UNIVERSITY OF NOTTINGHAM

Discussion Papers in Economics

Discussion Paper
No. 06/11

Simple, Robust and Powerful Tests of the Breaking Trend Hypothesis*

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July 2006

2006 DP 06/11

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Abstract

In this paper we develop a simple procedure which delivers tests for the presence of a broken trend in a univariate time series which do not require knowledge of the form of serial correlation in the data and are robust as to whether the shocks are generated by an $I(0)$ or an $I(1)$ process. Two trend break models are considered: the first holds the level fixed while allowing the trend to break, while the latter allows for a simultaneous break in level and trend. For the known break date case our proposed tests are formed as a weighted average of the optimal tests appropriate for $I(0)$ and $I(1)$ shocks. The weighted statistics are shown to have standard normal limiting null distributions and to attain the Gaussian asymptotic local power envelope, in each case regardless of whether the shocks are $I(0)$ or $I(1)$. In the unknown break date case we adopt the method of Andrews (1993) and take a weighted average of the statistics formed as the supremum over all possible break dates, subject to a trimming parameter, in both the $I(0)$ and $I(1)$ environments. Monte Carlo evidence suggests that our tests are in most cases more powerful, often substantially so, than the robust broken trend tests of Sayginsoy and Vogelsang (2004). An empirical application highlights the practical usefulness of our proposed tests.

Keywords: Broken trend; power envelope; unit root; stationarity tests.

JEL Classification: C22.

*We are grateful to Peter Phillips, Patrick Marsh and participants at the A.R. Bergstrom Memorial Conference held at the University of Essex, 24th and 25th May, 2006, for helpful comments on an earlier version of this paper. Correspondence to: Robert Taylor, School of Economics, University of Nottingham, University Park, Nottingham, NG7 2RD, U.K. E-mail: Robert.Taylor@nottingham.ac.uk

1 Introduction

The focus of this paper is on testing for structural change in the trend function of a univariate time series. This is an important practical problem because the typical macroeconomic series appears to be characterised by temporary ($I(0)$) or permanent ($I(1)$) shocks fluctuating around a broken (segmented) trend: see, *inter alia*, Stock and Watson (1996,1999,2005) and Perron and Zhu (2005). It is clearly important to adequately model the trend function and failure to do so will lead to inconsistent estimates and poor forecasts. A further interesting application of testing for structural change in the trend function is discussed in Sayginsoy and Vogelsang (2004) [SV, hereafter], and concerns the important empirical debate as to whether convergence in per capita incomes among U.S. regions levelled off in the mid-1970s, which can be explored by modelling the trend function in each region as having a slope shift in the mid-1970s; see SV for a number of key references in this literature. Segmented trends have also been fruitfully employed in the continuous time macroeconomic modelling literature by Nowan (1998), extending earlier work in Bergstrom *et al.* (1992).

Formal testing of whether a time series contains a broken trend function is greatly complicated by the fact that in practice it is not known whether the driving shocks are $I(0)$ or $I(1)$. If one knew that the shocks were $I(0)$ then one could test for structural change in the trend function based on the level of the data. Similarly, if it were known that the shocks were $I(1)$ then one could perform structural change tests on the first differences of the data (growth rates). However, tests based on growth rates display very poor power properties relative to those based on levels (see Theorems 1 and 3 below) when the shocks are in fact $I(0)$, as is discussed in a wider context in Vogelsang (1998). Moreover, as is shown later, the large sample null distributions of tests on the parameters of the trend function in levels data depend on whether the shocks are $I(0)$ or $I(1)$.

It is also well known that un-modelled trend breaks can bias unit root tests towards the non-rejection of the unit root hypothesis when the errors are $I(1)$ (see, *inter alia*, Perron, 1989), while including unnecessary broken trends greatly reduces power to reject the unit root null under $I(0)$ errors (see, for example, Marsh, 2005). Similarly, un-modelled trend breaks also cause spurious rejections in stationarity tests, such as that of Kwiatkowski *et al.* (1990)[KPSS, hereafter]. Where the potential trend break date is known, Perron (1989) shows that pivotal unit root inference can be achieved by including appropriate dummy variables in the relevant unit root regression. However, where the potential break date is unknown, as will usually be the case in practice, existing unit root tests which are based on search procedures, such as those of Zivot and Andrews (1992), are not similar, even asymptotically, (i.e. do not have pivotal limiting null distributions in the presence of trend breaks) with respect to the magnitude of the trend break, and often display poor power against $I(0)$ shocks; see, *inter alia*, Nunes *et al.* (1997) and Vogelsang and Perron (1998). A circular testing problem therefore arises between tests on the parameters of the trend function and unit root/stationarity tests, as might also be expected in the light of the theoretical results in Phillips (1998).

In this paper we propose powerful and serial correlation robust tests for the presence of a structural break in the trend function of a univariate time series process. Our proposed tests do not require knowledge of the form of serial correlation in the data; in particular, no prior knowledge is needed as to whether the shocks are $I(0)$ or $I(1)$, thereby breaking the circular testing problem discussed above. Our test statistics are formed as a weighted average of the regression t-statistics for a broken trend appropriate for the case of $I(0)$ and $I(1)$ shocks; that is, a weighted average of the trend break t-statistics from a regression in levels and a regression in growth rates. The weighting function we employ is based on the KPSS stationarity statistics applied to the levels and growth rate data. In the known break date case the trend function and KPSS statistics are based around the true break date and the resulting weighted statistics have standard normal limiting null distributions and achieve the relevant Gaussian asymptotic local power envelope under both $I(0)$ and $I(1)$ shocks. Where the break date is unknown we follow Andrews (1993) and take the supremum of the trend function t-statistics, calculated for all possible break dates (subject to trimming at the ends of the sample), for both the $I(0)$ and $I(1)$ environments. In this case the KPSS statistics used in the weighting function are evaluated using an estimator of the breakpoint which is consistent regardless of whether the shocks are $I(0)$ or $I(1)$. A correction, of the form used in Vogelsang (1998), is required in the unknown break date case to ensure that, for a given significance level, the weighted test has the same asymptotic critical value regardless of whether the shocks are $I(0)$ or $I(1)$. In both the known and unknown break date settings our proposed tests are made robust to short memory serial correlation in the shocks via the use of standard non-parametric long run [LR] variance estimators.

The remainder of the paper is organised as follows. Section 2 introduces our basic trend break model and outlines the assumptions underlying the model. Section 3 outlines our proposed test statistics for a broken trend, both for the known and unknown break date cases, and establishes the large sample properties of these statistics. In section 4 we extend the reference model of section 2 to allow for the possibility of a simultaneous break in level and trend, and develop corresponding test statistics for this case. Practical issues relating to the computation of our proposed statistics, including tabulations of relevant critical values and scaling constants, are discussed in section 5. In section 6 we present an evaluation of the finite sample size and power properties of our proposed tests, comparing these to the tests advocated in SV. Section 7 provides an empirical application to a variety of U.S. macroeconomic and financial data. Section 8 concludes. Proofs of our key results are gathered in a mathematical appendix.

In what follows we use the following notation: '$x := y$' ('$x =: y$') to indicate that x is defined by y (y is defined by x); $\lfloor \cdot \rfloor$ to denote the integer part of the argument; '$\xrightarrow{p}$' and '$\xrightarrow{d}$' denote convergence in probability and weak convergence, respectively, as the sample size diverges to positive infinity; $\mathbb{I}(\cdot)$ to denote the indicator function, and $N(a, b)$ to denote a Gaussian distribution with mean a and variance b. Finally, reference to a variable being $O_p(T^k)$ is taken to hold in its strict sense, meaning that the variable is not $o_p(T^k)$.

2 The Trend Break Model

Initially we consider the following trend break data generation process (DGP), referred to as "Model A" in what follows:

$$y_t = \alpha + \beta t + \gamma DT_t(\tau^*) + u_t, \quad t = 1, ..., T, \tag{1}$$
$$u_t = \rho u_{t-1} + \varepsilon_t, \quad t = 2, ..., T, \quad u_1 = \varepsilon_1 \tag{2}$$

where $DT_t(\tau^*) := \mathbb{I}(t > T^*)(t - T^*)$, with $T^* := \lfloor \tau^* T \rfloor$ the (potential) trend break date with associated break fraction $\tau^* \in (0, 1)$. The error process ε_t in (2) is taken to be an $I(0)$ process, satisfying Assumption 1 below. An extended version of this model which allows for a simultaneous break in level will be considered subsequently in section 4. In (1), a break in trend occurs at time T^* when $\gamma \neq 0$. The $I(0)$ scenario for u_t is represented by $|\rho| < 1$ in (2), while the $I(1)$ scenario obtains for $\rho = 1$. Our interest in this paper therefore centres on testing the null hypothesis $H_0 : \gamma = 0$ against the two-sided alternative hypothesis[1] $H_1 : \gamma \neq 0$, without assuming knowledge of whether u_t in (1) is $I(0)$ or $I(1)$.

We assume in what follows that ε_t in (2), satisfies Assumption 1 of SV (2005,pp.2-3); that is,

Assumption 1. *The stochastic process* $\{\varepsilon_t\}$ *is such that*

$$\varepsilon_t = c(L)\eta_t, \quad c(L) = \sum_{i=0}^{\infty} c_i L^i$$

with $c(1)^2 > 0$ *and* $\sum_{i=0}^{\infty} i|c_i| < \infty$*, and where* $\{\eta_t\}$ *is a martingale difference sequence with unit conditional variance and* $\sup_t E(\eta_t^4) < \infty$*.*

Remark 1. Under the conditions of Assumption 1, the LR variance of ε_t is given by $\omega_\varepsilon^2 := \lim_{T\to\infty} T^{-1} E(\sum_{t=1}^{T} \varepsilon_t)^2 = c(1)^2$. Moreover, in the $I(0)$ case the LR variance of u_t is given by $\omega_u^2 := \lim_{T\to\infty} T^{-1} E(\sum_{t=1}^{T} u_t)^2 = \omega_\varepsilon^2/(1-\rho)^2$. Both these LR variances play important roles in our subsequent analysis.

3 Tests for a Break in Trend

3.1 Known Break Fraction

In this section we consider first the case where the true break fraction, τ^*, is known. The case where the break fraction is unknown will be subsequently discussed in section 3.2. Under a known break fraction, we may partition H_1 into two scaled components:

[1] One-sided hypotheses can also be accommodated within our framework. However, we shall not discuss such tests further as it seems unlikely that the direction of any trend break would be known to the practitioner, *a priori*, particularly in the case of an unknown break date.

$H_{1,0} : \gamma = \kappa T^{-3/2}$ when u_t is $I(0)$, and $H_{1,1} : \gamma = \kappa T^{-1/2}$ when u_t is $I(1)$, where in each case κ is a finite non-negative constant. As we shall see below, these provide the appropriate Pitman drifts on γ under $I(0)$ and $I(1)$ errors, respectively. Notice that both $H_{1,0}$ and $H_{1,1}$ reduce to H_0 when $\kappa = 0$.

Consider first the case where u_t in (2) is known to be $I(0)$ with $\rho = 0$ in (2) and ε_t a Gaussian white noise. Here the optimal (uniformly most powerful unbiased) test of H_0 against H_1 rejects for large values of the absolute value of the t-ratio associated with γ when (1) is estimated using OLS. That is, $|t_0(\tau^*)|$ where[2]

$$t_0(\tau^*) := \frac{\hat{\gamma}(\tau^*)}{\sqrt{\hat{\sigma}^2(\tau^*) \left[\{ \sum_{t=1}^{T} x_{DT,t}(\tau^*) x_{DT,t}(\tau^*)' \}^{-1} \right]_{33}}}, \tag{3}$$

$$\hat{\gamma}(\tau^*) := \left[\left\{ \sum_{t=1}^{T} x_{DT,t}(\tau^*) x_{DT,t}(\tau^*)' \right\}^{-1} \sum_{t=1}^{T} x_{DT,t}(\tau^*) y_t \right]_3$$

with $x_{DT,t}(\tau^*) := \{1, t, DT_t(\tau^*)\}'$, $\hat{\sigma}^2(\tau^*) := T^{-1} \sum_{t=1}^{T} \hat{u}_t(\tau^*)^2$ and $\hat{u}_t(\tau^*) := y_t - \hat{\alpha} - \hat{\beta} t - \hat{\gamma}(\tau^*) DT_t(\tau^*)$.

Correspondingly, if u_t is known to be $I(1)$, so that $\rho = 1$ in (2), and Δu_t is a Gaussian white noise process, then the optimal test is based on the absolute value of the t-ratio associated with γ when (1) is estimated via OLS in first differenced form. That is, writing

$$\Delta y_t = \beta + \gamma DU_t(\tau^*) + \Delta u_t, \quad t = 2, ..., T \tag{4}$$

where $DU_t(\tau^*) := \mathbb{I}(t > T_1)$, the optimal test rejects for large values of $|t_1(\tau^*)|$, where

$$t_1(\tau^*) := \frac{\tilde{\gamma}(\tau^*)}{\sqrt{\tilde{\sigma}^2(\tau^*) [\{ \sum_{t=2}^{T} x_{DU,t}(\tau^*) x_{DU,t}(\tau^*)' \}^{-1}]_{22}}}, \tag{5}$$

$$\tilde{\gamma}(\tau^*) := \left[\left\{ \sum_{t=2}^{T} x_{DU,t}(\tau^*) x_{DU,t}(\tau^*)' \right\}^{-1} \sum_{t=2}^{T} x_{DU,t}(\tau^*) \Delta y_t \right]_2$$

with $x_{DU,t}(\tau^*) := \{1, DU_t(\tau^*)\}'$, $\tilde{\sigma}^2(\tau^*) := (T-1)^{-1} \sum_{t=2}^{T} \tilde{v}_t(\tau^*)^2$, and $\tilde{v}_t(\tau^*) := \Delta y_t - \tilde{\beta} - \tilde{\gamma}(\tau^*) DU_t(\tau^*)$.

In order to deal with more general $I(0)$ and $I(1)$ processes for u_t, as are allowed under Assumption 1, we need to replace $\hat{\sigma}^2(\tau^*)$ and $\tilde{\sigma}^2(\tau^*)$ in the definitions of $t_0(\tau^*)$ of (3) and $t_1(\tau^*)$ of (5) with corresponding non-parametric LR variance estimators,

[2] The notation $[.]_{jj}$ ($[.]_j$) is used to denote the jj'th (j'th) element of the matrix (vector) within the square brackets.

$\hat{\omega}^2(\tau^*)$ and $\tilde{\omega}^2(\tau^*)$, respectively, which are given by

$$\hat{\omega}^2(\tau^*) \;:=\; \hat{\gamma}_0(\tau^*) + 2\sum_{j=1}^{T-1} h(j/l)\hat{\gamma}_j(\tau^*), \;\; \hat{\gamma}_j(\tau^*) := T^{-1}\sum_{t=j+1}^{T} \hat{u}_t(\tau^*)\hat{u}_{t-j}(\tau^*) \quad (6)$$

$$\tilde{\omega}^2(\tau^*) \;:=\; \tilde{\gamma}_0(\tau^*) + 2\sum_{j=1}^{T-2} h(j/l)\tilde{\gamma}_j(\tau^*), \;\; \tilde{\gamma}_j(\tau^*) := (T-1)^{-1}\sum_{t=j+2}^{T} \tilde{v}_t(\tau^*)\tilde{v}_{t-j}(\tau^*). \quad (7)$$

In the context of (6) and (7), $h(\cdot)$ is a kernel function with associated bandwidth parameter ℓ. In what follows we shall make use of the Bartlett kernel for $h(\cdot)$, such that $h(j/\ell) := 1 - j/(\ell+1)$, with bandwidth parameter $\ell = O(T^{1/4})$.[3] In the sequel, unless otherwise stated, any reference to $t_0(\tau^*)$ or $t_1(\tau^*)$ will be taken to imply those based on the LR variance estimators in (6) and (7). Other choices of the kernel and bandwidth parameter could also be used, however, provided they satisfy standard regularity conditions, such as are outlined in Assumptions A3 and either A4 or A4' of Jansson (2002,pp.1450,1452), respectively.

The following Theorem establishes the asymptotic behaviour of the $|t_0(\tau^*)|$ and $|t_1(\tau^*)|$ statistics under both $H_{1,0}$ and $H_{0,1}$.

Theorem 1 *Let the time series process $\{y_t\}$ be generated according to (1) and (2), and let Assumption 1 hold.*

(i) *If u_t in (2) is $I(0)$ (i.e. $|\rho| < 1$), then: (a) $|t_0(\tau^*)| \xrightarrow{d} |L_{00}(\tau^*, \kappa)|$, where*

$$L_{00}(\tau^*, \kappa) := \frac{\kappa\{\int_0^1 RT(r,\tau^*)^2 dr\}^{1/2}}{\omega_u} + \frac{\int_0^1 RT(r,\tau^*)dW(r)dr}{\{\int_0^1 RT(r,\tau^*)^2 dr\}^{1/2}},$$

and (b) $|t_1(\tau^*)| = O_p\{(l/T)^{1/2}\}$.

(ii) *If u_t in (2) is $I(1)$ (i.e. $\rho = 1$), then: (a) $|t_0(\tau^*)| = O_p\{(T/l)^{1/2}\}$, and (b) $|t_1(\tau^*)| \xrightarrow{d} |L_{11}(\tau^*, \kappa)|$ where*

$$L_{11}(\tau^*, \kappa) := \frac{\kappa\{\int_0^1 RU(r,\tau^*)^2 dr\}^{1/2}}{\omega_\varepsilon} + \frac{\int_0^1 RU(r,\tau^*)dW(r)dr}{\{\int_0^1 RU(r,\tau^*)^2 dr\}^{1/2}}$$

where $W(r)$ is a standard Brownian motion on $[0,1]$, and $RT(r,\tau^*)$ is the continuous-time residual from the projection of $(r-\tau^*)\mathbb{I}(r > \tau^*)$ onto the space spanned by $\{1, r\}$, and $RU(r,\tau^*)$ is the residual from the projection of $\mathbb{I}(r > \tau^*)$ onto $\{1\}$.

