Figure 7:

vertical section at the same value of X, multiple equilibria would be possible, though the equilibrium X would still be unique.

In the remainder of this section, we relate these properties of share functions to geometric properties of the set $L_\imath$ (weak regularity) and then examine necessity of these properties for existence and uniqueness of equilibrium. In particular, we show that existence of a unique equilibrium is assured if one player is weakly regular and the rest are regular. Furthermore, no further weakening of these assumptions is possible, where such assumptions impose restrictions solely on the payoffs of individual players.

Definition 7.1 *Player i is weakly regular if*

1. *$B_i\left(X_{-i}\right)$ is a singleton for all $X_{-i} > 0$ and $B_i\left(0\right)$ is either a singleton or empty,*
2. *the set$\{x : (x, X) \in L_i\}$is convex for all $X > 0$,*
3. *the set $\{X : (\sigma X, X) \in L_i\}$ is convex for all $\sigma \in (0, 1]$.*

Since we count empty sets as convex, it follows from Lemma 3.2 that a regular player is also weakly regular. For weakly regular players, we can define a convex-valued *share correspondence* for any $X > 0$ by

$$S_i\left(X\right) = \left\{\frac{x}{X} : (x, X) \in L_i\right\}. \tag{23}$$

Note that S_i has a closed graph[24], except possibly at $X = 0$, and therefore Corollary 3.1 holds. It follows[25] that, if $S_i(X^0) \neq \varnothing$ and $X' > X^0$, then $S_i(X') \neq \varnothing$. Indeed, this is essentially the same argument used to prove that the domain of the replacement function is a semi-infinite (to the right) interval. Furthermore, we shall show that S_i is *decreasing* in the sense that there is at most one value of X satisfying $S_i(X) = 1$ and

$$\sigma^1 \in S_i\left(X^1\right), \sigma^2 \in S_i\left(X^2\right), X^2 > X^1 \Longrightarrow \sigma^2 \leq \sigma^1.$$

Note that, if Player i is weakly regular, either $v_i(X, X)$ is maximized at $\underline{X}_i > 0$, or (ii) $v_i(X, X)$ has no maximum in $X > 0$. As above, we refer to $\underline{X}_i$ as the participation value.

Proposition 7.1 *Weak regularity is a necessary and sufficient condition for the existence of a non-empty, convex-valued, decreasing share correspondence for Player i with domain $[\underline{X}_i, \infty)$ or $\mathbb{R}_{++}$. The former case occurs if and only if i has positive participation value $\underline{X}_i$ and $S_i(\underline{X}_i) = \{1\}$ and $\delta < 1$ for all $\delta \in s_i(X)$ with $X > \underline{X}_i$. In either case, either (a) there is $\overline{X}_i > 0$ such that $S_i(X) = \{0\}$ if and only if $X \geq \overline{X}_i$, or (b) $\max S_i(X) \longrightarrow 0$ as $X \longrightarrow \infty$.*

Proof. Suppose Player i is weakly regular and let the convex-valued, share correspondence be $S_i(X)$. The proof that the domain of S_i is $[\underline{X}_i, \infty)$ or $\mathbb{R}_{++}$ is established by a similar argument to that for share functions in Proposition 3.1; we omit the details. By assumption, Player i has at most one best response to $X_{-i} = 0$ and it follows that $S_i(X) = 1$ for at most one value of X. This can be established by a similar argument to that in Lemma 3.2. We shall prove that S_i is decreasing by contradiction, so suppose that we had $0 < X^1 < X^2$, $(x^1, X^1), (x^2, X^2) \in L_i$, $\sigma^1 = x^1/X^1$, $\sigma^2 = x^2/X^2$ and $\sigma^1 < \sigma^2$. The best response to $X_{-i} = (1 - \sigma^1) X^2$ is a point on the line of unit slope through $(\sigma^1 X^2, X^2)$. Writing (x^0, X^0) for this point, we have either A: $X^0 \leq X^2$ or B: $x^0 > \sigma^1 X^0$. In Case A, we can apply Corollary 3.1 to deduce that there exists $(x_i', X^2) \in L_i$ such that $X^2 - x_i' \geq X^0 - x_i^0 = (1 - \sigma^1) X^2$. Hence, $x_i' \leq \sigma^1 X^2$ and convexity of the set in Part 2. of the definition of weak regularity implies that

$$\left\{\left(x_i, X^2\right) : \sigma^1 X^2 \leq x_i \leq x^2\right\} \subset L_i. \tag{24}$$

Since $(\sigma^1 X^2, X^2) \in L_i$ by (24), convexity of the set in Part 3. of the definition implies that,

$$\left\{\left(\sigma^1 X, X\right) : X^1 \leq X \leq X^2\right\} \subset L_i. \tag{25}$$

[24] Since L_i is closed, except possibly at the origin.
[25] Put $\beta = 0$ in the corollary.

Hence, for small enough $\varepsilon > 0$, we have $(\sigma^1 X^2 + \varepsilon, X^2) \in L_i$ by (24) and $(\sigma^1 X, X) \in L_i$, where

$$X = X^2 - \frac{\varepsilon}{1 - \sigma^1},$$

by (25). But this means there are two distinct best responses to $X_{-i} = (1 - \sigma^1) X^2 - \varepsilon$, contradicting the assumption of a unique best response. In Case B, we can apply Corollary 3.1 to deduce that there exists $(x_i', X') \in L_i$ such that $x_i' = \sigma^1 X'$ and $X^{'} - x_i' \geq (1 - \sigma^1) X^2$. Hence, $X' \geq X^2$ and Part 3 of the definition of weak regularity implies (25). This shows that $(\sigma^1 X^2, X^2) \in L_i$ and, applying convexity again, (24). As we have seen, this contradicts uniqueness of best responses.

The converse result follows from the fact that

$$L_i = \left\{ (x, X) : \frac{x}{X} \in S_i(X) \right\}.$$

Since $S_i(X)$ is a convex set for all $X > 0$, Part 2 of the definition of weak regularity holds. To justify Part 3, note that

$$\{X : (\sigma X, X) \in L_i\} = \{X : \sigma \in S_i(X)\}.$$

If we had $X' < X'' < X'''$ with $\sigma \in S_i(X') \cap S_i(X''')$, since $X'' > X'$, $S_i(X'')$ is non-empty. If $\sigma'' \in S_i(X'')$, then $\sigma'' < \sigma$ would conflict with S_i being decreasing (for X'' to X'''). A similar conflict would hold if $\sigma'' > \sigma$ (for X' to X''). Hence, $\sigma \in S_i(X'')$, which shows that $\{X : \sigma \in S_i(X)\}$ is convex, proving Part 3. We prove Part 1 by contradiction, so suppose, to the contrary that best responses were not unique. Specifically, suppose we had $x_i, x_i' \in B_i(X_{-i})$ and $x_i < x_i'$ and let $X = x_i + X_{-i}$ and $X' = x_i' + X_{-i}$. Then $X_{-i} > 0$ would imply $x_i/X < x_i'/X'$, contradicting decreasing S_i, since $x_i/X \in S_i(X)$ and $x_i'/X' \in S_i(X')$. The same conclusion holds for $X_{-i} = 0$ from the supposition that at most one X satisfies $S_i(X) = 1$.

The final assertions can be established by a similar proof to that of Proposition 3.1; we omit the details. ■

Lemma 3.3 generalizes to a characterization of equilibria in terms of share correspondences.

Lemma 7.1 *There is a Nash equilibrium $\widehat{\mathbf{x}} \neq \mathbf{0}$ if and only if $\widehat{x}_i/\widehat{X} \in S_i\left(\widehat{X}\right)$ for all $i \in I$, where $\widehat{X} = \sum_{i \in I} \widehat{x}_i$.*

It follows that $\widehat{X}$ is an equilibrium value of the aggregate if and only if

$$1 \in \sum_{i \in I} S_i \left(\widehat{X} \right), \tag{26}$$

using conventional set addition. If all players are weakly regular, the aggregate share correspondence is decreasing but Proposition 7.1 does not exclude the possibility of more than one value of the equilibrium aggregate (although the set of such values is a closed interval). Even when the equilibrium aggregate $\widehat{X}$ is unique, if $S_i \left(\widehat{X} \right)$ is not a singleton for two or more i, multiple equilibria are possible. However, if all but one player is regular, it follows from Proposition 3.1 that the share functions of all players but one are strictly decreasing where positive. Since the share correspondence of the exceptional player is non-decreasing, the aggregate share correspondence is strictly decreasing where positive: $\sigma^1 \in S_i (X^1)$, $\sigma^2 \in S_i (X^2)$, $\sigma_2 > 0$ and $X^2 > X^1$ imply $\sigma^2 < \sigma^1$. Thus, there is at most one equilibrium value of X and, for any such value, the strategies of the regular players are uniquely determined, which implies a single equilibrium. Defining $\overline{\sigma}_i = \sup_{X>0} \max S_i (X)$, for weakly regular players, we have the following generalization of Theorem 3.6.

Theorem 7.2 *Suppose that all but one players in the aggregative game $\mathcal{G} = \left(I, \mathbf{w}, \{v_i\}_{i \in I} \right)$ are regular and the remaining player is weakly regular. If no player has a positive participation value, suppose further that $\sum_{i \in I} \overline{\sigma}_i > 1$. Then, $\mathcal{G}$ has a unique Nash equilibrium.*

If no player has a positive participation value, and (5) is invalid, $\mathcal{G}$ has no equilibrium.

