Rationalizable Variable-Population Choice Functions*

by

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Abstract

We analyze the rationalizability of variable-population social-choice functions in a welfarist framework. It is shown that fixed-population rationalizability and a weakening of congruence together are necessary and sufficient for rational choice, given a plausible dominance property that prevents the choice of alternatives involving low utility levels. In addition, a class of critical-level generalized-utilitarian choice functions is characterized. This result, which extends an earlier axiomatization of a related class of bargaining solutions to a variable-population setting, is the first axiomatization of critical-level generalized utilitarianism in a general choice-theoretic model.

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1. Introduction

Many policy issues require the choice of social alternatives where the composition and the size of the population may vary from one possible state of affairs to another. For example, policy choices involving the allocation of funds to population-control programs, prenatal care, or the intertemporal and intergenerational allocation of resources cannot be analyzed satisfactorily in a fixed-population framework.

Welfarist approaches to population ethics usually proceed by identifying social orderings with attractive properties that are capable of comparing states of affairs involving different populations and population sizes. Once a social ordering is established, social choices can be made by selecting the best states among those that are feasible provided that best elements exist.

Because variable-population considerations typically arise in choice problems, it may be more appropriate to analyze choices directly rather than to assume the existence of a rationalizing ordering from the outset. Thomson [1996a], for instance, makes this point in his discussion of Blackorby, Bossert, and Donaldson [1996c]. The purpose of this paper is to examine the requirements for the rationalizability of variable-population choice functions in a welfarist framework.

The rationalizability of choice functions has been analyzed extensively in the context of consumer theory—see, for example, Blackorby, Bossert, and Donaldson [1995a], Bossert [1993], Gale [1960], Houthakker [1950], Hurwicz and Richter [1971], Kihlstrom, Mas-Colell, and Sonnenschein [1976], Peters and Wakker [1994], Rose [1958], Samuelson [1938, 1948], and Uzawa [1960, 1971], among others. Extensions to more general choice problems can be found in Arrow [1959], Baigent and Gaertner [1996], Bossert [1995], Hansson [1968], Richter [1966, 1971], and Sen [1971, 1993]. Issues related to the rationalizability of various group-decision procedures (such as bargaining solutions) are discussed in Blackorby, Bossert, and Donaldson [1994, 1996a], Bossert [1994, 1998], Donaldson and Weymark [1988], Lensberg [1987], Ok and Zhou [1997a,b], Peters and Wakker [1991], and Zhou [1997].

Variable-population rational choice is investigated in Blackorby, Bossert, and Donaldson [1999b] in the context of a very simple choice problem—the pure population problem. In a pure population problem, a single resource has to be divided equally among the members of a society with population size and per-capita consumption as the objects of choice. In this simple setting, rationalizability is easy to obtain—see Blackorby, Bossert, and Donaldson [1999b] for details. However, pure population problems are rather specialized formulations of allocation problems, and the question arises whether the results obtained there can be extended to more general settings.

In this paper, we examine the rationalizability issue in a general welfarist framework, where the objects to be chosen are feasible utility vectors of variable dimension. Beginning with a choice-theoretic formulation of variable-population problems, we characterize classes of choice functions that are rationalizable by variable-population social orderings. Our approach is welfarist in the sense that we assume that the information relevant for a choice problem is summarized by a collection of feasible sets of utility vectors, one for each possible population size. In addition, in order to avoid utility levels associated with a very low standard of living, we impose a lower bound on chosen utilities. As a consequence of modelling choice problems in that way, the fixed-population restrictions of our choice functions can be interpreted as bargaining solutions. However, the variable-population aspect that is of particular importance in our model is quite different from that appearing in bargaining problems with a variable population that are analyzed, for example, in Blackorby, Bossert, and Donaldson [1996a], Lensberg [1987, 1988], Thomson [1983a,b, 1984, 1985, 1986], Thomson and Lensberg [1989]. In our formulation, population size is to be chosen as well, whereas the above-mentioned contributions analyze problems where a choice has to be made for each possible given population, and relationships between those choices across different compositions of the population are imposed. Thus, our results can also be interpreted in a generalized model of bargaining. In particular, our second main result is an extension of a characterization result by Lensberg [1987].

In Section 2, we present a formal definition of the variable-population choice problems considered in this paper. This class of problems constitutes the domain of the choice functions to be analyzed. We conclude the section by introducing an axiom guaranteeing that no individual's utility drops below a given lower bound in a chosen state of affairs.

Rationalizability is discussed in Section 3. We present a statement of a special case of Richter's [1966] result on the rationalizability of arbitrary choice functions adapted to our framework, and we characterize rationalizable variable-population choice functions. We show that, together with fixed-population rationalizability, a weakening of Richter's congruence axiom is necessary and sufficient for rational choice. This weakening rules out specific revealed-preference cycles involving alternating population sizes.

Section 4 introduces some further axioms, most of which are familiar from axiomatic approaches to bargaining. They are used to characterize a class of critical-level generalized-utilitarian choice functions. This characterization result is a variable-population extension of an axiomatization of a class of rationalizable bargaining solutions due to Lensberg [1987]. Section 5 concludes.

2. Variable-Population Choice Functions

The set of all positive integers is denoted by $\mathcal{Z}_{++}$, and the set of all (nonnegative, positive) real numbers is $\mathcal{R}$ ($\mathcal{R}_+$, $\mathcal{R}_{++}$). For $n \in \mathcal{Z}_{++}$, $\mathcal{R}^n$ ($\mathcal{R}^n_+$, $\mathcal{R}^n_{++}$) is the n-fold Cartesian product of $\mathcal{R}$ ($\mathcal{R}_+$, $\mathcal{R}_{++}$). $\mathbf{1}_n$ is the n-dimensional vector consisting of $n \in \mathcal{Z}_{++}$ ones.

Our notation for vector inequalities is $\geq$, $>$, $\gg$. Furthermore, $\Omega = \bigcup_{n \in \mathcal{Z}_{++}} \mathcal{R}^n$ and $\Omega_+ = \bigcup_{n \in \mathcal{Z}_{++}} \mathcal{R}^n_+$. The absolute value of a real number x is denoted by $|x|$.

For a nonempty set A, $\mathcal{P}(A)$ denotes the power set of A excluding the empty set, and $|A|$ is the cardinality of A. In addition, for $n \in \mathcal{Z}_{++}$, a set $A \subseteq \mathcal{R}^n$ is comprehensive if and only if, for all $x \in A$ and all $y \in \mathcal{R}^n$, $y \leq x$ implies $y \in A$. The comprehensive hull of $x \in \mathcal{R}^n$ is defined as $\mathcal{H}(x) = \{y \in \mathcal{R}^n \mid y \leq x\}$. $A \subseteq \mathcal{R}^n$ is bounded from above if and only if there exist $p \in \mathcal{R}^n_{++}$ and $M \in \mathcal{R}$ such that $px \leq M$ for all $x \in A$.

A variable-population choice problem is a feasible set S which is a collection of sets $\{S_n\}_{n \in \mathcal{Z}_{++}}$ where, for all $n \in \mathcal{Z}_{++}$, $S_n \subseteq \mathcal{R}^n$ is nonempty, convex, comprehensive, closed, and bounded from above. Furthermore, we assume that there exists $n \in \mathcal{Z}_{++}$ such that $S_n \cap \mathcal{R}^n_+ \neq \emptyset$. The set of all variable-population choice problems is denoted by Σ. For $n \in \mathcal{Z}_{++}$, the subset Σ^n of Σ is defined as follows. For all $S \in \Sigma$, $S \in \Sigma^n$ if and only if $S_m = \mathcal{H}(-\mathbf{1}_m)$ for all $m \in \mathcal{Z}_{++} \setminus \{n\}$.[1] In addition, $\bar{\Sigma}^n \subset \Sigma^n$ consists of all problems $S \in \Sigma^n$ satisfying $S_n \cap \mathcal{R}^n_{++} \neq \emptyset$. To simplify notation, we define $S_+ = \cup_{n \in \mathcal{Z}_{++}} (S_n \cap \mathcal{R}^n_+)$.

In the above definitions, $n \in \mathcal{Z}_{++}$ represents population size and S_n is the set of feasible vectors of lifetime utilities for the individuals labelled $1, \ldots, n$. This formulation could be generalized by allowing for different compositions of the population for a given population size. We use the above (more restrictive) formulation for simplicity and because an anonymity axiom is imposed in Section 4. It should be noted that the result of Section 3 could be generalized in order to allow for populations of size n that consist of different sets of individuals. The purpose of Σ^n is to represent fixed-population choice problems in the sense that, given the zero-dominance condition introduced below, utility vectors of size n only are chosen for all problems in Σ^n.