Remark 2. It is trivially seen from the results in Theorem 1 that under $H_0 : \kappa = 0$, $t_0(\tau^*) \xrightarrow{d} N(0,1)$ if u_t is $I(0)$, while $t_1(\tau^*) \xrightarrow{d} N(0,1)$ if u_t is $I(1)$. Consequently,

[3] Notice that $\hat{\gamma}_0(\tau^*) = \hat{\sigma}^2(\tau^*)$ and $\tilde{\gamma}_0(\tau^*) = \tilde{\sigma}^2(\tau^*)$.

with knowledge of the order of integration of u_t, the appropriate two-sided test can be implemented using critical values from the standard normal distribution.

Remark 3. From the results in part (i) of Theorem 1 it is seen that when u_t is $I(0)$ $|t_1(\tau^*)|$ converges in probability to zero, regardless of the value of κ, while $|t_0(\tau^*)|$ attains the Gaussian asymptotic local power envelope for this testing problem. Similarly, from the results in part (ii) of Theorem 1 it is seen that when u_t is $I(1)$, $|t_1(\tau^*)|$ achieves the $I(1)$ Gaussian asymptotic local power envelope, while $|t_0(\tau^*)|$ diverges irrespective of the value of κ. $\square$

In view of the above results, and given that the order of integration of u_t is not known in practice, it is a fairly natural step to consider constructing a procedure that employs some auxiliary routine which ensures that, asymptotically at least, the statistic $|t_0(\tau^*)|$ of (3) is selected when u_t is $I(0)$ while $|t_1(\tau^*)|$ of (5) is selected when u_t is $I(1)$, thereby ensuring that the asymptotically optimal test is selected in the limit. To that end we pursue an approach based on a data-dependent weighted average of $|t_0(\tau^*)|$ and $|t_1(\tau^*)|$ of the form

$$t_\lambda^* := \{\lambda(S_0(\tau^*), S_1(\tau^*)) \times |t_0(\tau^*)|\} + \{[1 - \lambda(S_0(\tau^*), S_1(\tau^*))] \times |t_1(\tau^*)|\} \tag{8}$$

In (8), $S_0(\tau^*)$ and $S_1(\tau^*)$ are auxiliary statistics chosen such that, as the sample size diverges to positive infinity, the weight function $\lambda(\cdot,\cdot)$ converges to unity when u_t is $I(0)$ and to zero when u_t is $I(1)$, such that t_λ^* will collapse to $|t_0(\tau^*)|$ when u_t is $I(0)$, and to $|t_1(\tau^*)|$ when u_t is $I(1)$. Because the auxiliary routine needs to be ambivalent between H_0 and H_1, the $S_0(\tau^*)$ and $S_1(\tau^*)$ statistics must also be invariant with respect to α, β and γ in (1).

We therefore need to chose appropriate auxiliary statistics, $S_0(\tau^*)$ and $S_1(\tau^*)$, and weight function, $\lambda(\cdot,\cdot)$. For the former we shall adopt the stationarity statistics of KPSS calculated from the residuals $\{\hat{u}_t(\tau^*)\}_{t=1}^T$ and $\{\tilde{v}_t(\tau^*)\}_{t=2}^T$, respectively, each of which are exact invariant to α, β and γ. Specifically,

$$S_0(\tau^*) := \frac{\sum_{t=1}^T \left(\sum_{i=1}^t \hat{u}_i(\tau^*)\right)^2}{T^2\hat{\omega}^2(\tau^*)}, \qquad S_1(\tau^*) := \frac{\sum_{t=2}^T \left(\sum_{i=2}^t \tilde{v}_t(\tau^*)\right)^2}{(T-1)^2\tilde{\omega}^2(\tau^*)} \tag{9}$$

where $\hat{\omega}^2(\tau^*)$ and $\tilde{\omega}^2(\tau^*)$ as as defined in (6) and (7) respectively. The relevant large sample properties of these two statistics are given in the following Lemma.

Lemma 1 *Let the conditions of Theorem 1 hold.*

(i) *If u_t is $I(0)$ then: (a) $S_0(\tau^*) = O_p(1)$, and (b) $S_1(\tau^*) = O_p(\ell/T)$.*

(ii) *If u_t is $I(1)$ then: (a) $S_0(\tau^*) = O_p(T/\ell)$, and (b) $S_1(\tau^*) = O_p(1)$.*

The results in Lemma 1 therefore suggest a weight function, $\lambda(\cdot,\cdot)$, of the form

$$\lambda(S_0(\tau^*), S_1(\tau^*)) := \exp[-\{g_1 S_0(\tau^*) S_1(\tau^*)\}^{g_2}] \tag{10}$$

where g_1 and g_2 are positive constants, since this will clearly converge to unity when u_t is $I(0)$ and to zero when u_t is $I(1)$, as required. Moreover, it does so at an exponential rate. Using the large sample results in Theorem 1 and Lemma 1, we are in a position to state the following Corollary.

Corollary 1 *Let the conditions of Theorem 1 hold.*

(i) *If u_t is $I(0)$, then $\lambda(S_0(\tau^*), S_1(\tau^*)) \xrightarrow{p} 1$ under both H_0 and $H_{1,0}$, and $t^*_\lambda = |t_0(\tau^*)| + o_p(1) \xrightarrow{d} |L_{00}(\tau^*, \kappa)|$.*

(ii) *If u_t is $I(1)$, then $\lambda(S_0(\tau^*), S_1(\tau^*)) \xrightarrow{p} 0$ under H_0 and $H_{1,1}$ and $t^*_\lambda = |t_1(\tau^*)| + o_p(1) \xrightarrow{d} |L_{11}(\tau^*, \kappa)|$.*

Remark 4. The results in Corollary 1 show that if u_t is $I(0)$, t^*_λ is asymptotically equivalent to $|t_0(\tau^*)|$, while if u_t is $I(1)$, t^*_λ is asymptotically equivalent to $|t_1(\tau^*)|$. Consequently, t^*_λ achieves the appropriate Gaussian asymptotic local power envelope regardless of whether u_t is $I(0)$ or $I(1)$. Moreover, under H_0, $t^*_\lambda \xrightarrow{d} |N(0,1)|$ irrespective of whether u_t is $I(0)$ or $I(1)$, so that a two-sided test can again be implemented using critical values from the standard normal distribution.

Remark 5. Notice from part (ii) of Corollary 1 that the product $\lambda(S_0(\tau^*), S_1(\tau^*)) \times |t_0(\tau^*)|$ is of $o_p(1)$ even though, as shown in part (ii) of Theorem 1, $|t_0(\tau^*)|$ diverges at rate $O_p\{(T/\ell)^{1/2}\}$. This result is due to our choice of weighting function $\lambda(S_0(\tau^*), S_1(\tau^*))$ of (10) which converges in probability to zero at an *exponential* rate in T when u_t is $I(1)$.

3.2 Unknown Break Fraction

We now consider the case where the true break fraction τ^* cannot be considered known, *a priori*. In this case we follow the approach of Andrews (1993) and consider statistics based on the maxima of the sequences of statistics[4] $\{|t_0(\tau)|, \tau \in \Lambda\}$ and $\{|t_1(\tau)|, \tau \in \Lambda\}$, where $\Lambda = [\tau_L, \tau_U]$, with $0 < \tau_L < \tau_U < 1$, where the quantities τ_L and τ_U will be referred to as the *trimming* parameters, and where it is assumed throughout that $\tau^* \in \Lambda$. Defining $\Lambda^* := \{\lfloor \tau_L T \rfloor, ..., \lfloor \tau_U T \rfloor\}$, these statistics are given by

$$t^*_0 := \sup_{s \in \Lambda^*} |t_0(s/T)| \tag{11}$$

and

$$t^*_1 := \sup_{s \in \Lambda^*} |t_1(s/T)|, \tag{12}$$

[4] Although we analyse tests based on the maxima of these sequences of statistics, tests based on the corresponding mean- or mean-exponential-type statistics of, *inter alia*, Hansen (1992) and Andrews and Ploberger (1994), respectively, could also be used.

with associated breakpoint estimators of τ^* given by $\hat{\tau} := \arg\sup_{s\in\Lambda^*} |t_0(s/T)|$ and $\tilde{\tau} := \arg\sup_{s\in\Lambda^*} |t_1(s/T)|$, respectively, such that $t_0^* \equiv |t_0(\hat{\tau})|$ and $t_1^* \equiv |t_1(\tilde{\tau})|$. The analogue of our t_λ^* statistic of (8) is then given by

$$t_\lambda := \{\lambda(S_0(\hat{\tau}), S_1(\hat{\tau})) \times t_0^*\} + m_\xi\{[1 - \lambda(S_0(\hat{\tau}), S_1(\hat{\tau}))] \times t_1^*\} \quad (13)$$

where m_ξ is a positive finite constant whose precise role is discussed below. Observe that both stationarity statistics are evaluated at the breakpoint estimator $\hat{\tau}$, this being a consistent estimator of τ^* regardless of whether u_t is $I(0)$ or $I(1)$.[5]

In the current context where the break fraction τ^* is unknown, it cannot be consistently estimated under the Pitman drift alternatives of the form considered in section 3.1. However, for the purposes of empirical work a rejection against a broken trend is clearly of rather limited use without a consistent estimate of where the break occurs. Consequently, we shall consider only fixed alternatives in this situation, where consistent estimation of the unknown break fraction is possible, establishing the consistency properties of our tests. However, in the case where u_t is $I(1)$ the test which rejects for large values of t_1^* has an equivalent critical region to the likelihood ratio-type test in a linear setting of Andrews (1993) and, as such, will possess the weak local optimality property of Andrews (1993, Equation (5.5)). This need not be true for a test based on t_0^* because of the presence of trending regressors.

We first establish the large sample behaviour of the t_0^* and t_1^* statistics under the null hypothesis, $H_0 : \gamma = 0$, when the shocks, u_t, are either $I(0)$ or $I(1)$.

Theorem 2 *Let the time series process $\{y_t\}$ be generated according to (1) and (2) under $H_0 : \gamma = 0$, and let Assumption 1 hold.*

(i) *If u_t is $I(0)$, then: (a) $t_0^* \xrightarrow{d} \sup_{\tau\in\Lambda} |L_{00}(\tau, 0)|$, and (b) $t_1^* = O_p\{(\ell/T)^{1/2}\}$.*

(ii) *If u_t is $I(1)$, then: (a) $t_0^* = O_p\{(T/\ell)^{1/2}\}$, and (b) $t_1^* \xrightarrow{d} \sup_{\tau\in\Lambda} |L_{11}(\tau, 0)|$.*

We now establish the consistency rates of these statistics under a fixed alternative of the form $H_1 : \gamma \neq 0$.

Theorem 3 *Let the time series process $\{y_t\}$ be generated according to (1) and (2) under $H_1 : \gamma \neq 0$, and let Assumption 1 hold.*

(i) *If u_t is $I(0)$, then: (a) $t_0^* = O_p(T^{3/2})$, and (b) $t_1^* = O_p\{(\ell T)^{1/2}\}$.*

(ii) *If u_t is $I(1)$, then: (a) $t_0^* = O_p(T/\ell^{1/2})$, and (b) $t_1^* = O_p(T^{1/2})$.*

Remark 6. It is interesting to note from the results in part (ii) of Theorem 3 that t_0^* diverges at a faster rate than t_1^* when u_t is $I(1)$, which may seem counterintuitive given that t_1^* would be thought of as the preferred test in this situation. However, it must

[5] An alternative, which makes no difference to the large sample results which follow, is to use the statistics $\inf_{s\in\Lambda^*} S_0(s/T)$ and $\inf_{s\in\Lambda^*} S_1(s/T)$ in place of $S_0(\hat{\tau})$ and $S_1(\tilde{\tau})$, respectively.

be borne in mind from part (ii) of Theorem 2 that t_0^* also diverges under H_0 when u_t is $I(1)$ while t_1^* has a well-defined critical region. □

In order to derive the asymptotic behaviour of the weighted statistic t_λ of (13) we must next establish the large sample behaviour of the $S_0(\hat{\tau})$ and $S_1(\hat{\tau})$ statistics. This is done in the following lemma, the proof of which is straightforward but tedious given results established in Lemma 1 and Theorems 2 and 3 and is therefore omitted in the interests of brevity.

Lemma 2 *Let the conditions of Theorem 1 hold.*

(i). *If u_t is $I(0)$, then: (a) $S_0(\hat{\tau}) = O_p(1)$, and (b) $S_1(\hat{\tau}) = O_p(\ell/T)$.*

(ii). *If u_t is $I(1)$, then: (a) $S_0(\hat{\tau}) = O_p(T/\ell)$, and (b) $S_1(\hat{\tau}) = O_p(1)$.*

An immediate corollary of the results in Lemma 2 is that, regardless of whether H_0 or H_1 holds, when u_t is $I(0)$, $\lambda(S_0(\hat{\tau}), S_1(\hat{\tau})) \xrightarrow{p} 1$, while if u_t is $I(1)$, $\lambda(S_0(\hat{\tau}), S_1(\hat{\tau})) \xrightarrow{p} 0$. Consequently, using the results in Theorems 2 and 3, we may state the following corollary concerning the large sample behaviour of our weighted statistic, t_λ of (13), which again exploits the fact that convergence in probability of $\lambda(S_0(\hat{\tau}), S_1(\hat{\tau}))$, either to unity or zero, occurs at an exponential rate.

Corollary 2 *Let the conditions of Theorem 1 hold.*

(i) *Let $H_0 : \gamma = 0$ hold. Then: (a) if u_t is $I(0)$, $t_\lambda = t_0^* + o_p(1) \xrightarrow{d} \sup_{\tau\in\Lambda} |L_{00}(\tau, 0)|$; (b) if u_t is $I(1)$, $t_\lambda = m_\xi t_1^* + o_p(1) \xrightarrow{d} m_\xi \sup_{\tau\in\Lambda} |L_{11}(\tau, 0)|$.*

(ii) *Let $H_1 : \gamma \neq 0$ hold. Then: (a) if u_t is $I(0)$, $t_\lambda = t_0^* + o_p(1) = O_p(T^{3/2})$; (b) if u_t is $I(1)$, $t_\lambda = m_\xi t_1^* + o_p(1) = O_p(T^{1/2})$.*

It is seen from the results in part (i) of Corollary 2 that, in contrast to the known breakpoint case considered in section 3.1, the asymptotic null distribution of the weighted statistic t_λ of (13) differs as to whether u_t is $I(0)$ or $I(1)$. Moreover, in neither case is this distribution standard normal. Similarly to Vogelsang (1998), however, we can choose the constant m_ξ in (13) such that, for a given significance level ξ under H_0, the critical value of $m_\xi \sup_{\tau\in\Lambda} |L_{11}(\tau, 0)|$ coincides with that of $\sup_{\tau\in\Lambda} |L_{00}(\tau, 0)|$. This then ensures that, for the chosen significance level, the asymptotic null critical value of t_λ is the same irrespective of whether u_t is $I(0)$ or $I(1)$. Under $H_1 : \gamma \neq 0$, it is seen from Corollary 2 that t_λ is consistent at rate $O_p(T^{3/2})$ when u_t is $I(0)$ and at rate $O_p(T^{1/2})$ when u_t is $I(1)$.

4 Allowing for a Simultaneous Break in Level

Although trend breaks are the central concern of this paper, we might also consider extending our analysis to allow (but not test for) a break in level occurring at the same time as the break in trend. To this end, consider replacing (1) with

$$y_t = \alpha + \beta t + \delta DU_t(\tau^*) + \gamma DT_t(\tau^*) + u_t, \quad t = 1, ..., T, \tag{14}$$

whose differenced form, corresponding to (4), is given by

$$\Delta y_t = \beta + \delta D_t(\tau^*) + \gamma DU_t(\tau^*) + \Delta u_t, \quad t = 2, ..., T, \tag{15}$$

where $D_t(\tau^*) := \mathbb{I}(t = T^*)$. The shocks, u_t, are still assumed to be generated according to (2). In what follows we will refer to (14) and (2) together as "Model B".

In section 4.1 we will initially consider the known break date case, with the unknown break date case subsequently discussed in section 4.2. In order to avoid unnecessarily complex notation, we will repeat the notation of section 3 for the quantities involved.

4.1 Known Break Fraction

For the known break fraction case we need place no restrictions on the value of δ under each of H_0, $H_{1,0}$ and $H_{1,1}$, these being defined exactly as in section 3.1. We consequently re-define $t_0(\tau^*)$ as follows:

$$\begin{aligned} t_0(\tau^*) &:= \frac{\hat{\gamma}(\tau^*)}{\sqrt{\hat{\omega}^2(\tau^*)\left[\{\sum_{t=1}^{T} x_{DT,t}(\tau^*)x_{DT,t}(\tau^*)'\}^{-1}\right]_{44}}}, \\ \hat{\gamma}(\tau^*) &:= \left[\left\{\sum_{t=1}^{T} x_{DT,t}(\tau^*)x_{DT,t}(\tau^*)'\right\}^{-1} \sum_{t=1}^{T} x_{DT,t}(\tau^*)y_t\right]_4 \end{aligned}$$

with $x_{DT,t}(\tau^*) := \{1, t, DU_t(\tau^*), DT_t(\tau^*)\}'$ and $\hat{\omega}^2(\tau^*)$ calculated as in (6) but using the OLS residuals $\hat{u}_t(\tau^*) := y_t - \hat{\alpha} - \hat{\beta}\, t - \hat{\delta} DU_t(\tau^*) - \hat{\gamma}(\tau^*)DT_t(\tau^*)$ from (14). Similarly, $t_1(\tau^*)$ is re-defined to be

$$\begin{aligned} t_1(\tau^*) &:= \frac{\tilde{\gamma}(\tau^*)}{\sqrt{\tilde{\omega}^2(\tau^*)\left[\{\sum_{t=2}^{T} x_{DU,t}(\tau^*)x_{DU,t}(\tau^*)'\}^{-1}\right]_{33}}}, \\ \tilde{\gamma}(\tau^*) &:= \left[\left\{\sum_{t=2}^{T} x_{DU,t}(\tau^*)x_{DU,t}(\tau^*)'\right\}^{-1} \sum_{t=2}^{T} x_{DU,t}(\tau^*)\Delta y_t\right]_3 \end{aligned}$$

with $x_{DU,t}(\tau^*) := \{1, D_t(\tau^*), DU_t(\tau^*)\}'$ and $\tilde{\omega}^2(\tau^*)$ calculated as in (7) but using the OLS residuals $\tilde{v}_t(\tau^*) := \Delta y_t - \tilde{\beta} - \tilde{\delta} D_t(\tau^*) - \tilde{\gamma}(\tau^*)DU_t(\tau^*)$ from (15).