A decreasing aggregate share correspondence at unit share value is clearly necessary for a unique equilibrium. However, this does not imply regularity or even weak regularity of the players; for example, an increase in one player's share function can be offset by a faster decrease in another's. However, if we rule out such interactions and impose conditions only on individual payoffs, Theorem 7.2 is best possible in the following sense. If there is at most one equilibrium of all games in which an individual plays against regular competitors is unique, then that individual is weakly regular. Similarly, if there is at most one equilibrium when an individual plays against competitors all but one of whom are regular and the exceptional player is weakly regular, then that individual is regular. It is enough to consider two-player games to justify these claims.

Proposition 7.2 *If every game played by a player against a weakly regular opponent with positive participation value has a unique equilibrium, then that* *player is regular. Similarly, if every game played by a player against a regular opponent with positive participation value has a unique equilibrium, then that player is weakly regular.*

Proof. We will make use of the fact that, given any $\overline{X}_i > 0$ and continuous, share function s_i defined on $[\underline{X}_i, \infty)$, which is strictly decreasing where positive and satisfies $s_i(\underline{X}_i) = 1$, there is a regular payoff function for which s_i is the share function. Indeed, we need only take $v_i(x, X)$ to be the negative of the distance from (x, X) to the set

$$L_i = \left\{ (Xs_i(X), X) : X \geq \overline{X}_i \right\}.$$

It is readily verified that this v_i has convex best responses and L_i satisfies the aggregate and radial crossing conditions for all $X > 0$ and $\sigma \in (0, 1]$, respectively. A similar argument shows that every correspondence satisfying the properties set out in Proposition 7.1, can be realized as the share correspondence of a weakly regular player.

Consider a player, to which we arbitrarily assign the label 1. Equation (23) defines a share correspondence S_1 for Player 1 and an application of Lemma 3.1 with ($\beta = -1$) shows that this correspondence is non-empty-valued on a semi-infinite (to the right) interval.

To prove the first assertion of the proposition, we start by showing that, under the first hypothesis, this correspondence must be strictly decreasing where positive. If, to the contrary, we had $X'' > X'$ and $\sigma'' \geq \sigma'$, $\sigma'' > 0$, where $\sigma' \in S_1(X')$, $\sigma'' \in S_1(X'')$, the argument in the first paragraph shows that there is a weakly regular player, 2, say, with share correspondence S_2 such that $S_2(X') = \{1 - \sigma'\}$ and $S_2(X'') = \{1 - \sigma''\}$. But then Lemma 7.1 leads to the contradiction of two equilibrium values of the aggregate: X' and X''. To complete the proof, we show that $S_1(X)$ must be single-valued. Suppose, *per contra*, we had $\sigma^*, \sigma^{**} \in S_1(X^*)$, with $\sigma^* < \sigma^{**}$. Then, there would exist a share correspondence for player 2 satisfying $S_2(X) = [1 - \sigma^{**}, 1 - \sigma^*]$ and, since $S_1(X) + S_2(X)$ is a non-degenerate interval containing 1, there are multiple equilibria (with aggregate X). This shows that S_1 is single-valued: Player 1 has a share function, which is strictly decreasing where positive. It follows, as in the first paragraph, that Player 1 is regular.

The second assertion is proved similarly; we only sketch the outline. Firstly, as above, we show that the share correspondence of the player is decreasing. To complete the proof, we need to establish that $S_1(X')$ is a (possibly degenerate) interval, where non-empty. Indeed, suppose we had $\sigma' < \sigma''$ such that $\sigma', \sigma'' \in S_1(X')$ and $\sigma''' \notin S_1(X')$ for all $\sigma''' \in \{\sigma', \sigma''\}$ and consider a share function s_2 with positive participation value such that $s_2(X') = 1 - \sigma'''$. Then, all elements of the aggregate share correspondence exceed 1 for $X < X'$, are less than 1 for $X > X'$ and are not equal to 1 at $X = X'$. Hence, this game has no equilibrium, contradicting the hypothesis of the proposition. These properties of S_1 imply that Player 1 is weakly regular. ■

As an illustration of weak regularity, we return to Cournot oligopoly and focus on the case of linear demand[26]: $p(X) = a - bX$ and a player, i, with a differentiable[27], but not necessarily convex, cost function, c_i. We shall show that, provided $c_i'(x) \geq 0$ and $\varphi_i(x) = bx + c_i'(x)$ is non-decreasing in $x \geq 0$, Player i is weakly regular. To see this, first note that, for fixed X_{-i}, Player i's payoff function:

$$ax_i - bx_i^2 - bx_iX_{-i} - c_i(x_i) \tag{27}$$

has slope $a - bx_i - bX_{-i} - \varphi_i(x_i)$. This is strictly decreasing in x_i, which shows that (27) is a continuous, strictly concave function of x_i and therefore has unique best responses: Part 1 of the definition of weak regularity. The first order conditions are that $(x_i, X) \in L_i$ if and only if

$$\varphi_i(x_i) \begin{cases} \geq a - bX & \text{if } x_i = 0, \\ = a - bX & \text{if } 0 < x_i < w_i, \\ \leq a - bX & \text{if } x_i = w_i. \end{cases}$$

Our assumption on φ_i verifies Part 2. of the definition of weak regularity. Furthermore, for any $\sigma \in (0, 1]$, the function $\varphi_i(\sigma X)$ is non-decreasing and $a - bX$ strictly decreasing in $X > 0$ and Part 3 is an immediate consequence. This establishes weak regularity. Since the assumption that φ_i is strictly decreasing implies regularity, if $bx + c_i'(x)$ is strictly increasing for all but one player and non-decreasing for the exceptional player, the game will have a unique equilibrium.

In this example, the share correspondence can be written down explicitly. For $X \geq \overline{X}_i$, where $\overline{X}_i$ is the dropout value:

$$\overline{X}_i = \frac{a - c'(0)}{b},$$

we have $S_i(X) = \{0\}$. For X between the monopoly value and $\overline{X}_i$, we have

$$S_i(X) = \frac{\varphi_i^{-1}(a - bX)}{X}, \tag{28}$$

[26]For expositional convenience, we permit negative p. However, such values will not arise in equilibrium.

[27]Strictly, in an open set containing $\mathbb{R}_+$.

where the correspondence φ_i^{-1} is the inverse of the function φ_i. Note that the graph of S_i can have vertical sections corresponding to intervals (of X) on which φ_i is constant. However, S_i as defined in (28) is strictly decreasing in X. It follows that, even if φ_i is non-decreasing for all players, although multiple equilibria are possible, the equilibrium value of X is unique.

8 Repeated payoffs

As we discussed in the previous section, regularity or even weak regularity is not necessary for a unique equilibrium, once we allow for interactions between payoffs. An example is provided by a game with n players with identical payoffs. Suppose that the share correspondence is single valued and strictly decreasing where positive, *for share values less than or equal to* $1/n$. The aggregate share correspondence is single valued and strictly decreasing for share values less than or equal to one, which implies that there is at most one equilibrium. Indeed, we can still infer uniqueness if the n-player symmetric game is augmented by additional regular players, provided conditions for existence are satisfied. An example of this is the n-fold replication of a general game, a special case of the type game discussed in Section 5, used to study the competitive limit. In this section, we formalize these observations and discuss their application.

Definition 8.1 *The share correspondence $S_i(X)$ of Player i is well behaved beyond $\overrightarrow{X}$ if*

1. $X < \overrightarrow{X}$ *and* $\sigma \in S_i(X)$ *imply* $\sigma > \overrightarrow{\sigma}$, *where*

$$\overrightarrow{\sigma} = \min S_i\left(\overrightarrow{X}\right), \tag{29}$$

2. $X > \overrightarrow{X}$ *implies S_i is single-valued and the function thereby defined is*

   (a) *strictly decreasing where positive*

   (b) *tends to* $\overrightarrow{\sigma}$ *as* $X \longrightarrow \overrightarrow{X}$

   (c) *is either asymptotic or eventually equal to zero*[28].

If the equilibrium aggregate exceeds $\overrightarrow{X}$ and the share correspondence of Player i is well-behaved beyond $\overrightarrow{X}$, the player can be considered as having a

[28]It is also continuous by the closed graph property of the correspondence.

well-behaved share function at the equilibrium. It is therefore useful to seek conditions on payoffs ensuring such a property of the share correspondence.

We shall say that Player i is *eventually regular* if the player satisfies the conditions of regularity for large enough aggregate and small enough shares.