We assume that individual utilities are normalized so that a negative level of utility represents an unacceptably low level of individual welfare from an ethical point of view. With this interpretation in mind, the assumption that there exists at least one population size so that everyone alive experiences a utility level of at least zero rules out degenerate cases where one would never want to choose any positive population size. This minimally acceptable level of utility can be chosen in many different ways: it could be equal to the utility level representing a *neutral* life but it could also be above or below.[2] Our results are independent of the interpretation of this minimally acceptable utility level. It should be mentioned, however, that if one deviates from the standard convention of using zero to represent neutrality, the usual definitions of population principles have to be amended accordingly, as is the case in Dasgupta [1993, 1994].[3] The zero normalization is of

[1] The choice of $-\mathbf{1}_m$ in the definition of Σ^n is arbitrary—any negative vector would do.

[2] A fully informed, rational, and selfish individual is indifferent between leading a life at neutrality and a life without any experiences. It is important to note that states of nonexistence do not have to be invoked in order to define neutrality. See, for example, Broome [1993] for a discussion.

[3] Dasgupta uses a negative utility level to represent a neutral life.

importance for one of our axioms that imposes a lower bound on chosen utilities, analogous to the individual-rationality condition in bargaining theory (this zero-dominance axiom is defined at the end of the section); clearly, a change in the normalization used in the definition of Σ would have to be accompanied by a corresponding modification of this axiom.

Note that we do not require the existence of at least one population size such that the corresponding feasible set of utility vectors contains a vector that strictly dominates the zero vector. This is another feature that distinguishes our approach from standard axiomatic models of bargaining, where such an assumption is usually made based on the view that there should be some potential gains for all participants in the bargaining process. In contrast, for the variable-population problems considered here, this does not appear to be a natural assumption. Given that one might very well choose a boundary point of $\mathcal{R}^n_+$ for some n, it seems inappropriate to exclude feasible sets which contain boundary points but no interior points of $\mathcal{R}^n_+$ from the outset. This difference is another consequence of considering a variable-population choice framework, and it is of importance for our results.

A variable-population choice function is a mapping $F\colon \Sigma \longrightarrow \mathcal{P}(\Omega)$ such that $F(S) \subseteq S$ for all $S \in \Sigma$. Formally, this is a generalization of a bargaining solution where, instead of merely selecting utility vectors for given population sizes, the objective is to choose population size (or sizes)[4] in addition.

As discussed above, we assume that zero represents the minimally acceptable level of utility from an ethical point of view. Therefore, we impose the following zero-dominance condition.

Zero Dominance: For all $S \in \Sigma$, for all $n \in \mathcal{Z}_{++}$, for all $x \in S_n$, if $x \in F(S)$, then $x \in \mathcal{R}^n_+$.

Zero dominance is consistent with the assumption that $F(S)$ is nonempty for all $S \in \Sigma$ because, by definition, S contains at least one population size n such that $S_n \cap \mathcal{R}^n_+$ is nonempty.

3. Rationalizability

As usual, rationalizability requires that choices can be generated by a binary relation in the sense that the choice function picks the best or maximal elements in the feasible set according to this relation. Although some approaches allow for rationalizability by more general relations, we follow the standard convention and require a rationalization to be an

[4] Note that $F(S)$ can contain more than one element.

ordering.[5] Formally, a variable-population choice function F is rationalizable if and only if there exists an ordering $\succeq$ on Ω such that, for all $S \in \Sigma$,

$$F(S) = \{x \in S \mid x \succeq y \text{ for all } y \in S\}. \tag{3.1}$$

In this case, we say that $\succeq$ rationalizes F or $\succeq$ is a rationalization of F. The strict preference relation and the indifference relation corresponding to $\succeq$ are denoted by $\succ$ and $\sim$ respectively.

The direct revealed preference relation R^d_F on Ω corresponding to the choice function F is defined as follows. For all $x, y \in \Omega$, $xR^d_F y$ if and only if there exists $S \in \Sigma$ such that $x \in F(S)$ and $y \in S$.

Richter [1966] shows that the following congruence axiom is necessary and sufficient for the rationalizability of a choice function with an arbitrary domain. Stated in terms of our variable-population choice problems, congruence is defined as follows.

Congruence: For all $S \in \Sigma$, for all $K \in \mathcal{Z}_{++} \setminus \{1\}$, for all $x^1, \ldots, x^K \in \Omega$, if $x^{k-1} R^d_F x^k$ for all $k \in \{2, \ldots, K\}$ and $x^K \in F(S)$ and $x^1 \in S$, then $x^1 \in F(S)$.

If there exist $S \in \Sigma$, $K \in \mathcal{Z}_{++} \setminus \{1\}$, and $x^1, \ldots, x^K \in \Omega$ such that $x^{k-1} R^d_F x^k$ for all $k \in \{2, \ldots, K\}$, $x^K \in F(S)$, and $x^1 \in S \setminus F(S)$, we say that there exists a revealed-preference cycle of length K. Therefore, an equivalent formulation of congruence is that there exists no revealed-preference cycle of length K for all $K \in \mathcal{Z}_{++} \setminus \{1\}$.

Richter's [1966] result formulated for variable-population choice problems is stated in the following theorem.

Theorem 1: *A variable-population choice function F is rationalizable if and only if F satisfies congruence.*

Richter's rationalizability result is very general in the sense that it does not require any assumptions regarding the domain of a choice function. In the context of variable-population choice problems as defined above, the question arises whether there may be weaker conditions that, due to the specific structure of the underlying domain, turn out to be sufficient for the existence of a rationalizing ordering. Clearly, the rationalizability of fixed-population restrictions of F is necessary in order to obtain full rationality. What is of particular interest is the set of additional restrictions on F imposed by variable-population considerations. Rationalizability can be obtained by adding an interesting weakening of congruence to fixed-population rationalizability, provided the variable-population choice function satisfies zero dominance. In particular, revealed-preference cycles involving four utility vectors of alternating population sizes have to be ruled out. To the best of our knowledge, this condition has not appeared in any other study of rationalizability on

[5] An ordering is a reflexive, transitive, and complete binary relation.

various domains and provides a useful insight into the additional requirements that need to be imposed when moving from fixed-population problems to choice problems with a variable population.

First, we provide a formal statement of fixed-population rationalizability. It requires that, for each population size n, there exists an ordering on $\mathcal{R}^n$ such that whenever $F(S)$ contains any n-dimensional vectors, then those must be best elements in S_n according to this ordering.

Fixed-Population Rationalizability: There exists a sequence of orderings $\{\succeq^n\}_{n\in\mathcal{Z}_{++}}$ with $\succeq^n\subseteq \mathcal{R}^n \times \mathcal{R}^n$ for all $n \in \mathcal{Z}_{++}$ such that, for all $S \in \Sigma$, for all $n \in \mathcal{Z}_{++}$, if $F(S) \cap S_n \neq \emptyset$, then $F(S) \cap S_n = \{x \in S_n \mid x \succeq^n y \text{ for all } y \in S_n\}$.

The weakening of congruence mentioned above is defined as follows. Note that only revealed-preference cycles with a very special pattern need to be ruled out; in the presence of zero dominance, the conjunction of this axiom and fixed-population rationalizability turns out to be sufficient to exclude any revealed-preference cycle (Theorem 2 below).

Weak Population Congruence: For all $S \in \Sigma$, for all $n, m \in \mathcal{Z}_{++}$ with $n \neq m$, for all $x^1, x^3 \in \mathcal{R}^n$, for all $x^2, x^4 \in \mathcal{R}^m$, if $x^{\ell-1} R_F^d x^\ell$ for all $\ell \in \{2, 3, 4\}$ and $x^4 \in F(S)$ and $x^1 \in S$, then $x^1 \in F(S)$.

Weak population congruence rules out revealed-preference cycles of length four where the elements of the cycle are of alternating dimension. This requirement is substantially weaker than congruence. Note that weak population congruence also rules out cycles of length two involving elements of different dimension because the vectors x^1 and x^3 in the definition of this axiom need not be distinct, and the same is true for x^2 and x^4.

In the presence of zero dominance and fixed-population rationalizability, weak population congruence is equivalent to a stronger version that rules out revealed-preference cycles involving alternating dimensions of any even size. This stronger axiom is defined as follows.