In Theorem 4 we now establish the asymptotic behaviour of $|t_0(\tau^*)|$ and $|t_1(\tau^*)|$ under both $H_{1,0}$ and $H_{1,1}$. The proof of Theorem 4 is a straightforward generalization of that of Theorem 1 and is therefore omitted.

Theorem 4 *Let the time series process $\{y_t\}$ be generated according to (14) and (2), and let Assumption 1 hold.*

(i) *If u_t is $I(0)$, then: (a) $|t_0(\tau^*)| \xrightarrow{d} |L_{U,00}(\tau^*,\kappa)|$ where*

$$L_{U,00}(\tau^*,\kappa) := \frac{\kappa\{\int_0^1 RT_U(r,\tau^*)^2 dr\}^{1/2}}{\omega_u} + \frac{\int_0^1 RT_U(r,\tau^*)dW(r)dr}{\{\int_0^1 RT_U(r,\tau^*)^2 dr\}^{1/2}},$$

and (b) $|t_1(\tau^*)| = O_p\{(\ell/T)^{1/2}\}$.

(ii) *If u_t is $I(1)$, then: (a) $|t_0(\tau^*)| = O_p\{(T/\ell)^{1/2}\}$, and (b) $|t_1(\tau^*)| \xrightarrow{d} |L_{11}(\tau^*,\kappa)|$ where*

$$L_{11}(\tau^*,\kappa) := \frac{\kappa\{\int_0^1 RU(r,\tau^*)^2 dr\}^{1/2}}{\omega_\varepsilon} + \frac{\int_0^1 RU(r,\tau^*)dW(r)dr}{\{\int_0^1 RU(r,\tau^*)^2 dr\}^{1/2}}$$

where $RT_U(r,\tau^*)$ is a continuous-time residual from the projection of $(r-\tau^*)1(r>\tau^*)$ onto the space spanned by $\{1, r, 1(r>\tau^*)\}$, and $W(r)$ and $RU(r,\tau^*)$ are as defined in Theorem 1.

Remark 7. As with the results in Theorem 1 for Model A, it is trivially seen that $L_{U,00}(\tau^*,\kappa)$ follows a Gaussian distribution, reducing to a standard normal distribution under H_0 and attaining the Gaussian asymptotic local power envelope under $H_{0,1}$.

Remark 8. Observe from the result given in part (ii)(b) of Theorem 4 that the limiting distribution of $|t_1(\tau^*)|$ from Model B is identical to that for Model A given in Theorem 1 (ii)(b). This is because the regressor $D_t(\tau^*)$ has an asymptotically negligible effect on $|t_1(\tau^*)|$. Consequently, the comments made in Remarks 2 and 3 relating to the $|t_1(\tau^*)|$ statistic in the context of Model A when u_t is $I(1)$ also apply under Model B. □

In order to extend our t_λ^* statistic of (8) to the case of a simultaneous break in level, we re-define $S_0(\tau^*)$ and $S_1(\tau^*)$, $\lambda(S_0(\tau^*), S_1(\tau^*))$, and t_λ^* to be constructed as in (9), (10) and (8), respectively, but constructed using the re-defined OLS residuals from (14) and (15). It is entirely straightforward to demonstrate that the orders given in Lemma 1 for $S_0(\tau^*)$ and $S_1(\tau^*)$ remain appropriate in the case of a simultaneous level break. We may therefore state the following corollary.

Corollary 3 *Let the conditions of Theorem 4 hold.*

(i) *If u_t is $I(0)$, then under both H_0 and $H_{1,0}$, $t_\lambda^* = |t_0(\tau^*)| + o_p(1) \xrightarrow{d} |L_{U,00}(\tau^*,\kappa)|$.*

(ii) *If u_t is $I(1)$, then under both H_0 and $H_{1,1}$, $t_\lambda^* = |t_1(\tau^*)| + o_p(1) \xrightarrow{d} |L_{11}(\tau^*,\kappa)|$.*

Remark 9. As with the results for the break in trend only case, t_λ^* achieves the appropriate Gaussian asymptotic local power envelope regardless of whether u_t is $I(0)$ or $I(1)$. Moreover, we again have the result that $t_\lambda^* \xrightarrow{d} |N(0,1)|$ under H_0, irrespective of whether u_t is $I(0)$ or $I(1)$; cf. Remark 4.

4.2 Unknown Break Fraction

We now consider the case where τ^* is unknown in Model B. Here we proceed as in section 3.1, appropriately re-defining the various statistics involved to be formed from the OLS residuals from either (14) or (15), as appropriate.

As in SV (2004), the null hypothesis H_0 must be re-stated as $H_0 : \gamma = \delta = 0$, in the current context in order to obtain a pivotal limiting null distribution for our test statistic. The following theorem, whose proof is entirely similar to that of Theorem 2 and, hence, is omitted, details the large sample behaviour of the re-defined t_0^* and t_1^* statistics under H_0.

Theorem 5 *Let the time series process $\{y_t\}$ be generated according to (14) and (2) under $H_0 : \gamma = \delta = 0$, and let Assumption 1 hold.*

(i) *If u_t is $I(0)$, then: (a) $t_0^* \xrightarrow{d} \sup_{\tau\in\Lambda} |L_{U,00}(\tau, 0)|$, and (b) $t_1^* = O_p\{(\ell/T)^{1/2}\}$.*

(ii) *If u_t is $I(1)$, then: (a) $t_0^* = O_p\{(T/\ell)^{1/2}\}$, and (b) $t_1^* \xrightarrow{d} \sup_{\tau\in\Lambda} |L_{11}(\tau, 0)|$.*

For fixed alternatives of the form $H_1 : \gamma \neq 0$, with δ now unrestricted, it can be shown that the rates of divergence given in Theorem 3 for t_0^* and t_1^* remain appropriate here also, as do the rates of divergence of $S_0(\hat{\tau})$ and $S_1(\tilde{\tau})$ given in Lemma 2. Consequently, for t_λ of (13) we find that under H_0, when u_t is $I(0)$

$$t_\lambda = t_0^* + o_p(1) \xrightarrow{d} \sup_{\tau\in\Lambda} |L_{U,00}(\tau, 0)|$$

while if u_t is $I(1)$

$$t_\lambda = m_\xi t_1^* + o_p(1) \xrightarrow{d} m_\xi \sup_{\tau\in\Lambda} |L_{11}(\tau, 0)| \,.$$

Under H_1, when u_t is $I(0)$, we obtain that t_λ is consistent at rate $O_p(T^{3/2})$, while if u_t is $I(1)$, t_λ is consistent at rate $O_p(T^{1/2})$. Again we must choose the constant m_ξ such that, for a significance level ξ under H_0, the critical value of $m_\xi \sup_{\tau\in\Lambda} |L_{11}(\tau, 0)|$ coincides with that of $\sup_{\tau\in\Lambda} |L_{U,00}(\tau, 0)|$. As before this ensures that the asymptotic null critical values of t_λ are the same regardless of whether u_t is $I(0)$ or $I(1)$.

5 Practical Implementation of the Test Procedures

Asymptotic critical values for our proposed t_λ tests for both Models A and B are provided in Table 1, along with the corresponding values of m_ξ. The values reported are for tests of the null of no break in trend against a two-sided alternative, with the potential break date unknown; that is, for testing $H_0 : \gamma = 0$ against $H_1 : \gamma \neq 0$ in the context of Model A, and $H_0 : \gamma = \delta = 0$ against $H_1 : \gamma \neq 0$ in the context of Model B. As in SV, 10% trimming was used, such that $\tau_L = 0.1$ and $\tau_U = 0.9$. The results were obtained by simulation of the appropriate limiting distributions using discrete approximations for $T = 1000$ and 50000 replications using the `rndKMn` random number generator of Gauss 6.0.

Table 1 about here

For the trend break tests to be operational, we also need to specify the constants g_1 and g_2 in (10). After Monte Carlo simulation of the finite sample size and power of the tests for a range of possible settings, we found that the choices $g_1 = 500$, $g_2 = 2$ and $\ell = \lfloor 4(T/100)^{1/4} \rfloor$ gave the best overall performance, and we therefore recommend use of these values for practical application of the new procedure. These choices apply to both Model A and Model B.

6 Numerical Results

In this section we use Monte Carlo methods to investigate the finite sample size and power performance of the t_λ tests using Monte Carlo simulation. We focus on the unknown break date case - arguably the case of most practical interest - and again employ 10% trimming ($\tau_L = 0.1$, $\tau_U = 0.9$) throughout. All the results reported in this section are for two-sided tests conducted at the 0.05 nominal asymptotic significance level, and were computed over 10000 replications, again using the `rndKMn` function of Gauss 6.0.

The data generating process [DGP] we use to conduct our simulations is a simplified form of (14) and (2) given by:

$$
\begin{aligned}
y_t &= \delta DU_t(\tau^*) + \gamma DT_t(\tau^*) + u_t, \quad t = 1, ..., T, & (16)\\
(1 - \rho L)u_t &= (1 - \theta L)\varepsilon_t, \quad t = 2, ..., T, \quad u_1 = \varepsilon_1 & (17)
\end{aligned}
$$

where $\varepsilon_t \sim IIN(0, 1)$. An investigation into the finite sample size properties of our proposed tests, relative to the tests advocated in SV, outlined immediately below, is provided first in section 6.1. The finite sample power properties of the tests are subsequently explored in section 6.2, again relative to the tests of SV. Results are reported for both Model A and Model B, for samples of size $T = 150$ and $T = 300$. Our proposed t_λ statistic is constructed as detailed in section 3.2 for the case of Model A, and as detailed in section 4.2 for the case of Model B. Recall that t_λ tests the null hypotheses $H_0 : \gamma = 0$ and $H_0 : \gamma = \delta = 0$ in the context of Models A and B, respectively.

In an unpublished paper, SV also propose tests for a break in trend that are robust to strong serial correlation in the data. For a given possible break date $T_a \in \Lambda^*$, consider a standard Wald statistic of the null hypothesis $H_0 : \gamma = 0$ for Model A, or $H_0 : \delta = \gamma = 0$ for Model B, with the implicit long-run variance estimator constructed using a Daniell kernel with bandwidth $M = \lfloor bT \rfloor$, $b \in (0, 1]$. Denoting this Wald statistic by $W^b(T_a)$, the superscript 'b' referring to the bandwidth fraction, the SV tests are based on one of the Andrews (1993) and Andrews and Ploberger (1994) approaches to overcoming the fact that the break date is unknown, i.e. one of the following quantities

$$
\begin{aligned}
MeanW^b &:= T^{-1} \sum_{T_a \in \Lambda^*} W^b(T_a) \\
SupW^b &:= \sup_{T_a \in \Lambda^*} W^b(T_a).
\end{aligned}
$$

To achieve robustness to the possibility of $I(1)$ shocks, these quantities are then multiplied by a correction factor based on a unit root statistic, following the approach of Vogelsang (1998). The unit root statistics considered are straightforward extensions of the variable addition statistic of Park (1990) and Park and Choi (1988), and the variance ratio statistic of Breitung (2002), where the underlying regressions are augmented with dummy variables to model the possible break. Denoting these unit root statistics for a given break date, $T_a \in \Lambda^*$, by $J(T_a)$ and $BG(T_a)$ respectively, the SV statistics are given by, where UR generically denotes either J or BG,

$$
\begin{aligned}
MeanW^b_{UR} &:= MeanW^b \exp\left(-\xi^{UR}_{\text{mean}} \inf_{T_a \in \Lambda^*} UR(T_a)\right) \\
SupW^b_{UR} &:= SupW^b \exp\left(-\xi^{UR}_{\text{sup}} \inf_{T_a \in \Lambda^*} UR(T_a)\right).
\end{aligned}
$$

The constants ξ^{BG}_{mean}, ξ^{J}_{mean}, ξ^{BG}_{sup} and ξ^{J}_{sup}, which are specific to each test, are chosen such that for the given test and a given significance level, ξ, the critical values for the test coincide under $I(0)$ and $I(1)$ errors. For Model A, SV advocate using (i) $MeanW^{0.02}_{BG}$ if u_t is $I(0)$, (ii) $SupW^{0.10}_{BG}$ if u_t is $I(1)$, while (iii) $SupW^{0.06}_{BG}$ is recommended where it is not known if u_t is $I(0)$ or $I(1)$, it giving the best overall power across both the $I(0)$ and $I(1)$ cases. For Model B, the corresponding tests are (i) $MeanW^{0.02}_{BG}$, (ii) $SupW^{0.36}_{J}$, and (iii) $MeanW^{0.18}_{BG}$. Critical values and associated values of the ξ^{BG}_{mean}, ξ^{J}_{mean}, ξ^{BG}_{sup} and ξ^{J}_{sup} constants are reported by SV for each of these tests.

6.1 Size Properties

Table 2 reports the empirical rejection frequencies (sizes), for the t_λ test together with the recommended tests from SV for each of Models A and B. These were obtained by setting $\delta = \gamma = 0$ in (16). The AR and MA noise parameters in (17) were varied over $\rho = 1 - (c/T)$ for $c \in \{0, 10, 20, T\}$ and $\theta \in \{0, \pm 0.4, \pm 0.8\}$, respectively. Notice that for $c = T$, u_t is a pure $MA(1)$ process.

Table 2 about here

In the case of $I(1)$ shocks ($c = 0$) we see that the t_λ test is somewhat oversized in finite samples. This is a finite sample effect, as can be seen on comparing results for $T = 150$ and $T = 300$, suggesting that the asymptotic distributions can be poor approximations in relatively small samples. Conversely, in cases where the shocks are

$I(0)$ ($c > 0$), the t_λ test tends to be slightly under-sized (some exceptions are seen where $\theta = 0.8$), particularly for $c = T$. The over-sizing effect under $I(1)$ shocks is in most cases (an exception occurs for $\theta = 0.8$) little different between Models A and B, but the under-sizing seen in the $I(0)$ case tends to be somewhat less pronounced under Model B. Interestingly, for a given value of c, the size of t_λ does not appear particularly sensitive to the choice of θ in the case of Model A, although in the case of Model B there does appear to be some sensitivity in the case of $\theta = 0.8$. The size properties of the SV tests are also sensitive to the choice of c and θ. For the case of $I(1)$ shocks these tests can be seriously over-sized when $\theta = 0.8$, but are better sized than t_λ when $\theta \leq 0$. For $I(0)$ shocks the pattern is mixed, depending on the value of θ and on the particular test involved, but in most cases a larger degree of under-size is seen in the SV tests than for t_λ.

6.2 Power Properties

Figures 1–3 and 5–7 present results for the report the empirical rejection frequencies (powers) of the tests based on Models A and B respectively. The data were generated according to (16) for a grid of γ values, covering the range $[0, 1]$ in steps of 0.02. Under Model A, $\delta = 0$ in all experiments, while for Model B we set $\delta = 5\gamma$. We consider three break timings[6] $\tau^* \in \{0.25, 0.5, 0.75\}$, and focus on cases where there is most likely to be some ambiguity as to the order of integration by using $\rho = 1 - (c/T)$ with $c \in \{0, 10, 20\}$ in (17). In order to economise on space, we only report results for $\theta = 0$. The results for $\theta \in \{\pm 0.4, 0, 8\}$ are qualitatively similar and are available on request.

Consider first Figures 1 and 5 which relate to the case of $I(0)$ shocks ($c = 0$) in Models A and B respectively. Here we can see that t_λ dominates the SV tests on power, enjoying a marked power advantage right across the range of the power function over the recommended tests of SV for both Model A and Model B and for all of the values of τ^* considered. For example, for $T = 150$ and $\tau^* = 0.25$, while t_λ effectively has power of unity for $\gamma = 1$ under both Models A and B, none of the SV tests have power in excess of 25 % (15 %) under Model A (Model B). The power of the SV tests are also sensitive to the location of the trend break: specifically, their power is much lower, other things equal, for cases where the break is located away from the middle of the sample. In contrast, the power functions for t_λ do not appear to depend to any noticeable degree on the location of the break. Comparing results between Models A and B we also see that, other things equal, the power functions of t_λ are virtually identical for Models A and B, as predicted by the asymptotic theory; cf. Remark 8.

Figures 1 – 8 about here

Because the t_λ tests are a little over-sized in the $I(1)$ case (see the discussion in section 6.1), we also report size-adjusted powers for this case. The size-adjusted power curves are given in Figures 4 and 8 for Models A and B, respectively. The qualitative

[6] SV only report results relating to a mid-sample breakpoint, $\tau^* = 0.5$.

conclusions drawn from Figures 4 and 8 are largely the same as those from Figures 1 and 5, with t_λ clearly dominating all of SV's advocated tests on power.

Consider next Figures 2 and 3 which graph, for Model A, the power curves of the various tests for $c = 10$ and $c = 20$, respectively. In each case, a comparison at the origin ($\gamma = 0$) shows that all of the tests are under-sized, with the recommended tests of SV generally more so than t_λ; cf. Table 2. For both early and late break dates ($\tau^* = 0.25$ and $\tau^* = 0.75$) the t_λ test again dominates all of the SV tests on power for both $c = 10$ and $c = 20$. Where a break occurs in the middle of the sample ($\tau^* = 0.5$) the t_λ test again dominates the SV tests for $T = 150$, but for $T = 300$, it is seen that a small region occurs in the power curves where the SV tests display slightly higher power than t_λ. For $c = 10$ and for $SupW_{BG}^{0.06}$ (which is the appropriate test to compare t_λ with) this region is approximately from $\gamma = 0.1$ to $\gamma = 0.2$, while for $c = 20$ it is around $\gamma = 0.1$ to $\gamma = 0.25$. Much the same patterns are also seen under Model B in Figures 6 and 7, except that the $SupW_J^{0.36}$ test is always dominated by t_λ.