Definition 8.2 *Player i is eventually regular with threshold aggregate value $\overrightarrow{X}$ if*

1. *the set $[0, \overrightarrow{w}_i] \cap B_i(X_{-i})$ is convex for any $X_{-i} \geq (1 - \overrightarrow{\sigma})\overrightarrow{X}$, where $\overrightarrow{w}_i = \min\{w_i, \overrightarrow{\sigma} X_{-i}/(1 - \overrightarrow{\sigma})\}$[29],*
2. *best responses satisfy ACC for all $X > \overrightarrow{X}$,*
3. *best responses satisfy RCC for all σ satisfying $0 < \sigma \leq \overrightarrow{\sigma}$, where $\overrightarrow{\sigma}$ is defined in (29).*

It is convenient to refer to $\overrightarrow{\sigma}$ as the *threshold share.*

Proposition 8.1 *If a player is eventually regular with threshold value $\overrightarrow{X}$, then that player's share correspondence is well behaved beyond $\overrightarrow{X}$.*

Proof. Label the player i. The first step is to prove that best responses to $X_{-i} \geq (1 - \overrightarrow{\sigma})\overrightarrow{X}$ cannot exceed $\overrightarrow{w}_i$, which implies that $B_i(X_{-i})$ is equal to the set of best responses in $[0, \overrightarrow{w}_i]$. The proof is by contradiction, so suppose we had $x_i^0 \in B_i\left(X_{-i}^0\right)$ satisfying $\overrightarrow{w}_i < x_i^0 \leq w_i$ and $X_{-i}^0 \geq (1 - \overrightarrow{\sigma})\overrightarrow{X}$. Then, we would have $(x_i^0, X^0) \in L_i$ with $x_i^0 > \overrightarrow{\sigma} X^0$. Since L_i is closed and the set $M_i = \{(x, X) \,|x \geq \overrightarrow{\sigma} X, 0 \leq x \leq \min\{w_i, X\}\}$ is compact, there will be $(x^*, X^*) \in L_i \cap M_i$ satisfying $X^* - x^* \geq X - x$ for all $(x, X) \in L_i \cap M_i$. There are two possibilities; A: $X^* - x^* > X^0 - x^0$, and B: $X^* - x^* = X^0 - x^0$. In Case A, we must have

$$x^* > \overrightarrow{\sigma} X^* \tag{30}$$

to avoid violating RCC (since $\left(\overrightarrow{\sigma}\overrightarrow{X}, \overrightarrow{X}\right) \in L_i$). If

$$x \in B_i(X^* - x^* + \varepsilon) \tag{31}$$

for some $\varepsilon > 0$, we must have $x \leq \overrightarrow{\sigma} X$ (by definition of (x^*, X^*)) and $X > X^*$ for all $x \in B_i(X_{-i})$. This latter inequality can be justified by

[29] Geometrically, $\overrightarrow{w}_i$ is the x value of the intersection of the line $X - x = X_{-i}$ and the ray through the origin with slope $\overrightarrow{\sigma}$. Note that, any x_i in the intersection satisfies $x_i \leq \overrightarrow{\sigma} X$; we are focusing attention on share values below $\overrightarrow{\sigma}$.

noting that, were it not to hold, we could apply Corollary 3.1 to deduce the existence of $(x', X^*) \in L_i$ such that $X^* - x' \geq X_{-i}$, which implies $x' \leq \overrightarrow{\sigma} X^*$ and violates ACC at $X = X^*$. Note that $x \leq \overrightarrow{\sigma} X$ and $X > X^*$ imply

$$X^* < X \leq \frac{X^* - x^* + \varepsilon}{1 - \overrightarrow{\sigma}} < X^* + \frac{\varepsilon}{1 - \overrightarrow{\sigma}},$$

where we have also used (30) for the final inequality. Hence, for any sequence of ε that vanishes in the limit, the corresponding sequence of X approaches X^* and, since L_i is closed, we deduce the existence of $(x'', X^*) \in L_i$ such that $x'' \leq \overrightarrow{\sigma} X^*$, which violates ACC at $X = X^*$. In Case B, we use the fact that all best responses x to $X_{-i} > X^* - x^* \geq (1 - \overrightarrow{\sigma}) \overrightarrow{X}$ satisfy $x \leq \overrightarrow{\sigma} X$ to use an obvious modification of the proof of Lemma 3.2 to deduce that $B_i(X_{-i})$ is a singleton for all such X_{-i} and therefore defines a function $b_i(X_{-i})$. Furthermore, this function has a closed graph and is therefore continuous. An application of the intermediate value theorem shows that there is $(x_i, X^*) \in L_i$ such that $x_i \leq \overrightarrow{\sigma} X^*$. This violates ACC at $X = X^*$. This completes the proof that eventually regular players have best reponses in $[0, \overrightarrow{w}_i]$. A straightforward modification of the proof of Lemma 3.2 then shows that $B_i(X_{-i})$ is a singleton for $X_{-i} \geq (1 - \overrightarrow{\sigma}) \overrightarrow{X}$.

The remainder of the proof is an adaptation of arguments in Section 3. To avoid lengthy repetition, we sketch only an outline, omitting most of the details. The argument following Lemma 3.2 is easily modified to establish that $S_i(X)$ is a singleton for all $X > \overrightarrow{X}$ and we write $\overrightarrow{s}_i$ for the function thus defined. A similar modification shows that $\overrightarrow{s}_i$ is strictly decreasing where positive in $\left(\overrightarrow{X}, \infty\right)$ and either approaches or equals zero as $X \longrightarrow \infty$. Since L_i is a closed set,

$$\lim_{X \longrightarrow \overrightarrow{X}} \overrightarrow{s}_i(X) \in S_i\left(\overrightarrow{X}\right).$$

If the limit exceeded $\overrightarrow{\sigma}$, RCC would be violated at $\sigma = \overrightarrow{\sigma}$ and we may conclude that $\overrightarrow{s}_i(X) \longrightarrow \overrightarrow{\sigma}$ as $X \longrightarrow \overrightarrow{X}$. This establishes requirement (ii) of eventual regularity. Condition (i) can be proved by supposing we had $\sigma \in (0, \overrightarrow{\sigma}]$ satisfying $\sigma \in S_i(X)$, where $X < \overrightarrow{X}$ and using Corollary 3.1 to derive a contradiction. Such a contradiction (of RCC at $\overrightarrow{\sigma}$) is immediate if $\sigma = \overrightarrow{\sigma}$. For $\sigma < \overrightarrow{\sigma}$, we have shown that there is $X' > \overrightarrow{X}$ such that $\sigma \in S_i(X')$ and this contradicts RCC at σ, completing the proof. ■

Figure 8 shows a game in which players are eventually well-behaved. Combining the results of Proposition 8.1 with Lemma 7.1 gives the following theorem in which we take the threshold value of a regular player to be the participation value if it exists and, otherwise, zero.

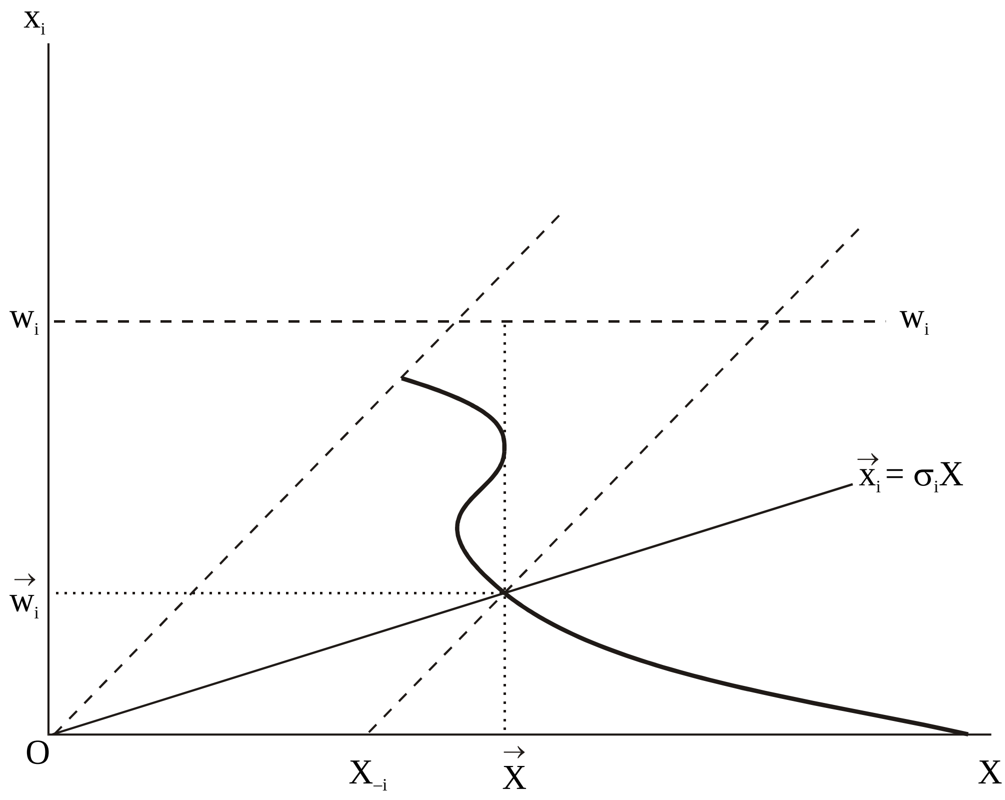

Figure 8:

Theorem 8.3 *Suppose that all players in the aggregative game $\mathcal{G} = \left(I, \mathbf{w}, \{v_i\}_{i \in I}\right)$ are regular or eventually regular and let $\overleftrightarrow{X}$ be the maximum threshold value. If*

$$\sum_{i \in I} \min S_i \left(\overleftrightarrow{X} \right) \geq 1,$$

$\mathcal{G}$ has a unique non-null Nash equilibrium.

Note that for any player whose threshold value is less than $\overleftrightarrow{X}$ (ie. not a maximizer of threshold values) the share correspondence is single-valued in equilibrium.