Strong Population Congruence: For all $S \in \Sigma$, for all $n, m \in \mathcal{Z}_{++}$ with $n \neq m$, for all even $L \in \mathcal{Z}_{++}$, for all $x^1, x^3, \ldots, x^{L-1} \in \mathcal{R}^n$, for all $x^2, x^4, \ldots, x^L \in \mathcal{R}^m$, if $x^{\ell-1} R_F^d x^\ell$ for all $\ell \in \{2, \ldots, L\}$ and $x^L \in F(S)$ and $x^1 \in S$, then $x^1 \in F(S)$.

If there exist $S \in \Sigma$, $n, m \in \mathcal{Z}_{++}$ with $n \neq m$, an even $L \in \mathcal{Z}_{++}$, $x^1, x^3, \ldots, x^{L-1} \in \mathcal{R}^n$, and $x^2, x^4, \ldots, x^L \in \mathcal{R}^m$ such that $x^{\ell-1} R_F^d x^\ell$ for all $\ell \in \{2, \ldots, L\}$, $x^L \in F(S)$, and $x^1 \in S \setminus F(S)$, we say that there exists an alternating-population revealed-preference cycle of length L.

The next lemma states the above-mentioned equivalence between the two versions of population congruence.

Lemma 1: *Suppose a variable-population choice function F satisfies zero dominance and fixed-population rationalizability. F satisfies weak population congruence if and only if F satisfies strong population congruence.*

Proof: That strong population congruence implies weak population congruence is obvious. Now suppose F satisfies zero dominance, fixed-population rationalizability, and weak population congruence. We proceed by induction on L. By weak population congruence, the conclusion of strong population congruence applies to the cases $L = 2$ and $L = 4$. Now suppose the claim is true for all even $\overline{L} \leq L$ where $L \geq 4$. By way of contradiction, suppose there exists an alternating-population revealed-preference cycle of length $L+2$. That is, there exist $S \in \Sigma$, $n, m \in \mathcal{Z}_{++}$ with $n \neq m$, $x^1, x^3, \ldots, x^{L+1} \in \mathcal{R}^n$, and $x^2, x^4, \ldots, x^{L+2} \in \mathcal{R}^m$ such that $x^{\ell-1} R_F^d x^\ell$ for all $\ell \in \{2, \ldots, L+2\}$, $x^{L+2} \in F(S)$, and $x^1 \in S \setminus F(S)$. Let $S^3 \in \Sigma$ be such that $x^3 \in F(S^3)$ and $x^4 \in S^3$. Define $\overline{S} \in \Sigma$ by letting $\overline{S}_n = S^3_n$, $\overline{S}_m = S_m$, and $\overline{S}_r = \mathcal{H}(-\mathbf{1}_r)$ for all $r \in \mathcal{Z}_{++} \setminus \{n, m\}$.

If $x^3 \in F(\overline{S})$, it follows that $x^1 R_F^d x^2$, $x^2 R_F^d x^3$, $x^3 R_F^d x^{L+2}$, $x^{L+2} \in F(S)$, and $x^1 \in S \setminus F(S)$. This implies that there exists an alternating-population revealed-preference cycle of length four, a contradiction.

If $x^3 \notin F(\overline{S})$, zero dominance and fixed-population rationalizability imply $x^{L+2} \in F(\overline{S})$. This, in turn, implies that we have $x^3 R_F^d x^4, \ldots, x^{L+1} R_F^d x^{L+2}$, $x^{L+2} \in F(\overline{S})$, and $x^3 \in \overline{S} \setminus F(\overline{S})$, which establishes the existence of an alternating-population revealed-preference cycle of length L, again a contradiction. ■

We now obtain

Theorem 2: *Suppose a variable-population choice function F satisfies zero dominance. F is rationalizable if and only if F satisfies fixed-population rationalizability and weak population congruence.*

Proof: Clearly, rationalizability implies fixed-population rationalizability and weak population congruence. Conversely, suppose F satisfies zero dominance, fixed-population rationalizability, and weak population congruence. We show by induction on K that F satisfies congruence which, by Theorem 1, is sufficient to complete the proof.

Let $S \in \Sigma$, $n, m \in \mathcal{Z}_{++}$, $x^1 \in \mathcal{R}^n$, and $x^2 \in \mathcal{R}^m$ be such that $x^1 R_F^d x^2$, $x^2 \in F(S)$, and $x^1 \in S$. If $n = m$, fixed-population rationalizability implies $x^1 \in F(S)$ because the restriction of R_F^d to $\mathcal{R}^n$ must be a subrelation of $\succeq^n$ (see Samuelson [1938] and Richter [1971]). If $n \neq m$, $x^1 \in F(S)$ follows from weak population congruence. Therefore, there exists no revealed-preference cycle of length two.

Now suppose there exists no revealed-preference cycle of length $\overline{K} \leq K$ with $K \geq 2$. By way of contradiction, suppose there exists a revealed-preference cycle of length $K+1$. That is, there exist $S \in \Sigma$ and $x^1, \ldots, x^{K+1} \in \Omega$ such that $x^{k-1} R_F^d x^k$ for all

$k \in \{2, \ldots, K+1\}$, $x^{K+1} \in F(S)$, and $x^1 \in S \setminus F(S)$. For all $k \in \{1, \ldots, K\}$, let $S^k \in \Sigma$ be such that $x^k \in F(S^k)$ and $x^{k+1} \in S^k$. Furthermore, let $S^{K+1} = S$. Clearly, one of the following three cases must occur.

(a) There exist $n \in \mathcal{Z}_{++}$ and $k \in \{1, \ldots, K\}$ such that $x^k, x^{k+1} \in \mathcal{R}^n$;

(b) There exist pairwise distinct $n, m, r \in \mathcal{Z}_{++}$ and $k \in \{2, \ldots, K\}$ such that $x^{k-1} \in \mathcal{R}^n$, $x^k \in \mathcal{R}^m$, and $x^{k+1} \in \mathcal{R}^r$;

(c) There exist distinct $n, m \in \mathcal{Z}_{++}$ such that $x^k \in \mathcal{R}^n$ for all odd k and $x^k \in \mathcal{R}^m$ for all even k.

Case (a): If $x^j \in \mathcal{R}^n$ for all $j \in \{1, \ldots, K+1\}$, we obtain a contradiction to fixed-population rationalizability. Therefore, n and k can be chosen such that at least one of the following subcases must occur.

(i) There exists $m \in \mathcal{Z}_{++} \setminus \{n\}$ such that $x^{k-1} \in \mathcal{R}^m$;

(ii) there exists $m \in \mathcal{Z}_{++} \setminus \{n\}$ such that $x^{k+2} \in \mathcal{R}^m$.

In subcase (i), let $\overline{S} \in \Sigma$ be such that $\overline{S}_m = S^{k-1}_m$, $\overline{S}_n = S^k_n$, and $\overline{S}_r = \mathcal{H}(-\mathbf{1}_r)$ for all $r \in \mathcal{Z}_{++} \setminus \{n, m\}$.

If $F(\overline{S}) \cap \overline{S}_m \neq \emptyset$, $x^{k-1} \in F(\overline{S})$ by fixed-population rationalizability. Because $x^{k+1} \in \overline{S}$, there exists a revealed-preference cycle of length K, a contradiction.

If $F(\overline{S}) \cap \overline{S}_m = \emptyset$, zero dominance implies $F(\overline{S}) \cap \overline{S}_n \neq \emptyset$, and by fixed-population rationalizability, $x^k \in F(\overline{S})$. Because $x^{k-1} \in \overline{S} \setminus F(\overline{S})$, there exists a revealed-preference cycle of length two, a contradiction.

In subcase (ii), let $\overline{S} \in \Sigma$ be such that $\overline{S}_m = S^{k+2}_m$, $\overline{S}_n = S^k_n$, and $\overline{S}_r = \mathcal{H}(-\mathbf{1}_r)$ for all $r \in \mathcal{Z}_{++} \setminus \{n, m\}$.

If $F(\overline{S}) \cap \overline{S}_n \neq \emptyset$, $x^k \in F(\overline{S})$ by fixed-population rationalizability. Because $x^{k+2} \in \overline{S}$, there exists a revealed-preference cycle of length K, a contradiction.

If $F(\overline{S}) \cap \overline{S}_n = \emptyset$, zero dominance implies $F(\overline{S}) \cap \overline{S}_m \neq \emptyset$, and by fixed-population rationalizability, $x^{k+2} \in F(\overline{S})$. Because $x^{k+1} \in \overline{S} \setminus F(\overline{S})$, there exists a revealed-preference cycle of length two, a contradiction.