7 Empirical Application

We now consider applying the trend break tests to recent US macroeconomic time series data. Specifically, we examine quarterly seasonally adjusted real GDP observed for 1970Q1 to 2003Q4 (136 observations), and six monthly series observed for 1970M1 to 2003M12 (408 observations): seasonally adjusted unemployment rate, seasonally adjusted money supply M2, 3-month Treasury bill interest rate, commodities spot price index, seasonally adjusted consumer price index and average hourly earnings. The real GDP data was obtained from www.economagic.com, and the monthly series were taken from the database compiled by Stock and Watson (2005). All the variables are measured in logarithms and are plotted in Figure 9.

Table 3 and Figure 9 about here

Table 3 reports the results from application of the tests to these series at the $\xi = 0.01$ and $\xi = 0.05$ significance levels. Consider first the tests applied using Model A. Our proposed t_λ test fails to reject the null of no break in trend for real output and unemployment, but rejects at the 0.05-level for the other five series, of which the null hypothesis can be rejected at the 0.01-level for the money supply, consumer prices and earnings. Where rejections (at the 0.05-level or lower) are obtained, the estimated break dates suggested by the procedure are also reported in Table 3. These were obtained by taking a weighted average of the break date estimates $\hat{\tau}$ and $\tilde{\tau}$, using the weight function $\lambda(S_0(\hat{\tau}), S_1(\hat{\tau}))$ employed in the t_λ test, i.e. $\lambda(S_0(\hat{\tau}), S_1(\hat{\tau}))\hat{\tau} + \{1 - \lambda(S_0(\hat{\tau}), S_1(\hat{\tau}))\}\tilde{\tau}$. The estimated break dates are superimposed on the plots in Figure 9, and correspond well to break timings suggested by visual inspection. In contrast, none of the SV tests can reject the null at either significance level for any of the series considered.

Turning now to Model B, the t_λ results correspond very closely with those obtained using Model A. Rejections in favour of a break in trend at the 0.05-level are observed for the money supply, interest rate, commodity prices, consumer prices and earnings, at the 0.01-level for the money supply, consumer prices and earnings, while the null is not rejected for real output and unemployment. Moreover, for the five series where a break is detected, the estimated break dates coincide with those found for Model A, except for consumer prices and earnings where they differ only by one observation. On the other hand, the SV tests again do not reject the null in general, especially when the more reliable $MeanW_{BG}^{0.18}$ test is used. Rejections are obtained at the 0.05-level for the money supply and consumer prices series using the $SupW_J^{0.36}$ test, but not elsewhere, and it should be stressed that this test is recommended for practical use by SV only where it is known that the shocks are $I(1)$.

8 Conclusions

In this paper we have proposed new tests for a broken trend, with or without a simultaneous break in level, in a univariate time series process which do not require knowledge of the form of serial correlation in the data and are robust to whether the shocks are $I(0)$ or $I(1)$. Our proposed tests are based on simple data-dependent weighted averages of the absolute values of two conventional regression t-ratios, one appropriate for when the data are generated by an $I(0)$ process and the other when the data are $I(1)$. Under a known break date our proposed tests have standard normal limiting null distributions and achieve the relevant Gaussian power envelope, in both $I(0)$ and $I(1)$ environments. For the more realistic case of an unknown break date we employ a supremum-based approach as in, *inter alia*, Andrews (1993) and establish the large sample properties of the resulting tests which are shown to have pivotal null distributions and to deliver consistent tests, again regardless of whether the shocks are $I(0)$ or $I(1)$. Monte Carlo evidence was reported which suggested that our tests are in most cases more powerful, and often substantially so, than the recommended robust broken trend tests proposed in an unpublished paper by Sayginsoy and Vogelsang (2004). An empirical example to a variety of US macroeconomic data highlighted the practical usefulness of our proposed tests, uncovering significant evidence of trend breaks in the majority of the series analysed.

To conclude we suggest three topics for possible further research. First, as discussed in the introduction, extant unit root tests which allow for trend breaks have the undesirable property of not being similar, even asymptotically, with respect to the magnitude of the trend break. Moreover, they lose power if unnecessary trend break dummies are included in the vector of deterministic variables used to de-trend the data prior to computing the unit root statistic. It would therefore be interesting to conduct a formal analysis of the properties of unit root tests where the inclusion of trend break dummies or otherwise in the vector of deterministic variables was specified on the basis of the outcomes of the tests considered in this paper, in particular to establish whether similar unit root tests can be obtained. Second, we have focused in this paper on the case of a single break in trend. It would also be useful to extend our analysis to the case of multiple trend breaks. For the case of known break dates this would be a trivial extension of the results in this paper. An analysis of the unknown break dates case would be likely to be considerably more involved. However, we conjecture that it should be feasible using a sequential testing strategy, along the lines of that considered in Bai and Perron (1998,2003). Thirdly, we have restricted attention in this paper to the case where the shocks are either $I(0)$ or $I(1)$. An interesting extension of this paper would be to consider the more general scenario where the shocks are either $I(0)$ (short memory) or fractionally integrated of order d, $I(d)$, $0 < d \leq 1$ (long memory). In this case a bounds-type procedure, based on the results presented in this paper for the polar $I(0)$ and $I(1)$ cases, might be usefully explored.

Appendix

In what follows, due to invariance of the statistics concerned, we can set $\alpha = \beta = 0$ without loss of generality.

PROOF OF THEOREM 1.
(i) We first establish the result in (a). Using the Frisch-Waugh-Lovell Theorem (FWLT) we can write $t_0(\tau^*)$ in the form

$$t_0(\tau^*) = \left\{ \kappa + \frac{T^{-3/2} \sum RT_t(\tau^*) u_t}{T^{-3} \sum RT_t(\tau^*)^2} \right\} \times \frac{1}{\sqrt{\hat{\omega}^2(\tau^*)/T^{-3} \sum RT_t(\tau^*)^2}}$$

where $RT_t(\tau^*)$, $t = 1, ..., T$, are the OLS residuals from regressing $DT_t(\tau^*)$ onto 1 and t. Entirely standard results, including the fact that $\hat{\omega}^2(\tau^*) \xrightarrow{p} \omega_u^2$, allow us to establish the weak convergence result,

$$t_0(\tau^*) \xrightarrow{d} \left\{ \kappa + \frac{\omega_u \int_0^1 RT(r, \tau^*) dW(r) dr}{\int_0^1 RT(r, \tau^*)^2 dr} \right\} \times \frac{1}{\sqrt{\omega_u^2 / \int_0^1 RT(r, \tau^*)^2 dr}}, \tag{A.1}$$

where $W(r)$ is a standard Brownian motion, defined *via*, $\omega_u^{-1} T^{-1/2} \sum_{t=1}^{\lfloor Tr \rfloor} u_t \xrightarrow{d} W(r)$. The result in (a) is then established on rearranging (A.1).

Turning to the result in (b), and again using the FWLT, we can write $t_1(\tau^*)$ as

$$(T/\ell)^{1/2} t_1(\tau^*) = \left\{ \kappa T^{-1/2} + \frac{\sum RU_t(\tau^*) \Delta u_t}{T^{-1} \sum RU_t(\tau^*)^2} \right\} \times \frac{1}{\sqrt{\ell \tilde{\omega}^2(\tau^*)/T^{-1} \sum RU_t(\tau^*)^2}}$$

where $RU_t(\tau^*)$, $t = 2, ..., T$, are the OLS residuals from a regression of $DU_t(\tau^*)$ onto 1. In fact, $RU_t(\tau^*)$ can be written in the simple form

$$RU_t(\tau^*) = \begin{cases} \tau^* - 1, & t \leq T^* \\ \tau^*, & t > T^* \end{cases}$$

which implies that $\sum RU_t(\tau^*)\Delta u_t = \tau^* u_T + u_{T^*} + (1-\tau^*)u_1 = O_p(1)$, since u_t is $I(0)$. Moreover, $\tilde{\omega}^2(\tau^*) \xrightarrow{p} \lim_{T\to\infty} T^{-1}E(\sum_{t=2}^T \Delta u_t)^2 = 0$, since Δu_t is over-differenced when u_t is $I(0)$. However, it follows from Leybourne *et al.* (2004) that $\ell\tilde{\omega}^2(\tau^*) \xrightarrow{p} -2\sum_{s=1}^\infty s\gamma'_s$ where $\gamma'_s = E(\Delta u_t \Delta u_{t-s})$. Consequently, $(T/\ell)^{1/2}t_1(\tau^*) = O_p(1)$, which establishes the result in (b).

(ii) In order to establish the result in (a), notice first that

$$(\ell/T)^{1/2}t_0(\tau^*) = \left\{\kappa + \frac{T^{-5/2}\sum RT_t(\tau^*)u_t}{T^{-3}\sum RT_t(\tau^*)^2}\right\} \times \frac{1}{\sqrt{(\ell T)^{-1}\hat{\omega}^2(\tau^*)/T^{-3}\sum RT_t(\tau^*)^2}}. \tag{A.2}$$

A simple extension of the results in KPSS (pp.168-169) establishes the result that $(\ell T)^{-1}\hat{\omega}^2(\tau^*) \xrightarrow{d} \omega_\varepsilon^2 \int_0^1 H(r,\tau^*)^2 dr$ where $H(r,\tau^*)$ is a continuous-time residual from the projection of $W(r)$ onto the space spanned by $\{1, r, (r-\tau^*)1(r>\tau^*)\}$. Consequently, and since all the other stochastic terms appearing in (A.2) are of $O_p(1)$ with non-generate limiting distributions, it follows that $(\ell/T)^{1/2}t_0(\tau^*) = O_p(1)$, from which the result in (a) follows.

In order to establish the result in (b), observe that

$$\begin{aligned} t_1(\tau^*) &= \left\{\kappa + \frac{T^{-1/2}\sum RU_t(\tau^*)\varepsilon_t}{T^{-1}\sum RU_t(\tau^*)^2}\right\} \times \frac{1}{\sqrt{\tilde{\omega}^2(\tau^*)/T^{-1}\sum RU_t(\tau^*)^2}} \\ &\xrightarrow{d} \left\{\kappa + \frac{\omega_\varepsilon \int_0^1 RU(r,\tau^*)dW(r)dr}{\int_0^1 RU(r,\tau^*)^2 dr}\right\} \times \frac{1}{\sqrt{\omega_\varepsilon^2/\int_0^1 RU(r,\tau^*)^2 dr}} \end{aligned} \tag{A.3}$$

using standard results, including the fact that $\tilde{\omega}^2(\tau^*) \xrightarrow{p} \omega_\varepsilon^2$. Rearranging (A.3) delivers the stated result in (b).

Proof of Lemma 1. The proof of Lemma 1 follows from trivial extensions to the results in KPSS (pp.164-169) and, for (i)(b), results in Leybourne *et al.* (2004). The proof is therefore omitted in the interests of brevity.

Proof of Theorem 2. Consider first the proof of the results in (i)(a) and (b)(ii). Here the joint distributions of the sequences of statistics used in forming the statistics t_0^* and t_1^* follow from the fixed τ representations given in Theorem 1, using arguments proved in Zivot and Andrews (1992). The stated results in (i)(a) and (b)(ii) then follow directly from Theorem 1 (i)(a) and (ii)(b), respectively, using applications of the CMT, noting the continuity of the sup function. The result in (i)(b) follows from Theorem 1 (i)(b) and the result that $\ell\tilde{\omega}^2(\tau) \xrightarrow{p} -2\sum_{s=1}^\infty s\gamma'_s$ uniformly in τ. Hence, $t_1(\tau) = O_p\{(\ell/T)^{1/2}\}$ uniformly in τ. Finally, for the result in (ii)(a) we appeal to Theorem 1 (ii)(a) and the fact that because $(\ell T)^{-1}\hat{\omega}^2(\tau) \xrightarrow{d} \omega_\varepsilon^2 \int_0^1 H(r,\tau)^2 dr$ uniformly in τ. So, $t_0(\tau) = O_p\{(T/\ell)^{1/2}\}$ uniformly in τ.

Proof of Theorem 3.

As we are only concerned with establishing the orders in probability of the statistics under H_1, for technical expediency we omit the constant and trend regressors from (1) and the constant regressor from (4). These particular regressors have no effect on any of the orders involved, but just introduce uninformative algebraic complexities.

(i) To establish the result in part (a) we must first derive the order of $t_0(\tau^*)$ under the fixed alternative $H_1 : \gamma \neq 0$. To that end, observe first that

$$\begin{aligned} T^{-3/2} t_0(\tau^*) &= \left\{ \gamma + \frac{\sum DT_t(\tau^*) u_t}{\sum DT_t(\tau^*)^2} \right\} \times \frac{1}{\sqrt{\hat{\omega}^2(\tau^*)/T^{-3} \sum_{t=1}^{T} DT_t(\tau^*)^2}} \\ &\xrightarrow{p} \frac{\gamma}{\sqrt{\omega_u^2 3(1-\tau^*)^{-3}}} \end{aligned}$$

where we have used the results that $\hat{\omega}^2(\tau^*) \xrightarrow{p} \omega_u^2$, and that $T^{-3} \sum_{t=1}^{T} DT_t(\tau^*)^2 \to (1-\tau^*)^3/3$. Now, it is straightforward to establish that for any $\tau \in \Lambda$, we may write

$$T^{-3/2} \left| t_0(\tau) \right| = \sqrt{\left(\frac{T^{-3} \sum y_t^2}{\hat{\sigma}^2(\tau)} - T^{-2} \right) \times \frac{\hat{\sigma}^2(\tau)}{\hat{\omega}^2(\tau)}},$$

from which it is clear that the stated result will hold if both $\hat{\sigma}^2(\hat{\tau}) - \hat{\sigma}^2(\tau^*)$ and $\hat{\omega}^2(\hat{\tau}) - \hat{\omega}^2(\tau^*)$ are asymptotically negligible. Considering the first of these, it is straightforward to show that

$$\hat{\sigma}^2(\hat{\tau}) - \hat{\sigma}^2(\tau^*) = -T^{-1} \left[\frac{\{\sum DT_t(\hat{\tau}) y_t\}^2}{\sum DT_t(\hat{\tau})^2} - \frac{\{\sum DT_t(\tau^*) y_t\}^2}{\sum DT_t(\tau^*)^2} \right] \tag{A.4}$$

from which it is easy to demonstrate that the dominant term of the right member of (A.4) is of the form

$$-\gamma^2 T^{-1} \left[\frac{\{\sum DT_t(\hat{\tau}) DT_t(\tau^*)\}^2}{\sum DT_t(\hat{\tau})^2} - \sum DT_t(\tau^*)^2 \right]. \tag{A.5}$$

After some straightforward but lengthy manipulations, the dominant term of (A.5) can be shown to be given by

$$-\gamma^2 \frac{(\hat{c}T)^2}{36} (\hat{\tau} - 1)^3 (4\tau^* - \hat{\tau} - 3) \tag{A.6}$$

where $\hat{c} = \tau^* - \hat{\tau}$. Our break fraction estimator $\hat{\tau}$ can be shown to have the same rate of consistency as the minimum sum of squares break fraction estimator of Perron and Zhu (2005). Consequently, from Theorem 3 of Perron and Zhu (2005, p.75) we have that $\hat{c} = O_p(T^{-3/2})$, and, hence, that (A.6) is $O_p(T^{-1})$. Thus, $\hat{\sigma}^2(\hat{\tau}) - \hat{\sigma}^2(\tau^*) = O_p(T^{-1})$. Turning to the difference between the estimated LR variances, we have that

$$\begin{aligned} \hat{\omega}^2(\hat{\tau}) - \hat{\omega}^2(\tau^*) &= \hat{\sigma}^2(\hat{\tau}) - \hat{\sigma}^2(\tau^*) + 2 \sum_{j=1}^{T-1} h(j/\ell) \{\hat{\gamma}_j(\hat{\tau}) - \hat{\gamma}_j(\tau^*)\} \\ &= O_p(T^{-1}) + O_p(T^{-1}) O(\ell) = O_p(\ell T^{-1}) \end{aligned}$$

which follows since $\hat{\gamma}_j(\hat{\tau})-\hat{\gamma}_j(\tau^*)$ is uniformly bounded by an $O_p(T^{-1})$ variate. Hence, it also holds that $\hat{\omega}^2(\hat{\tau})-\hat{\omega}^2(\tau^*) \xrightarrow{p} 0$, and, as a consequence, $T^{-3/2}\{|t_0(\hat{\tau})|-|t_0(\tau^*)|\} \xrightarrow{p} 0$, which establishes the result in (a).

In order to establish the result in (b) we again must first determine the order of $t_1(\tau^*)$ under H_1. Observe first that,

$$\begin{aligned}(\ell T)^{-1/2}t_1(\tau^*) &= \left\{\gamma + \frac{\sum DU_t(\tau^*)\Delta u_t}{\sum DU_t(\tau^*)^2}\right\} \times \frac{1}{\sqrt{\ell\tilde{\omega}^2(\tau^*)/T^{-1}\sum DU_t(\tau^*)^2}} \\ &= \{\gamma + o_p(1)\}O_p(1) = O_p(1).\end{aligned}$$

For any $\tau \in \Lambda$ it can be shown that

$$(\ell T)^{-1/2}\left|t_1(\tau)\right| = \sqrt{\left(\frac{T^{-1}\sum \Delta y_t^2}{\tilde{\sigma}^2(\tau)} - 1\right) \times \frac{\tilde{\sigma}^2(\tau)}{\ell\tilde{\omega}^2(\tau)}}$$

so that, as in proof of part (a), we must establish the behaviour of the difference between the OLS and LR variance estimators evaluated at τ^* and $\tilde{\tau}$. Considering the difference between the OLS variance estimators first, we have that

$$\tilde{\sigma}^2(\tilde{\tau}) - \tilde{\sigma}^2(\tau^*) = -T_*^{-1}\left[\frac{\{\sum DU_t(\tilde{\tau})\Delta y_t\}^2}{\sum DU_t(\tilde{\tau})^2} - \frac{\{\sum DU_t(\tau^*)\Delta y_t\}^2}{\sum DU_t(\tau^*)^2}\right], \qquad \text{(A.7)}$$

the dominant term in the right member of which can be shown to be given by

$$-\gamma^2\frac{\tilde{c}(\tau^*-1)}{(2\tau^*-\tilde{\tau}-1)} \qquad \text{(A.8)}$$

where $\tilde{c} = \tau^* - \tilde{\tau}$. Consequently, $\tilde{\sigma}^2(\tilde{\tau}) - \tilde{\sigma}^2(\tau^*) = O_p(T^{-1/2})$ owing to the fact that $\tilde{c} = O_p(T^{-1/2})$, as is straightforward but tedious to establish. As regards the difference between the LR variance estimators, we have that

$$\begin{aligned}\ell\{\tilde{\omega}^2(\tilde{\tau}) - \tilde{\omega}^2(\tau^*)\} &= \ell\{\tilde{\sigma}^2(\tilde{\tau}) - \tilde{\sigma}^2(\tau^*)\} + 2\ell\sum_{j=1}^{T-2} h(j/\ell)\{\tilde{\gamma}_j(\tilde{\tau}) - \tilde{\gamma}_j(\tau^*)\} \\ &= \ell O_p(T^{-1/2}) + \ell O_p(T^{-1/2})O(\ell) = O_p(1)\end{aligned}$$

since $\ell = O(T^{1/4})$. Consequently, $(\ell T)^{-1/2}\{|t_1(\tilde{\tau})| - |t_1(\tau^*)|\} = O_p(1)$, establishing (b).

