Production Networks

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Abstract

In this paper, we model the economy as a *production network of competitive firms* that interact in a general-equilibrium setup. First, we find that, at the unique Walrasian equilibrium, the profit of each active firm is proportional to (a suitable generalization of) its Bonacich centrality. We also determine consumer welfare at equilibrium and characterize efficient networks. Then we proceed to conduct a broad range of comparative-static analyses. These include the effect on profits and welfare of:
(a) *distortions* (e.g. tax/subsidies) imposed on the whole economy or specific firms;
(b) *structural* changes such as the addition of links and the elimination of nodes;
(c) *productivity* and *preference* changes.

We discover that the induced effects are in general *non-monotone*, depend on *global* network features, and impinge on each sector depending on the pattern of *incentralities* displayed by its input providers and output users. Furthermore, the inter-sector "linkages" underlying these effects can usually be decomposed – following the heuristic dichotomy proposed by Hirschman (1958) – into a *forward* (push) component and a *backward* (pull) one. Finally, we undertake some preliminary analysis of firm dynamics and illustrate that, when evaluating policies of support and shock mitigation from a dynamic viewpoint, the reliance on strict *market-based criteria* can be quite *misleading* in terms of social welfare.

1 Introduction

Any modern economy is a complex network of interfirm buyer-seller relationships that constitute its production structure. There is a growing interest among economists in adopting this network perspective to study a wide variety of economic phenomena: trade, intermediation, innovation, technological diffusion, learning, or the transmission of shocks. This, of course, has prominent precursors in the celebrated work of John von Neumann, Wassily Leontief or Albert O. Hirschmann, all of whom stressed the importance of interindustry relationships (or linkages) for a proper understanding of some of the key characteristics of an economic system. In this tradition, the present paper proposes a general-equilibrium model of the economy that, despite being standard in almost every respect, highlights the network of interfirm relationships on its production side. In a nutshell, our objective is to obtain a precise understanding of how the details of the *production network topology* (on which no *a priori* restrictions are imposed) shapes profits, welfare, and the effects (static and dynamic) induced by wide variety of changes and policy interventions.

In order to focus our analysis on the production structure of the economy, the demand side is modeled through a representative consumer while the technology of each firm is assumed of the Cobb-Douglas type. In general, however, each firm uses a diverse range of inputs, whose productivities are individually specific. Under these conditions, our analysis starts by showing that a unique equilibrium exists that displays a very sharp relationship to the network structure of the economy. Specifically, we find that, at equilibrium, the profitability of any given firm is *proportional* to its network centrality, as given by a suitable generalization of the well-known measure of centrality proposed by Bonacich (1987)[1]. On the other hand, we also determine the consumer's welfare at equilibrium, providing closed expressions for how it depends on the different parameters of the model (in particular, the density and symmetry of the production structure).

Next, the paper turns to studying how the equilibrium is affected by a wide variety of different changes in the environment. First, we consider the impact of distortions, formulated in the idealized form of price wedges – either general ones that affect uniformly the whole economy, or individual ones applying to single firms. Then, we turn our attention to structural changes in the production network and consider both the effect of *global* transformations that affect either connectivity or the number of nodes, as well as *local* changes that affect only single nodes or individual links. We then close this part of our analysis with a comparative study of supply-channeled changes (such as technological improvements) with those that operate through variations in demand (e.g. a rise or fall

[1]For example, as compared to the received notion of Bonacich centrality that considers the paths of different lengths that join any given pair of nodes, in our case only those that connect a firm node to a consumption node are considered. The weight attributed to each of these paths reflects not only its length (as in the standard Bonacich centrality, they are discounted by their respective length) but also the relative importance that the consumer's utility function attributes to the end good in question (while in the standard notion all end nodes are given a uniform weight).

in the consumer's relative preference for a particular consumption good).

For all of the cases listed above, we provide formal expressions that capture the total impact of the change in an explicit form. These expressions are conceptually quite simple and should prove helpful in evaluating empirically alternative measures of economic policy. Furthermore, many of them can be understood in a quite parallel fashion, in that they involve a similar integration of constituent effects. Thus, on the one hand, they can be regarded as consisting of a first-order impact on the immediate agent experiencing the change (e.g. the firm subject to an individual tax or subsidy), followed by a spread of such a first-order impact throughout the whole network as captured by a suitable matrix of incentralities. On the other hand, the impact of many of those changes can also be decomposed in terms of a push effect that operates downstream and a pull effect that does so upstream. This dichotomy is reminiscent of the distinction between forward and backward linkages often considered in policy analysis and notably proposed by Hirschman (1958). Here we show how to distinguish precisely between them and also compare their implications: while forward linkages induce resource reallocation downstream but have no effect on profits, revenues, or input demands because of the entailed adjustment of prices, backward linkages alter significantly all those variables – i.e, not only prices but also revenues, profits, and input demands.

The analysis advanced so far is inherently static, i.e. it compares how different changes in the parameters of the model affect *equilibrium magnitudes*. By building on it, however, we can also address some genuinely dynamic issues. In this paper, we simply outline the problem, leaving for future work an exhaustive study of it. We consider, specifically, the problem of how to guide the distribution of firm support when, in the absence of it, some incumbent firms may go bankrupt and disappear. The key dynamic concern here is that, when several firms are at stake and not all can be supported, what particular firms are chosen may lead to subsequent effects that are drastically different. For, as one firm is protected but other fails, the latter may generate a cascade of ensuing failures that has major longer-run effects. The question then is: what is the best criterion to use if the objective is to minimize the overall (intertemporal) impact? In particular, we may ask: are current prices and induced profits the right market signal? In brief, the answer we provide is that, in some cases, profitability may by itself be a very misleading criterion and that other network-based criteria (embodying considerations of intercentrality) will, in general, be much more appropriate.

We end this short introduction with a brief review of related literature. From a methodological viewpoint, our approach is close to recent work by Acemoglu et al. (2012) who, building upon a Cobb-Douglas model proposed by Long and Plosser (1983), study how/whether microeconomic shocks on individual agents or sectors may aggregate into generating significant aggregate effects at the macroscopic level. Previous papers by Horvath (1998); Gabaix (2011) are motivated by a similar concern. As already explained, our objective in this paper is very different and so are some of the key assumptions and questions asked. Thus, just to mention one significant difference, we do not make the assumption of constant returns in production since we are interested in how profit performance is affected by network structure. Besides, our analysis focuses on the effect of distortions, structural changes, or different kinds of demand and support policies rather than the spread of shocks, which are very different kind of phenomena.

Our work is also related to Jones (2011) who, building as well on the framework introduced by Long and Plosser (1983), studies how misallocation at the sector level affects GDP. More generally, our paper relates to the vast literature that has attempted to understand the intersectorial basis of economic development and the sectorial) policies that can mitigate either misallocation or/and coordination problems. The workhorse in this literature has been the input-output methodology originally formulated by Leontief (1936), which has spawned a huge body of work, both theoretical and empirical (see e.g. the monograph Miller and Blair (2009) for a recent account). Early on, building upon this rise of input-output analysis, the influential work of Hirschman (1958) highlighted the importance of some of the notions (e.g. forward and backward linkages) that will help us understand key forces underlying our model. In contrast with the traditional input-output literature, Hirschman's approach emphasizes unbalanced growth rather than equilibrium as the primary tool of *dynamic* analysis. To adopt such a perspective is also our eventual objective, even though in the present paper our analysis is still mostly static.

However, the preliminary dynamic analysis of evolutionary market forces undertaken in the last part of the paper (cf. the summary above) embodies some of the considerations implicitly highlighted by Hirschman's work. It also hints at the role that network-based processes of propagation – e.g. of default through failure cascades – should have in a proper assessment of economic policy in a complex interconnected economy. This, in turn, leads us to the rich and diverse literature on contagion – for example, financial contagion – that has experienced a significant impetus in recent years (see the recent Handbook edited by Bramoulle et al. (2016) and the recent interesting work by Baqaee (2015) on the specific context of production networks). A different, and complementary, aspect in the dynamic study of economic networks concerns the processes of endogenous (payoff-guided) network formation. The network-formation literature is still in too-preliminary a state to provide much help in this endeavour. Our analysis of how certain structural changes on nodes and links affect agents' payoffs is a very preliminary step in this direction. The recent paper by Oberfield (2012) studies a stylized version of the problem where agents endogenously select their production techniques, conceived as links that connect them to a unique supplier.

The rest of the paper is organized as follows. First, in Section 2 we present the benchmark model, followed in Section 3 by the introduction of a collection of basic results. These results include, specifically, an account of how the topological features of the production network shape the corresponding outcomes (i.e. profits of the firms and utility of the consumer). Then, in Section 4 we study how different kinds of distortions (uniform or firm-specific) affect the allocation of resources and relative firm performance. In Section 5 we investigate how does the *global* structure of production network determine the production and the welfare potential of the economy and how *local* perturbations in the network structure impinge on the equilibrium outcomes. Section 6 discusses the influential *push-pull* dichotomy stressed by Hirschman (1958) through the lense of our model. In Section 7 we provide a preliminary exploration of issues pertaining to firm dynamics and point out that market signals can be quite misleading from the point of the long run welfare. We summarize our contribution and conclude in Section 8.

In Appendix A we include the formal proofs of our main results. Then, in the (online) Appendix B, we present the general case with unrestricted heterogeneity, while in Appendix C (also available online) we revisit the comparative-statics analysis on firm distortions when the system is closed and thus the monetary flows involved must be balanced. Finally, (online) Appendix D collects a set of mathematical results that are useful, or directly invoked, at different points in the paper.

2 Benchmark model

We model an economy consisting of a finite set of firms $N = \{1, 2, ..., n\}$ and a single representative consumer. Our focus, therefore, is on the interfirm production relationships – i.e. the production network – through which the economy eventually delivers the net amounts of consumption goods enjoyed by the consumer. In principle, we allow that any specific good may be consumed, used as an intermediate input in the production of some other good, or display both roles simultaneously.

The goods that provide some utility to the consumer are labeled consumption goods, and are included in the set $M \subset N$, with $m = |M|$ and $\boldsymbol{c} = (c_1, c_2, ..., c_m)$ representing a typical consumption bundle. For simplicity, leisure is not in the set M and thus yields no utility. Thus the consumer's endowment of labor (which is normalized to unity) will be inelastically supplied by the consumer and hence only has an instrumental role as a source of income. The consumer's preferences over consumption bundles are represented by a Cobb-Douglas utility function $U(\cdot)$ of the form

$$U(\boldsymbol{c}) = \prod_{i=1}^{m} c_i^{\gamma_i} \tag{1}$$

where each γ_i represents the weight that the consumer's preferences attributes to consumption good i.

Concerning production, we assume that there is a one-to-one correspondence between firms and goods (thus, in particular, we rule out joint production).[2] The production of each good k takes place under decreasing returns to scale, and requires both labour and intermediate produced inputs. The set of intermediate inputs that firm k uses in its production is denoted by N_k^+, with $n_k = |N_k^+|$. Let l_k stand for the amount of labor used in the production of good k and let $(z_{jk})_{j\in N_k^+}$ be the associated amount of intermediate goods. Then the amount y_k of good k produced by the homonymous firm is determined by the production function $f_k : \mathbb{R}^{n_k} \times \mathbb{R} \to \mathbb{R}$ which is taken to display the following Cobb-Douglas form:

$$y_k = f_k\left((z_{jk})_{j\in N_k^+}; l_k\right) = A_k l_k^{\beta_k} \left(\prod_{j\in N_k^+} z_{jk}^{g_{jk}}\right)^{\alpha_k} \tag{2}$$

where the vector $(g_{jk})_{k\in N}$ reflects the intensity (assumed positive)[3] with which firm k uses/requires its different inputs, and $\alpha_k > 0$ and $\beta_k > 0$ are the output elasticities of labor and intermediate inputs.[4]

It is convenient to view the intensities g_{jk} of input use as reflecting *relative* magnitudes, so unless mentioned otherwise we shall normalize the corresponding vector to satisfy $\sum_{j\in N} g_{jk} = 1$ – in other words, we assume that the matrix G is column-stochastic. On the other hand, we assume that the production technologies exhibit decreasing returns, hence we posit that $\alpha_k + \beta_k < 1$. It is worth noting that the model allows for full heterogeneity across firms $k \in N$, not only in their pattern of input use $(g_{jk})_{k\in N}$ but also in their production elasticities α_k and β_k. The latter heterogeneity, however, does not raise particularly interesting issues and, therefore, for the sake of formal simplicity, in the main text we shall focus throughout on the case where $\alpha_k = \alpha$ and $\beta_k = \beta$ for some common values α and β. A detailed analysis for the fully heterogeneous case may be found in the online Appendix B.