The theorem can be applied to a *type game* with a finite set of types T as formalized in Section 5, from which we also adopt the notation. For each type $t \in T$, let I_t denote the set of players of that type and n_t be the number of players in I_t. Here, $\{I_t\}_{t \in T}$ is a partition of the player set I and all players in I_t have the same payoffs. Suppose players of type $t \in T$ are eventually regular and denote the threshold aggregate and share values by $X_{(t)}$ and $\sigma_{(t)}$. If types are labelled so that $X_{(1)} \geq X_{(t)}$ for all $t \in T$ then $n_{(1)}\sigma_{(1)} \geq 1$ implies the inequality in the theorem and therefore a unique equilibrium. In particular, we note the following corollary.

Corollary 8.1 *A type game in which, for all $t \in T$, players of type t are eventually regular with threshold share value at least $1/n_t$, has a unique non-null Nash equilibrium.*

The corollary can obviously be extended to include some players which are regular rather than eventually regular.

As an application of the corollary, consider the Cournot oligopoly game considered earlier. We shall prove that, provided the cost function of firms of type t is convex and inverse demand p is continuous, and, where positive, p is twice differentiable and satisfies

$$p'(X) < 0 \text{ and } \left(\vartheta^{-1} + 1\right) p'(X) + Xp''(X) < 0,$$

for some $\vartheta \in (0, 1)$ then players of type t are eventually regular with threshold share ϑ. We first establish that, for any $X_{-i} > 0$ the revenue of Player i is a strictly concave function of own strategy for x_i satisfying

$$0 \leq x_i \leq \max\left\{\frac{\vartheta X_{-i}}{1-\vartheta}, w_i\right\}.$$

For $x_i > 0$,

$$\frac{d^2}{dx_i^2} x_i p\left(x_i + X_{-i}\right) = 2p'(X) + x_i p''(X).$$

If $p''(X) \leq 0$, the right hand side is negative. If $p''(X) > 0$,

$$\frac{d^2}{dx_i^2} x_i p\left(x_i + X_{-i}\right) \leq (1-\vartheta)\, p'(X) + \vartheta\left[\left(\vartheta^{-1}+1\right) p'(X) + Xp''(X)\right] < 0,$$

using $x_i \leq \vartheta X$.

The aggregate crossing condition holds for all $X > 0$; this is immediate from (7). The derivative of the left hand side of (8) with respect to X can be written

$$\left(1 - \frac{\sigma_i}{\vartheta}\right) p'(X) + \sigma_i\left[\left(\vartheta^{-1}+1\right) p'(X) + Xp''(X)\right]$$

and this is negative for $\sigma_i \leq \vartheta$, which verifies the radial crossing condition for such σ_i. If we label the types so that

$$n_1 \leq n_2 \leq \cdots \leq n_T$$

and

$$p'(X) < 0 \text{ and } (n_1 + 1)\, p'(X) + Xp''(X) < 0, \tag{32}$$

holds for positive demand, then firms of type t are eventually regular with threshold share value at least $1/n_t$ and Corollary 8.1 is applicable.

For example, suppose demand has constant elasticity η and all firms have the same costs. Direct application of Theorem 3.6, using (6) restricts the elasticity to exceed unity. However, applying Corollary 8.1 with a single type, using (32) and noting that $n_1 = n$, extends the result to inelastic demand. In particular, (32) holds if and only if $\eta > 1/n$. Note also that in the inelastic case, $\overline{\sigma}_i = \eta$ and the same inequality is equivalent to (5). It follows from Corollary 8.1, that there is a unique (and therefore symmetric) equilibrium.

The case $|T| = 2$ and linear costs is addressed by Collie [6]. It is straightforward to check that (32) is equivalent to the conditions given by Collie, of which our results for Cournot games can be seen as a generalization[30].

Condition (32) suggests that, if the game has a finite competitive limit, then for sufficiently many players of the type with the largest dropout point, the second inequality is implied by the first. This will certainly be true if the (negative of the) elasicity of marginal demand: $-Xp''(X)/p'(X)$ is bounded above in the region where demand is positive. Indeed, if we write $\overline{\eta}$ for this upper bound, (32) holds for positive demand, provided $n_1 > \overline{\eta} - 1$. Note that boundedness of marginal demand holds in common cases such as linear and constant elasticity demand. For a sufficiently large game, Theorem 8.3 implies the existence of a unique equilibrium. If we suppose further that $p(X) \longrightarrow 0$ as $X \longrightarrow \infty$ and $c'_{(t)}(0) > 0$, players of type t will have a dropout

[30] Collie considered two groups of firms with m in the first and n in the second. All firms in the same group have identical, constant marginal costs. Demand satisfies $p'(X) < 0$ for all $X > 0$ and

$$\begin{aligned} (m+1)\, p'(X_1 + X_2) + X_1 p''(X_1 + X_2) &< 0 \text{ for all } X_1, X_2 > 0, \\ (n+1)\, p'(X_1 + X_2) + X_2 p''(X_1 + X_2) &< 0 \text{ for all } X_1, X_2 > 0, \\ (m+n+1)\, p'(X) + Xp''(X) &< 0 \text{ for all } X > 0. \end{aligned}$$

Choosing labels so that $m \leq n$ and taking the limit $X_2 \longrightarrow 0$ in the first inequality, gives (32) with a weak inequality. A modification of our arguments can be used to establish eventual regualarity directly from Collie's inequalities.

point $\overline{X}_{(t)}$ which is the unique solution of $p(X) = c'_{(t)}(0)$. Combined with Theorem 5.1 on the large-game limit, this validates the competitive limit of Cournot oligopoly under the weaker assumptions that marginal demand is negative and has an upper bound on its elasticity. Specifically, provided all types have positive and non-decreasing marginal costs, the equilibrium aggregate output approaches $\max_{t \in T} \overline{X}_{(t)}$ and aggregate profits fall to zero as the game becomes large.

9 Smooth payoffs

Establishing regularity by direct application of the aggregate and radial crossing conditions may require some ingenuity. When payoffs are sufficiently smooth, these conditions can be tested by examining the properties of marginal payoffs. In this section, we describe and justify the relevant inequalities as well as discussing comparative statics, the competitive limit under smoothness assumptions and eventual regularity.

Throughout this and the next section, we shall assume that $\pi_i(\mathbf{x})$, the payoff of each player $i \in I$, is a continuously differentiable function of $x_i \in (0, w_i)$ for all $\mathbf{x}_{-i} \in \mathcal{S}_{-i}$. For an aggregative game in which $\pi_i = v_i(x_i, X)$, we shall write $\gamma_i(x_i, X)$ for the marginal payoff with respect to own strategy and note that, if $(x, X) \in \mathrm{int}\widetilde{S}_i$, where the latter denotes the interior of $\widetilde{S}_i$, then

$$\gamma_i(x, X) = \frac{\partial v_i}{\partial x_i}(x, X) + \frac{\partial v_i}{\partial X}(x, X).$$

Note that L_i is a (possibly strict) subset of the set of zeroes of γ_i in $\mathrm{int}\widetilde{S}_i$. We shall further assume that γ_i is a continuously differentiable function of (x_i, X) in $\mathrm{int}\widetilde{S}_i$ and refer to payoffs satisfying these differentiability assumptions as *smooth*.

The conditions we shall study are as follows.

A1 If $(x, X) \in \mathrm{int}\widetilde{S}_i$ and $\gamma_i(x, X) = 0$, then

$$\frac{\partial \gamma_i}{\partial x}(x, X) < 0.$$

We shall show that this assumption implies ACC at any $X > 0$. Similarly, the following assumption implies that RCC holds for any $\sigma \in (0, 1]$.

A2 If $(x, X) \in \widetilde{S}_i$, $0 < x < w_i$ and $\gamma_i(x, X) = 0$, then

$$x\frac{\partial \gamma_i}{\partial x_i}(x, X) + X\frac{\partial \gamma_i}{\partial X}(x, X) < 0.$$

Note that, when $x = X$, it is necessary to interpret the left hand side of this inequality as $x\partial^2\pi_i/\partial x_i^2$.

These two conditions are sufficient for regularity.

Proposition 9.1 *If a player has smooth payoffs satisfing **A1** and **A2**, then that player is regular. Furthermore, the share function s_i is differentiable except possibly at the dropout point and, if $s_i(X) > 0$, then $s_i'(X) < 0$.*

Proof. Convexity of best responses follows from quasi-concavity of payoffs in own strategy and the latter follows from the observation that, if $(x, X) \in \text{int}\widetilde{S}_i$ and $\gamma_i(x, X) = 0$, then **A1** and **A2** imply

$$\frac{\partial^2\pi_i}{\partial x_i^2} = X^{-1}\left[(X - x)\frac{\partial\gamma_i}{\partial x} + x\frac{\partial\gamma_i}{\partial x} + X\frac{\partial\gamma_i}{\partial X}\right] < 0.$$

This inequality also holds for $x = X$ by direct application of **A2**. This shows that $\pi_i(x_i, \mathbf{x}_{-i})$ is a continuous function of $x_i \in [0, w_i]$, has no local minima in $(0, w_i)$ and is therefore strictly quasi-concave. Note that this implies that, for $(x, X) \in \widetilde{S}_i$ with $0 < x < w_i$, we have $(x, X) \in L_i$ if and only if $\gamma_i(x, X) = 0$.