(ii) In order to prove (a), we again establish first the behaviour of $t_0(\tau^*)$ under H_1. Observing first that

$$\begin{aligned}(\ell^{1/2}/T)t_0(\tau^*) &= \left\{\gamma + \frac{\sum DT_t(\tau^*)u_t}{\sum DT_t(\tau^*)^2}\right\} \times \frac{1}{\sqrt{(\ell T)^{-1}\hat{\omega}^2(\tau^*)/T^{-3}\sum_{t=1}^{T} DT_t(\tau^*)^2}} \\ &= \{\gamma + o_p(1)\}O_p(1) = O_p(1).\end{aligned}$$

Again, for any $\tau \in \Lambda$, we may write

$$(\ell^{1/2}/T)\,|t_0(\tau)| = \sqrt{\left(\frac{T^{-3}\sum y_t^2}{T^{-1}\hat{\sigma}^2(\tau)} - T^{-1}\right) \times \frac{T^{-1}\hat{\sigma}^2(\tau)}{(\ell T)^{-1}\hat{\omega}^2(\tau)}}$$

so again we need to establish the behaviour of the difference between the OLS and LR variance estimators evaluated at τ^* and $\tilde{\tau}$. Using (A.4)-(A.6) we obtain that the dominant term of $T^{-1}\{\hat{\sigma}^2(\hat{\tau}) - \hat{\sigma}^2(\tau^*)\}$ in this case is given by

$$-\gamma^2 \frac{\hat{c}^2 T}{36} (\hat{\tau} - 1)^3 (4\tau^* - \hat{\tau} - 3)\,.$$

Next, again utilizing the results from Theorem 3 of Perron and Zhu (2005), we may show that $\hat{c} = O_p(T^{-1/2})$, and, hence, we obtain that $T^{-1}\{\hat{\sigma}^2(\hat{\tau}) - \hat{\sigma}^2(\tau^*)\} = O_p(1)$. As regards the difference between the LR variance estimators, evaluated at τ^* and $\hat{\tau}$, we have that

$$\begin{aligned}
(\ell T)^{-1}\{\hat{\omega}^2(\hat{\tau}) - \hat{\omega}^2(\tau^*)\} &= (\ell T)^{-1}\{\hat{\sigma}^2(\hat{\tau}) - \hat{\sigma}^2(\tau^*)\} + 2(\ell T)^{-1}\sum_{j=1}^{T-1} h(j/\ell)\{\hat{\gamma}_j(\hat{\tau}) - \hat{\gamma}_j(\tau^*)\} \\
&= O_p(\ell^{-1}) + \ell^{-1} O_p(1) O(\ell) \\
&= O_p(1)
\end{aligned}$$

so it follows that $(\ell^{1/2}/T)\{|t_0(\hat{\tau})| - |t_0(\tau^*)|\} = O_p(1)$, establishing (a).

Turning finally to the result in (b), we note first that the statistic evaluated at the true break point τ^* can be written as

$$\begin{aligned}
T^{-1/2} t_1(\tau^*) &= \left\{\gamma + \frac{\sum DU_t(\tau^*)\varepsilon_t}{\sum DU_t(\tau^*)^2}\right\} \times \frac{1}{\sqrt{\tilde{\omega}^2(\tau^*)/T^{-1}\sum DU_t(\tau^*)^2}} \\
&\xrightarrow{p} \gamma \frac{(1-\tau^*)^{1/2}}{\omega_\varepsilon}.
\end{aligned}$$

Notice again that for any $\tau \in \Lambda$ we may write

$$T^{-1/2}\,|t_1(\tau)| = \sqrt{\left(\frac{T^{-1}\sum \Delta y_t^2}{\tilde{\sigma}^2(\tau)} - 1\right) \times \frac{\tilde{\sigma}^2(\tau)}{\tilde{\omega}^2(\tau)}}$$

so again we need to establish the behaviour of the difference between the OLS and LR variance estimators evaluated at τ^* and $\tilde{\tau}$. Now from (A.7) and (A.8) it follows that the dominant term of $\tilde{\sigma}^2(\tilde{\tau}) - \tilde{\sigma}^2(\tau^*)$ is given by $-(\gamma^2\tilde{c}(\tau^* - 1))/\left(2\tau^* - \tilde{\tau} - 1\right)$. Consequently, $\tilde{\sigma}^2(\tilde{\tau}) - \tilde{\sigma}^2(\tau^*) = O_p(T^{-1/2})$ owing to the fact that as in the case of $I(0)$ data, $\tilde{c} = O_p(T^{-1/2})$. Moreover, for the difference between the LR variance estimators, we have that

$$\begin{aligned}
\tilde{\omega}^2(\tilde{\tau}) - \tilde{\omega}^2(\tau^*) &= \tilde{\sigma}^2(\tilde{\tau}) - \tilde{\sigma}^2(\tau^*) + 2\sum_{j=1}^{T-2} h(j/\ell)\{\tilde{\gamma}_j(\tilde{\tau}) - \tilde{\gamma}_j(\tau^*)\} \\
&= O_p(T^{-1/2}) + O_p(T^{-1/2})O(\ell) \\
&= O_p(\ell T^{-1/2})
\end{aligned}$$

so that $\tilde{\omega}^2(\tilde{\tau}) - \tilde{\omega}^2(\tau^*) \xrightarrow{p} 0$. Hence, $T^{-1/2}\{|t_1(\tilde{\tau})| - |t_1(\tau^*)|\} \xrightarrow{p} 0$, which establishes the result in (b).

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Table 1. Asymptotic critical values and m_ξ values for the t_λ tests.

	Model A		Model B	
ξ	Critical value	m_ξ	Critical value	m_ξ
0.10	2.284	0.835	2.904	1.062
0.05	2.563	0.853	3.162	1.052
0.01	3.135	0.890	3.654	1.037

Table 2. Empirical sizes of nominal 0.05-level tests.

Panel A. Model A

		$T=150$				$T=300$			
c	θ	t_λ	$Mean\,W_{BG}^{0.02}$	$Sup\,W_{BG}^{0.10}$	$Sup\,W_{BG}^{0.06}$	t_λ	$Mean\,W_{BG}^{0.02}$	$Sup\,W_{BG}^{0.10}$	$Sup\,W_{BG}^{0.06}$
0	−0.8	0.161	0.035	0.043	0.041	0.114	0.038	0.040	0.039
	−0.4	0.155	0.037	0.045	0.042	0.110	0.039	0.041	0.040
	0.0	0.139	0.046	0.050	0.050	0.098	0.043	0.045	0.044
	0.4	0.094	0.093	0.076	0.084	0.063	0.071	0.062	0.065
	0.8	0.133	0.304	0.201	0.251	0.112	0.309	0.184	0.239
10	−0.8	0.030	0.013	0.011	0.008	0.016	0.015	0.011	0.009
	−0.4	0.030	0.014	0.011	0.009	0.016	0.016	0.012	0.009
	0.0	0.030	0.018	0.013	0.011	0.016	0.018	0.013	0.010
	0.4	0.032	0.042	0.025	0.025	0.020	0.033	0.019	0.018
	0.8	0.060	0.126	0.066	0.085	0.077	0.188	0.068	0.098
20	−0.8	0.025	0.009	0.008	0.005	0.016	0.010	0.007	0.004
	−0.4	0.025	0.009	0.009	0.005	0.016	0.011	0.008	0.005
	0.0	0.023	0.014	0.010	0.007	0.017	0.013	0.009	0.005
	0.4	0.029	0.035	0.018	0.016	0.023	0.025	0.013	0.011
	0.8	0.032	0.048	0.042	0.040	0.075	0.134	0.042	0.056
T	−0.8	0.018	0.009	0.025	0.016	0.025	0.017	0.032	0.023
	−0.4	0.018	0.012	0.028	0.019	0.026	0.019	0.035	0.026
	0.0	0.015	0.018	0.036	0.028	0.022	0.028	0.038	0.034
	0.4	0.005	0.014	0.046	0.040	0.009	0.033	0.046	0.043
	0.8	0.000	0.000	0.089	0.055	0.000	0.004	0.082	0.073

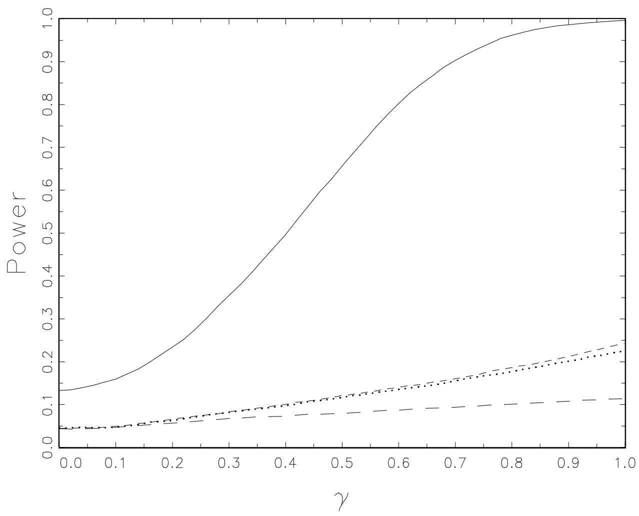
Panel B. Model B

		$T=150$				$T=300$			
c	θ	t_λ	$Mean\,W_{BG}^{0.02}$	$Sup\,W_{J}^{0.36}$	$Mean\,W_{BG}^{0.18}$	t_λ	$Mean\,W_{BG}^{0.02}$	$Sup\,W_{J}^{0.36}$	$Mean\,W_{BG}^{0.18}$
0	−0.8	0.161	0.034	0.057	0.039	0.114	0.037	0.055	0.040
	−0.4	0.156	0.036	0.056	0.040	0.110	0.039	0.055	0.041
	0.0	0.140	0.045	0.053	0.045	0.099	0.044	0.053	0.044
	0.4	0.108	0.099	0.047	0.066	0.068	0.076	0.047	0.055
	0.8	0.229	0.395	0.045	0.150	0.230	0.429	0.043	0.136
10	−0.8	0.032	0.012	0.029	0.018	0.018	0.016	0.029	0.019
	−0.4	0.032	0.013	0.028	0.019	0.018	0.016	0.029	0.019
	0.0	0.035	0.020	0.027	0.021	0.021	0.020	0.028	0.020
	0.4	0.050	0.056	0.025	0.032	0.034	0.043	0.027	0.026
	0.8	0.128	0.170	0.029	0.064	0.213	0.308	0.029	0.062
20	−0.8	0.028	0.009	0.024	0.016	0.020	0.008	0.027	0.015
	−0.4	0.030	0.010	0.023	0.017	0.020	0.009	0.027	0.015
	0.0	0.035	0.013	0.023	0.020	0.026	0.012	0.026	0.017
	0.4	0.060	0.041	0.022	0.031	0.049	0.034	0.025	0.023
	0.8	0.067	0.050	0.024	0.048	0.197	0.198	0.026	0.049
T	−0.8	0.030	0.005	0.024	0.034	0.039	0.012	0.031	0.038
	−0.4	0.032	0.007	0.024	0.036	0.042	0.015	0.032	0.040
	0.0	0.032	0.013	0.026	0.042	0.042	0.024	0.032	0.043
	0.4	0.014	0.010	0.028	0.047	0.017	0.030	0.039	0.046
	0.8	0.001	0.000	0.021	0.033	0.000	0.003	0.044	0.036

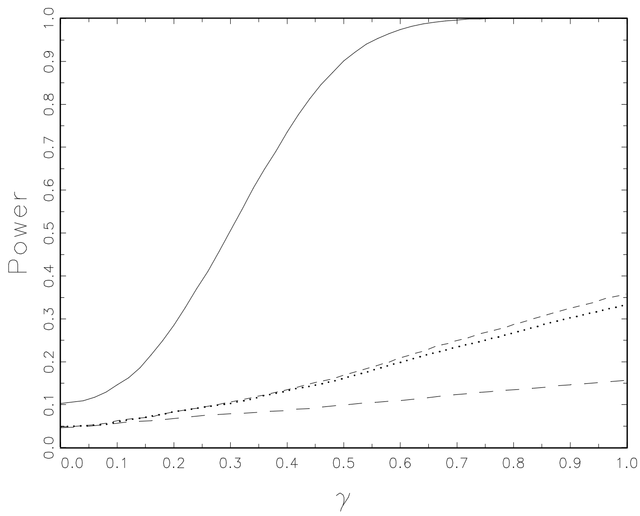
Table 3. Application of tests to US macroeconomic time series.

		Model A				Model B			
Series	ξ	t_λ	$Mean\,W_{BG}^{0.02}$	$Sup\,W_{BG}^{0.10}$	$Sup\,W_{BG}^{0.06}$	t_λ	$Mean\,W_{BG}^{0.02}$	$Sup\,W_{J}^{0.36}$	$Mean\,W_{BG}^{0.18}$
Real output	0.05	1.085	0.024	0.113	0.076	1.350	0.082	671.358	0.581
	0.01	1.130	0.010	0.050	0.033	1.334	0.029	513.746	0.310
		(-)				(-)			
Unemployment	0.05	2.417	0.516	2.310	1.598	2.967	0.600	2153.949	6.227
	0.01	2.522	0.217	1.053	0.715	2.925	0.224	1526.300	3.463
		(-)				(-)			
Money supply	0.05	5.710**	0.006	0.719	0.246	7.047**	0.003	14410.803**	0.752
	0.01	5.957***	0.000	0.068	0.022	6.947***	0.000	163.638	0.121
		(1986M12)				(1986M12)			
Interest rate	0.05	2.884**	0.339	1.981	1.100	3.665**	0.264	473.080	2.399
	0.01	3.009	0.113	0.731	0.396	3.612	0.072	268.033	1.110
		(2000M7)				(2000M7)			
Commodity prices	0.05	2.603**	1.894	10.543	6.471	3.215**	1.368	1185.617	10.041
	0.01	2.716	0.808	4.864	2.932	3.169	0.478	742.748	5.371
		(1974M2)				(1974M2)			
Consumer prices	0.05	7.235**	0.332	9.717	4.501	8.925**	0.158	12890.790**	4.313
	0.01	7.549***	0.064	2.169	0.970	8.798***	0.020	4344.766	1.243
		(1982M7)				(1982M6)			
Earnings	0.05	9.536**	0.024	3.721	1.534	11.284**	0.637	2657.756	12.101
	0.01	9.950***	0.002	0.423	0.166	11.123***	0.103	1532.787	4.086
		(1982M1)				(1981M12)			

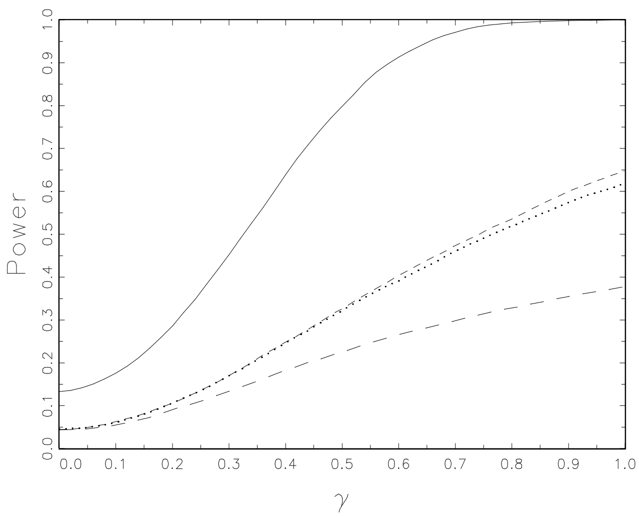
Notes: ** and *** denotes rejection at the 0.05-level and 0.01-level respectively. Where rejections are obtained for the t_λ test, the estimated break date is reported in parentheses.