It is common in economic models to posit that more advanced technologies employ a wider range of intermediate inputs, this being conceived as a reflection of higher "production complexity." Here we choose to formalize this idea in the way suggested by Benassy (1998) (see also Acemoglu et al. (2007)), setting the pre-factor of the production function

[2] Conceptually, we can think of each good as a "sector" consisting of many identical firms. Thus, from this perspective, what our model describes is the behavior of a typical firm in its corresponding sector, all firms belonging to a given sector producing perfectly substitute goods. This, of course, is a classical way of rationalizing, and providing foundations for, competitive behavior in equilibrium, as postulated below (see also Definition 2 in Appendix A).

[3] If $g_{jk} = 0$ for some input j used in the production of a certain good k, such an input is neither useful nor required in k's production. Therefore, it might as well be ignored altogether and the corresponding link jk eliminated from the production network.

[4] This formulation implies that labor and all intermediate inputs in N_k^+ are essential in production. When the production technology of $N_k^+ = \emptyset$, we interpret $\prod_{j\in N_k^+} z_{jk}^{g_{jk}} \equiv 1$, thus no production is possible because we then have $A_k = 0$ (see below for details).

as

$$A_k = n_k^{\alpha+\nu} \tag{3}$$

with $\alpha + \nu > 0$. With this formulation, it is easy to see that a positive value of the parameter ν corresponds to the case where a wider input range enhances productivity, while a negative value reflects the opposite situation.[5]

The inputs $j \in N_k^+$ used in the production of good k can be viewed as the in-neighbours of node k in the production directed network $\Gamma = \{N, L\}$, where the set N of its vertices is identified with the set of firms and there is a directed edge $(i, j) \in L$ whenever j uses good i as an input, i.e. iff $g_{ji} > 0$. This is a discrete (binary) network that only provides a qualitative account of the *production structure* of the economy. A full-fledged description of this structure is provided by the matrix of production intensities $G = (g_{ij})_{i,j \in N}$, which in turn can be regarded as the *adjacency matrix* of the *directed weighted network* that describes completely the production structure. The matrix G, together with the elasticity parameters α and β, and the utility function $U(\cdot)$ of the representative consumer jointly provide a full description of the economy. To study its performance we shall focus on the standard notion of Walrasian (or Competitive) Equilibrium (WE), which consists of a collection of prices and quantities that satisfy the usual optimality and market clearing conditions. More precisely, a WE is an array $[(\boldsymbol{p}^*, w^*), (\boldsymbol{c}^*, \boldsymbol{y}^*, \boldsymbol{Z}^*, \boldsymbol{l}^*)]$ such that the following conditions hold:

1. The consumption plan $\boldsymbol{c}^* = (c_1^*, c_2^*, ..., c_m^*)$ maximizes $U(\boldsymbol{c})$ subject to the budget constraint given by the wage and profit income the consumer earns.
2. The production plans given by the outputs $\boldsymbol{y} = (y_1^*, y_2^*, ..., y_n^*)$, the demands for produced inputs $Z^* = (z_{ij}^*)_{i,j \in N}$, and the labor demands $\boldsymbol{l}^* = (l_1^*, l_2^*, ..., l_n^*)$ are all technologically feasible and maximize, for each firm $i \in N$, its respective profit.
3. The labor market and the markets for produced goods clear (i.e. supply equals demand).

A rigorous formalization of WE is provided by Definition 2 in Appendix A. Since the economy satisfies the usual properties contemplated by General Equilbrium Theory, existence of a WE readily follows.

[5]To understand this heuristically, suppose that firm k has a total amount of money M to spend among n_k intermediate inputs used in the production of good k. Then, if each input enters production symmetrically and also bears the same price, firm k should split M equally among the n_k inputs. Fixing the amount of labour used, this implies that production must be proportional to $n_k^{\alpha+\nu} M^\alpha \left(\frac{1}{n_k}\right)^\alpha = n_k^\nu M^\alpha$. Thus, if $\nu > 0$ there are benefits from increasing the range of inputs used in production, and the magnitude of ν quantifies precisely this effect.

3 Basic results

The form of the production function (2) has the implication that (provided α and β are both positive) labor and at least one intermediate input are essential in the production of any good. No firm, therefore, can be active (i.e. achieve a positive production) unless it relies on some other active firm. This imposes a natural requirement of "systemic balance" on an economic system if all its firms are to be active. And such a condition not only has static implications but, as we shall see in Section 7, it entails dynamic consequences as well. Similar considerations have been found to be important in the evolution of many other systems – biological, ecological, or chemical – where some suitable notion of systemic balance is also key to its static stability and dynamic evolution.[6]

However, an important feature of economic systems that has no clear counterpart in other contexts is that, in a market economy, the "source of value" is not just internal to the production network but, crucially, is also "externally" dependent on consumers' preferences. Preferences provide the standard to measure welfare and also determine the market value that shapes firm performance. To discuss this issue, it is useful to extend the directed production network $\Gamma = \{N, L\}$ with an additional node c that stands for our representative consumer. This node c has an in-link (j, c) originating in every node $j \in M$ that produces a consumption good. Such an extended network is denoted by $\hat{\Gamma}$.

As a preliminary step in our analysis, we want to understand when, given a particular (extended) production structure $\hat{\Gamma}$, a particular firm i can be active at a WE. To this end, we rely on the notion of *Strongly-Connected Component* (SCC). Let the notation $j \twoheadrightarrow k$ indicate that there is a directed path originating in j and ending in k. Then, a subset of nodes $Q \subset N$ is said to be a SCC if, from every pair of nodes $j, k \in Q$, $j \twoheadrightarrow k$. The following result specifies necessary conditions for some particular node to be active at a WE.

Proposition 1. *Consider any firm $i \in N$ that is active at some WE. Then we have:*
(a) $\exists Q \subset N$ that defines a SCC of the (extended) directed network $\hat{\Gamma}$ s.t. $\forall j \in Q$, $j \twoheadrightarrow i$;
(b) $i \twoheadrightarrow c$.

Proof. See Appendix A.

A simple illustration of the necessary conditions contemplated in Proposition 1 is provided by the production network shown in Figure 1. First we note that in this network the four firms/nodes coded in red – i.e. 1 to 4 – do not satisfy the necessary conditions (a)-(b) specified in Proposition 1 and thus cannot be active at any WE. The violation of these conditions occur because either they do *not* have any intermediate input to rely on (Firm 4) or there is *no* direct or indirect connection to the consumer node (Firms 1 to 3).

[6]See, for example, the interesting work of Jain and Krishna (1998), who stress the importance of "autocatalytic balance" (the analogue of what we have called systemic balance above) in processes of growth in biological systems.

In the first case, the issue is one of feasibility (production of good 4 is not at all possible), while in the second case the problem is one of incentives (if the goods were produced, they would fetch no market value and hence lead to a zero profit).

In contrast, the remaining blue nodes 5-11 satisfy the aforementioned conditions and, in principle, could be active at a WE. For nodes 5-7 this follows from the following two-fold observation: they define a SCC (so they jointly define a feasible production structure), and they also have an (indirect) connection to the consumer node (hence they may contribute to market value). Instead, firms 8 to 11 do not form part of a SCC, and thus have to depend on "external inputs" to undertake production. They can, however, obtain those inputs from the aforementioned SCC and then access market value through firm 11, which is connected to consumer node c (i.e. produces a consumer good).

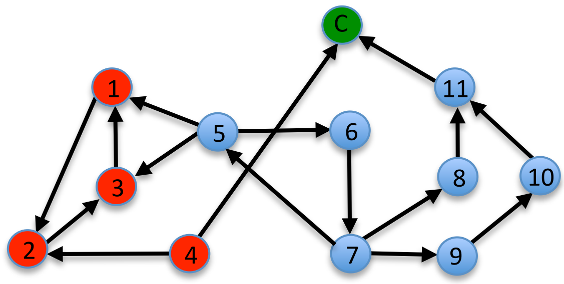

Figure 1: An extended production network $\hat{G}$ that includes the consumer node. The blue nodes are those that satisfy conditions (a)-(b) in Proposition 1, while the red ones do not. The green node represents the consumer demand.

Next, we address the question of whether there are conditions (possibly more stringent than those specified in Proposition 1) that are not only necessary but also sufficient, i.e. *characterize* when a firm is active at a WE. Building upon the ideas underlying Proposition 1 and the Cobb-Douglas specification of the model, we arrive at the following result.

Corollary 1. *Consider any given firm $i \in N$. This firm is active at any WE if, and only if, it satisfies (a) and (b) in Proposition 1 and so happens as well for all other firms $j \in N_i^+$ providing inputs to it.*[7]

Proof. See Appendix A.

In view of the previous result, throughout this paper we shall assume that all existing firms satisfy (a)-(b). For short, this will be labeled Condition (A). Clearly, this condition can be assumed without loss of generality, for any other firm can be simply ignored in the analysis.

[7]Note that, clearly, if a good i satisfies (b) in Proposition 1, then all its inputs satisfy it as well. Hence the only relevant requirement concerning i's inputs is that they be supported from the production side, i.e. that they satisfy (a).

Of course, the identification of what firms are active at a WE provides only a very partial description of the situation. In general, there will be a wide heterogeneity in performance across firms (specifically, in terms of sales and profits). And if firms are symmetric in every respect except for their network position, it is the overall topology of the production network that should be used to explain such heterogeneity. Can we map the network-performance relationship in a sharp and insightful manner? To answer this question we turn to our first main result of the paper, Proposition 2 below.

Proposition 2. *There exists a unique WE for which the vector of equilibrium revenues* $\boldsymbol{s}^* = (s_i^*)_{i=1}^n \equiv (p_i^* \cdot y_i^*)_{i=1}^n$ *is given by*

$$\boldsymbol{s}^* = \frac{w(1-\alpha)}{\beta}(I - \alpha G)^{-1}\boldsymbol{\gamma} \qquad (i = 1, 2, ..., n),$$

while the corresponding equilibrium profits $\boldsymbol{\pi}^* = (\pi_i^*)_{i=1}^n = (1 - \alpha - \beta)\boldsymbol{s}^*$.

Corollary 2. *Assume* $M = N$ *(i.e. all goods are consumed). Then, the* relative *revenues and profits of the different firms at the WE satisfy:*

$$\left(\frac{s_i^*}{\sum_{j=1}^n s_j^*}\right)_{i=1}^n = \left(\frac{\pi_i^*}{\sum_{j=1}^n \pi_j^*}\right)_{i=1}^n = (1-\alpha)(I - \alpha G)^{-1}\boldsymbol{\gamma}.$$

Proofs. See Appendix A.

Proposition 2 establishes that – under the maintained condition (A) – the unique WE induces an equilibrium revenue (as well as profit) for each firm i that is proportional to a suitable measure of network centrality of this firm given by $(I - \alpha G)^{-1}\boldsymbol{\gamma}$. As we explain next, this centrality notion integrates two main factors:

(i) the utility weight of each of the consumption goods whose production the firm's output contributes to, either directly or/and indirectly;

(ii) a weighted discounted measure of the direct and indirect ways in which the aforementioned contribution takes place.

The measure of network centrality that arises from our analysis is a variation of the widely-used concept of Bonacich centrality. Since this notion of centrality is defined in the literature in slightly different forms, let us start our discussion by introducing the particular version to which we are referring here.

Definition 1. *Consider a directed weighted network whose adjacency* $(n \times n)$*-matrix is* G. *Let* $\delta \in (0, 1)$ *be a discount factor and* $\zeta > 0$ *a scale factor. Then, the associated vector of* Bonacich centralities *is given by* $\boldsymbol{v}(G, \delta, \zeta) = \zeta(I - \delta G)^{-1}\mathbf{1}$, *where* $\mathbf{1}$ *is a suitable column vector whose components are all equal to* 1.