We now show that **A1** leads to ACC at all $X > 0$. Define

$$\begin{aligned}\underline{\mu}(X) &= \inf\{x \in (0, w_i) : \gamma_i(x, X) < 0\} \\ &= \sup\{x \in (0, w_i) : \gamma_i(x, X) > 0\},\end{aligned}$$

where we take the infimum of an empty set to be w_i and the supremum to be 0. The equality of the two definitions is a consequence of the fact that, given X, $\gamma_i(x, X)$ changes sign at most once as x increases in $(0, w_i)$ and such a change must be from positive to negative. Note also that $\underline{\mu}$ is a continuous function on $X > 0$. Indeed, compactness of the range of $\underline{\mu}$ implies that, if it were discontinuous at X^0, there would be a sequence $\{X^n\}$ convergent to X^0 on which $\underline{\mu}(X^n) \longrightarrow \mu^+ \neq \underline{\mu}(X^0)$ as $n \longrightarrow \infty$. If, say $\mu^+ > \underline{\mu}(X^0)$, we would then have $\underline{\mu}(X^n) > \mu^{++} = \left[\mu^+ + \underline{\mu}(X^0)\right]/2$ for all large enough n. Hence, $\gamma_i(\mu^{++}, X^n) > 0$ for all large enough n, which, because of the continuity of γ implies that $\gamma_i(\mu^{++}, X^0) \geq 0$ implying $\mu^{++} \leq \underline{\mu}(X^0)$, a contradiction. A contradiction can be derived similarly if $\mu^+ < \underline{\mu}(X^0)$. Verification of ACC is completed by showing that L_i is a subset of the graph of $\underline{\mu}$, for then ACC is immediate. So suppose that $(x', X') \in L_i$ and $X' > 0$. If A: $x' \in (0, w_i)$, we have already noted that $\gamma_i(x', X') = 0$, which is readily seen to imply $x' = \underline{\mu}(X')$. If B: $x' = 0$ and there is a neighborhood of X' such that $(0, X) \in L_i$ for all X in the neighborhood, strict quasi-concavity of best responses implies that $\gamma_i(x, x + X) < 0$ for all $x \in (0, w_i)$. By considering all $X < X'$ in the neighborhood, we deduce that $\gamma_i(x, X') < 0$ for all small enough x which entails $\underline{\mu}(X) = 0$ (using the first definition of $\underline{\mu}$). If C: $x' = 0$ and there exists X arbitrarily close to X' such that $(x, X) \in L_i$ with $x > 0$, we know that $x = \underline{\mu}(X)$ by (i) and can deduce that $\underline{\mu}(X') = 0$ from the closedness of L' and continuity of $\underline{\mu}$. Finally, if $x' = w_i$, a similar argument (using the second definition of $\underline{\mu}$) shows that $\underline{\mu}(X') = w_i$. In all cases, $\underline{\mu}(X') = x'$ as required.

To complete the proof, we need to show that **A2** implies RCC for all $\sigma \in (0, 1]$. This is done by first observing that, for any such σ and $X > 0$ with $\gamma_i(\sigma X, X) = 0$, we have

$$\frac{\partial}{\partial X}\gamma_i(\sigma X, X) = \sigma X \frac{\partial \gamma_i}{\partial x_i}(\sigma X, X) + X \frac{\partial \gamma_i}{\partial X}(\sigma X, X) < 0,$$

by **A2**. This implies that $\gamma_i(\sigma X, X)$ changes sign at most once as X increases from 0 to w_i/σ and such a change must be from positive to negative. This observation can be used to modify the proof for ACC to show that **A2** implies RCC. We shall omit the details.

Differentiability of the replacement function (and therefore the share function) when $0 < s_i(X) < w_i/X$ follows from applying the implicit function theorem to the first order condition $\gamma_i(X s_i(X), X) = 0$. To justify this application, we note that $\partial \gamma_i / \partial x_i \neq 0$, by **A1**. Furthermore,

$$s_i'(X) = \left[x_i \frac{\partial \gamma_i}{\partial x_i} + X \frac{\partial \gamma_i}{\partial X_i} \right] / X^2 \frac{\partial \gamma_i}{\partial x_i},$$

evaluated at $(x_i, X) = (X s_i(X), X)$. By **A1** and **A2,** $s_i'(X) < 0$. If $s_i(X) = w_i/X$ we must have $\gamma_i(X s_i(X), X) \geq 0$. If we had $\gamma_i = 0$, the same argument would hold. If we had $\gamma_i > 0$, then $s_i(X') = w_i/X'$ in a neighborhood of X and $s_i' < 0$ is immediate. ■

In a Cournot oligopoly game, if demand is twice continuously differentiable for $X > 0$ and Player i's cost function is twice continuously differentiable for $0 < x < w_i$, then

$$\gamma_i(x, X) = p(X) + x p'(X) - c_i'(x) \tag{33}$$

for all such (x, X). For **A1**, we require

$$p'(X) < c_i''(x)$$

if $(x, X) \in \operatorname{int}\widetilde{S}_i$ and $\gamma_i(x, X) = 0$, and for **A2**, we require

$$x p'(X) - x c_i''(x) + X p'(X) + x X p''(X) < 0$$

if $(x, X) \in \widetilde{S}_i$, $0 < x < w_i$ and $\gamma_i(x, X) = 0$. If $p'(X) < 0$ for all $X > 0$, a sufficient condition for both inequalities is

$$c_i''(x) > \max\{p'(X), 2p'(X) + Xp''(X)\} \tag{34}$$

for $0 < x \leq X$. Note that this inequality permits some concavity in demand functions without losing a unique equilibrium. For example, if demand is linear[31] with slope $-b$, then we require $c_i''(x) > -b$ for $0 < x < w_i$.

Comparative statics can also be studied when payoffs are smooth. Obviously, a sufficient condition for $v_i(x_i, X)$ to be strictly increasing [decreasing] in X is $\partial v_i/\partial X > [<]0$ for $0 < x_i < X$. A sufficient condition for supposition (iii) in Theorem 4.2 is that $\gamma_{\hat{\imath}}^1(x_i, X) < \gamma_{\hat{\imath}}^2(x_i, X)$ whenever $0 < x_i < X$. This can be proved using the fact that, if $0 < x_i \leq X$, then L_i coincides with the set of zeroes of $\gamma_{\hat{\imath}}(x, X)$. Suppose, $\left(x^k, X\right) \in L_{\hat{\imath}}^k$ for $k = 1, 2$. If $x^2 > 0$, then

$$\gamma_{\hat{\imath}}^1\left(x^2, X\right) < \gamma_{\hat{\imath}}^2\left(x^2, X\right) = 0.$$

We can conclude from **A1** and the continuity of $\gamma_{\hat{\imath}}^1$ that $x^1 < x^2$. (Recall that $x^k = 0$ if and only if $\gamma_{\hat{\imath}}^k(x, X) \leq 0$ for all $x \in (0, w_{\hat{\imath}})$.) If $x^2 = 0$, then $\gamma_{\hat{\imath}}^2(x, X) \leq 0$ for all $x \in (0, w_i)$ and therefore $\gamma_{\hat{\imath}}^1(x, X) < 0$ for all such x, which implies $x^1 = 0$.

In the Cournot game,

$$\frac{\partial v_i}{\partial X} = xp'(X) < 0$$

and an idiosyncratic increase in the marginal costs of an active players $\hat{\imath}$ reduces $\gamma_{\hat{\imath}}$. It follows from Theorem 4.2 that such a change leads to a fall in aggregate output and reduces the profits of other active players.

Comparative statics results are stronger when the game has decreasing or increasing best responses. With smooth payoffs, interior best responses to X_{-i}, satisfy

$$\gamma_i(b_i(X_{-i}), X_{-i} + b_i(X_{-i})) = 0 \tag{35}$$

and the implicit function theorem allows us to deduce that the best response function $b_i(X_{-i})$ is differentiable at X_{-i} provided $\partial\gamma_i/\partial x_i + \partial\gamma_i/\partial X \neq 0$. Furthermore,

$$b_i'(X_{-i}) = \frac{-\partial\gamma_i/\partial X}{\partial\gamma_i/\partial x_i + \partial\gamma_i/\partial X}, \tag{36}$$

[31]The assumption of differentiability holds for linear demand only if the demand curve reaches the axis no later than w_i. Dominance considerations show that we can make this assumption without loss of generality.

where the right hand side is evaluated at $(b_i(X_{-i}), X_{-i} + b_i(X_{-i}))$. Under **A1** and **A2**, we have seen that the denominator in (36) is strictly negative when (35) holds. Hence, the following condition is sufficient for decreasing best responses.

A2* If $(x, X) \in \mathrm{int}\widetilde{S}_i$ and $\gamma_i(x, X) = 0$, then

$$\frac{\partial \gamma_i}{\partial X}(x, X) < 0. \tag{37}$$

Note that **A1** and **A2*** together imply **A2**, except possibly when $x = X$. The latter case is covered if (37) holds for $x = X$.

Similarly, a sufficient condition for increasing best responses is.

A3 If $(x, X) \in \mathrm{int}\widetilde{S}_i$ and $\gamma_i(x, X) = 0$, then

$$\frac{\partial \gamma_i}{\partial X}(x, X) > 0.$$

At first sight this may appear to conflict with **A2** at least when x is small. Note, however, that the inequalities in **A2** and **A3** are required to hold only when $\gamma_i = 0$. That this restriction permits both **A2** and **A3** is illustrated in the first application in the following subsection.