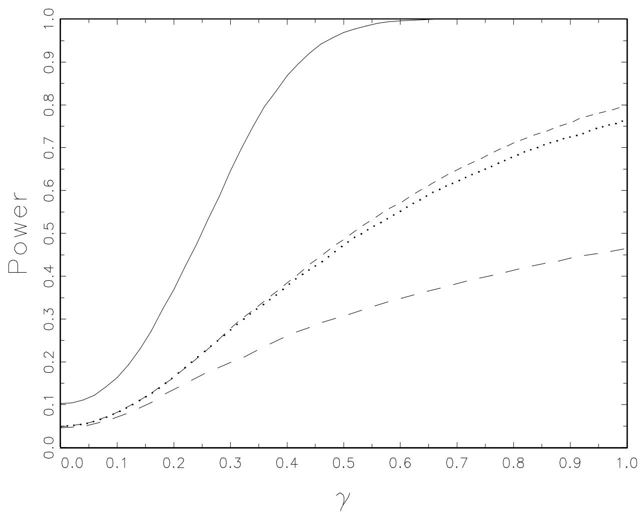
(a) $T = 150$, $\tau^* = 0.25$



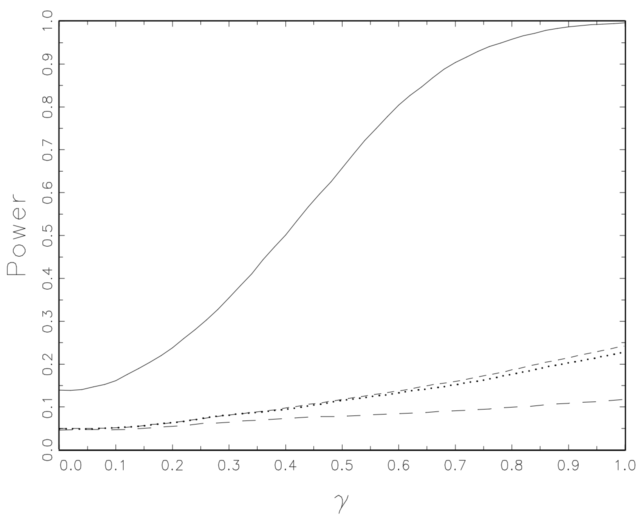
(b) $T = 300$, $\tau^* = 0.25$



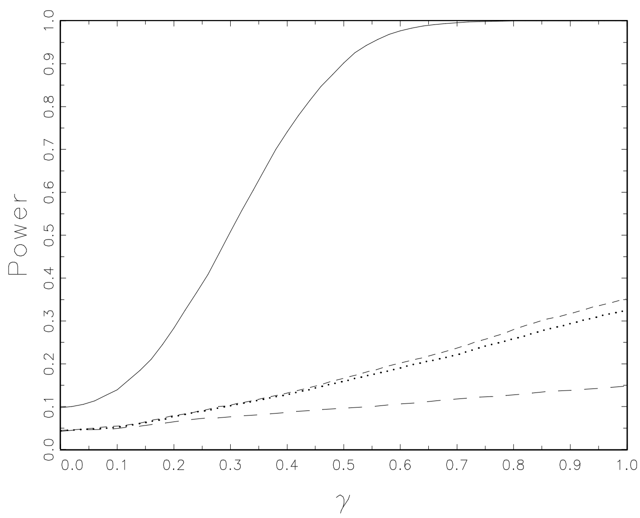
(c) $T = 150$, $\tau^* = 0.5$



(d) $T = 300$, $\tau^* = 0.5$

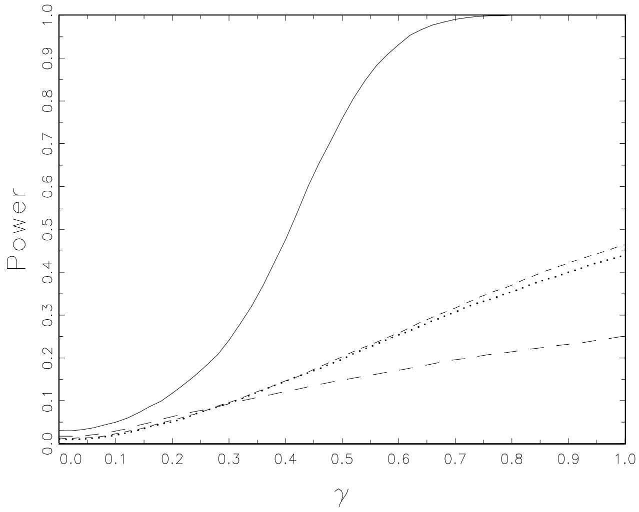


(e) $T = 150$, $\tau^* = 0.75$

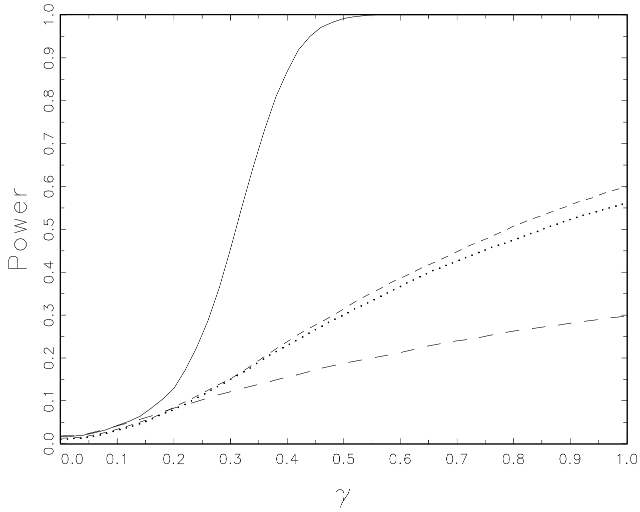


(f) $T = 300$, $\tau^* = 0.75$

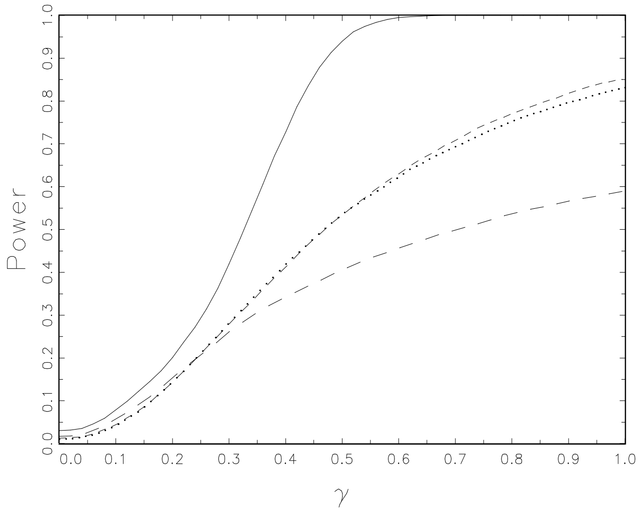
Figure 1. Power: Model A, $c = 0$, t_λ: —— , $Mean\, W_{BG}^{0.02}$: $--$, $Sup\, W_{BG}^{0.10}$: - - - , $Sup\, W_{BG}^{0.06}$: $\cdots$



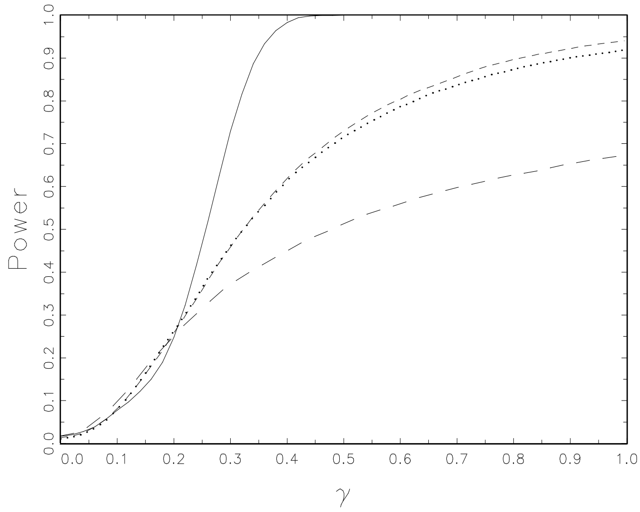
(a) $T = 150$, $\tau^* = 0.25$



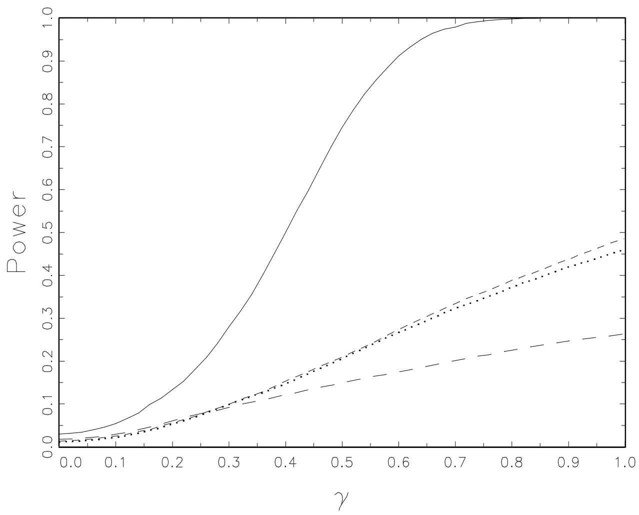
(b) $T = 300$, $\tau^* = 0.25$



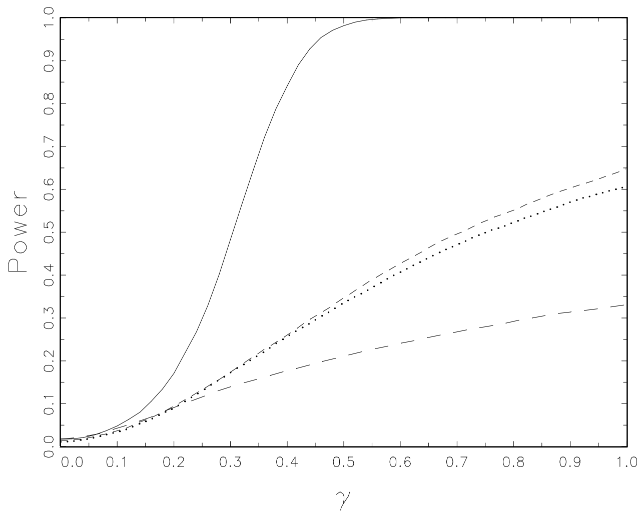
(c) $T = 150$, $\tau^* = 0.5$



(d) $T = 300$, $\tau^* = 0.5$

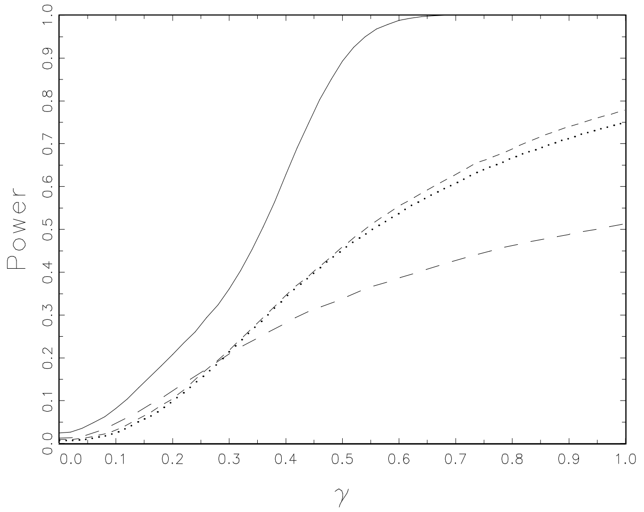


(e) $T = 150$, $\tau^* = 0.75$

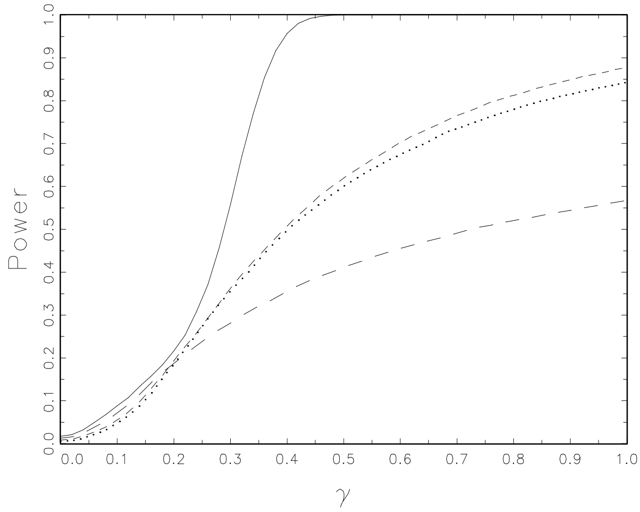


(f) $T = 300$, $\tau^* = 0.75$

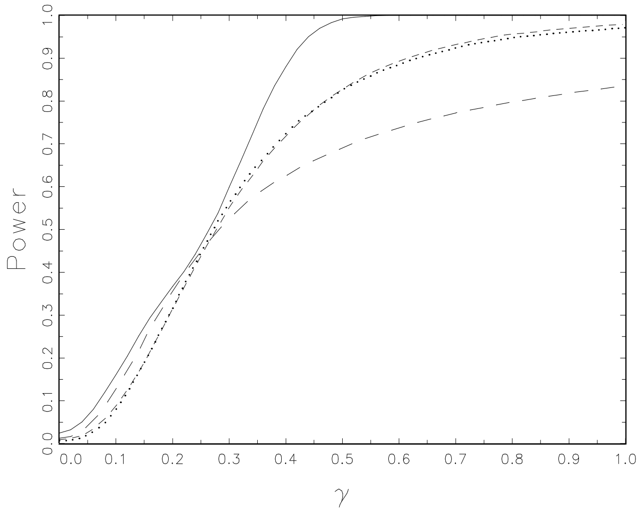
Figure 2. Power: Model A, $c = 10$, t_λ: ——, $Mean\,W_{BG}^{0.02}$: – –, $Sup\,W_{BG}^{0.10}$: - - -, $Sup\,W_{BG}^{0.06}$: $\cdots$



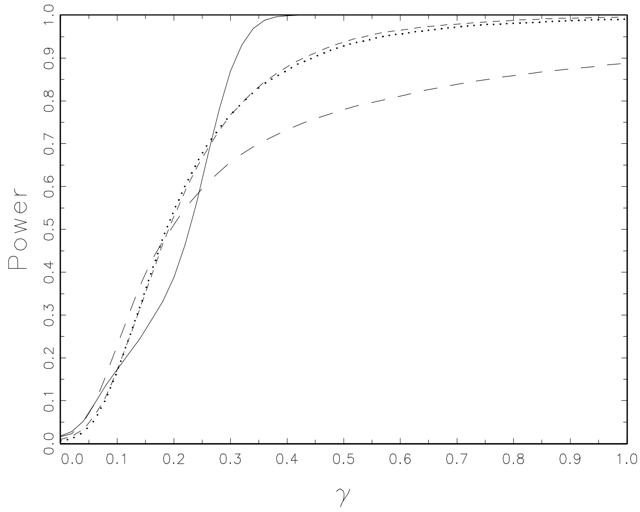
(a) $T = 150$, $\tau^* = 0.25$



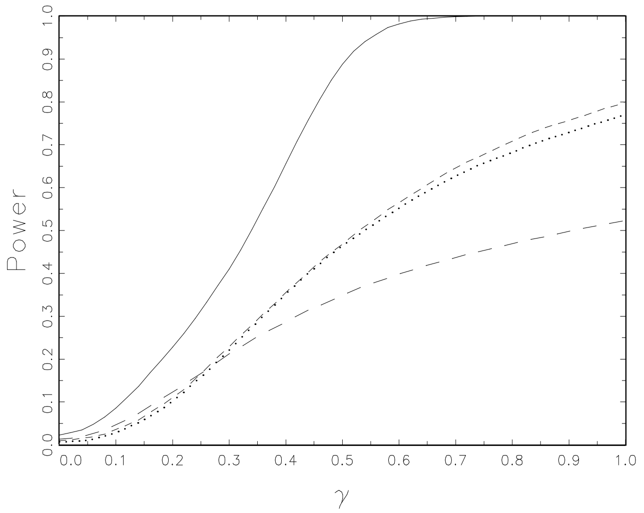
(b) $T = 300$, $\tau^* = 0.25$



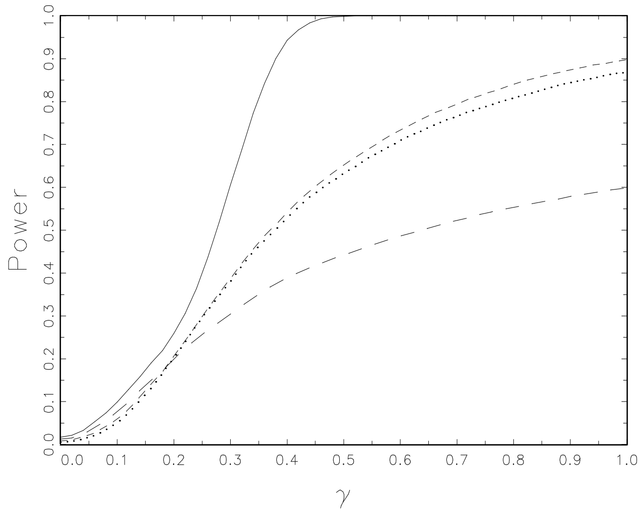
(c) $T = 150$, $\tau^* = 0.5$



(d) $T = 300$, $\tau^* = 0.5$

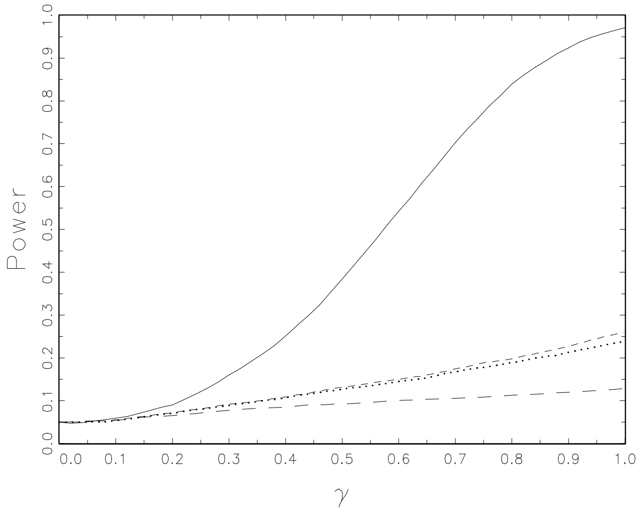


(e) $T = 150$, $\tau^* = 0.75$

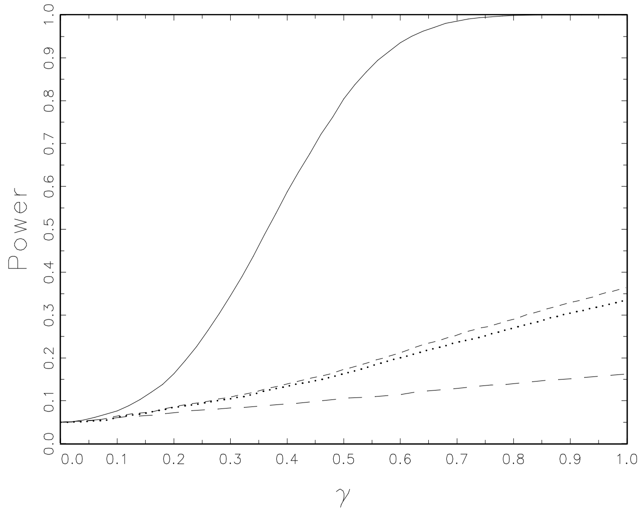


(f) $T = 300$, $\tau^* = 0.75$

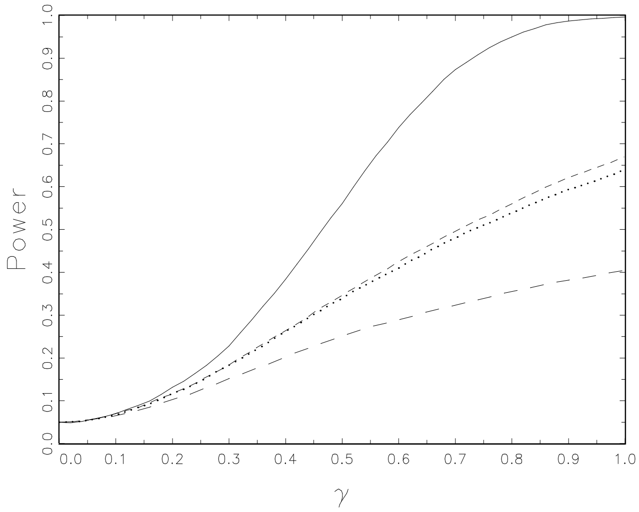
Figure 3. Power: Model A, $c = 20$, t_λ: ——, $Mean\,W_{BG}^{0.02}$: – –, $Sup\,W_{BG}^{0.10}$: - - -, $Sup\,W_{BG}^{0.06}$: $\cdots$