To understand intuitively the notion of Bonacich centrality,[8] suppose first that the

[8] The measure introduced in Bonacich (1987) is defined by the expression $c(G, a, b) = b(I - \alpha G)^{-1}G\mathbf{1}$ while Ballester et al. (2006) use instead $b(G, a) = (I - \alpha G)^{-1}\mathbf{1}$ as the measure they call Bonacich centrality.

network is binary, so that links either have a zero or unit weight, i.e. $g_{ij} \in \{0,1\}$ for every $i,j \in N$. Then, since the matrix G is assumed column-stochastic, its different columns have to include exactly one unit entry and the other entries must be equal to zero. In this simplified case, it is easy to show that the centrality of a particular node i can be interpreted as (is proportional to) the average of the *discounted number of paths* that start at node i and reach all n nodes (including itself) in r steps ($r = 1, 2, ...$). The discount factor used is $\delta \in (0,1)$, and the total discount imposed on any given path is tailored to its length, i.e. it is δ^r if its length is r. To see this, rewrite the expression for centrality introduced in Definition 1 as follows:

$$\boldsymbol{v}(G,\delta,\zeta) = (v_i(G,\delta,\zeta))_{i=1}^n = \zeta \left[\sum_{r=0}^{\infty} \delta^r G^r\right] \mathbf{1} \tag{4}$$

so that, for each $i \in N$, its corresponding centrality is given by

$$v_i(G,\delta,\zeta) = \zeta \sum_{j=1}^{n} \left[\sum_{r=0}^{\infty} \delta^r g_{ij}^{[r]}\right]$$

where each $g_{ij}^{[r]}$ represents the ij-th entry of the matrix G^r. The suggested interpretation is then a consequence of the fact that $g_{ij}^{[r]}$ simply counts the number of paths of *exact* length r that start at node i and end at node j. As we have seen (cf. Proposition 2), the discount factor to be used in our case is $\delta = \alpha$ (where α is the output elasticity of intermediate inputs). Therefore, it turns out to be convenient to have a scale factor $\zeta = (1-\alpha)/n$. Since the matrix G (and therefore every power G^r) is column-stochastic, this scaling amounts to normalizing the centrality vector $\boldsymbol{v}$ to lie in the $n-1$ simplex, i.e. $\sum_{i=1}^n v_i = 1$.

More generally, when the matrix G is a general column-stochastic matrix with its entries $g_{ij} \in [0,1]$, an analogous interpretation can be provided but the "number of paths" must then be replaced by the "normalized intensity" that flows between pairs of nodes for every possible path lengths. Applied to our production context, such intensity simply corresponds to the product of the input-demand flows between any given firm i and the firms j that use the former's good as input, directly or indirectly.

The relationship between a firm's performance and its Bonacich centrality is especially stark in the context considered in Corollary 2. There we abstract from any asymmetry associated to the consumption side: all produced goods are assumed to be consumption goods and equivalent for the consumer, which implies that $\boldsymbol{\gamma} = \frac{1}{n}\mathbf{1}$. Then, this result establishes that the *relative* performance of firms is *exclusively* determined by their Bonacich centrality in the production network for a discount factor $\delta = \alpha$ and scale factor $(1-\alpha)$, where recall that α is the production elasticity of intermediate inputs. In this sense, we may say that, given this elasticity, the profitability of a firm is the reflection of purely "topological" features of the production network.

In the general case where the vector of preferences $\boldsymbol{\gamma}$ is arbitrary and possibly $M \neq N$ (not all goods are necessarily consumption goods), matters must be correspondingly adapted and the notion of centrality considered has to account for those asymmetries. Then, the relevant measure of centrality associates to the paths that arrive to any particular node j the weight γ_j that the consumer attributes to the respective good in her utility function (cf. Proposition 2). Thus, in particular, if good j is not a consumption good, paths of any given length r that connect some node i to such a node j do *not* contribute directly to the centrality (and therefore the profitability) of firm i. They may only do so indirectly to the extent that such paths can be constituent subpaths of longer ones that eventually connect i to some consumption good k (in which case they would be weighted by γ_k and affected by a lower discount factor).

Having characterized the situation on the production side of the economy, now we turn to studying its consumption side. Specifically, our aim is to understand how the features of the network impinge on the consumer's welfare at the WE. Again we find that the vector of centralities plays a prominent role, although in this case additional technological parameters are also important. We start the analysis by providing an explicit expression for the *equilibrium utility* in our next result.

Proposition 3. *At the WE the consumer's utility is given by*

$$\log U(\boldsymbol{c}) = \sum_{i=1}^{n} \gamma_i \log \gamma_i + \log(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} + \frac{1}{1-\alpha}$$
$$\left((\nu + \alpha) \sum_i v_i \log n_i - (1-\alpha-\beta) \sum_{i \in N} v_i \log v_i + \alpha \sum_{i \in N} \sum_{j \in N_i^+} v_i g_{ji} \log g_{ji} \right) \tag{5}$$

where recall that n_i is the number of inputs used in the production of good i, ν is the parameter modulating the economies of scope through the factor $A_i = n_i^{\alpha+\nu}$, and $\boldsymbol{v}(G, \alpha, 1-\alpha) = (v_i)_{i=1}^n$ is the corresponding vector of Bonacich centralities for $\delta = \alpha$ and $\zeta = 1-\alpha$.

Proof. See Appendix A.

The previous result highlights the three *endogenous* magnitudes that shape welfare in our context:

1. The weighted average (log-)connectivity, $\sum_i v_i \log n_i$, where the degree of each node/firm is weighted by its respective centrality.
2. A measure of heterogeneity/dispersion among firms, as reflected by the entropy of their centralities, $(-\sum_{i \in N} v_i \log v_i)$.
3. The average heterogeneity/entropy in input use (i.e. $-\sum g_{ji} \log g_{ji}$) displayed by the technologies of the different firms $i \in N$, each of these individual magnitudes weighted by the centrality of the respective firm i.
4. The heterogeneity/entropy across goods, $\sum_{i=1}^n \gamma_i \log \gamma_i$, displayed by the preferences of the representative consumer.

Specifically, we find that consumer welfare is increasing in the average log-connectivity of firms and in *inter*-firm symmetry, while it is decreasing in the *intra*-firm symmetry displayed by their technologies. These different magnitudes are aggregated through a simple affine function whose coefficients are given by the underlying technological parameters.

Next, to obtain a sharper characterization of the situation, it is useful to reduce the degrees of freedom by postulating the following two symmetry assumptions:

(S1) For any given firm $i \in N$, its different inputs play a symmetric role in its production technology, i.e. $g_{ji} = g_{ki}$ for all $j, k \in N_i^+$.

(S2) All goods are consumption goods and therefore $\gamma_i = 1/n$ for every firm $i \in N$.

Under the previous conditions, the intra-firm heterogeneity of each firm i is dependent on n_i alone, i.e. on the *number* of inputs it uses (which, heuristically, could be understood as the "complexity" of its production technology). Then, the expression in (5) is substantially simplified, as stated by the following corollary.

Corollary 3. *Assume (S1) and (S2) above. Then, the utility of the consumer at the WE is given by*

$$\log U = -logn + \log(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} + \frac{1}{1-\alpha}\left(\nu \sum_{i=1}^{n} v_i \log n_i - (1-\alpha-\beta)\sum_{i=1}^{n} v_i \log v_i\right) \tag{6}$$

Proof. See Appendix A.

Thus, under input symmetry, connectivity and inter-firm entropy are the *sole considerations* that impinge on consumer's welfare. As we shall discuss in Subsection 5.1, this stark conclusion will allow us as well to obtain a similarly sharp assessment of what production structures are welfare optimal in different technological scenarios.

4 Price distortions

In this section, we focus on the study of what could be interpreted as taxes, distortions, or policy/price interventions of different sorts. An obvious, but still important, characteristic of an interconnected economy is that any such distortion or intervention cannot be studied just locally. In general, it is to be expected that its first-order impact could turn out to be quite different from its overall effect on the economy, once the complete chain of indirect effects is taken into account. Such a full-fledged analysis of the situation, however, can be very complex and we need effective tools to carry out a proper analysis of the situation. Here we provide a step in this direction.

Specifically, in this section we consider two different cases. First, in Subsection 4.1, we consider a price distortion that applies directly to *all* firms in the economy in a uniform manner. Then, in Subsection 4.2, we turn to studying distortions that apply directly to just one firm in the economy, so the effects on all others are only indirect. In both cases, uniform or individualized, we focus on situations that, at an abstract level, can be conceived as inducing "price wedges." More concretely, this is an approach that can taken to capture a variety of different cases: government policies that favor a particular sector, *ad-valorem* taxes or subsidies, or the reflection of market power.[9] To stress the intended generality, we shall simply speak of them as "distortions."

A common idea that arises in the analysis of both uniform and individual distortion is that centrality-based measures are key in determining the size and direction of the induced effects. Centrality being an inherently global measure, the point is then that the impact of any change or intervention, no matter how "local" it might appear, must be evaluated globally. In fact, variants of the same general idea will reappear as well in much of the analysis conducted throughout the paper, again an indication of the unavoidable global nature of the issues being studied. In essence, the crucial network magnitudes involved in the analysis will turn out to be what we shall call (bilateral) in- and out-centralities. These are the elements of the matrix $\mathcal{M} = (m_{ij})_{i,j=1}^{n} \equiv (I - \alpha G)^{-1}$ included in the definition of (Bonacich) centrality – cf. Definition 1.

Specifically, for every pair of nodes i and j, the ij-incentrality is identified with m_{ij}. By writing

$$v_i(G, \alpha, \zeta) = \frac{1-\alpha}{n} \sum_{j=1}^{n} \left[\sum_{r=0}^{\infty} \alpha^r g_{ij}^{[r]} \right] = \frac{1-\alpha}{n} \sum_{j=1}^{n} m_{ij} \tag{7}$$

it is indeed apparent that m_{ij} represents (or, more precisely, is proportional to) the contribution of node j to the Bonacich centrality of i. Reciprocally, we refer to the entry m_{ji} as the ij-outcentrality. Note that, within our general economic model (cf. Proposition 2), for each $i \in N$ the entries m_{ij} $(j = 1, 2, ..., n)$ are the weights given to the preference weights γ_j of each good produced in the determination of the equilibrium profits of firm i through the expression

$$\pi_i^* = (1 - \alpha - \beta) \frac{w(1-\alpha)}{\beta} \sum_{j=1}^{n} m_{ij} \gamma_j. \tag{8}$$

Thus, as explained, the contribution to i's centrality provided by an intermediate product j that is not consumed (whose $\gamma_j = 0$) has no effect on i's equilibrium profits.

[9] See, for example, the paper by Hsieh and Klenow (2009) or Jones (2011) for an elaboration on such a general interpretation of price distortions.

4.1 Uniform price distortion

We start our analysis of distortionary effects by considering the implications of a uniform price distortion τ imposed on all firms of the economy. Its effect is to draw a proportional wedge between the price p_i the consumer pays for each good i and the price $(1-\tau)p_i$ received by the firm selling it. In principle, the value of τ might be negative, in which case it could be interpreted, for example, as a subsidy (hence amounting to a proportional increase in the revenue earned from the firm). An issue that arises here is whether the monetary payments (or proceeds) entailed should be distributed back to (or subtracted from) the revenue available to the agents of the economy – that is, whether the system is to be conceived as closed to those monetary flows. In general, the answer must depend on the specific interpretation attributed to those flows (e.g. on whether they are redistributive, or purely distortionary and hence wasteful). Here, in the main text, we shall consider the formally simpler case where they are a pure outflow (or inflow) of resources, while referring the reader to the online Appendix C for a consideration of the alternative closed-system version. None of our results are qualitatively affected by the scenario being considered.

Our first observation (see Appendix A for details) is that, under a uniform τ applied to all goods of the economy, the expression that characterizes the vector of equilibrium profits is generalized to

$$\boldsymbol{\pi}^*(\tau) = (1-\tau)(1-\alpha-\beta)\frac{w(1-\alpha)}{\beta}\left(I-\alpha(1-\tau)G\right)^{-1}\boldsymbol{\gamma} \tag{9}$$

This readily implies that, as expected, the effect on equilibrium profits of an increase in τ is unambiguously negative – the reason, of course, is simply that, in our context, any distortion is always detrimental.[10] Indeed, if we consider the marginal effect of increasing τ, we find:

$$\frac{\mathrm{d}\boldsymbol{\pi}^*}{\mathrm{d}\tau}(\tau) = -(1-\alpha-\beta)\boldsymbol{s}^*(\tau) - \alpha(I-\alpha(1-\tau)G)^{-1}G\boldsymbol{\pi}^*(\tau) < 0$$

where $\boldsymbol{\pi}^*(\tau)$ stands for the full vector of profits earned under τ by all firms.

Having settled the question of how a uniform distortion affects firms' profits, next we turn our attention to the much more complex issue of how it can *diversely* impinges on the relative profits of the different firms, as a function of their individual position in the network. To account for this relative performance in a convenient manner, it is useful to focus on the normalized (equilibrium) profits $(\hat{\pi}_j)_{i=1}^n$ of each firm i obtained by setting the nominal wage $w(\tau)$ so that $\sum_{j\in N}\hat{\pi}_j = 1$. A formal characterization of the (marginal) implications of changing τ on the relative profit performance of the different firms is provided by the following result.