Case (b): Let $\overline{S} \in \Sigma$ be such that $\overline{S}_n = S^{k-1}_n$, $\overline{S}_m = S^k_m$, $\overline{S}_r = S^{k+1}_r$ and $\overline{S}_t = \mathcal{H}(-\mathbf{1}_t)$ for all $t \in \mathcal{Z}_{++} \setminus \{n, m, r\}$. By zero dominance, $F(\overline{S}) \cap (\overline{S}_n \cup \overline{S}_m \cup \overline{S}_r) \neq \emptyset$, and by fixed-population rationalizability, we must have

(i) $F(\overline{S}) \cap \overline{S}_n \neq \emptyset \implies x^{k-1} \in F(\overline{S})$;

(ii) $F(\overline{S}) \cap \overline{S}_m \neq \emptyset \implies x^k \in F(\overline{S})$;

(iii) $F(\overline{S}) \cap \overline{S}_r \neq \emptyset \implies x^{k+1} \in F(\overline{S})$.

If $x^{k+1} \in F(\overline{S})$, it follows that $x^k \in F(\overline{S})$ because there exists no revealed-preference cycle of length two. Analogously, if $x^k \in F(\overline{S})$, it follows that $x^{k-1} \in F(\overline{S})$. Therefore, $x^{k-1} \in F(\overline{S})$ in all possible cases and, because $x^{k+1} \in \overline{S}$, we have $x^{k-1} R_F^d x^{k+1}$. Therefore, there exists a revealed-preference cycle of length K, a contradiction.[6]

Case (c): There are two subcases.

(i) K is odd;

(ii) K is even.

In case (i), $K+1$ is even, and we obtain a contradiction to strong population congruence and, by Lemma 1, to weak population congruence.

In case (ii), it follows that $x^1, x^{K+1} \in \mathcal{R}^n$. Let $\overline{S} \in \Sigma$ be such that $\overline{S}_n = S_n^{K+1}$, $\overline{S}_m = S_m^2$, and $\overline{S}_r = \mathcal{H}(-\mathbf{1}_r)$ for all $r \in \mathcal{Z}_{++} \setminus \{n, m\}$.

If $F(\overline{S}) \cap \overline{S}_m \neq \emptyset$, fixed-population rationalizability implies $x^2 \in F(\overline{S})$. Because $x^{K+1} \in F(S^{K+1})$ and $x^1 \in S^{K+1} \setminus F(S^{K+1})$, fixed-population rationalizability implies $x^1 \in \overline{S} \setminus F(\overline{S})$. Therefore, there exists a revealed-preference cycle of length two, a contradiction.

If $F(\overline{S}) \cap \overline{S}_m = \emptyset$, zero dominance implies $F(\overline{S}) \cap \overline{S}_n \neq \emptyset$. By fixed-population rationalizability, $x^{K+1} \in F(\overline{S})$. Because $x^2 \in \overline{S} \setminus F(\overline{S})$, we obtain a revealed-preference cycle of length K, a contradiction. ■

Weak population congruence cannot be weakened to an axiom that merely rules out alternating-population revealed-preference cycles of length two. This is illustrated in the following example. Let $x^1 = (1, 0)$, $x^2 = (1, 0, 0)$, $x^3 = (0, 1)$, and $x^4 = (0, 1, 0)$. Define the sets $B_2 = \{x^1, x^3\}$, $B_3 = \{x^2, x^4\}$, $C_2 = \big((\mathcal{H}(x^1) \cup \mathcal{H}(x^3)) \setminus B_2\big) \cap \mathcal{R}^2_+$, $C_3 = \big((\mathcal{H}(x^2) \cup \mathcal{H}(x^4)) \setminus B_3\big) \cap \mathcal{R}^3_+$, $A_2 = \mathcal{R}^2_+ \setminus (B_2 \cup C_2)$, and $A_3 = \mathcal{R}^3_+ \setminus (B_3 \cup C_3)$. Clearly, $\{A_2, B_2, C_2\}$ is a partition of $\mathcal{R}^2_+$ and $\{A_3, B_3, C_3\}$ is a partition of $\mathcal{R}^3_+$. Define a relation $\succeq$ on Ω as follows.

(i) For all $x \in \Omega$, $x \sim x$;
(ii) for all $x \in \Omega_+$ and all $y \in \Omega \setminus \Omega_+$, $x \succ y$;
(iii) for all $x, y \in \Omega \setminus \Omega_+$, $x \sim y$;
(iv) for all $x \in \mathcal{R}^2_+ \cup \mathcal{R}^3_+$ and all $y \in \bigcup_{n \in \mathcal{Z}_{++} \setminus \{2,3\}} \mathcal{R}^n_+$, $x \succ y$;
(v) for all $x \in \mathcal{R}^n_+$ and all $y \in \mathcal{R}^m_+$ with $n, m \in \mathcal{Z}_{++} \setminus \{2, 3\}$, $x \succeq y$ if and only if $\sum_{i=1}^n x_i \geq \sum_{i=1}^m y_i$;
(vi) for all $x \in A_2 \cup A_3$ and all $y \in B_2 \cup C_2 \cup B_3 \cup C_3$, $x \succ y$;
(vii) for all $x \in A_2 \cup B_2 \cup A_3 \cup B_3$ and all $y \in C_2 \cup C_3$, $x \succ y$;

[6] The proof of case (b) is analogous to Sen's [1971] proof of the observation that if the domain of a choice function contains all subsets of cardinality three or less of the universal set under consideration, then the direct revealed-preference relation is an ordering.

(viii) for all $x \in A_n$ and all $y \in A_m$ with $n, m \in \{2, 3\}$, $x \succeq y$ if and only if $\sum_{i=1}^{n} x_i \geq \sum_{i=1}^{m} y_i$;

(ix) for all $x \in C_n$ and all $y \in C_m$ with $n, m \in \{2, 3\}$, $x \succeq y$ if and only if $\sum_{i=1}^{n} x_i \geq \sum_{i=1}^{m} y_i$;

(x) $x^1 \succ x^2$, $x^1 \succ x^3$, $x^2 \succ x^3$, $x^2 \succ x^4$, $x^3 \succ x^4$, $x^4 \succ x^1$.

Now define F by letting $F(S) = \{x \in S \mid x \succeq y \text{ for all } y \in S\}$ for all $S \in \Sigma$. It can be checked that F is well-defined even though $\succeq$ is not an ordering. That F satisfies zero dominance is easy to verify. To see that fixed-population rationalizability is satisfied, note that the restriction of $\succeq$ to $\mathcal{R}^n$ is an ordering for all $n \in \mathcal{Z}_{++}$. F violates weak population congruence because there exists an alternating-population revealed-preference cycle of length four involving the points $\{x^1, x^2, x^3, x^4\}$. However, no alternating-population revealed-preference cycle of length two exists, which demonstrates that weak population congruence cannot be weakened in the suggested fashion.

Fixed-population rationalizability and weak population congruence are independent (even in the presence of zero dominance). The above example shows that zero dominance and fixed-population rationalizability do not imply weak population congruence, and a choice function satisfying zero dominance and weak population congruence that violates fixed-population rationalizability is presented in the next section.

4. Critical-Level Generalized Utilitarian Choice Functions

In this section, we provide a generalization of Lensberg's [1987] characterization of bargaining solutions with an additively separable rationalization. Specifically, we characterize a class of critical-level generalized-utilitarian variable-population choice functions. Whereas critical-level generalized-utilitarian orderings are axiomatized in several frameworks (see, for example, Blackorby, Bossert, and Donaldson [1995b, 1998a] and Blackorby and Donaldson [1984]), this is the first characterization in a general choice-theoretic setting. Throughout this section, it is assumed that, for all variable-population choice problems S and all population sizes n, the choice function F selects at most one element from the set S_n of feasible utility vectors of dimension n. This axiom is analogous to the single-valuedness assumption that is usually employed in bargaining models.

Fixed-Population Single-Valuedness: For all $S \in \Sigma$, for all $n \in \mathcal{Z}_{++}$, $|F(S) \cap S_n| \leq 1$.

We want to include fixed-population choice problems as special cases in our model. For that purpose, we need to specify fixed feasible sets of utility vectors for population sizes other than the one under consideration. To formulate these fixed-population axioms, we use one fixed feasible set for each remaining population size, where all utilities are below zero. In conjunction with zero dominance, this formulation leads to weak versions of the axioms under consideration. Other formulations could be employed but we have chosen to work with the following weak variants. Due to the presence of fixed-population single-valuedness, we can formulate our axioms so that their conclusions apply to fixed-population single-valued choices only.

The next axiom we introduce is Pareto optimality, suitably adapted to our domain.