The application of these conditions is often simplified when γ_i can be factorized:

$$\gamma_i(x_i, X) = \phi_i(x_i, X)\widetilde{\gamma}_i(x_i, X) \text{ for all } (x_i, X) \text{ satisfying } 0 < x_i < X,$$

where $\phi_i(x_i, X) > 0$ if $(x, X) \in \widetilde{S}_i$ and $0 < x < w_i$. In this case, **A1**, **A2**, **A2*** and **A3** hold for γ_i if and only if they hold for $\widetilde{\gamma}_i$. For, $\gamma_i = 0 \Leftrightarrow \widetilde{\gamma}_i = 0$ so that

$$\frac{\partial \gamma_i}{\partial x_i} = \phi_i \frac{\partial \widetilde{\gamma}_i}{\partial x_i} \text{ and } \frac{\partial \gamma_i}{\partial X} = \phi_i \frac{\partial \widetilde{\gamma}_i}{\partial X}$$

when these derivatives are evaluated where $\gamma_i = 0$. We will use this 'factorization principle' in several of the applications evaluated below.

9.1 Applications of smooth games

In this section, we discuss these conditions for the applications covered in Section 6.

9.1.1 Search games

In the search game discussed in Subsection 6.1, suppose that c_i is convex and twice continuously differentiable for positive arguments. Then,

$$\gamma_i(x, X) = \theta(X - x) - c_i'(x).$$

Assume that, for all $x \in (0, w_i)$,

$$0 < c_i'(x) < xc_i''(x).$$

Then,

$$\frac{\partial \gamma_i}{\partial x}(x, X) = -\theta - c_i''(x) < 0,$$

so **A1** holds. If $\gamma_i(x, X) = 0$, we also have

$$\begin{aligned} x\frac{\partial \gamma_i}{\partial x_i}(x, X) + X\frac{\partial \gamma_i}{\partial X}(x, X) &= \theta(X - x) - xc_i''(x) \\ &= c_i'(x) - xc_i''(x) \\ &< 0 \end{aligned}$$

for $(x, X) \in \widetilde{S}_i$, $0 < x < w_i$. Thus **A2** holds.

Finally, $\partial\gamma_i/\partial X = 1 > 0$, so **A3** holds, which shows that best responses are increasing; the game is supermodular. Since an increase in θ increases γ_i, condition (iii) of Theorem 4.2 applies and we can deduce by sequential application of the theorem, that the search intensity of all players with equilibrium in $(0, w_i)$ increases.

9.1.2 Smash-and-Grab games

In the Smash-and-Grab games discussed in Subsection 6.2, suppose that utility u_i is twice continuously differentiable for positive arguments and the probability function $h_i(X)$ is twice continuously differentiable for $X > 0$ for which h_i is positive. Then,

$$\gamma_i(x, X) = u_i'(x) h_i(X) + u_i(x) h_i'(X).$$

Suppose that for each i we have $u_i'(x_i) > 0$ and $u_i''(x_i) \leq 0$ if $x_i > 0$ and $h_i'(X) < 0$, $[h_i'(X)]^2 > h_i(X) h_i''(X)$ for all $X > 0$. If $0 < x < X$, then

$$\frac{\partial \gamma_i}{\partial x}(x, X) = u_i''(x) h_i(X) + u_i'(x) h_i'(X) < 0.$$

If, in addition, $\gamma_i(x, X) = 0$, then

$$\begin{aligned}\frac{\partial \gamma_i}{\partial X}(x, X) &= u_i'(x)\,h_i'(X) + u_i(x)\,h_i''(X) \\ &= u_i'(x)\left[h_i'(X) - \frac{h_i(X)}{h_i'(X)}h_i''(X)\right] < 0.\end{aligned}$$

This establishes **A1** and **A2***. Hence, under these assumptions, players in a Smash-and-Grab game are regular and have decreasing best responses.

9.1.3 Public good games

In the public good contribution games discussed in Subsection 6.3, suppose that u_i is twice continuously differentiable for positive arguments. Then,

$$\gamma_i(x, X) = -\frac{\partial u_i}{\partial q}(m - x, X) + \frac{\partial u_i}{\partial X}(m - x, X).$$

If $\partial u_i/\partial X > 0$ for all positive arguments, we can apply the factorization principle with $\phi_i = \partial u_i/\partial X$ to divide by ϕ_i, which gives

$$\widetilde{\gamma}_i(x, X) = 1 - MRS_i(m - x, X), \tag{38}$$

where $MRS_i = [\partial u_i/\partial q] / [\partial u_i/\partial X]$. Now suppose further that

$$\frac{\partial\, MRS_i}{\partial q} < 0, \frac{\partial\, MRS_i}{\partial X} > 0 \tag{39}$$

for positive arguments. Then **A1** and **A2*** follow immediately from (38) and (39). Hence, under these assumptions, players in a public good contribution game are regular and have decreasing best responses.

9.1.4 Sharing games

In the sharing games discussed in Subsection 6.4, suppose that F, C and u_i are twice differentiable for positive arguments. Then

$$\begin{aligned}\gamma_i(x, X) &= MF\left(\frac{x}{X}, X\right)\frac{\partial u_i}{\partial f}\left(\frac{x}{X}F(X), \frac{x}{X}C(X)\right) \\ &\quad + MC\left(\frac{x}{X}, X\right)\frac{\partial u_i}{\partial c}\left(\frac{x}{X}F(X), \frac{x}{X}C(X)\right),\end{aligned}$$

where MF and MC are given by (19) and (20). If $\partial u_i/\partial c > 0$, we can apply the factorization principle with $\phi_i = MF\,\partial u_i/\partial c$ to divide by ϕ_i, which gives

$$\widetilde{\gamma}_i(x, X) = MRS_i\left(\frac{x}{X}F(X), \frac{x}{X}C(X)\right) - \frac{MC(x/X, X)}{MF(x/X, X)}.$$

Now suppose further that $F'(X) > 0$, $F''(X) < 0$, $C'(X) > 0$, $C''(X) > 0$ for all $X > 0$. Suppose, in addition, that u_i is concave and $\partial MRS_i / \partial f < 0$ and $\partial MRS_i / \partial c$. Then, if $0 < x < X$,

$$\frac{\partial MRS_i}{\partial x}\left(\frac{x}{X}F(X), \frac{x}{X}C(X)\right) = AF\frac{\partial MRS_i}{\partial f} + AC\frac{\partial MRS_i}{\partial c} < 0,$$

where AF and AC are the average product and average cost, and

$$\begin{aligned} & x\frac{\partial MRS_i}{\partial x}\left(\frac{x}{X}F(X), \frac{x}{X}C(X)\right) + X\frac{\partial MRS_i}{\partial X}\left(\frac{x}{X}F(X), \frac{x}{X}C(X)\right) \\ = \; & xF'\frac{\partial MRS_i}{\partial f} + xC'\frac{\partial MRS_i}{\partial c} < 0. \end{aligned}$$

Furthermore,

$$\frac{\partial}{\partial x}\left[\frac{MC(x/X, X)}{MF(x/X, X)}\right] = \frac{(C' - AC)\ MF + (AF - F')\ MC}{X\,MF^2} > 0.$$

Note that the cost function is strictly convex ($C'' > 0$) and therefore average cost is less than marginal cost. Similarly average product exceeds marginal product. After some manipulation, we find

$$\begin{aligned} & MF^2\left\{x\frac{\partial}{\partial x}\left[\frac{MC(x/X, X)}{MF(x/X, X)}\right] + X\frac{\partial}{\partial X}\left[\frac{MC(x/X, X)}{MF(x/X, X)}\right]\right\} \\ = \; & \left\{\left[\frac{x}{X}C'' + \left(1 - \frac{x}{X}\right)AC'\right]MF - \left[\frac{x}{X}F'' + \left(1 - \frac{x}{X}\right)AF'\right]MC\right\} > 0. \end{aligned}$$

The sign is justified as average cost is non-decreasing and average product is non-increasing. The first and third inequalities above show that assumption **A1** is satisfied (for $\widetilde{\gamma}$ and therefore for γ.) Similarly, the second and fourth inequalities justify **A2**.

Hence, under these assumptions, players in sharing games are regular.

9.1.5 Contests

In the contests discussed in Subsection 6.5, suppose that f_i is twice continuously differentiable for positive arguments and $f_i' > 0$, $f_i'' < 0$ for all positive arguments. Then a calculation shows that

$$\gamma_i(x, X) = \exp\{\alpha_i g_i(x)\}\left(\frac{X - \beta_i x}{X}\right)\widetilde{\gamma}_i(x, X),$$

where α_i is the (constant) coefficient of risk aversion, g_i is the inverse function of f_i and

$$\begin{aligned}\widetilde{\gamma}_i(x,X) &= \frac{1}{X} - \frac{1-\beta_i}{X-\beta_i x} - \alpha_i g_i'(x)\text{, and}\\ \beta_i &= 1-\exp\{-\alpha_i R\} < 1.\end{aligned}$$

Since $X - \beta_i x > 0$, we may apply the factorisation principle by dividing by the first two terms in the expression for γ. If $0 < x < X$,

$$\frac{\partial\widetilde{\gamma}_i(x,X)}{\partial x} = -\frac{\beta_i(1-\beta_i)}{(X-\beta_i x)^2} - \alpha_i g_i''(x) < 0,$$

where we have used the fact that $g_i'' > 0$, a consequence of our assumptions on f_i'. This verifies **A1** and

$$x\frac{\partial\widetilde{\gamma}_i(x,X)}{\partial x} + X\frac{\partial\widetilde{\gamma}_i(x,X)}{\partial X} = -\frac{\beta_i(X-x)}{X(X-\beta_i x)} - \alpha_i x g_i''(x) < 0$$

verifies **A2**.