[10] An analogous conclusion obtains for the effect of τ on consumer's welfare, which can be shown to be always negative. Details are available upon request.

Proposition 4. *Given any given distortion $\tau \in [0,1]$, let $(\hat{\pi}_j(\tau))_{i\in N}$ be the corresponding profile of* normalized *profits and denote $(\tilde{m}_{ij}(\tau))_{i,j=1}^n \equiv (I - \alpha(1-\tau)G)^{-1}$. The marginal effects on profits induced by a change on τ are determined by the following system of differential equations:*

$$\frac{d\hat{\boldsymbol{\pi}}}{d\tau}(\tau) = \frac{1}{1-\tau}\sum_{j=1}^{n} \tilde{m}_{ij}(\tau)\left(\gamma_j - \hat{\pi}_j(\tau)\right) \qquad (i = 1, 2, ..., n), \tag{10}$$

and hence, at a situation with no distortion ($\tau = 0$), the marginal impact of introducing *it is:*

$$\left.\frac{d\hat{\boldsymbol{\pi}}}{d\tau}\right|_{\tau=0} = \sum_{j=1}^{n} m_{ij}\left(\gamma_j - \hat{\pi}_j(0)\right) \qquad (i = 1, 2, ..., n), \tag{11}$$

where recall that $\mathcal{M} = (m_{ij})_{i,j=1}^n$ is the matrix of incentralities.

Proof. See Appendix A.

The above result highlights that the firms most *negatively* affected in their *relative* profit performance by the introduction of the distortion are those whose centrality is heavily dependent on firms with high original profits. A particularly stark manifestation of this idea is displayed in (11), which applies to the case where the distortion is just being marginally introduced. This expression can be heuristically understood as follows. The effect on the relative profit standing of any given firm changes as prescribed by an average composition/multiplication of two magnitudes:

(a) the first-order impacts experienced by each one of the firms in the economy, whose sign and size is captured for each firm $j \in N$ by $(\gamma_j - \hat{\pi}_j(0))$ – a comparison of firm j's relative profit and the (relative) weight of its output in consumer's preferences;

(b) the extent to which those impacts are transmitted to the firm i in question, as captured by its respective vector of ij-incentralities $(m_{ij})_{j=1}^n$.

When the base distortion is not zero but is some given $\tau > 0$, this two-fold mechanism exhibits the general form given by (10), which has an analogous interpretation. An interesting feature of this expression is that the sensitivity of relative equilibrium profits to interfirm asymmetries grows steeply as τ approaches unity. Another interesting consideration worth highlighting is that, because the dependence of profits on τ is non-linear, there is the potential for complex (and, in particular, nonmonotonic) behavior as the distortion changes within its full range $[0, 1]$. The following simple example illustrates that this is indeed a possibility.

Example 1. *Consider the simple production network with nine firms depicted in Figure 2, where it is assumed that all firms produce a consumption good and hence the arrows only indicate the flow of input use. Figure 3 traces the relative profits for possible values of*

$\tau \in [0, 1]$ and a subset of firms (for the sake of readability, not all firms are included). The second diagram shows that, as τ grows (and, naturally, the profit spectrum across firms narrows) there are rank changes in the relative position of some firms. Thus, while Firm 6 remains the highest-profit firm throughout, as τ grows the position of Firm 8 deteriorates, from second to fourth position. Indeed, we find that it is not just the ranking partially changes but even the relative profit of an individual firm may evolve in non-monotonic way as τ rises. Specifically, as shown more clearly in Figure 4, this happens to Firm 1, whose profit first grows and then decreases.

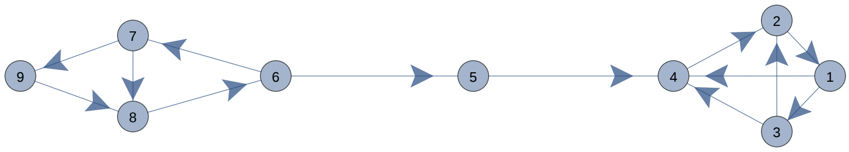

Figure 2: A simple production network used to illustrate the complex dependence of relative profits on a common distortion experienced uniformly by all firms.

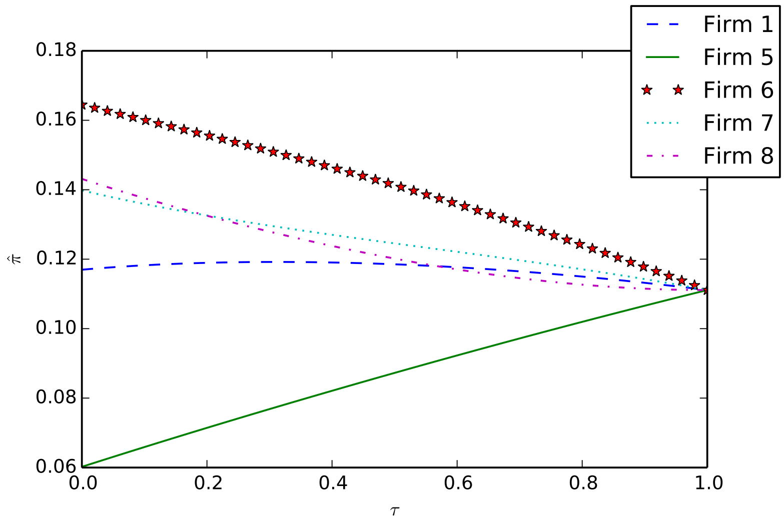

Figure 3: The relative profits of all firms in the production network displayed in Figure 2 for all values of $\tau \in [0, 1]$.

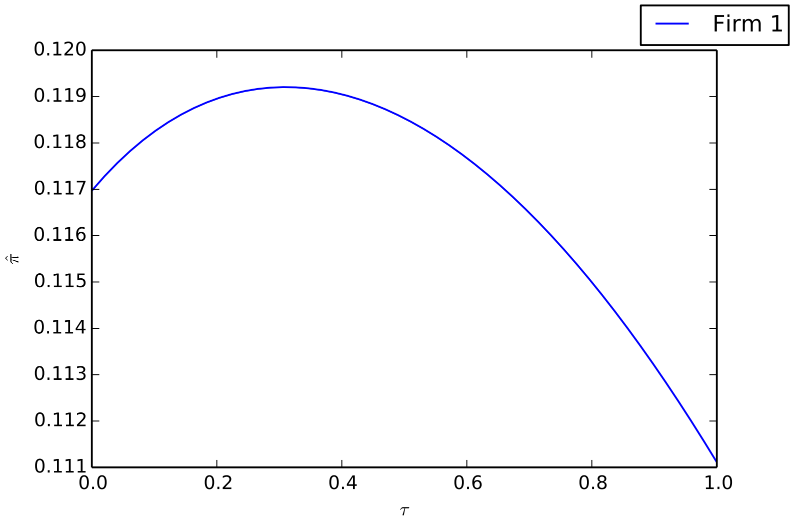

Figure 4: The non-monotonic behavior of the relative profit of firm1, as τ changes, in the production network displayed in Figure 2for all values of $\tau \in [0, 1]$.

4.2 Individual price distortion

In this subsection, our focus turns from a common distortion that affects uniformly all firms to one that impinges only on a specific firm k. As shown in Appendix A, such a firm-specific price distortion, denoted by τ_k, introduces just a simple modification in the equilibrium expressions derived for the benchmark model, but one that is of a quite different nature from those obtained for a uniform distortion. For example, in contrast with (9), the induced equilibrium profits $\boldsymbol{\pi}^*(\tau_k^o) = [\pi_i^*(\tau_k)]_{i=1}^n$ are now given by:

$$\boldsymbol{\pi}^*(\tau_k) = (1 - \alpha - \beta)\frac{w(1 - \alpha)}{\beta}\left(I - \alpha\hat{G}(\tau_k)\right)^{-1}\boldsymbol{\gamma}. \tag{12}$$

where $\hat{G}$ represents a modified matrix that replaces the original one, G. These matrices only differ in their respective kth columns, $\hat{\mathbf{g}}_k$ and $\mathbf{g}_k$, which satisfy $\hat{\mathbf{g}}_k = (1 - \tau_k)\mathbf{g}_k$.

The difficulty lies in studying the inverse in (12) to obtain its dependence on τ_k. To do this, it is useful to write the modified matrix $\hat{G}(\tau_k)$ as follows:

$$\hat{G}(\tau_k) = G - \tau_k \mathbf{g}_k \mathbf{e}_k', \tag{13}$$

where $\mathbf{e}_k'$ is the n-dimensional row vector that has a 1 in its kth position and 0 elsewhere. This then allows us to rely on the following result in Linear Algebra (cf. Sherman and Morrison (1949) or Hager (1989)):

Sherman-Morrison Formula. Let A be a nonsingular n-dimensional real matrix, and

$\boldsymbol{c}$, $\boldsymbol{d}$ two real n-dimensional column vectors such that $1 + \boldsymbol{d}'A^{-1}\boldsymbol{c} \neq 0$. Then,

$$(A + \boldsymbol{c}\boldsymbol{d}')^{-1} = A^{-1} - \frac{A^{-1}\boldsymbol{c}\boldsymbol{d}'A^{-1}}{1 + \boldsymbol{d}'A^{-1}\boldsymbol{c}}.$$

Applying the above result to our case – with the particularization $A = I - \alpha G$, $c = \tau_k \boldsymbol{g}_k$, and $d = \boldsymbol{e}_k$ – we arrive at the following full characterization of how the k-specific distortion affects equilibrium profits.

Proposition 5. *Consider a distortion τ_k imposed on firm k and let $(\pi_i^*(\tau_k))_{i=1}^n$ stand for the corresponding equilibrium profits. The marginal effects on profits induced by a change on τ_k are determined by the following system of differential equations:*

$$\frac{\mathrm{d}\pi_i^*}{\mathrm{d}\tau_k}(\tau_k) = -\pi_k^*(0)\frac{m_{ik}}{(1 - \tau_k(1 - m_{kk}))^2} \tag{14}$$

and hence, at a situation with no distortion ($\tau_k = 0$), the marginal impact of introducing *it is:*

$$\left.\frac{\mathrm{d}\pi_i^*}{\mathrm{d}\tau_k}\right|_{\tau_k=0} = \pi_k^*(0)m_{ik} \qquad (i = 1, 2, ..., n), \tag{15}$$

Proof. See Appendix A.

The previous result states that the profit decrease experienced by any firm i due to the direct distortion impinging on another firm k depends on

- how profitable k was prior to the distortion;
- the importance of k in determining the centrality of i;
- how much of k's centrality "feeds into" itself.

As intuition would suggest, while the first two considerations increases the profit loss on firm i due to distortion on k, the last one decreases it (with the only exception considered in (15) when there is no distortion to start with). Again we observe that incentralities (as captured by the m_{ik} and m_{kk} mentioned in the last two items) play a prominent role in shaping the overall global effect. It is also interesting to observe from (14) that the function mapping k's distortion into i's profit is convex so that, just as in the uniform-distortion case, the marginal effect of increasing τ_k grows with the level of it. Another feature that was noted for the previous uniform case is that the induced nonlinearities can lead to somewhat paradoxical conclusions. This phenomenon arises as well here, as illustrated below by through two examples.

Example 2. *Consider the production network depicted in Figure 5, where the corresponding profile of Bonacich centralities are also shown. Again, for simplicity, all produced goods are assumed to be not only intermediate inputs but consumption goods as well. Firm 1 and then Firm 2 are those with the highest centrality in the production network.*

Therefore, under no distortion, they are also the firms enjoying the highest profits at the Walrasian equilibrium.