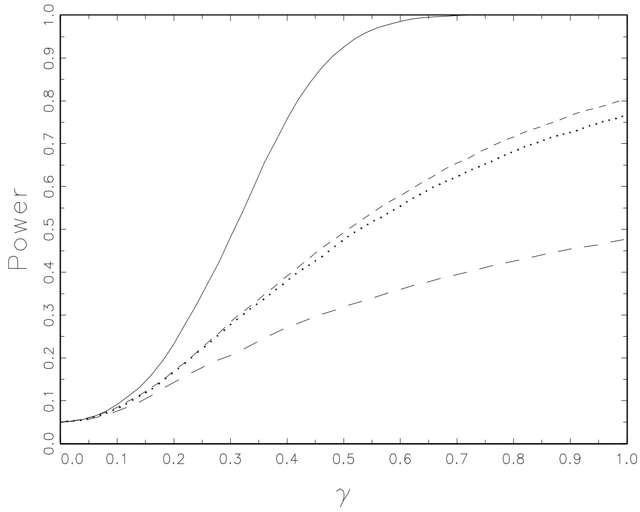
(a) $T = 150$, $\tau^* = 0.25$



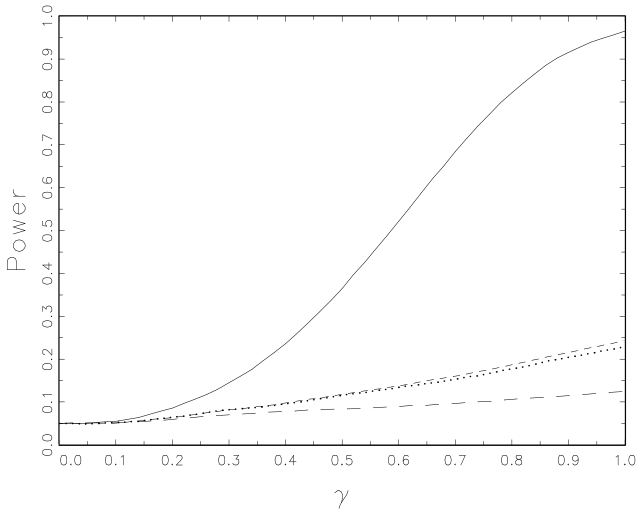
(b) $T = 300$, $\tau^* = 0.25$



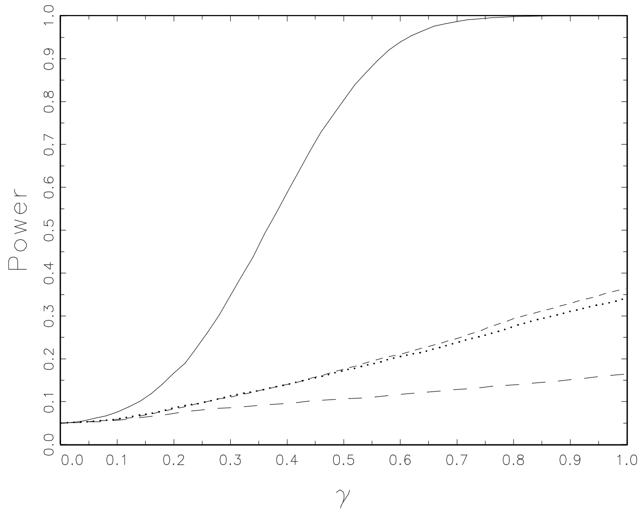
(c) $T = 150$, $\tau^* = 0.5$



(d) $T = 300$, $\tau^* = 0.5$

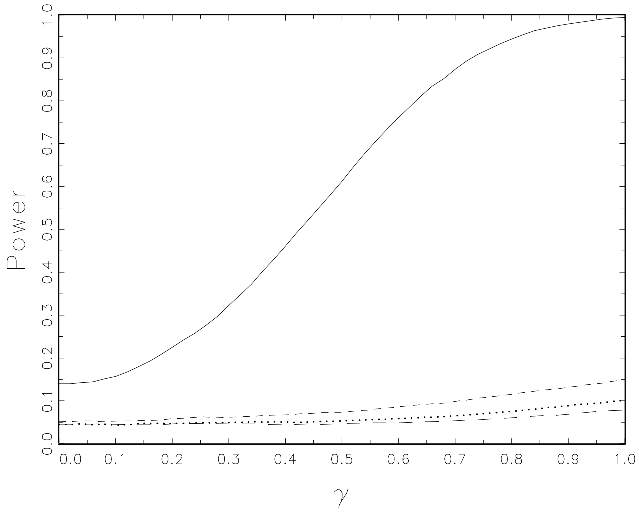


(e) $T = 150$, $\tau^* = 0.75$

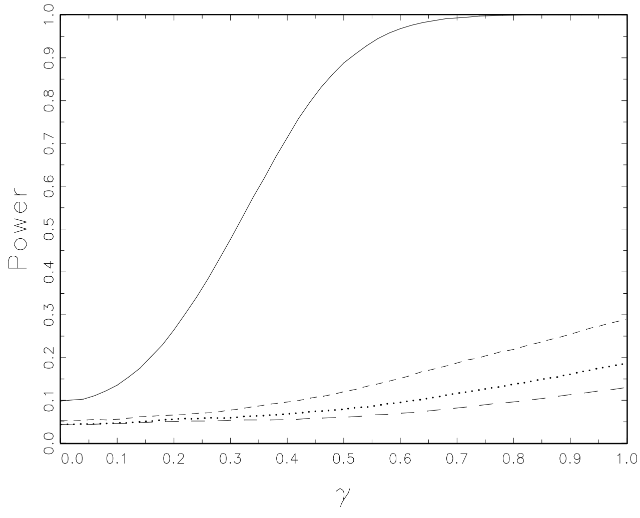


(f) $T = 300$, $\tau^* = 0.75$

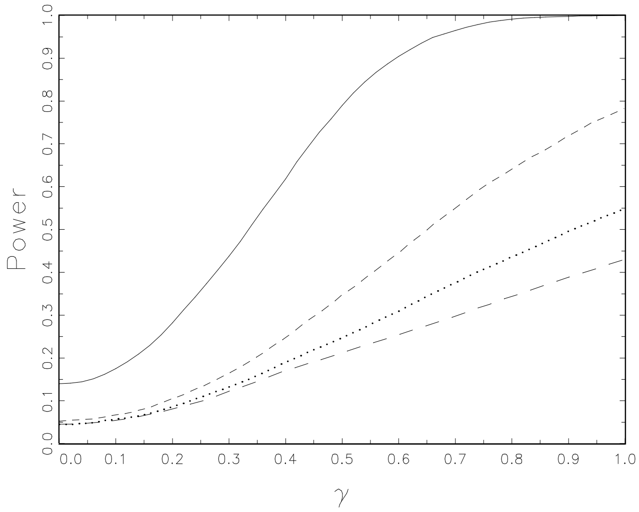
Figure 4. Size-adjusted power: Model A, $c = 0$, t_λ: —— , $Mean\,W_{BG}^{0.02}$: $--$, $Sup\,W_{BG}^{0.10}$: - - - , $Sup\,W_{BG}^{0.06}$: $\cdots$



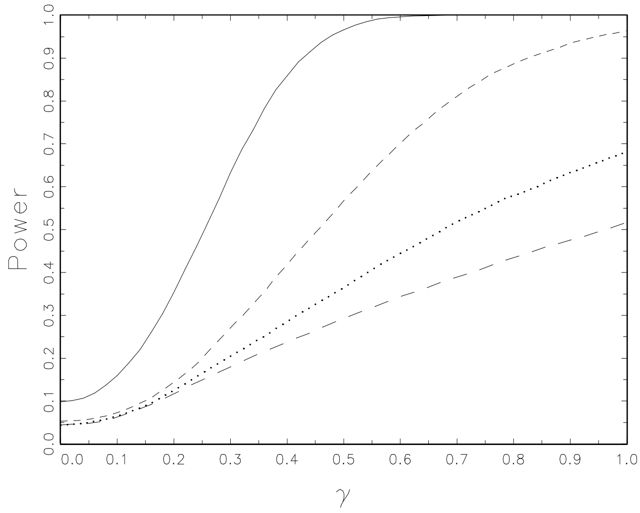
(a) $T = 150$, $\tau^* = 0.25$



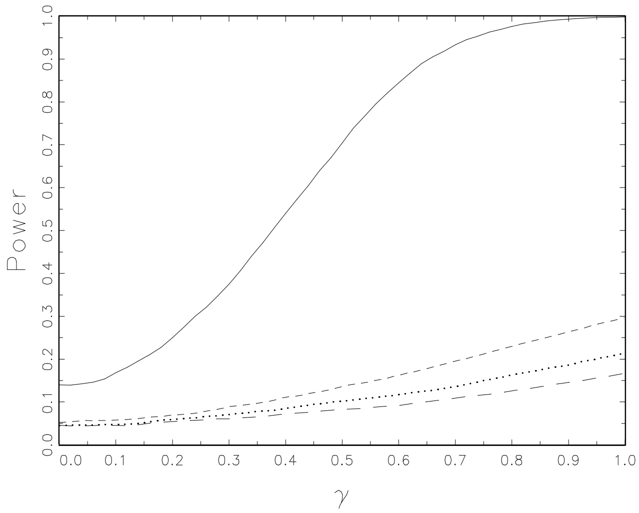
(b) $T = 300$, $\tau^* = 0.25$



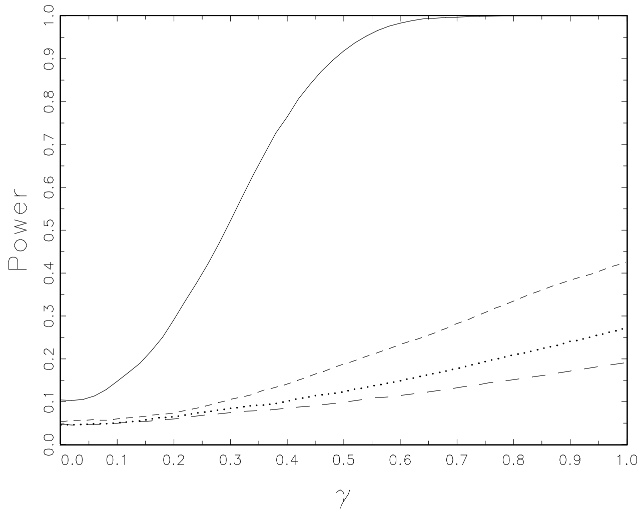
(c) $T = 150$, $\tau^* = 0.5$



(d) $T = 300$, $\tau^* = 0.5$

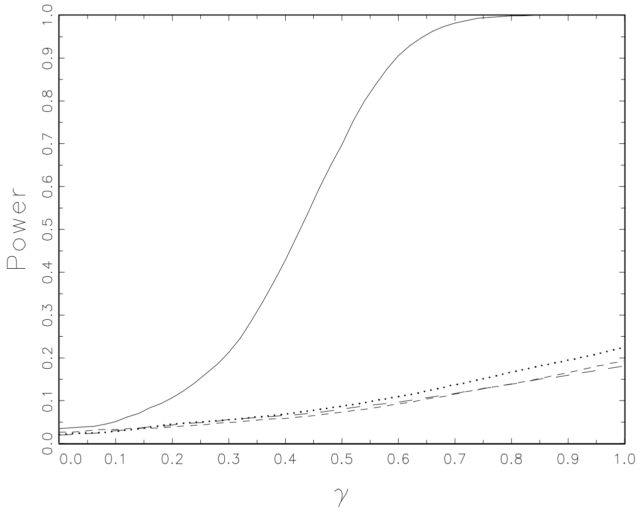


(e) $T = 150$, $\tau^* = 0.75$

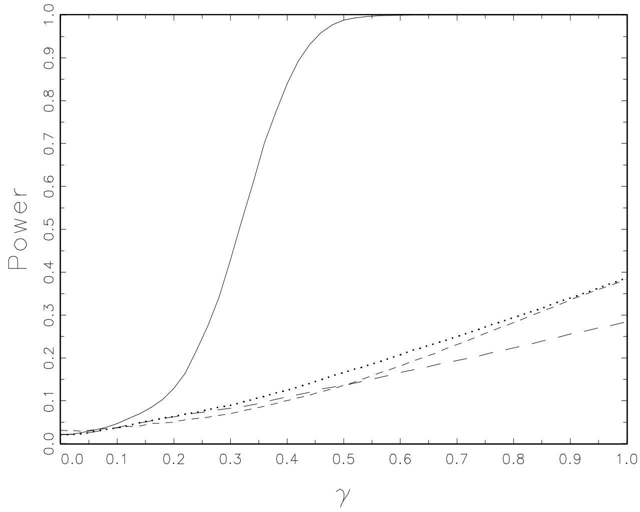


(f) $T = 300$, $\tau^* = 0.75$

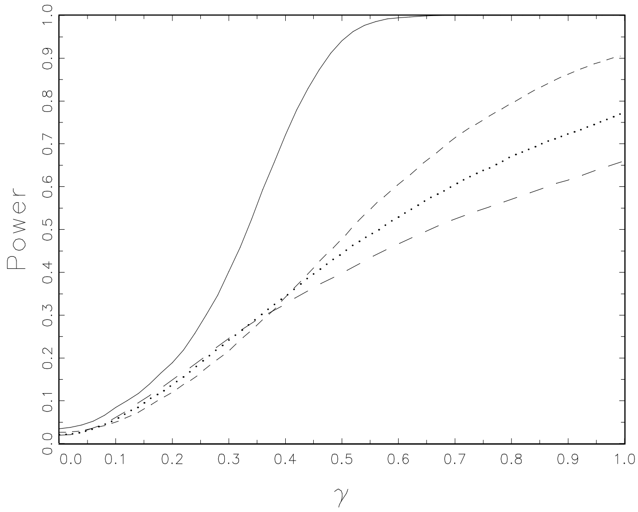
Figure 5. Power: Model B, $c = 0$, t_λ: ——, $Mean\,W_{BG}^{0.02}$: $--$, $Sup\,W_J^{0.36}$: - - - , $Mean\,W_{BG}^{0.18}$: $\cdots$