Pareto Optimality: For all $S \in \Sigma$, for all $n \in \mathcal{Z}_{++}$, for all $x \in \mathcal{R}^n$, if $S_m = \mathcal{H}(-\mathbf{1}_m)$ for all $m \in \mathcal{Z}_{++} \setminus \{n\}$ and $F(S) \cap S_n = \{x\}$, then $x \in \{y \in S_n \mid \nexists z \in S_n \text{ such that } z > y\}$.

As usual, continuity requires that 'small' changes in the description of a problem only lead to 'small' changes in the outcome. For the following definition, convergence is defined in terms of the Hausdorff topology.

Continuity: For all $S \in \Sigma$, for all sequences $\{S^k\}_{k \in \mathcal{Z}_{++}}$ with $S^k \in \Sigma$ for all $k \in \mathcal{Z}_{++}$, for all $n \in \mathcal{Z}_{++}$, for all $\{x^k\}_{k \in \mathcal{Z}_{++}}$ with $x^k \in \mathcal{R}^n$ for all $k \in \mathcal{Z}_{++}$, for all $x \in \mathcal{R}^n$, if $S_m = S_m^k = \mathcal{H}(-\mathbf{1}_m)$ for all $m \in \mathcal{Z}_{++} \setminus \{n\}$ and for all $k \in \mathcal{Z}_{++}$, $F(S^k) \cap S_n^k = \{x^k\}$ for all $k \in \mathcal{Z}_{++}$, $F(S) \cap S_n = \{x\}$, and $\lim_{k \to \infty} S_n^k = S_n$, then $\lim_{k \to \infty} x^k = x$.

Anonymity requires that the choice function selects outcomes impartially, paying no attention to the agents' identities. To define this axiom, we need more notation. Let $n \in \mathcal{Z}_{++}$, and let $\pi\colon \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ be a bijection. For $x \in \mathcal{R}^n$ and $A \subseteq \mathcal{R}^n$, let $\pi(x) = (x_{\pi(1)}, \ldots, x_{\pi(n)})$ and $\pi(A) = \{y \in \mathcal{R}^n \mid \exists z \in A \text{ such that } y = \pi(z)\}$.

Anonymity: For all $S, T \in \Sigma$, for all $n \in \mathcal{Z}_{++}$, for all bijections $\pi\colon \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$, for all $x \in \mathcal{R}^n$, if $S_m = T_m = \mathcal{H}(-\mathbf{1}_m)$ for all $m \in \mathcal{Z}_{++} \setminus \{n\}$ and $T_n = \pi(S_n)$ and $F(S) \cap S_n = \{x\}$, then $F(T) \cap T_n = \{\pi(x)\}$.

The next axiom is a suitably adapted version of Nash's [1950] well-known independence of irrelevant alternatives.

Independence of Irrelevant Alternatives: For all $S, T \in \Sigma$, for all $n \in \mathcal{Z}_{++}$, for all $x \in \mathcal{R}^n$, if $S_m = T_m = \mathcal{H}(-\mathbf{1}_m)$ for all $m \in \mathcal{Z}_{++} \setminus \{n\}$ and $T_n \subseteq S_n$ and $F(S) \cap T_n = \{x\}$, then $F(T) \cap T_n = \{x\}$.

As shown in Lemma 3, zero dominance and independence of irrelevant alternatives imply that the outcomes selected by a choice function are independent of the points in the feasible set that do not dominate the zero vector. This property is analogous to the axiom independence of nonindividually rational alternatives, which is a well-known property of many bargaining solutions (see, for example, Peters [1992]).

Independence of Dominated Alternatives: For all $S, T \in \Sigma$, for all $n \in \mathcal{Z}_{++}$, for all $x \in \mathcal{R}^n$, if $S_m = T_m = \mathcal{H}(-\mathbf{1}_m)$ for all $m \in \mathcal{Z}_{++} \setminus \{n\}$ and $S_n \cap \mathcal{R}_+^n = T_n \cap \mathcal{R}_+^n \neq \emptyset$ and $F(S) \cap S_n = \{x\}$, then $F(T) \cap T_n = \{x\}$.

The consistency principle requires that if an agent leaves with her or his payoff, the choice of a utility vector for the remaining agents is unchanged.[7] This axiom has remarkably strong consequences. In addition to encompassing a separability condition, it plays a crucial role in establishing the rationalizability of bargaining solutions; see Lensberg [1987] and Thomson and Lensberg [1989]. To define a version of this axiom that is appropriate for our framework, the following notation is used. For $T \in \Sigma$, $n \in \mathcal{Z}_{++}$, $x \in T_{n+1}$, and $i \in \{1, \ldots, n+1\}$, let $t_i^x(T_{n+1}) = \{y \in \mathcal{R}^n \mid (y_1, \ldots, y_{i-1}, x_i, y_i, \ldots, y_n) \in T_{n+1}\}$ and $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$.

Consistency: For all $S, T \in \Sigma$, for all $n \in \mathcal{Z}_{++}$, for all $x \in \mathcal{R}^{n+1}$, for all $i \in \{1, \ldots, n+1\}$, if $S_m = \mathcal{H}(-\mathbf{1}_m)$ for all $m \in \mathcal{Z}_{++} \setminus \{n\}$ and $T_m = \mathcal{H}(-\mathbf{1}_m)$ for all $m \in \mathcal{Z}_{++} \setminus \{n+1\}$ and $F(T) \cap T_{n+1} = \{x\}$ and $S_n = t_i^x(T_{n+1})$, then $F(S) \cap S_n = \{x_{-i}\}$.

The above axioms are well-known in the context of resource-allocation mechanisms and bargaining solutions. In addition, we use a weak fixed-population independence condition that guarantees fixed-population rationalizability if added to some other axioms. It is the counterpart of independence of irrelevant alternatives when applied to specific feasible sets across population sizes.

Restricted Choice Independence: For all $S, T \in \Sigma$, for all $n \in \mathcal{Z}_{++}$, for all $x \in \mathcal{R}^n$, if $S_m = \mathcal{H}(-\mathbf{1}_m)$ for all $m \in \mathcal{Z}_{++} \setminus \{n\}$ and $F(S) \cap S_n = \{x\}$ and $T_n = S_n$ and $F(T) \cap T_n \neq \emptyset$, then $F(T) \cap T_n = \{x\}$.

Finally, to provide a link between choices involving different population sizes, we impose the critical-level principle. It requires that there exists a fixed critical level such that, if a chosen utility vector is augmented by an individual utility at this critical level and the resulting vector is feasible, then the augmented vector should be chosen as well. This is the choice-theoretic version of Blackorby and Donaldson's [1984] critical-level principle for social orderings. Blackorby and Donaldson's [1984] axiom requires that there exists a critical level of utility such that, for any given utility vector of any given population size, augmenting that vector with a utility level equal to the critical level is a matter of indifference according to a variable-population social ordering.

Critical-Level Principle: There exists $\alpha \in \mathcal{R}$ such that, for all $x \in \Omega$, for all $S \in \Sigma$, if $x \in S$ and $(x, \alpha) \in S$, then $x \in F(S)$ if and only if $(x, \alpha) \in F(S)$.

The members of the class of variable-population choice functions characterized in this section are rationalized by critical-level generalized-utilitarian orderings. To introduce these orderings, several further definitions are required. Let G be the set of continuous,

[7] See Thomson [1990, 1996b] for a general discussion and Blackorby, Bossert, and Donaldson [1996a] and Lensberg [1987] for versions that are analogous to the one employed here.

increasing, and strictly concave functions $g\colon \mathcal{R}_+ \longrightarrow \mathcal{R}$. For $g \in G$ and $n \in \mathcal{Z}_{++}$, define the ordering $\succeq^n_g$ on $\mathcal{R}^n_+$ by letting, for all $x, y \in \mathcal{R}^n_+$,

$$x \succeq^n_g y \iff \sum_{i=1}^{n} g(x_i) \geq \sum_{i=1}^{n} g(y_i). \tag{4.1}$$