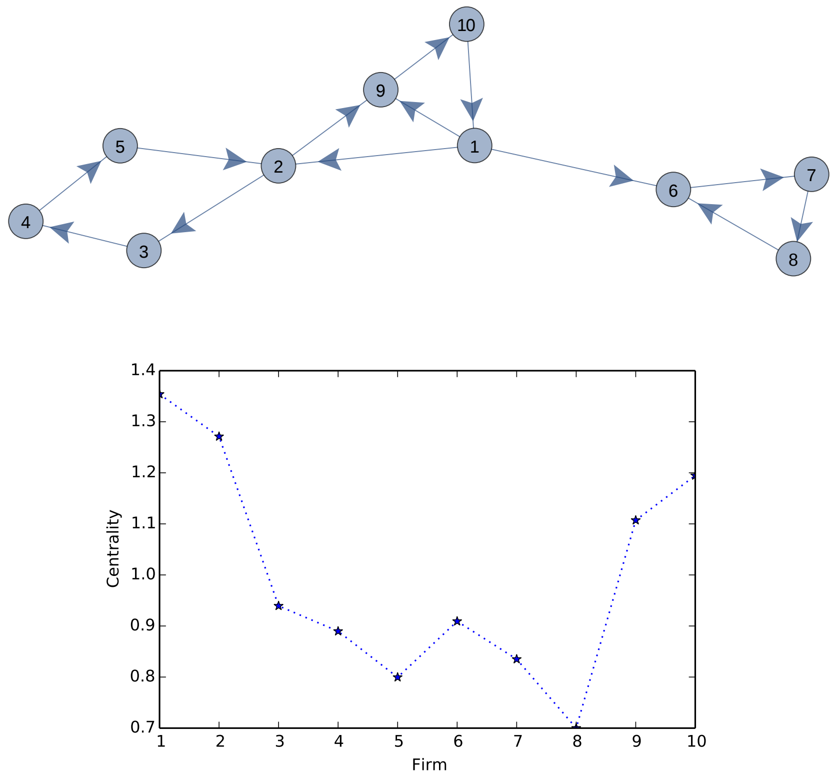

Figure 5: A production network with the corresponding profile of Bonacich centralities for each of the nodes.

However, as shown in Figure 6, the situation can change in interesting ways if our focus turns to relative profits (normalized to add up to one) and the distortion τ_2 *experienced by Firm* 2 *varies. (Note that here we are allowing* τ_2 *to be negative, varying in the range* $\tau_2 \in [-1, 1]$ *and thus playing possibly the role of a subsidy.) For* $\tau_2 = 0$ *(no distortion) the profit ranking exactly mimics that of centralities, as already explained. In contrast, as* τ_2 *grows and becomes positive, we observe the somewhat paradoxical fact that the relative profit of Firm 2 monotonically grows and, eventually, if* τ_2 *is high enough, even surpasses that of Firm 1 and becomes the highest. The opposite state of affairs is found if Firm 2 is subject to a negative* τ_2*. Then, a higher absolute value for it induces a lower profit for this firm, eventually leading it to fall below that of Firm 10. Why does this happen? The reason is that, as the distortion varies, the effect of this change – which is modulated by the various incentralities – impinges most strongly on the those firms i whose i*2*-incentrality is highest, which then alters the relative position of Firm 2 in the direction opposite to the change of* τ_2*. A further illustration of this role of incentralities, which is particularly simple and stark, is provided by our next example.*

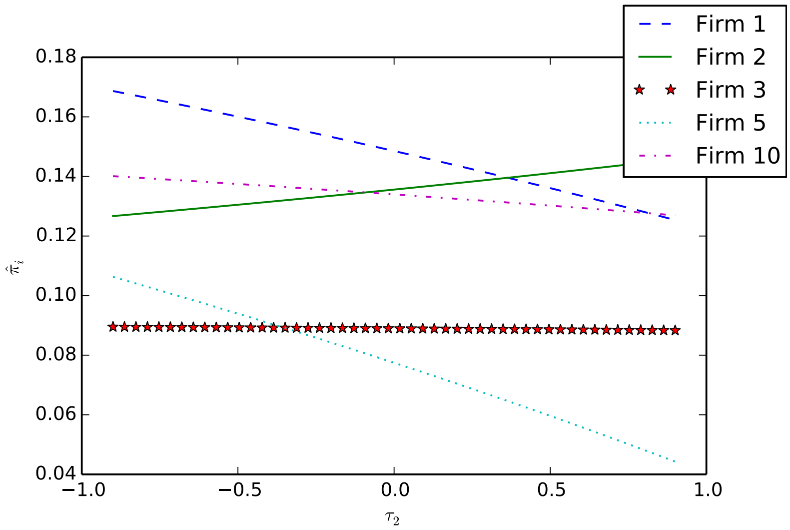

Figure 6: The change in relative profits among the firms placed in the production network depicted in Figure 5 as the distortion experienced by Firm 2 changes in the range $[-1, 1]$.

Example 3. *Suppose that ten firms are arranged into a ring production network with each firm* $i = 1, 2, ..., 10$ *using the good produced by Firm* $i - 1$ *as the sole intermediate input and having its produced good be the sole intermediate input in the production of Firm* $i + 1$ *(of course, indices* 0 *and* 11 *are interpreted as* 10 *and* 1*, respectively). As in the previous examples, all produced goods are assumed to be valuable to the consumer. Consider now a distortion experienced by Firm* 1*,* $\tau_1 \in [-1, 1]$*.*

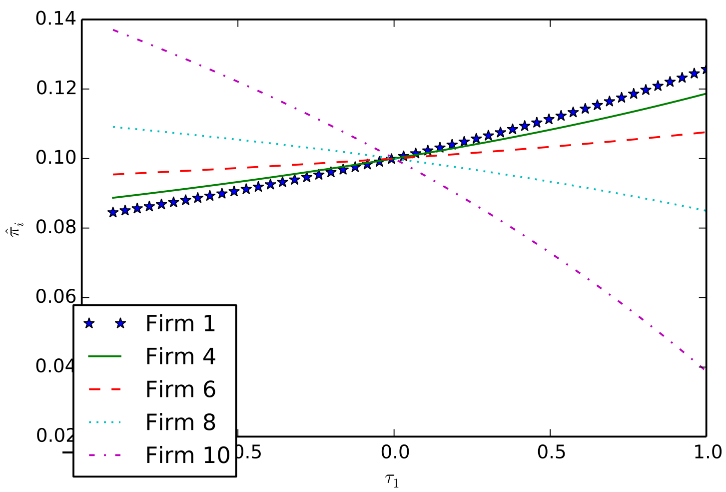

Figure 7: The change in relative profits among the firms placed in the ring production network as the distortion experienced by Firm 1 changes in the range $[-1, 1]$. In such a ring network, the sole input links define the cycle $1 \to 2 \to 3 \to ... \to 10 \to 1$.

Figure 7 depicts how the relative profits of firms change over the whole range of variation of τ_1*. The most affected is Firm* 10*, either negatively if* $\tau_1 > 0$ *or positively if*

$\tau_1 < 0$. *The reason is that this firm is the one with the highest* $i1$*-incentrality. Following it, the firm whose bilateral incentrality relative to* 1 *is highest is Firm* 9*, and thus it it is this firm the one which is the second most affected by changes in* τ_1*, and so on along the ring. In the end, we find that, indeed, the firm that is least affected is Firm 1, the firm that is directly subject to the distortion. Of course, this stark conclusion is an artifact of the extreme production network considered but should help clarify the key mechanism underlying global effects in our context.*

5 Network structure

In line with our core concern of understanding how economic structure affects performance, here we undertake an analysis of the following issues. First, in Subsection 5.1, we study what features of the production structure (e.g. whether it is more or less connected, or its degree of heterogeneity) impinge on the production and welfare potential of the economy. Then, in Subsections 5.2 and 5.3, our focus turns to studying the overall effect of single "local perturbations" on a pre-existing production structure. Specifically, we consider two of them: (i) the creation (or elimination) of a link ij in the production structure; (ii) the elimination of an existing node/firm, or the creation of a new one. As we shall explain, these changes can be understood as reflecting the operation of alternative economic phenomena.

5.1 Optimal network structure

The structure of production networks can be studied along many different dimensions. Here we show that, as far as its impact on (consumer) welfare is concerned, the *internode symmetry* displayed by the network and its *connectivity* stand out as two key features. To highlight their effect, it is useful to simplify the analysis by focusing on networks that satisfy the assumptions (S1)-(S2) introduced at the end of Section 3. That is, we postulate that (a) the inputs involved in the production of every good play a symmetric role, and (b) all goods are consumption goods. We also suppose that the number of goods in the economy is given, while postponing to Section 7 an analysis of what are the welfare implications of a change in this feature of the economy.Under these conditions, we can build upon Corollary 3 to arrive readily at the following conclusion.

Proposition 6. *Assume (S.1)-(S.2) in Section 3. Then, the production structure that maximizes consumer utility is a regular network (i.e. all firms have the same number of inputs). Furthermore, if* $\nu > 0$ *the optimal network is complete* (*i.e. for all* $i \in N$, $n_i = |N_i^+| = n - 1$)*, while if* $\nu < 0$ *the optimal network is given by a ring* (*i.e. for all* $i \in N, n_i = 1$)*.*

Proof. See Appendix A.

The previous result follows from inspection of the expression determining equilibrium utility in (6), as a function of the the parameters of the economy and the induced profile of firm centralities. On the one hand, the last term of the expression calls for the maximization of the entropy of the simplex-normalized vector of centrality profiles given by $-\sum_{i\in N} v_i \log v_i$. This is achieved when all nodes have the same centrality, which in turn requires that the network be node symmetric and hence regular. Thus all firms must display the same input complexity, i.e. it should use the same *number* of inputs. Then, whether such common complexity should be maximal or minimal sharply depends on the sign of the parameter ν in (3), which reflects the nature of the "economies of input scope" in production. If $\nu > 0$, complexity is beneficial and hence the optimal production structure should be complete, i.e. all other goods should be used in the production of every one of them. Instead, if $\nu < 0$, the exact opposite applies and the optimal production structure should display the minimum complexity consistent with viability (cf. Corollary 2). That is, it should constitute a ring.

In principle, of course, it is far-fetched to entertain the possibility that the production structure of the economy might be "designed." Instead, it is more natural to conceive this structure to be a reflection of a wide number of different forces (e.g. technological change) whose magnitude and direction can hardly be controlled. Therefore, the nature of the discussion in this subsection has to interpreted in a different vein, mostly as a conceptual exercise. That is, our objective here is to gain some understanding of what features of the production structure of an economy have an important impact on welfare. And, from this perspective, we have found that (abstracting from some other considerations) two characteristics arise as key: symmetry across all production processes and extreme exploitation/avoidance of all economies/diseconomies of scope.

5.2 Link creation

As advanced, in the remaining part of this section, we adopt a local perspective to the analysis of changes in the production structure. First, in this subsection, we focus on the impact of a change affecting a single link – for concreteness, we consider the case where the link is created. A preliminary and immediate point to note is that if a link ij is formed, the firm i who consequently sees an expansion in the uses of its product cannot have its *relative* profits decrease. The reason is that its centrality cannot decrease by this change – intuitively, the set of paths that access consumer nodes can only grow. Signing the effect for other firms, however, is much more difficult to assess, as the following result shows. Indeed, it is even possible – see Example 4 below for an illustration – that the firm j whose range of inputs is widened by the link ij sees its relative profit decrease after this (single) link is added to the production network.

Proposition 7. *Consider an initial production network with adjacency matrix G and suppose that a link ij is added to it. Denote by $\tilde{G}$ the resulting adjacency matrix and let*

$\boldsymbol{q} = (q_k)_{k=1}^n \in \mathbb{R}^n$ ($\sum_{k\in N} q_k = 0$) be the real vector that reflects the entailed adjustment across the two adjacency matrices, i.e. their respective jth columns, $(g_{kj})_{k=1}^n$ and $(\tilde{g}_{kj})_{k=1}^n$, satisfy $\tilde{g}_{kj} = g_{kj} + q_k$ for all $k = 1, 2, ..., n$ with $\tilde{g}_{ij} = q_i > 0$. The change on equilibrium profits, $(\Delta\pi_k^*)_{k=1}^n$, induced by the new link is given by:

$$\Delta\pi_k^* = \alpha\,\pi_j^*\,\frac{m_{ki}\,q_i + \sum_{\ell\in N_j^+} m_{k\ell}\,q_\ell}{1 + q_i\,m_{ji} + \alpha\sum_{\ell\in N_j^+} m_{j\ell}\,q_\ell}, \tag{16}$$

where π_j^* is the original profit of firm j under adjacency matrix G.

Proof. See Appendix A.

The previous result establishes that if a new link ij is added to the production network, the effect on the equilibrium profit of any particular firm j is proportional to two essential factors:

- The profit originally obtained by the firm j that provides a new use to good i. Intuitively, this magnitude reflects the importance of the firm that generates the direct value accessed by the new link.
- The impact on the centrality of k of each of the inputs involved in the production of j (including the new input i), as captured by the respective in-centralities, m_{ki} and $(m_{k\ell})_{\ell\in N_j^+}$. Each of these in-centralities is weighted by the factor q_ℓ that indicates the adjustment made (positive or negative) in transforming the adjacency matrix G into $\tilde{G}$.