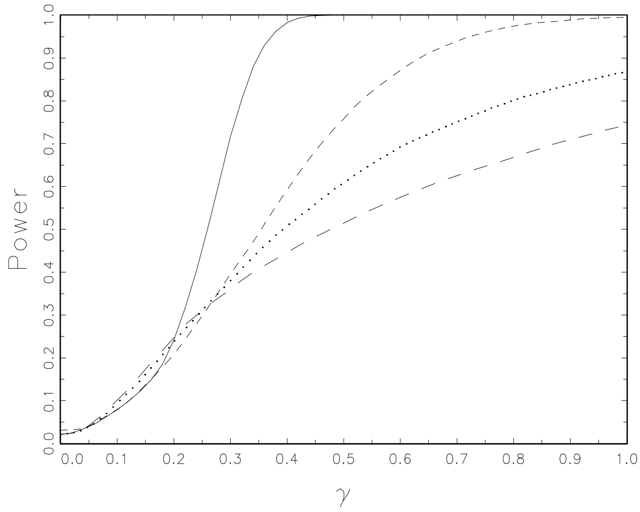
(a) $T = 150$, $\tau^* = 0.25$



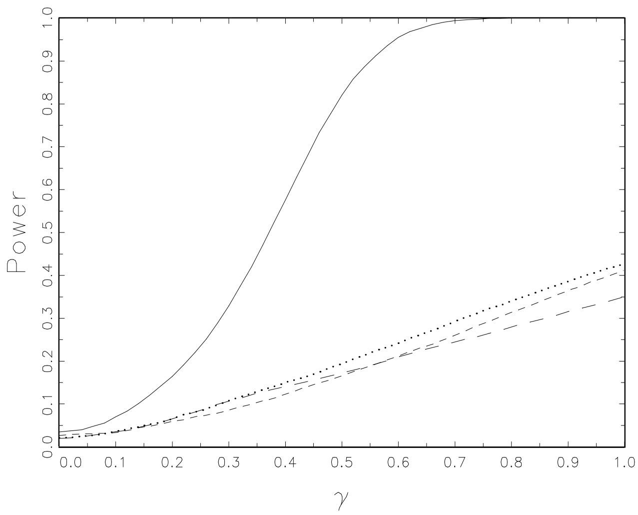
(b) $T = 300$, $\tau^* = 0.25$



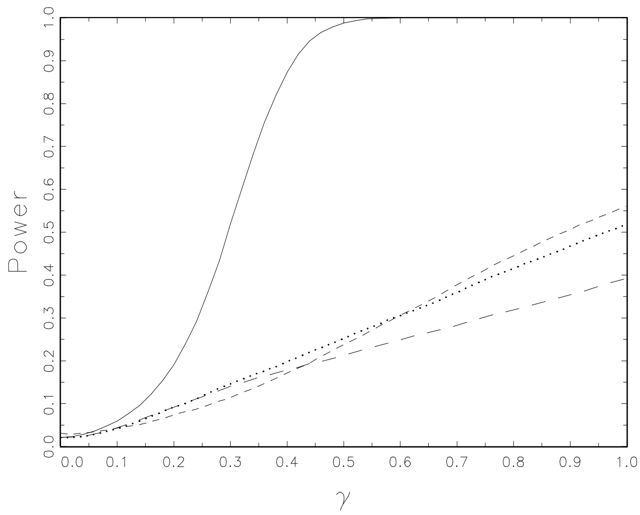
(c) $T = 150$, $\tau^* = 0.5$



(d) $T = 300$, $\tau^* = 0.5$

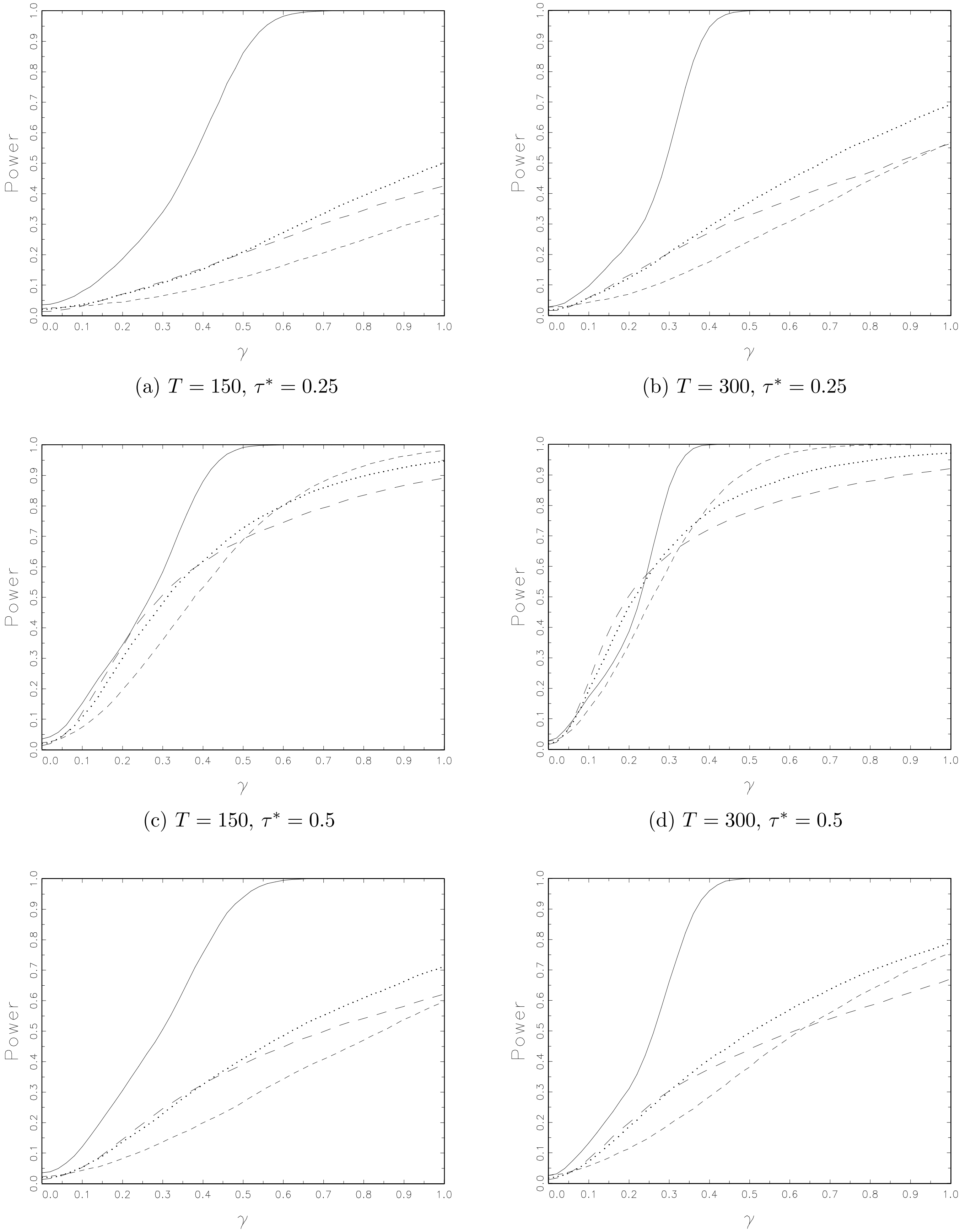


(e) $T = 150$, $\tau^* = 0.75$



(f) $T = 300$, $\tau^* = 0.75$

Figure 6. Power: Model B, $c = 10$, t_λ: ——, $Mean\,W_{BG}^{0.02}$: $--$, $Sup\,W_J^{0.36}$: - - - , $Mean\,W_{BG}^{0.18}$: $\cdots$



(a) $T = 150$, $\tau^* = 0.25$

(b) $T = 300$, $\tau^* = 0.25$

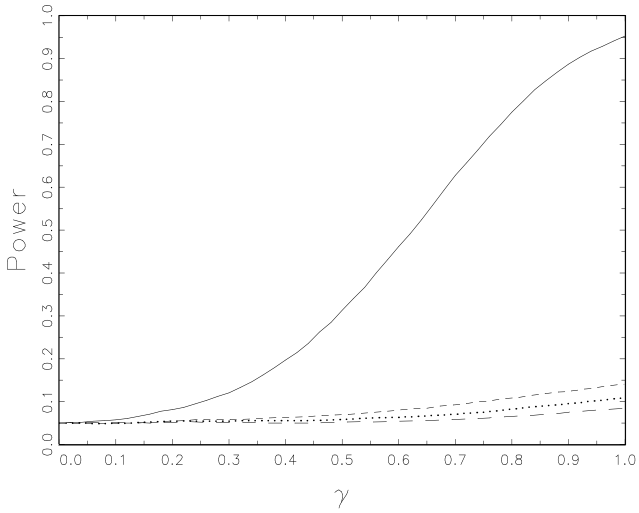
(c) $T = 150$, $\tau^* = 0.5$

(d) $T = 300$, $\tau^* = 0.5$

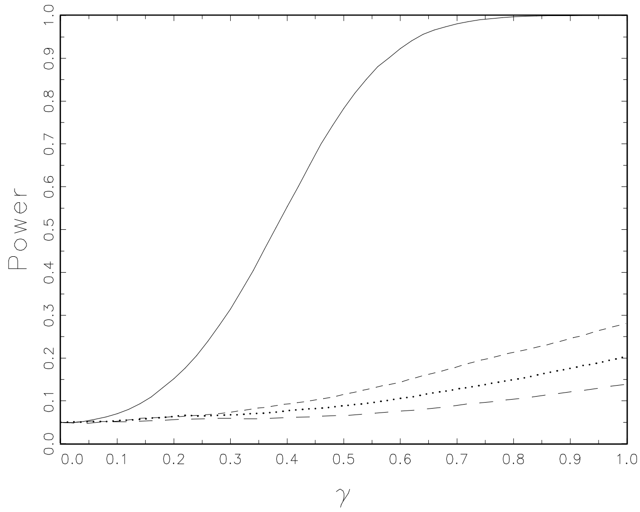
(e) $T = 150$, $\tau^* = 0.75$

(f) $T = 300$, $\tau^* = 0.75$

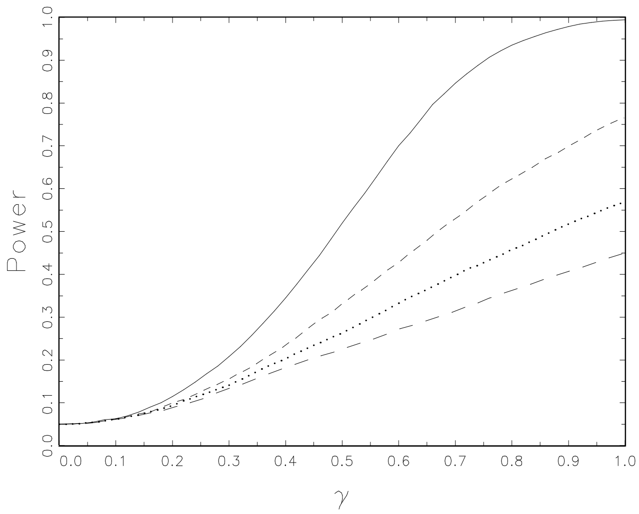
Figure 7. Power: Model B, $c = 20$, t_λ: ——, $Mean\, W_{BG}^{0.02}$: $--$, $Sup\, W_J^{0.36}$: - - -, $Mean\, W_{BG}^{0.18}$: $\cdots$



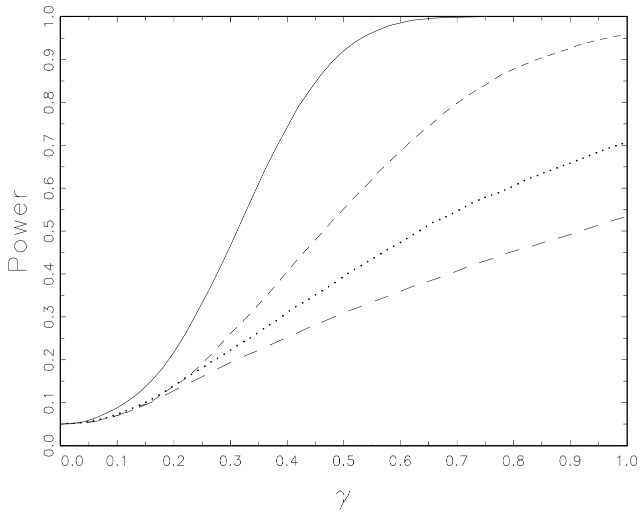
(a) $T = 150$, $\tau^* = 0.25$



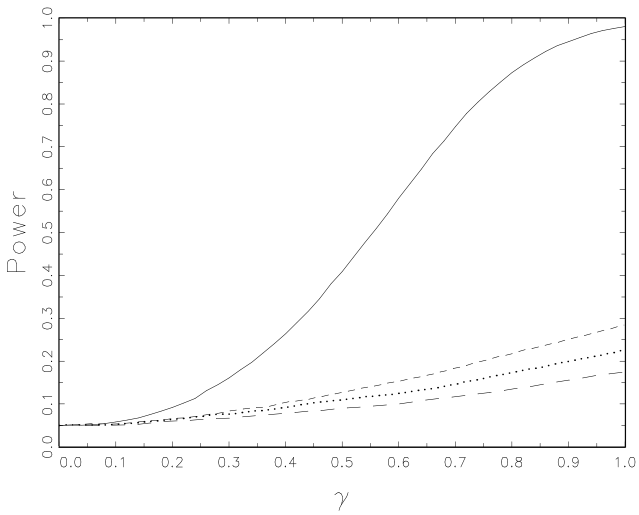
(b) $T = 300$, $\tau^* = 0.25$



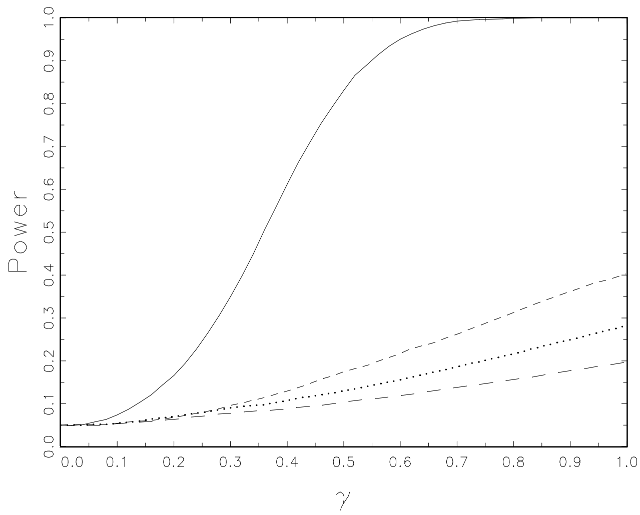
(c) $T = 150$, $\tau^* = 0.5$



(d) $T = 300$, $\tau^* = 0.5$

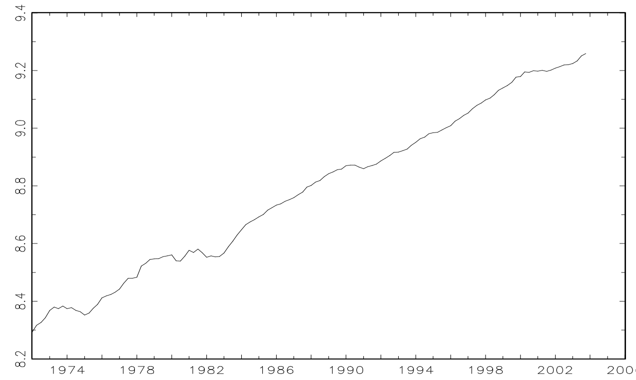


(e) $T = 150$, $\tau^* = 0.75$

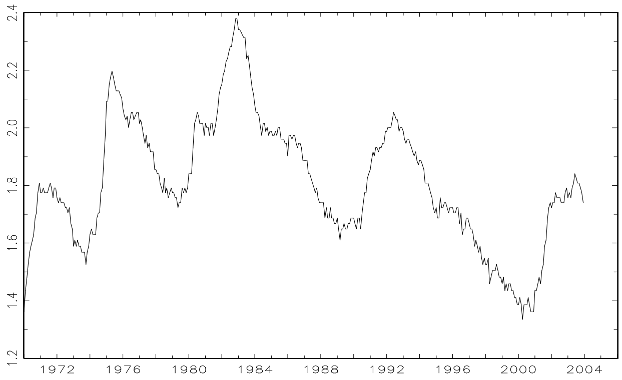


(f) $T = 300$, $\tau^* = 0.75$

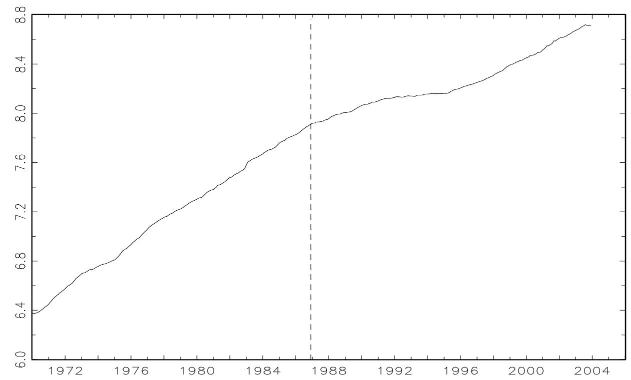
Figure 8. Size-adjusted power: Model B, $c = 0$, t_λ: ——, $Mean\, W_{BG}^{0.02}$: – –, $Sup\, W_J^{0.36}$: - - -, $Mean\, W_{BG}^{0.18}$: $\cdots$



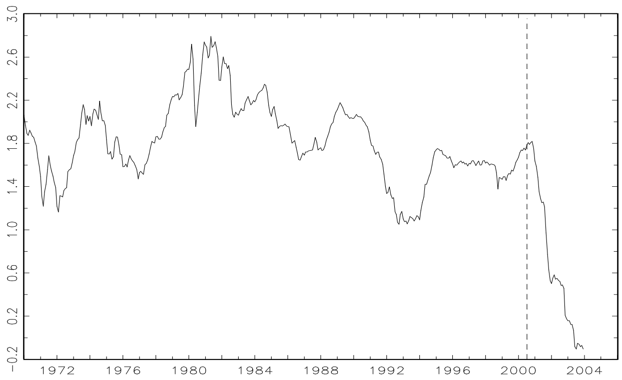
(a) Real output



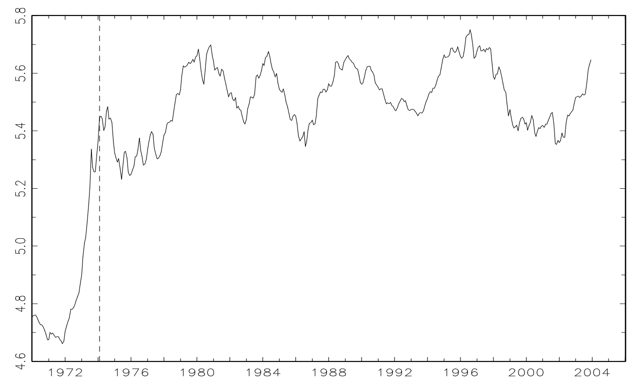
(b) Unemployment



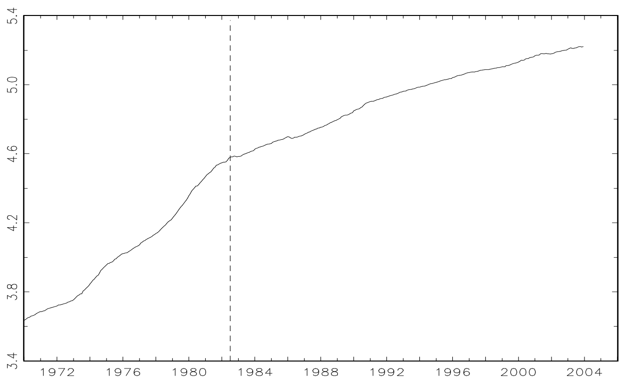
(c) Money supply



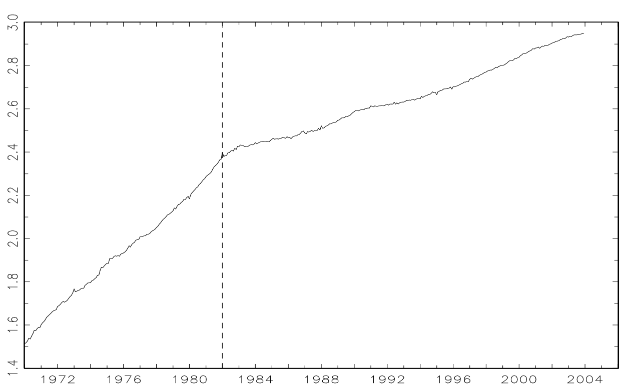
(d) Interest rate



(e) Commodity prices



(f) Consumer prices



(g) Earnings

Figure 9. US macroeconomic time series and estimated Model A break dates.