Now consider any continuous, increasing, and strictly concave function $g\colon \mathcal{R}_+ \longrightarrow \mathcal{R} \cup \{-\infty\}$ such that $g(0) = -\infty$. For all $p \in \mathcal{R}^2_{++}$, let

$$\bar{x}^g(p) = \arg\max\{g(x_1) + g(x_2) \mid px \leq 1\}. \tag{4.2}$$

Note that the properties of g ensure that $\bar{x}^g(p)$ is well-defined and unique and, furthermore, $\bar{x}^g(p) \in \mathcal{R}^2_{++}$ for all $p \in \mathcal{R}^2_{++}$. Let G_0 be the set of all continuous, increasing, and strictly concave functions $g\colon \mathcal{R}_+ \longrightarrow \mathcal{R} \cup \{-\infty\}$ such that $g(0) = -\infty$ and

$$\lim_{p_2 \to \infty} \bar{x}^g(p) = (1/p_1, 0) \tag{4.3}$$

for all $p_1 \in \mathcal{R}_{++}$. For all $n \in \mathcal{Z}_{++}$ and all $x \in \mathcal{R}^n_+$, let $N_+(x) = \{i \in \{1, \ldots, n\} \mid x_i > 0\}$ and $n_+(x) = |N_+(x)|$. Let $g \in G_0$ and $n \in \mathcal{Z}_{++}$, and define the ordering $\succeq^n_g$ on $\mathcal{R}^n_+$ by letting, for all $x, y \in \mathcal{R}^n_+$,

$$\begin{aligned} x \succeq^n_g y \iff & [n_+(x) > n_+(y)] \text{ or} \\ & [n_+(x) = n_+(y) \text{ and } \sum_{i \in N_+(x)} g(x_i) \geq \sum_{i \in N_+(y)} g(y_i)]. \end{aligned} \tag{4.4}$$

Let $\alpha \in \mathcal{R}_+$ and $g \in G \cup G_0$. The ordering $\succeq_{g,\alpha}$ on Ω_+ is defined by letting, for all $n, m \in \mathcal{Z}_{++}$, $x \in \mathcal{R}^n_+$, $y \in \mathcal{R}^m_+$,

$$x \succeq_{g,\alpha} y \iff \begin{cases} x \succeq^n_g y \text{ if } n = m; \\ x \succeq^n_g (y, \alpha \mathbf{1}_{n-m}) \text{ if } n > m; \\ (x, \alpha \mathbf{1}_{m-n}) \succeq^n_g y \text{ if } n < m. \end{cases} \tag{4.5}$$

F is a critical-level generalized-utilitarian choice function if and only if there exist $\alpha \in \mathcal{R}_+$ and $g \in G \cup G_0$ such that, for all $S \in \Sigma$,

$$F(S) = \{x \in S_+ \mid x \succeq_{g,\alpha} y \text{ for all } y \in S_+\}. \tag{4.6}$$

A few remarks concerning this class of choice functions are in order. First, note that, due to the strict-concavity assumption, the class of orderings $\{\succeq_{g,\alpha}\}_{g \in G \cup G_0, \alpha \in \mathcal{R}_+}$ does not contain the critical-level utilitarian ordering which results from choosing g to be linear. This particular ordering is excluded because the choice function rationalized by it fails to satisfy fixed-population single-valuedness.

Second, in the case where g is an element of G, the definition of $\succeq_{g,\alpha}$ can alternatively be written as follows.[8] Let $n, m \in \mathcal{Z}_{++}$, $x \in \mathcal{R}^n_+$, and $y \in \mathcal{R}^m_+$. If $n = m$, it is clear that

$$x \succeq_{g,\alpha} y \iff \sum_{i=1}^{n} \big[g(x_i) - g(\alpha)\big] \geq \sum_{i=1}^{m} \big[g(y_i) - g(\alpha)\big] \tag{4.7}$$

because $\sum_{i=1}^{n} g(\alpha) = \sum_{i=1}^{m} g(\alpha)$ in this case. If $n > m$, we obtain

$$\begin{aligned} x \succeq_{g,\alpha} y &\iff x \succeq (y, \alpha \mathbf{1}_{n-m}) \\ &\iff \sum_{i=1}^{n} g(x_i) \geq \sum_{i=1}^{m} g(y_i) + (n-m)g(\alpha) \\ &\iff \sum_{i=1}^{n} g(x_i) - ng(\alpha) \geq \sum_{i=1}^{m} g(y_i) - mg(\alpha) \\ &\iff \sum_{i=1}^{n} \big[g(x_i) - g(\alpha)\big] \geq \sum_{i=1}^{m} \big[g(y_i) - g(\alpha)\big], \end{aligned} \tag{4.8}$$

and the same reasoning applies to the case $n < m$. Thus, (4.7) is valid for all $x, y \in \Omega_+$, provided g is in G.

Finally, the restriction (4.3) in the definition of G_0 is needed to ensure that the resulting choice functions are continuous and satisfy Pareto optimality. Note that (4.3) is not satisfied by all continuous, increasing, and strictly concave functions g such that $g(0) = -\infty$; for example, logarithmic functions fail to possess this property. However, G_0 is not empty. For example, the function $g: \mathcal{R}_+ \longrightarrow \mathcal{R} \cup \{-\infty\}$ defined by letting

$$g(z) = \begin{cases} -\big(-\ln(z)\big)^q & \text{if } 0 \leq z \leq e^{q-1}; \\ -(1-q)^{1+q} + q(1-q)^{q-1}\ln(z) & \text{if } z > e^{q-1} \end{cases} \tag{4.9}$$

is an element of G_0 for all parameter values $q \in (0, 1)$.[9]

We now turn to a characterization of these variable-population choice functions. The following lemmas are used in the proof of this characterization result. The proof of Lemma 2 is analogous to the proof of Lensberg's [1987] Lemma 1 formulated for bargaining solutions and is omitted.

Lemma 2: *If a variable-population choice function F satisfies fixed-population single-valuedness, Pareto optimality, continuity, and consistency, then F satisfies independence of irrelevant alternatives.*

Lemma 3 shows that independence of dominated alternatives is implied by some of our other axioms.

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[8] See Blackorby and Donaldson [1984] and Blackorby, Bossert, and Donaldson [1998a].

[9] We thank James Redekop for proving the nonemptiness of G_0 by providing this example.

Lemma 3: *If a variable-population choice function F satisfies zero dominance and independence of irrelevant alternatives, then F satisfies independence of dominated alternatives.*

Proof: Suppose F satisfies zero dominance and independence of irrelevant alternatives. Let $S, T \in \Sigma$, $n \in \mathcal{Z}_{++}$, and $x \in \mathcal{R}^n$ be such that $S_m = T_m = \mathcal{H}(-\mathbf{1}_m)$ for all $m \in \mathcal{Z}_{++} \setminus \{n\}$ and $S_n \cap \mathcal{R}^n_+ = T_n \cap \mathcal{R}^n_+ \neq \emptyset$ and $F(S) \cap S_n = \{x\}$. Let $W \in \Sigma$ be such that $W_m = \mathcal{H}(-\mathbf{1}_m)$ for all $m \in \mathcal{Z}_{++} \setminus \{n\}$ and $W_n = S_n \cap T_n$. By zero dominance, $F(S) \cap S_n \cap \mathcal{R}^n_+$ and $F(T) \cap T_n \cap \mathcal{R}^n_+$ are nonempty and, because $S_n \cap \mathcal{R}^n_+ = T_n \cap \mathcal{R}^n_+$, $W_n \cap \mathcal{R}^n_+ = S_n \cap \mathcal{R}^n_+ = T_n \cap \mathcal{R}^n_+$. Therefore, $F(S) \cap W_n$ and $F(T) \cap W_n$ are nonempty, and independence of irrelevant alternatives implies $F(W) \cap W_n = F(S) \cap S_n = \{x\}$ and $F(W) \cap W_n = F(T) \cap T_n$. Hence, $F(T) \cap T_n = F(S) \cap S_n = \{x\}$. ■

The next lemma provides an extension of Lensberg's [1987] Theorem 1 to a larger domain. In particular, Lensberg considers bargaining problems where, for each possible population size, a utility vector is to be selected but population size is not a characteristic to be chosen. The domain considered in Lensberg [1987] is the standard domain of bargaining solutions with a normalized disagreement point satisfying individual rationality, which is analogous to our zero-dominance condition. A crucial difference between that domain and the one considered here is that Lensberg's is restricted to problems S_n such that $S_n \cap \mathcal{R}^n_{++} \neq \emptyset$—that is, each fixed-population problem contains a strictly positive vector. As we argued earlier, this restriction is not a natural one to impose on the choice problems considered here and, therefore, we have to examine the consequences of extending Lensberg's [1987] characterization result to our domain.