Again, therefore, we can understand the changes resulting from the new link as consisting of a composition of two kinds of effects: a first-order effect whose magnitude is associated to the equilibrium-induced importance of the directly affected nodes; a second-order effect that involves the transmission of the first-order effect as embodied by the full array of network-based interactions displayed in the matrix of in-centralities.

Example 4. *As advanced, in this example we illustrate that the addition of a new link ij can deteriorate the relative position of one of the firms involved – namely, the one whose inputs possibilities are enlarged the new link. Consider the production networks depicted in Figure 8, where the smaller network gives rise to the larger one by the addition of the link* 14. *Let us assume that goods are consumer goods and have an equal weight in the utility function of the consumer. Then, the vectors of normalized profits for each them, respectively denoted by $\hat{\pi}^\dagger$ and $\hat{\pi}^\ddagger$, are as follows:*

$$\hat{\boldsymbol{\pi}}^\dagger = \begin{pmatrix} 0.1000 \\ 0.3265 \\ 0.2960 \\ 0.2775 \end{pmatrix} \qquad \hat{\boldsymbol{\pi}}^\ddagger = \begin{pmatrix} 0.1790 \\ 0.2863 \\ 0.2718 \\ 0.2630 \end{pmatrix} \tag{17}$$

Thus one finds that firm 4 obtains a lower relative profit after the link 14 is added to the original network. Its market possibilities are enhanced relative to those of other firms (which remain unchanged) but its relative profits deteriorate. Why is this the case? The simple reason for this is that, as explained, centrality (and hence profitability) is associated to the consumption value a product delivers, not to the consumption value it allows other firms to attain. That is, profitability is gathered downstream rather that upstream.

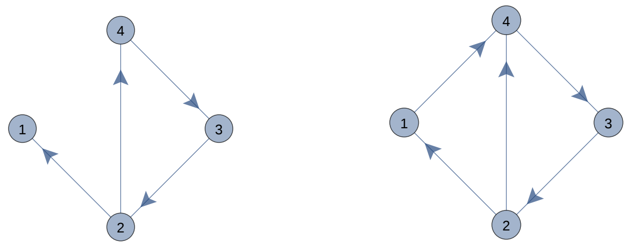

Figure 8: Two different production networks, whose only difference concerns the existence, or not, of the link 14. All nodes/firms are supposed to produce a different consumption good.

5.3 Node deletion

In this section, we study the polar (local) change in the production structure where it is a (single) node, rather than a link, that is added or removed. In fact, for concreteness, we focus our attention on the case of node removal, which is the specific situation that will be considered in Section 7 when studying the chain implications of a process of node/firm selection that eliminates firms when they fail to meet a certain survival criterion – e.g. some predetermined level of profits.

A modeling question that needs to be addressed is how the setup is to be renormalized when the set of nodes changes. This concerns both the utility and the production functions, which all deliver a value of zero when any of its arguments is equal to zero. For the utility function, the renormalization we choose is the natural one: we restrict the its arguments to the range of consumer goods which can actually be produced given the prevailing set of active firms. In Subsection 7.1, we provide a specific motivation for this choice in a context where the number of firms changes due to a process firm selection.

Concerning the renormalization implemented on the production functions when there is a change in the inputs available, we adopt a different route when there is an increase in those inputs as compared to the case when there is a decrease. This contrast is based on a different interpretation of both cases. On the one hand, the availability of a new input is conceived as a change that entails gradual adaptation and is actually implemented only when the adjustments required are largely in place. This means that, at the time it becomes effective, the production function embodies a balanced set of weights among all its inputs (old and new), which therefore add up to unity. This, in fact, was the assumption made in Subsection 5.2 when we considered an analogous phenomenon where the new input became available by the creation of an additional link.

Instead, when a node/firm disappears, the consequences are thought to materialize quite differently: abruptly and, in a sense, unexpectedly. Therefore, important consequences – such as the bankruptcy – may well happen before the firm is able to find the way to adjust and a suitable adaptation can be implemented. In this case, therefore, we choose not to normalize the input-use requirements of the firms affected by the change.[11] Admittedly, one should not expect real-world situations to be so markedly dichotomous. The contrast, however, is plausible and our sharp modeling of it underscores the differences between the the appearance and removal of input possibilities. In view of the preceding discussion, it is quite clear that the elimination of a node can only be detrimental to the profits of all firms in the economy.[12] A precise specification of the magnitude of the effect applying to each of them is given by the following result.

Proposition 8. *Consider an initial production network and adjacency matrix G and suppose that node $i \in N = \{1, 2, ..., n\}$ is removed from it. Denote by $\tilde{G}$ the resulting adjacency matrix and $\left(\Delta\pi_j^*\right)_{j \neq i}$ the change in equilibrium profits of the remaining firms. Then,*

$$\Delta\pi_j^* = -\pi_i^* \frac{m_{ji}}{m_{ii}} \qquad (j =, 1, 2, ..., n;\ j \neq i), \tag{18}$$

where π_i^* denotes the equilibrium profits of firm i and $(m_{ki})_{k=1}^n$ is the vector of out-centralities of firm i (in both cases, before the removal of i) .

Proof. See Appendix A.

The previous result indicates that the effect induced by the removal of a particular firm i on the profits of the remaining firms $j \neq i$ exhibits the usual pattern: a first-order effect (that is quantified by the prominence/profitability of the firm directly affected, i.e. firm i) composed with the effect that the directly affected firm has on the centrality of any other firm (measured by the corresponding in-centrality). In the present case, however, the in-centrality terms m_{ji} are "normalized" by the effect m_{ii} that firm i has on its own centrality – in this sense, the relevant magnitude scales the out-centralities of i by the extent to which this firm's centrality effect feeds into itself.

[11] In essence, this is equivalent to unit-normalizing the input-use parameters $(g_{ij})_{j=1}^n$ of any firm j affected by the elimination of a firm i, together with a corresponding downward adjustment of i's intermediate-input elasticity α_i. The analysis of the situation would require relying on the extended framework considered in the online Appendix B, where full firm heterogeneity is allowed (in particular, on the that elasticity). To dispense with this need here, we simply maintain the non-normalized input-use parameters and the original common α for the affected firm.

[12] Incidentally, we advance that this unambiguous negative effect will not necessarily occur in the context studied in Subsection 7.1, where we shall contemplate the possibility of fixed labor costs. In this case, the removal of a firm may have beneficial effects on some of the other firms, due to the relaxation of labor feasibility constraint induced by a lower number of active firms.

As we shall explain in Section 7, the network considerations underlying firm removal go beyond the purely static ones specified in Proposition 8. Indeed, another aspect in which the network can play a non-trivial role is dynamic and relates to the chain reactions possibly induced by the concatenation of node removals. We shall find, for example, that "market criteria" for firm removal/selection (e.g. current profits) may be very misleading if the primary objective is to sustain a welfare-maximizing production structure.

6 Forward and backward linkages

As explained in the Introduction, one of the useful features of the model is that it allows a transparent formalization of the notions of forward and backward linkages (push and pull effects) that, as stressed by Hirschman (1958), can be important in assessing, for example, public interventions and economic policies. In essence, all the comparative static results considered so far in Subsections 4.1 to 5.3 can be understood along these lines. More specifically, the overall effects of the exogenous changes under consideration can be decomposed into a forward linkage that operates downstream on the production structure and a backwards one that works upstream. Rather that revisiting all our results in this light, it will prove more useful at this point to present these two forces in the context of two particularly clear-cut instances: a (Hicks-neutral) pattern of technological improvements on the production technologies of the different firms; an arbitrary change in the preferences of the consumer across the different goods. As we shall explain, while the first case can be seen as exerting a set of pure "pushing" forces on the equilibrium the second one represents an array of pulling forces (some positive and others negative).

6.1 Technological change

Consider a change in the pre-factors $(A_k)_{k \in N}$ of the production functions of each firm k in the economy (cf.2-3), which become $(\tilde{A}_k)_{k \in N}$. We shall refer to them as the productivities of the respective firms. The key observation to make from Proposition 2 is that the equilibrium conditions are such that *equilibrium sales* are independent of those productivities. This readily leads to the following two conclusions:

(i) The equilibrium input demands $[(z^*_{jk})_{j \neq k}]_{k \in N}$ by each firm k (which are proportional to sales) remain unaltered by the change in productivities. Thus, in this sense, there are *no pull effects operating upstream* along the production structure.

(ii) The equilibrium prices of the goods produced by the different firms decrease at least in the same proportion as their respective productivities. This in turn triggers a downstream push over all sectors that use those goods directly or indirectly, with lower prices all along inducing more output being sold (and hence produced).

Item (i) is quite clear and requires no further elaboration. It may be simply added that, for the same reason why input demands are unaffected by the change, profits are unaffected as well. Productivity changes, therefore, have no effect of the inter-firm distribution of profits. This, of course, is crucially dependent on our Cobb-Douglas formulation, and the fact that exactly absorb all quantity changes resulting from increased productivities. Admittedly, this is a quite extreme implication of the postulated functional forms but has the advantage of highlighting the contrast between forward and backward linkages in a clear-cut manner.

Indeed, the fact that the linkages across firms resulting from the change propagate upstream is well underscored by the way in which prices are affected by it. This is the focus of item (ii) above. A good way of grasping the idea is provided by the following result

Proposition 9. *Consider a set of changes in firm productivities such that the new levels, $(\tilde{A}_i)_{i\in N}$, are related to the former ones as follows:*

$$\tilde{A}_i = \rho_i \, A_i \qquad (\rho_i > 0, \; i = 1, 2, ..., n). \tag{19}$$

Under a suitable normalization, the corresponding equilibrium prices, $(\tilde{p}_i^*)_{i\in N}$ and $(p_i)_{i\in N}^*$, satisfy:

$$-\left(\log \tilde{p}_i^* - \log p_i^*\right)_{i=1}^n = -(I - \alpha G')^{-1} \left(\log \rho_i\right)_{i=1}^n$$

or

$$\frac{\tilde{p}_i^*}{p_i^*} = \prod_{k=1}^n \rho_k^{-m_{ki}} \qquad (i = 1, 2, ..., n), \tag{20}$$

where G' stands for the transpose of the adjacency matrix of the production network and, for each firm i, $(m_{ki})_{k\in N}$ is the vector of its out-centralities.

The key implication of the previous result is that, as explained in (ii), the inter-firm linkages in this case are mediated through prices and quantities alone (i.e. not revenues nor profits) and operate downstream. That is, the first-order effect on each firm k, as embodied by its corresponding ρ_k, propagates through the out-centralities of this firm. The effect on prices is as described in (9). On the other hand, in view of the constancy of equilibrium revenues, the corresponding effect on quantities is simply given by

$$\frac{\tilde{y}_i^*}{y_i^*} = \prod_{k=1}^n \rho_k^{m_{ki}} \qquad (i = 1, 2, ..., n). \tag{21}$$

6.2 Preference changes

Here we illustrate the nature of the backward linkages by focusing on a characteristic instance, namely, those induced by changes in preferences, as captured by the weight vector γ determining the utility function of the consumer – cf. (1). Recall that, as established by Proposition 2, equilibrium revenues and profits are given by

$$\boldsymbol{s}^* = (s_i^*)_{i=1}^n \equiv (p_i^* \cdot y_i^*)_{i=1}^n = \frac{w(1-\alpha)}{\beta}(I - \alpha G)^{-1}\boldsymbol{\gamma} \qquad (i = 1, 2, ..., n),$$

and

$$\boldsymbol{\pi}^* = (\pi_i^*)_{i=1}^n = (1 - \alpha - \beta)\boldsymbol{s}^*.$$

Hence they are both linear functions of the preference weights $\boldsymbol{\gamma} = (\gamma_i)_{i \in N}$. This trivially implies the following statement, which we formulate formally for the sake of clarity.