Hence, under our assumptions on f_i, contestants with constant absolute risk aversion are regular.

10 Stability

In this section, we investigate local stability of equilibria of smooth aggregative games under a version of the Cournot tatonnement process. This offers a crude model of learning under bounded rationality and asymptotic stability may be viewed as supporting the robustness of Nash equilibrium as a solution concept. Furthermore, the stability conditions are closely related to conditions giving benign comparative statics.

We focus on the continuous-time version of best-response dynamics. Assuming that all players are regular, Player i has a well-defined best response function, $b_i(X_{-i})$, and the dynamics can be written

$$\dot{x}_i = \kappa_i[b_i(X_{-i}) - x_i] \text{ for all } i \in I, \tag{40}$$

where $\kappa_i > 0$ is a measure of the speed of response of i and the initial point satisfies $\mathbf{x}(0) \geq \mathbf{0}$. (See Moulin [39], for example.) Hahn [29] established asymptotic stability of this process to the unique equilibrium of the Cournot oligopoly game under conditions which ensure decreasing replacement functions. These results were extended by Al-Nowaihi and Levine [1] and, more recently, by Dastidar [20]. Discrete best-response dynamics for aggregative games have recently been analyzed by Kukushkin [34] for finite strategy spaces and "better-response" dynamics[32] by Dindoš and Mezetti [23] for interval strategy spaces.

Any strategy profile $\widehat{\mathbf{x}}$ is a Nash equilibrium if and only if it is a rest point of the dynamics (40). Furthermore, if $\widehat{X}_{-i}$ exceeds the dropout value of Player i, then $b_i(X_{-i}) = 0$. Within any small enough neighbourhood of the equilibrium, $x_i(t) = x_i(0)\exp(-\kappa_i t)$. In particular, $x_i(t)$ is non-negative and approaches $\widehat{x}_i = 0$. This observation allows us to focus our analysis on players which are active in equilibrium, with the exception of any inactive player i for which $\widehat{x}_i$ is exactly equal to their dropout value, $\overline{X}_i$. In such a case, it is also possible for the right hand side of (40) to have discontinuous derivatives at $\widehat{\mathbf{x}}$ and this, in turn, may induce multiple solutions of the differential equations. We call any strategy profiles for which $\widehat{x}_i = \overline{X}_i$ for some $i \in I$ *critical.* It is conventional to analyze such cases by solving the equations either side of the point and stitching the solutions together at the boundary. To avoid the consequent complications, we confine our discussion to non-critical profiles.

Qualitative properties of a trajectories (40) in the neighborhood of a regular strategy profile $\mathbf{x}^*$ can be obtained by linearization about that point. In particular, provided the Jacobian $\mathbf{J}(\mathbf{x}^*)$ of the right hand side of (40) is non-singular, asymptotic stability of the linearized system implies asymptotic stability of (40). In the next proposition, we give conditions on the replacement functions to ensure this. Recall that regular player i of a smooth game has a replacement functions that is differentiable in $(\underline{X}_i, \infty)$, except possibly at $X = \overline{X}_i$, if this is finite.

Proposition 10.1 *Suppose that all players have a differentiable replacement function at the non-critical strategy profile $\mathbf{x}^*$, then*

$$\sum_{j \in I} r_j'(X^*) \neq 1$$

implies that $\mathbf{J}(\mathbf{x}^*)$ is non-singular.

The proof uses the following lemma.

Lemma 10.1 *If $a_j \neq -1$ for $j = 1, \ldots, m$ and*

$$\sum_{j=1}^{m} \frac{a_j}{1 + a_j} \neq 1, \tag{41}$$

[32]A randomly chosen player randomly selects a strategy and switches to it if and only if it results in an improved payoff for that player.

the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & a_1 & \cdots & a_1 \\ a_2 & -1 & \cdots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_m & a_m & \cdots & -1 \end{pmatrix}$$

is non-singular.

Proof. If $\mathbf{A}$ were singular, there would be a non-zero m-vector $\mathbf{z}$ satisfying $\mathbf{Az} = \mathbf{0}$. These equations can be written

$$z_j = \frac{a_j}{1 + a_j} Z, \text{ for } j = 1, \ldots, m,$$

where $Z = \sum_{j=1}^{m} z_j$. Summing over j and using (41) would give $Z = 0$ and hence $\mathbf{z} = \mathbf{0}$, a contradiction. ■

Proof of Proposition 10.1. Under the suppositions of the proposition, $\mathbf{J}(\mathbf{x}^*)$ is given by $\mathbf{A}$ in the lemma, where the off-diagonal elements in row i are $b_i'\left(\sum_{j \neq i} x_j^*\right)$. Note that

$$r_i(X) = b_i(X - r_i(X))$$

and differentiating with respect to X at $\mathbf{x}^*$ and rearranging gives

$$b_i'\left(\sum_{j \neq i} x_j^*\right) = \frac{r_i'(X^*)}{1 - r_i'(X^*)}.$$

It follows that $b_i'\left(\sum_{j \neq i} x_j^*\right) \neq -1$ for all i and the sum in (41) is $\sum_{j \in I} r_j'(X^*)$. The proposition is now an immediate application of the lemma. ■

Note that

$$r_i'(X) = X s_i'(X) + s_i(X). \tag{42}$$

If $\widehat{\mathbf{x}}$ is an equilibrium, we can sum over i and use Lemma 3.3 to deduce that, if the slopes of all share functions are non-positive and strictly negative for at least one player,

$$\sum_{j \in I} r_j'\left(\widehat{X}\right) < 1, \tag{43}$$

so linearization can be applied. But this holds at any non-critical equilibrium of a smooth game with players whose payoffs satisfy **A1** and **A2**, since then $s_i'\left(\widehat{X}\right) \leq 0$ for all i and $s_i\left(\widehat{X}\right) > 0$ for at least one i, which implies that $s_i'\left(\widehat{X}\right) < 0$. We use this linearization in the proof of the next theorem which gives a sufficient condition for asymptotic stability.

Theorem 10.1 *Suppose $\widehat{\mathbf{x}}$ is a non-critical equilibrium of a smooth aggregative game $\mathcal{G} = \left(I, \mathbf{w}, \{v_i\}_{i\in I}\right)$ in which **A1** and **A2** hold for all players, of which m are active. If $mr_i'\left(\widehat{X}\right) < 1$ for all $i \in I$, then $\widehat{\mathbf{x}}$ is locally asymptotically stable.*

The proof uses the following lemma.

Lemma 10.2 *The matrix*

$$\mathbf{B} = \begin{pmatrix} -b_1 & c & \cdots & c \\ c & -b_2 & \cdots & c \\ \vdots & \vdots & \ddots & \vdots \\ c & c & \cdots & -b_m \end{pmatrix}$$

is negative definite if

1. *$c = -1$ and $b_1, \ldots, b_m > 1$, or*
2. *$c = 1$ and $b_1, \ldots, b_m > m - 1$.*

Proof. Suppose that $\mathbf{z} \neq \mathbf{0}$.

1. We can write

$$\mathbf{z}^T\mathbf{B}\mathbf{z} = -\sum_{i=1}^{m} (b_i - 1)\, z_i^2 - \left(\sum_{i=1}^{m} z_j\right)^2 < 0.$$

2. In this case, we have

$$\begin{aligned}\mathbf{z}^T\mathbf{B}\mathbf{z} &= \left(\sum_{i=1}^{m} z_i\right) - \sum_{i=1}^{m} (b_i + 1)\, z_i^2 \\ &< \left(\sum_{i=1}^{m} z_i\right)^2 - m\sum_{i=1}^{m} z_i^2 \\ &= -\sum_{\substack{i,j=1 \\ i\neq j}}^{m} (z_i - z_j)^2 \leq 0.\end{aligned}$$

In either case, $\mathbf{B}$ is negative definite. ■

Proof of Theorem 10.1. We have already observed that, if $\widehat{x}_i = 0$, then $x_i(t) \longrightarrow \widehat{x}_i$ as $t \longrightarrow \infty$, in a neighborhood of $\widehat{\mathbf{x}}$. This solution can be substituted in the right hand side of (40) to give a time-dependent system which approaches the system with x_1 eliminated as $t \longrightarrow \infty$. We use the result that, provided the process with x_1 replaced by its limiting value is asymptotically convergent the same is true of the full process. In this way we can remove all such variables and assume, without loss of generality, that $a_i > 0$ for all $i = 1, \ldots, m$ and $a_i = 0$ for $i > m$, if $|I| > m$. The linearization of the process involving the first m components can be written $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ where $\mathbf{A}$ is in Lemma 10.1 with $a_i = b_i'\left(\widehat{X}\right)$ and $\mathbf{y} = \mathbf{x} - \widehat{\mathbf{x}}$. If $m = 1$, convergence of these equations is obvious, so assume that $m \geq 2$.