Lensberg [1987] shows that suitably reformulated versions of Pareto optimality, continuity, and consistency characterize, under an implicit individual-rationality assumption, the class of bargaining solutions that are rationalized by a family of orderings (one for each population size) with a specific additively separable representation. Due to the assumption that there exists a strictly positive vector in each bargaining problem, it is possible that the functions generating this representation have the value $-\infty$ at zero. If this assumption is dropped, the only functions of that type consistent with our axioms are those satisfying (4.3). An interesting consequence of this observation is that the adaptations of the (weighted) Nash solutions to our framework are no longer among the resulting choice functions because they cannot be extended continuously to the domain Σ^n. Furthermore, we add anonymity to the list of axioms, which implies that the functions used in the additive representation can be chosen to be the same for all agents. We obtain

Lemma 4: *If a variable-population choice function F satisfies zero dominance, fixed-population single-valuedness, Pareto optimality, continuity, anonymity, and consistency, then there exists a function $g \in G \cup G_0$ such that, for all $n \in \mathcal{Z}_{++}$, for all $S \in \Sigma^n$,*

$$F(S) = \left\{x \in S_n \cap \mathcal{R}^n_+ \mid x \succeq_g y \text{ for all } y \in S_n \cap \mathcal{R}^n_+\right\}. \tag{4.10}$$

Proof: By Lemmas 2 and 3, the axioms in the statement of Lemma 4 imply that Lensberg's [1987] Theorem 1 can be invoked to conclude that there exists a continuous, increasing, and strictly concave function $g\colon \mathcal{R}_+ \longrightarrow \mathcal{R} \cup \{-\infty\}$ such that, for all $n \in \mathcal{Z}_{++}$ and all $S \in \bar{\Sigma}^n$, (4.10) is satisfied. Note that anonymity implies that the functions g_i in Lensberg's result can be chosen to be identical for all $i \in \mathcal{Z}_{++}$, and we can therefore write $g = g_i$ for all $i \in \mathcal{Z}_{++}$.

It is easy to see that, in order to accommodate problems in $\Sigma^n \setminus \bar{\Sigma}^n$, functions with $g(0) = -\infty$ must satisfy (4.3) in order to ensure that the resulting choice functions satisfy continuity and Pareto optimality. Therefore, only functions in $G \cup G_0$ are admissible. To complete the proof, note that, given the properties of g, $F(S)$ must consist of the (unique) best element in S according to $\succeq_g$ for all $S \in \Sigma^n$. ■

The last preliminary result before stating the characterization theorem of this section shows that fixed-population rationalizability is implied by some of our other axioms.

Lemma 5: *If a variable-population choice function F satisfies zero dominance, fixed-population single-valuedness, Pareto optimality, continuity, anonymity, consistency, and restricted choice independence, then F satisfies fixed-population rationalizability.*

Proof: By Lemma 4, there exists a function $g \in G \cup G_0$ such that (4.10) is satisfied for all $n \in \mathcal{Z}_{++}$ and all $S \in \Sigma^n$. Let $n \in \mathcal{Z}_{++}$ and define the ordering $\succeq^n$ on $\mathcal{R}^n$ as follows.

(i) For all $x, y \in \mathcal{R}^n_+$, $x \succeq^n y$ if and only if $x \succeq^n_g y$;
(ii) for all $x, y \in \mathcal{R}^n \setminus \mathcal{R}^n_+$, $x \sim^n y$;
(iii) for all $x \in \mathcal{R}^n_+$ and all $y \in \mathcal{R}^n \setminus \mathcal{R}^n_+$, $x \succ^n y$.

By Lemma 4 and restricted choice independence, it follows that F satisfies fixed-population rationalizability with the sequence $\{\succeq^n\}_{n \in \mathcal{Z}_{++}}$. ■

We now obtain

Theorem 3: *A variable-population choice function F satisfies zero dominance, weak population congruence, fixed-population single-valuedness, Pareto optimality, continuity, anonymity, consistency, restricted choice independence, and the critical-level principle if and only if F is a critical-level generalized-utilitarian choice function.*

Proof: That the critical-level generalized-utilitarian choice functions satisfy the required axioms can be verified easily.[10] Now suppose F satisfies all of the axioms. By weak population congruence, Theorem 2, and Lemma 5, F is rationalizable by an ordering $\succeq$ on Ω. By zero dominance, only nonnegative utility vectors can be chosen and, therefore, we can restrict attention to vectors in Ω_+. Consider any $n \in \mathcal{Z}_{++}$ and $x \in \mathcal{R}^n_+$, and let $\alpha \in \mathcal{R}$ be as in the definition of the critical-level principle. Let $S \in \Sigma$ be such that $S_m = \mathcal{H}(-\mathbf{1}_m)$ for all $m \in \mathcal{Z}_{++} \setminus \{n, n+1\}$, $S_n = \mathcal{H}(x)$, and $S_{n+1} = \mathcal{H}((x, \alpha))$. By zero dominance and Pareto optimality, $x \in F(S)$ or $(x, \alpha) \in F(S)$. By the critical-level principle, $x \in F(S)$ and $(x, \alpha) \in F(S)$. By zero dominance, $\alpha \geq 0$. By definition of a rationalization,

$$x \sim (x, \alpha). \tag{4.11}$$

Let $n, m \in \mathcal{Z}_{++}$, $x \in \mathcal{R}^n_+$, and $y \in \mathcal{R}^m_+$. If $n = m$, Lemmas 4 and 5 imply that there exists a function $g \in G \cup G_0$ such that

$$x \succeq y \iff x \succeq^n_g y \iff x \succeq_{g,\alpha} y. \tag{4.12}$$

If $n > m$, repeated application of (4.11) implies

$$x \succeq y \iff x \succeq (y, \alpha\mathbf{1}_{n-m}) \iff x \succeq^n_g (y, \alpha\mathbf{1}_{n-m}) \iff x \succeq_{g,\alpha} y \tag{4.13}$$

and, analogously, if $n < m$, we obtain

$$x \succeq y \iff (x, \alpha\mathbf{1}_{m-n}) \succeq y \iff (x, \alpha\mathbf{1}_{m-n}) \succeq^n_g y \iff x \succeq_{g,\alpha} y \tag{4.14}$$

which completes the proof. ■

The axioms used in Theorem 3 are independent. For each of the following examples, the axiom that is indicated is violated, and all other axioms are satisfied.

Zero Dominance: Let $g \colon \mathcal{R} \longrightarrow \mathcal{R}$ be continuous, increasing, and strictly concave. Define the ordering $\succeq$ on Ω by letting, for all $n, m \in \mathcal{Z}_{++}$, $x \in \mathcal{R}^n$, $y \in \mathcal{R}^m$,

$$x \succeq y \iff \sum_{i=1}^{n} g(x_i) \geq \sum_{i=1}^{m} g(y_i). \tag{4.15}$$

Let, for all $n \in \mathcal{Z}_{++}$, $\mathcal{R}^n_* = \bigcup_{n \in \mathcal{Z}_{++}} \{x \in \mathcal{R}^n \mid x \geq (-1/2)\mathbf{1}_n\}$ and define, for all $S \in \Sigma$,

$$F(S) = \left\{ x \in \bigcup_{n \in \mathcal{Z}_{++}} (S_n \cap \mathcal{R}^n_*) \,\middle|\, x \succeq y \text{ for all } y \in \bigcup_{n \in \mathcal{Z}_{++}} (S_n \cap \mathcal{R}^n_*) \right\}. \tag{4.16}$$

Weak Population Congruence: Let Σ^1 be the subset of Σ defined as follows. For all $S \in \Sigma$, $S \in \Sigma^1$ if and only if there exists $n \in \mathcal{Z}_{++}$, $x \in S_n$, and $i \in \{1, \ldots, n\}$ such that $x_i \geq 1$.

[10] Note that (4.3) ensures that continuity and Pareto optimality are satisfied if $g \in G_0$.