Proposition 10. *Suppose that the preference weights change from $\boldsymbol{\gamma}$ to $\tilde{\boldsymbol{\gamma}}$, with $\gamma_k = 0$ for all $k \notin M$ and $\sum_{j \in N} \Delta\gamma_i \equiv \sum_{j \in N} \tilde{\gamma}_i - \gamma_i = 0$. Then, the induced change in equilibrium profits, $(\Delta\pi_i^*)_{i \in N}$, satisfies:*

$$\Delta\pi_i^* \propto \sum_{j \in N} m_{ij}\, \Delta\gamma_j \qquad (i = 1, 2, ..., n). \tag{22}$$

Proof. Omitted □

Thus, in contrast with the pure forward linkages channeling the allocational adjustments induced by changes in firms' productivities, any variation in preferences not only has consequences on the allocation of resources but also distributional implications on firms' profits. Specifically, we have that the change in the profit of any given firm i is proportional to a weighted average of the changes (positive or negative) experienced by the relative importance that the consumer's utility attributes to each consumption good. Naturally, the weight that a particular consumption good j has in the computation of such an average effect is given by the in-centrality m_{ij} that captures the direct and indirect impact of good j on the centrality of firm i. The overall effect of the whole set of firms is purely redistributive, in the sense that the total profits are unaffected by the changes in $\boldsymbol{\gamma}$. The following corollary – which is an immediate consequence of the fact that the matrix of in-centralities $\mathcal{M} = (m_{ij})_{i,j=1}^n \equiv (I - \alpha G)^{-1}$ is column stochastic – states it explicitly.

Corollary 4. *Under the conditions specified in Proposition 10, any change in the preference weights $\boldsymbol{\gamma}$ induces a change in equilibrium profits, $(\Delta\pi_i^*)_{i \in N}$, that satisfies $\Sigma_{i \in N} \Delta\pi_i^* = 0$.*

The effects of technology change and preference change are examples of pure push and pull effects respectively. A price distortion of a single firm, as considered in Subsection 4.2 exhibits both effect. The pull effect comes from the fact that the demand of distorted firm k for inputs decreases (as the revenue decreases). As firms demand less inputs, the production decreases, creating push effect downstream through the network. The pull effect is visible from equation (42), while the push effect is proportional to a technology shock which would change A_k to $(1 - \tau_k)A_k$.

7 Firm dynamics

To study the issue of firm dynamics, we start our discussion by introducing in Subsection 7.1 a theoretical framework that extends the basic setup considered so far (cf. Section 2) along two complementary directions. First, we incorporate fixed labor costs. This has the main effect of allowing for the possibility that, even at equilibrium, a firm can be subject to losses. The interpretation is that the firm in question becomes then bankrupt and thus, in principle, liable to disappear. The second extension allows for a varying number of goods in the economy and hence admits changes in the size of the production network. One can then assess the welfare impact of changes in the number of active firms in the economy and, in particular, study the welfare implications of different market selection forces over time.

Once we extend our benchmark framework in the two aforementioned directions, in Subsection 7.2 we conduct a preliminary exploration of firm dynamics. Our focus, specifically, is on understanding what are the welfare implications of alternative criteria that may be used to single out firms that are worth of support against, say, shocks or some threat to their survival. We show that selecting those firms on the basis of strict market-based indicators (e.g. profits), instead of using other network-based criteria, can be quite misleading from the viewpoint of long-run welfare.

7.1 A generalized framework: fixed costs and varying network size

7.1.1 Fixed costs

Suppose that, as long as any given firm i remains active, it has to pay some fixed cost $f_i \geq 0$, which is interpreted as some given labor requirements that are independent on the scale of production, This induces two different changes in the benchmark framework described in Section 2. First, given the wage w, prices $(p_j)_{j=1}^n$ for the intermediate inputs, and the production plan $[y_i, l_i, (z_{ji})_{j=1}^n)]$, the profit of firm i is given by

$$\pi = p_i y_i - \sum_{j=1}^{n} p_j z_{ji} - w(l_i + f_i).$$

Second, while the demand functions for labor and intermediate inputs do not change by the introduction of fixed costs, the market-clearing condition for the labor market does change since, in this case, the amount of labor available for production decreases with the number of active firms. Specifically, the new market-clearing condition of the labor market becomes:

$$\sum l_i = 1 - nf \tag{23}$$

where we assume, for simplicity, that all active firms are subject to the same fixed cost f.

The former modifications on the theoretical framework in turn induce some changes on the equilibrium magnitudes. As a simple counterpart of former Proposition 2, we have the following generalization:

$$\boldsymbol{s}^* = \frac{w(1-nf)(1-\alpha)}{\beta}(I - \alpha G)^{-1}\boldsymbol{\gamma} \qquad (i = 1, 2, ..., n),$$

while the corresponding equilibrium profits continue to satisfy the proportionality condition $\boldsymbol{\pi}^* = (\pi_i^*)_{i=1}^n = (1 - \alpha - \beta)\boldsymbol{s}^* - wf$.

Proposition 11. *There exists a unique WE for which the vector of equilibrium revenues* $\boldsymbol{s}^* = (s_i^*)_{i=1}^n \equiv (p_i^* \cdot y_i^*)_{i=1}^n$ *is given by*

$$\boldsymbol{s}^* = \frac{w(1-nf)(1-\alpha)}{\beta}(I - \alpha G)^{-1}\boldsymbol{\gamma} \qquad (i = 1, 2, ..., n),$$

while the corresponding equilibrium profits $\boldsymbol{\pi}^* = (\pi_i^*)_{i=1}^n = (1 - \alpha - \beta)\boldsymbol{s}^* - wf\,\boldsymbol{e}$*, where* $\boldsymbol{e}^\top = (1, 1, ..., 1)$.

Proof. See Appendix A. □

Naturally, equilibrium profits are one of the standards we would like to measure firm performance in a market-environment such as the one considered here. More specifically, the requirement that a firm incur no losses is a commonly used criterion to assess firm "viability." Then, if we assume for simplicity that all goods are consumption goods, the implications of such a profit-based criterion are formally spelled out in the following immediate corollary.

Corollary 5. *In the context considered in Proposition 11, the individual firm profits obtained at the WE satisfy:*

$$\pi_i^* \geq 0 \quad \iff \quad v_i \equiv (1-\alpha)\sum_{j=1}^{n} m_{ij}\gamma_j \geq \frac{\beta}{1-\alpha-\beta}\frac{f}{1-nf} \tag{24}$$

where $(m_{ij})_{j=1}^n$ *is the* i*th row of the matrix of incentralities* $\mathcal{M}$ *(cf. (7)).*

Proof. Omitted □

The above Corollary expresses the clear-cut requirement that, in order for a firm i to meet a profit-based (non-negativity) viability condition, its network centrality must be no lower than a certain given magnitude, which only depends on technological/cost parameters, the size of the economy, and the fixed costs. As advanced, in Subsection 7.2 below we compare this requirement with other criteria that are also of the same nature – i.e. depend solely on the topological features of the production network, technological/cost parameters, and the population size – but instead embody in a more effective manner the effects that the survival (or not) of a given firm has on the subsequent ability to survive of the other firms. Since our approach to evaluating the situation will be dynamic, such considerations are indeed. For, obviously, they bear on the likely size of "bankruptcy cascades" and hence on the overall final effect of alternative courses of action.

A suitable account for the aforementioned effects needs to extend former Proposition 8 – which determined the effect of node/firm removal on the profits of all other firms – to the consideration of fixed costs. This is the purpose of the following result.

Proposition 12. *Consider an initial production network and adjacency matrix G and suppose that node $i \in N = \{1, 2, ..., n\}$ is removed from it. Denote by $\tilde{G}$ the resulting adjacency matrix and $\left(\Delta\pi_j^*\right)_{j\neq i}$ the change in equilibrium profits of the remaining firms. Then,*

$$\Delta\pi_j^* = \frac{f}{1-nf}\,\pi_j^* - \pi_i^*\,\frac{m_{ji}}{m_{ii}} \qquad (j =, 1, 2, ..., n;\ j \neq i), \tag{25}$$

where $(\pi_k^*)_{k=1}^n$ denotes the equilibrium profits of all firms and $(m_{ki})_{k=1}^n$ the vector of out-centralities of firm i (in both cases, before the removal of i) .

Proof. See Appendix A.

We observe, therefore, that in contrast with the case with no fixed costs, the removal of a firm i could conceivably have positive effects on the profits of some other firms $j \neq i$. This occurs because in addition to the negative effect on the profits of other firms that is caused by a decrease on their network centrality (also at work before), when there are fixed labor costs the elimination of a firm relaxes the labor feasibility constraint. The extent to which this is important for a particular firm j depends on the scale s_j^* at which this firms operated (captured by its former profits $\pi_j^* \propto s_j^*$) as well as on the magnitude f of the fixed costs and the market pressure that these costs was imposing on the labor market, i.e. the size of the residual market given by $(1 - nf)$.

7.1.2 Network size

Now we focus on the modeling issue of how to measure (consumer) welfare across situations where the network size changes due to variations in the number of firms active in the economy. To this end, we need a formulation of preferences that is of course consistent with our Cobb-Douglas specification (1) but is also able to accommodate a varying number of consumption goods. A useful route to do so is provided by the so-called

Ethier-Dixit-Stiglitz (EDS) preferences, which are represented by the utility function:

$$\tilde{U}(\mathbf{c}\,;\ \rho, N) = \sum_{i \in N} c_i^{\rho} \tag{26}$$

for some $\rho > 0$, where N is the universe of all possible goods (for simplicity, all assumed suitable for consumption). An equivalent representation of the same preferences is embodied by the monotone transformation of $\tilde{U}(\cdot)$ that induces its CES-format counterpart given by:

$$\hat{U}(\mathbf{c}\,;\ \rho, N) = \left[\sum_{i \in N} c_i^{\rho}\right]^{\frac{1}{\rho}}.$$

Clearly, given any ρ, the above formulation can be adapted to any non-empty subset of goods $M \subset N$ by simply changing the set of goods under consideration. Moreover, as it is well-known, the corresponding function $\hat{U}(\cdot\,;\ \rho, M)$ converges to the Cobb-Douglas (CD) utility function (1) with equal weights $\gamma_i = \frac{1}{m}$ in the limit of $\rho \to 0$. It is in this sense that, for *any* given set M of consumption goods, we may view our CD formulation as a representation of EDS preferences with an elasticity of substitution $\frac{1}{1-\rho}$ converging to 1. Moreover, consumer optimality in this limit context requires that all goods considered in the corresponding set M be consumed in some positive quantity, so we can make this assumption without loss of generality when focusing on WE.

The former considerations suggest that one can suitably compare the welfare of the consumer for different sets of goods being consumed, say M and M', by resorting to the functions $\tilde{U}(\cdot\,;\ \rho, M)$ and $\tilde{U}(\cdot\,;\ \rho, M')$ respectively, defined as suitable extensions of (26). Then, again taking the limit on ρ for both of them, we arrive at the conclusion that

$$\lim_{\rho \to 0^+} \tilde{U}(\mathbf{c}\,;\ \rho, M) = |M| \equiv m; \qquad \lim_{\rho \to 0^+} \tilde{U}(\mathbf{c}'\,;\ \rho, M') = |M'| \equiv m' \tag{27}$$

where $\mathbf{c} \in \mathbb{R}^m$ and $\mathbf{c}' \in \mathbb{R}^{m'}$ represent *any* pair of *equilibrium* consumption vectors (whose components are all positive, as argued above). Thus, in the end, we conclude that the welfare comparison of two equilibrium allocations with different sets of available consumption goods, M and M', can be viewed as boiling down, in the Cobb-Douglas limit, to comparing their respective cardinalities, m and m'.

7.2 Firm dynamics: an illustrative example

Here we rely on the formalization and results presented in the former Subsection to undertake a preliminary and merely illustrative exploration of firm dynamics that focuses on the extent to which alternative firm-selection criteria may have different welfare implications. To fix ideas, we shall consider the specific example given by the production network with 12 firms depicted in Figure 9.

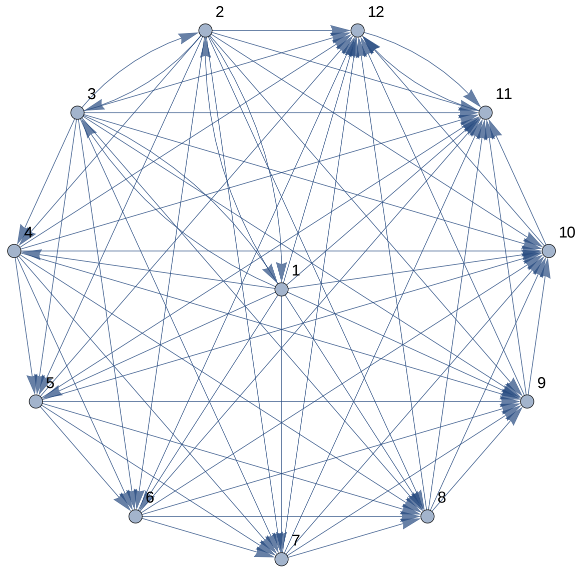

Figure 9: Initial production network prevailing at the beginning of the firm dynamics. It consists of 12 nodes with firms $\{1,2,3\}$ and $\{11,12\}$ defining two completely connected cliques.