Consider the Lyapunov function

$$V(\mathbf{z}) = \frac{1}{2}\sum_{i=1}^{m} z_i^2 / \kappa_i |a_i|.$$

It is clear that $V(\mathbf{z}) = 0$ if and only if $\mathbf{z} = \mathbf{0}$ and we complete the proof by showing that V is decreasing on trajectories. This is done by noting that

$$\dot{V} = \nabla V(\mathbf{y}) \frac{d\mathbf{y}}{dt} = \mathbf{y}^T \mathbf{D} \mathbf{y},$$

for $\mathbf{y} \neq \mathbf{0}$, where

$$\mathbf{D} = \begin{pmatrix} -1/|a_1| & \cdots & -1 & -1 & \cdots & -1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -1 & \cdots & -1/|a_{m'}| & -1 & \cdots & -1 \\ 1 & \cdots & 1 & -1/|a_{m'+1}| & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 1 & \cdots & -1/|a_m| \end{pmatrix}$$

and we have arranged the rows of $\mathbf{A}$ so that $a_i < 0$ for $i \leq m$ and $a_i > 0$ for $i > m$. Note that

$$\frac{\mathbf{D}+\mathbf{D}^{\mathbf{T}}}{2} = \begin{pmatrix} \mathbf{D}^- & \mathbf{0} \\ \mathbf{0}^T & \mathbf{D}^+ \end{pmatrix},$$

where $\mathbf{0}$ is an $m' \times (m - m')$ matrix of zeroes,

$$\mathbf{D}^- = \begin{pmatrix} -1/|a_1| & \cdots & -1 \\ \vdots & \ddots & \vdots \\ -1 & \cdots & -1/|a_{m'}| \end{pmatrix}$$

and

$$\mathbf{D}^{+} = \begin{pmatrix} -1/\left|a_{m'+1}\right| & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & -1/\left|a_m\right| \end{pmatrix}.$$

It follows that $\dot{V} < 0$ for all $\mathbf{y} \neq \mathbf{0}$ if and only if $\mathbf{D}^{-}$ and $\mathbf{D}^{+}$ are negative definite. (Obvious modifications can be made to cover the case when all off-diagonal elements of $\mathbf{A}$ have the same sign.)

To establish negative definiteness for $\mathbf{D}^{-}$, observe that, if $i \leq m'$, then

$$a_i = r_i'\left(\widehat{Y}\right) / \left[1 - r_i'\left(\widehat{Y}\right)\right] < 0$$

which implies that $a_i < -1$, since $r_i'\left(\widehat{Y}\right) < 1/2$ by supposition. Hence, $1/\left|a_i\right| > 1$ and Part 1 of Lemma 10.2 can be applied. For $\mathbf{D}^{+}$, if $i \geq m'+1$, then $a_i > 0$ and $r_i'\left(\widehat{Y}\right) < 1/m$ implies that $0 < a_i < 1/(m-1)$. This allows us to apply the second part of the lemma to $\mathbf{D}^{+}$ and completes the proof. ■

The condition in Theorem 10.1 is satisfied *a fortiori* if $r_i'\left(\widehat{X}\right) \leq 0$ and this is entailed by decreasing best responses. Thus, equilibria of smooth Smash and Grab games as well as public good contribution games are locally stable. In a Cournot oligopoly game with twice continuously differentiable demand and costs, $r_i'\left(\widehat{X}\right) \leq 0$ holds if $c_i''(x) \leq 0$ for all $x > 0$ and, for all $X > 0$, $p'(X) < 0$ and

$$p'(X) + Xp''(X) < 0. \tag{44}$$

These are the conditions used by Hahn [29], but they are quite restrictive. For example, constant-elasticity demand is excluded as is rent-seeking, even if players are risk neutral.

In the case of Cournot oligopoly, (44) can be weakened. Indeed, a sufficient condition for $r_i'(X) < 1/m$ is $p'(X) < 0$ for all $X > 0$ and the following modification of condition (34) for regularity:

$$c_i''(x) > \max\left\{p'(X), (m+1)p'(X) + mXp''(X)\right\} \tag{45}$$

for $0 < x \leq X$. To see this, note that, if positive, $r_i(X)$ is the unique x_i satisfying $\gamma_i(x_i, X) = 0$, where γ_i is given by (33). Differentiating with respect to X and solving for r_i', we have

$$r_i'(X) = \frac{p'(X) + r_i(X)\, p''(X)}{c_i''(r_i(X)) - p'(X)}$$

and (45) implies $r_i'(X) < 1/m$. For example, if the inverse demand function is $p(X) = X^{-1/\eta}$ and the cost function is convex, (45) holds if $\eta > m$.

Note however, that $r_i'(X) < 1/m$ is only required to hold at equilibrium, which can expand the set of parameters for which that equilibrium is stable. With constant-elasticity demand, suppose Player i has constant marginal cost c_i for all i. Then, the replacement function satisfies

$$r_i(X) = \max\left\{\eta\left[X - c_i X^{(1+\eta)/\eta}\right], 0\right\}.$$

If there are m active players, the equilibrium condition says that

$$\eta\left[m\widehat{X} - \left(\sum_{i=1}^{m} c_i\right)\widehat{X}^{(1+\eta)/\eta}\right] = \widehat{X}, \tag{46}$$

where we label the costs of the active players $c_1, \ldots, c_m$. Further, the condition in Theorem 10.1 can be written:

$$r_i'\left(\widehat{X}\right) = \eta - (1+\eta)\, c_i \widehat{X}^{1/\eta} < \frac{1}{m}. \tag{47}$$

Solving (46) for $\widehat{X}$, substituting in (47) and simplifying the resulting inequality gives:

$$c_i > \frac{\eta\bar{c}}{1+\eta},$$

for active players i, where $\bar{c}$ is the average marginal cost over all active players. Thus, provided marginal costs of active players do not vary too much, equilibria are stable.

When all active players have the same payoffs, even weaker conditions apply. To see this, observe that share functions of players with payoffs satisfying **A1** and **A2** have negative slope, that shares sum to one in equilibrium and sum (42) over active players to get $mr_i'(X) < 1$. Hence, regularity is sufficient for stability in such a game.

Corollary 10.1 *Suppose that **A1** and **A2** hold in a smooth game and all active players are identical, then the equilibrium is locally asymptotically stable.*

Consider a game with a continuous aggregate share function. Were this to be non-decreasing at an equilibrium, there must be at least one more equilibrium. For such games, uniqueness entails a strictly decreasing aggregate share function at equilibrium, which must therefore be stable. Note that the assumption of identical active players may be a good one for large games in which players fall into a finite set of types, all of which have distinct finite dropout points, for then, once there are enough players of the type with largest dropout point, only they are active and the corollary applies. It is interesting to compare these observations with the result of Dastidar [20] that the equilibrium of almost all symmetric Cournot oligopoly games with unique equilibria is stable.

11 Conclusions

We have explored sufficient conditions on payoffs in aggregative games which ensure a unique (non-null) equilibrium, benign comparative statics and desirable large game limits and illustrated the application of these results to several classes of aggregative games. These conditions are almost the weakest possible requirements on individual payoffs. We have also demonstrated how these conditions can be tested when payoffs are sufficiently smooth and investigated the stability of the unique equilibrium for such payoffs. The main tool in our approach is the share function. In fact share functions and correspondences have wider applicability than our use of them in this paper would suggest. For example, [12] studies rent dissipation in a sequential game with entry costs and [13] examines efficient rules for sharing the surplus of a joint production game. In both cases, share functions are the essential analytical tool for deriving the results. In some cases, share functions have to be replaced with correspondences. For example, an application to the analysis of (multiple) equilibria of Tullock rent-seeking contests where the "production function" exhibits increasing returns to scale is given in [15].

Since the present treatment has focussed particularly on the task of identifying well-behaved games, we should emphasize two features of our treatment. First, the use of replacement and share correspondences is much more widely applicable. Any game with aggregative structure can be – and, in our view, is most easily – modelled using our approach. This observation applies not only to well-behaved, but also to 'badly-behaved', games that possess multiple pure strategy equilibria. In particular, our approach can handle games of strategic substitutes, games with strategic complements, and those that fit into neither of these categories. In contrast to other treatments, we do not need to provide separate treatments of these families of games.

Another potential extension is to games in which payoffs depend on the strategies of rivals through some (common) function other than the sum of all strategies. In some cases, a transformation of strategy spaces and payoffs can restore aggregativity. An application is given above in Subsection 6.5 for the case of contests in which the production function f_i is non-linear. Indeed, in the case of a smooth game, it can be shown that for replacement and share functions to exist, such a transformation must be possible. However, where there are kinks in payoffs (as in weakest-link problems, where payoffs depend on own strategy and the minimum of all strategies) there may be no share function. Nevertheless, share correspondences may still be used to analyze such games and, indeed, completely characterize the set of equilibria in both weakest-link and best-shot games. More general aggregation functions are also considered by Dubey et al [26], who, however, use (pseudo-)potential functions to conduct their analysis.

Finally, in some games payoffs depend on more than one aggregative function. It may still be possible to adapt the methods used above. In particular, by isolating aggregative sub-games which can be analyzed as above and then imposing consistency in the overall game, existence, uniqueness, comparative statics, and large-game limits can be studied. Hartley and Dickson [30] apply this approach to obtain a number of novel results in market games with a single product and Cornes et al [16] consider games in which groups contribute to "local" public goods that also contribute to a global public good entering the payoffs of all players.

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