Let Σ^2 be the complement of Σ^1 in Σ. Let $g\colon \mathcal{R}_+ \longrightarrow \mathcal{R}$ be continuous, increasing, and strictly concave. Define the orderings $\succeq^1$ and $\succeq^2$ on Ω_+ as follows. For all $n, m \in \mathcal{Z}_{++}$, $x \in \mathcal{R}^n_+$, $y \in \mathcal{R}^m_+$,

$$x \succeq^1 y \iff \sum_{i=1}^{n} \big[g(x_i) - g(1)\big] \geq \sum_{i=1}^{m} \big[g(y_i) - g(1)\big] \tag{4.17}$$

and

$$x \succeq^2 y \iff [n > m] \text{ or } [n = m \text{ and } \sum_{i=1}^{n} g(x_i) \geq \sum_{i=1}^{m} g(y_i)]. \tag{4.18}$$

Let, for all $S \in \Sigma^1$,

$$F(S) = \{x \in S_+ \mid x \succeq^1 y \text{ for all } y \in S_+\} \tag{4.19}$$

and, for all $S \in \Sigma^2$,

$$F(S) = \{x \in S_+ \mid x \succeq^2 y \text{ for all } y \in S_+\}. \tag{4.20}$$

Fixed-Population Single-Valuedness: Define the ordering $\succeq$ on Ω_+ by letting, for all $n, m \in \mathcal{Z}_{++}$, $x \in \mathcal{R}^n_+$, $y \in \mathcal{R}^m_+$,

$$x \succeq y \iff \sum_{i=1}^{n} x_i \geq \sum_{i=1}^{m} y_i. \tag{4.21}$$

Let, for all $S \in \Sigma$,

$$F(S) = \{x \in S_+ \mid x \succeq y \text{ for all } y \in S_+\}. \tag{4.22}$$

Pareto Optimality: Let, for all $S \in \Sigma$,

$$F(S) = \{0\mathbf{1}_n \mid S_n \cap \mathcal{R}^n_+ \neq \emptyset\}. \tag{4.23}$$

Continuity: Let $\succeq^n_L$ be the Leximin ordering on $\mathcal{R}^n$, and define the ordering $\succeq$ on Ω_+ by letting, for all $n, m \in \mathcal{Z}_{++}$, $x \in \mathcal{R}^n_+$, $y \in \mathcal{R}^m_+$,

$$\begin{aligned} x \succeq y \iff & [n = m \text{ and } x \succeq^n_L y] \text{ or} \\ & [n > m \text{ and } x \succeq^n_L (y, 0\mathbf{1}_{n-m})] \text{ or} \\ & [n < m \text{ and } (x, 0\mathbf{1}_{m-n}) \succeq^m_L y. \end{aligned} \tag{4.24}$$

Let, for all $S \in \Sigma$,

$$F(S) = \{x \in S_+ \mid x \succeq y \text{ for all } y \in S_+\}. \tag{4.25}$$

See also Blackorby, Bossert, and Donaldson [1996b] for the above-defined variable-population extensions of the Leximin orderings.

Anonymity: For all $i \in \mathcal{Z}_{++}$, let $g_i\colon \mathcal{R}_+ \longrightarrow \mathcal{R}$ be continuous, increasing, and strictly concave. Define these functions so that there exist $i, j \in \mathcal{Z}_{++}$ such that $g_i \neq g_j$. Define the ordering $\succeq$ on Ω_+ by letting, for all $n, m \in \mathcal{Z}_{++}$, $x \in \mathcal{R}^n_+$, $y \in \mathcal{R}^m_+$,

$$x \succeq y \iff \sum_{i=1}^{n} \big[g_i(x_i) - g_i(0)\big] \geq \sum_{i=1}^{m} \big[g_i(y_i) - g_i(0)\big]. \tag{4.26}$$

Let, for all $S \in \Sigma$,

$$F(S) = \{x \in S_+ \mid x \succeq y \text{ for all } y \in S_+\}. \tag{4.27}$$

Consistency: Define the ordering $\succeq$ on Ω_+ by letting, for all $n, m \in \mathcal{Z}_{++}$, $x \in \mathcal{R}^n_+$, $y \in \mathcal{R}^m_+$,

$$x \succeq y \iff \sum_{i=1}^{n} \big[1 - e^{-x_i}\big] - e^{-\sum_{i=1}^{n} x_i} \geq \sum_{i=1}^{m} \big[1 - e^{-y_i}\big] - e^{-\sum_{i=1}^{m} y_i}. \tag{4.28}$$

Let, for all $S \in \Sigma$,

$$F(S) = \{x \in S_+ \mid x \succeq y \text{ for all } y \in S_+\}. \tag{4.29}$$

Restricted Choice Independence: Let Σ^1 be the subset of Σ defined as follows. For all $S \in \Sigma$, $S \in \Sigma^1$ if and only if there exists a unique $n \in \mathcal{Z}_{++}$ such that $S_n \cap \mathcal{R}^n_+ \neq \emptyset$. Let Σ^2 be the complement of Σ^1 in Σ. Let $g^1\colon \mathcal{R}_+ \longrightarrow \mathcal{R}$ and $g^2\colon \mathcal{R}_+ \longrightarrow \mathcal{R}$ be two different continuous, increasing, and strictly concave functions. In addition, define the orderings $\succeq^1$ and $\succeq^2$ on Ω_+ as follows. For all $n, m \in \mathcal{Z}_{++}$, $x \in \mathcal{R}^n_+$, $y \in \mathcal{R}^m_+$,

$$x \succeq^1 y \iff \sum_{i=1}^{n} g^1(x_i) \geq \sum_{i=1}^{m} g^1(y_i) \tag{4.30}$$

and

$$x \succeq^2 y \iff \sum_{i=1}^{n} g^2(x_i) \geq \sum_{i=1}^{m} g^2(y_i). \tag{4.31}$$

Let, for all $S \in \Sigma^1$,

$$F(S) = \{x \in S_+ \mid x \succeq^1 y \text{ for all } y \in S_+\} \tag{4.32}$$

and, for all $S \in \Sigma^2$,

$$F(S) = \{x \in S_+ \mid x \succeq^2 y \text{ for all } y \in S_+\}. \tag{4.33}$$

This example can also be used to complete the proof that fixed-population rationalizability and weak population congruence are independent (see Theorem 2)—it is straightforward to verify that F violates fixed-population rationalizability.

Critical-Level Principle: Let $g\colon \mathcal{R}_+ \longrightarrow \mathcal{R}$ be continuous, increasing, and strictly concave. Define the ordering $\succeq$ on Ω_+ by letting, for all $n, m \in \mathcal{Z}_{++}$, $x \in \mathcal{R}^n_+$, $y \in \mathcal{R}^m_+$,

$$x \succeq y \iff (1/n)\sum_{i=1}^{n} g(x_i) \geq (1/m)\sum_{i=1}^{m} g(y_i). \tag{4.34}$$

Let, for all $S \in \Sigma$,

$$F(S) = \{x \in S_+ \mid x \succeq y \text{ for all } y \in S_+\}. \tag{4.35}$$

5. Concluding Remarks

Given fixed-population rationalizability of choice functions, two axioms are necessary and sufficient for rationalizability in models in which, in addition to individual utility levels, population size is to be chosen. The axioms are weak population congruence, which is a significant weakening of Richter's congruence axiom, and zero dominance, which rules out choices with very low utility levels. In addition, these axioms, along with fixed-population single-valuedness, Pareto optimality, continuity, anonymity, consistency, restricted choice independence and the critical-level principle, characterize the critical-level generalized utilitarian choice functions.

Critical-level generalized-utilitarian orderings are usually defined for all utility vectors including those with negative components. The norm in the zero-dominance condition is arbitrary, however: it can be set, for example, at a utility level that represents a neutral life, a wretched life, or a life that is worse than any person might reasonably experience. Consequently, the ethical judgement that it represents is flexible. In addition, it is reasonable to deal with the problem of utility levels that are 'too low' with constraints (lower bounds on individual utilities) rather than a modification of the social objective function. This is a position that is analogous to the one advocated in Blackorby, Bossert, and Donaldson [1998b] in an examination of the ethical foundations of the common practice of discounting the utilities of the members of future generations. Moreover, imposing a lower bound on individual utilities allows us to respond to a criticism that can be directed toward all generalized utilitarian orderings. They allow sufficiently high utility levels to compensate for arbitrarily low utilities of other individuals. If a condition such as zero dominance is imposed, the objection loses much of its force.[11]

Although fixed-population choices are unaffected by the critical level, its value has important consequences for the resulting variable-population social ordering and choice

[11] Norms such as those employed in zero dominance can be shown to be useful in formulating informational restrictions on variable-population social-choice mechanisms, as is illustrated in Blackorby, Bossert, and Donaldson [1999a]. See also Tungodden [1998] for a discussion of independent norm levels in fixed-population social choice.

function. In the choice-theoretic framework analyzed in this paper, the critical level cannot be below the minimally acceptable utility level represented by zero. Because the norm level can be chosen arbitrarily, however, this restriction may not produce reasonable social orderings. The classical generalized utilitarian principles employ a critical level that is equal to the utility level representing neutrality and all of them lead to the *repugnant conclusion.*[12] Any principle that implies the repugnant conclusion declares every alternative in which each person has a utility level above neutrality to be inferior to one in which each person in a suitably large population has a utility level above but arbitrarily close to neutrality. Critical-level generalized utilitarian principles with critical levels above neutrality avoid this unreasonable trade-off.

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[12] See Parfit [1976, 1982, 1984]. Further discussions of critical levels and the repugnant conclusion can be found, for example, in Arrhenius [1997], Blackorby, Bossert, and Donaldson [1997], Blackorby, Bossert, Donaldson, and Fleurbaey [1998], Blackorby and Donaldson [1991], and Ng [1989].

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