To understand the role played by each node i in the network, a natural way to do it is to add its bilateral in-centralities, $m_i^I \equiv \sum_{j\in N} m_{ij}$, and their bilateral out-centralities, $m_i^O \equiv \sum_{j\in N} m_{ji}$. As we know, the first magnitude – aggregate in-centrality – is proportional to equilibrium profits and thus can be seen as a measure of performance, as evaluated by the market at any given point in time.[13] On the other hand, the second one – aggregate out-centrality – measures the reciprocal notion of how important any given firm is for the aggregate (in-)centrality of all others. In our case, however, where the focus will be on the effect on others induced by a possible elimination of the node in question, it follows from Propositions 8 and Proposition 12 that the relevant magnitude to be considered is the out-centrality *normalized* by the contribution of the node to its own centrality. Therefore, as a measure of external influence of any given node i we shall consider the adjusted (aggregate) out-centrality defined as $\hat{m}_i^O \equiv \frac{1}{m_{ii}} \sum_{j\in N} m_{ji}$. Figure 10 depicts the profiles $\left[m_i^I\right]_{i\in N}$ and $\left[\hat{m}_i^O\right]_{i\in N}$ across all 12 nodes of the production network considered. It is apparent, as suggested above, that there is an acute contrast between the in-centralities and adjusted out-centralities of all nodes, of a polar nature in the sets $\{1,2,3\}$ and $\{4,5,...,12\}$.[14]

Given the aforementioned contrast and the conflicting ranking of prominence it entails in our example, the question arises as to what criterion should be use to evaluate the

[13]Assuming symmetric taste vector $\boldsymbol{\gamma}$

[14]We show in Appendix D that the node whose elimination affects the vector of centralities and the GDP the most is the node with the highest value of $\frac{m_i^O s_i}{m_{ii}}$ which is the result parallel to the key player result in Ballester et al. (2006). It is clear that this does not mean that the elimination of this node will create the largest cascade.

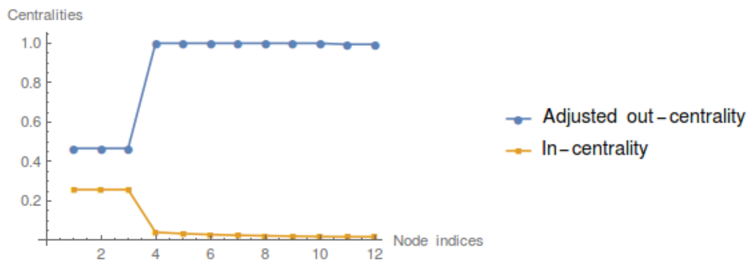

Figure 10: Graphical representation of the in-centrality profile and the profile of adjusted out-centrality of the 12 firms of the economy.

"systemic value" of the different firms. To fix ideas, let us pose the following question, which is somewhat artificial but has the benefit of posing the issue in stark and transparent terms. Suppose that two firms such as, say, Firm 2 and 12 are hit by a shock that, in the absence of any outside support, is sure to lead to their bankruptcy and hence disappearance from the economy. Further assume that the funds that may be used to provide such support are limited and only one of the two firms can be helped. Which of the two should it be?

Of course, the answer to the previous question depends on what is the objective function to be optimized. In line with the discussion undertaken in Subsection 7.1.2, let us suppose that the final objective is to maintain the largest possible number of active firms in the economy. One possible criterion to judge what firm should be saved from bankruptcy is given by what the market identifies as most and least valuable. According to it, the support should be directed towards Firm 2 and hence this firm survives while Firm 12 will go under and . Once this has implemented, interpreted as as a once and for all intervention, the market forces alone are left to operate unrestrained. We then find that Firm 11 becomes inviable since it depended on Firm 12 as the supplier of its only intermediate input of production. Next, it is Firm 10 that, despite being productively viable, in the Walrasian Equilibrium (WE) prevailing in the economy with the set of firms restricted to $i = 1, 2, ..., 10$, it incurs losses. Thus it becomes inactive. An analogous situation then applies to Firm 9, and then Firm 8, 7, 6, 5, and 4. At the point where the set of firms is reduced to $\{1, 2, 3\}$, the corresponding WE allows the three of them to obtain positive profits and thus all three survive and the situation remain stationary thereafter, in absence of any further shock or some subsequent entry of new firms. This is, in the end, the final outcome attained when the criterion used to select the firms to be supported is based on how the market ranks them, i.e. their relative profits. The process is graphically illustrated in Figure 11, which displays the production networks prevailing in stages 2 through 9, up to the point where the process reaches its stationary state.

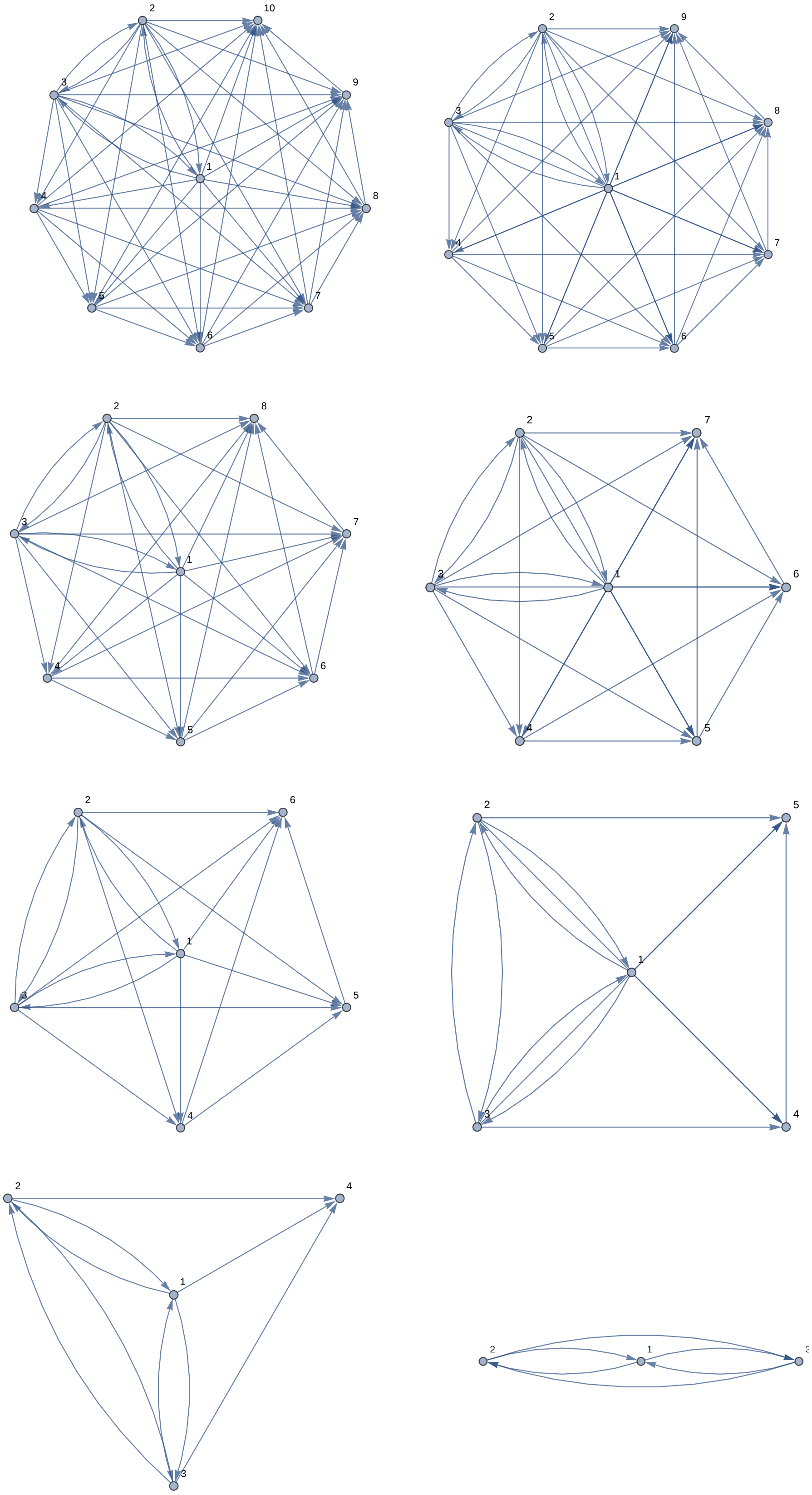

Figure 11: Firm dynamics induced by the first elimination of Firm 2, followed by subsequent steps where the firms that incur losses (i.e. obtain negative profits) are eliminated in turn. The diagram starts with step 2 (depicted in the upper-left corner) where firms 1-10 are still in the system. Thereafter, one firm is eliminated in each of the following six stages (first Firm 9, then 8, 7, 6, 5, 4) until the clique {1, 2, 3} remains and no further eliminations are required.

Alternatively, the criterion for selecting the firm to be supported may single out the firm that, if it were to disappear, would have the highest (detrimental) effect on other firms. Then, if one quantifies such an effect by the adjusted (aggregate) out-centrality, the support against the shock should be enjoyed by Firm 12 rather than Firm 2. And once this choice is implemented, the immediate consequence is that Firm 2 vanishes from the system. In the resulting situation, where the other eleven firms are still active, all of them enjoy positive profits at the Walrasian Equilibrium. The system, therefore, reaches a stationary state that dominates (in the sense explained in 7.1.2) the alternative state that would have been reached by a selection criterion that was exclusively profit-based. This helps illustrates in a stark manner what has been the leading theme of this paper, namely, that a networked production economy cannot be properly understood, nor have suitable policies implemented, unless the analysis adopts a genuine network viewpoint.

8 Summary and conclusions

In this paper, we have studied a general equilibrium model of an economy that focuses primarily on the inter-firm network of input-output relationships that underlie its production structure. Is this modeling approach useful to understand economic performance? A first, and particularly sharp, illustration of its usefulness derive from the connection we have identified between market equilibrium outcomes and an intuitive measure of centrality that combines topological and preference information. Specifically, we have shown that, at equilibrium, the profits of the different firms are directly proportional to their corresponding centralities.

The paper has then turned to extending this approach into the investigation of a number of important comparative-statics questions:
(a) the effects of distortions (local or global);
(b) the implications of alternative network changes (on nodes or links);
(c) the impact of non-network fundamentals (changes in preferences or productivities).
While the specifics of each of these cases is quite different, the conclusions obtained exhibit a common format. In all cases, the overall effect derives from a *composition* of a first-order impact on the firms directly affected, followed by a diffusion of this impact throughout the economy as determined by the matrix of cross-firm inter-centralties. A different, but complementary, perspective is gained if we decompose matters into the effects that spread upstream along the network (backward linkages) and those that do it downstream (forward linkages). This alternative decomposition has been found useful, for example, when comparing the changes that affect productivities and those that impinge on preferences.

Finally, we have illustrated the dynamic consequences of incorporating network considerations into the evaluation of simple policy dilemmas – in particular, we have studied alternative criteria for firm support in the face of exogenous shocks that threaten firm survival. In particular, we have seen that tailoring those supporting decisions only to current market-based information (thus ignoring forward-looking network-based considerations) may be quite misleading and lead to disappointing results. For example, the number of firms *eventually* affected by the shock may be much larger than would have been optimal.

Clearly, the model proposed in this paper is quite stylized and hence leaves ample room for extensions. By way of illustration, interesting possibilities would be to allow for less stringent assumptions on the competitive behavior of agents, or to generalize the Cobb-Douglas formulation of preferences and technologies posited here. In fact, a theoretical framework with the aforementioned features has been formulated in the paper by Baqaee (2015) already mentioned in the Introduction. Specifically, that paper considers a context where the set of firms are involved in monopolistic competition, and has preferences and technologies be of the CES type. (On the other hand, it also allows for adjustments in the extensive margin, through the endogenous entry and exit of firms.) Baqaee's concern, however, is similar to that of Acemoglu et al. (2012) and thus focuses his analysis on how microeconomic shocks aggregate at the level of the whole economy. An extension of our analysis to the richer scenario studied in Baqaee (2015) would be very interesting indeed, and is one of our objectives for future research.

More generally, a primary objective for future research should be to exploit the network-based approach formulated here to inform and shape a variety of measures of economic policy. But to this end, of course, a thorough empirical testing of the model must be conducted first. This, in turn, requires the use of data that are rich and granular enough at the microeconomic level to account for the intricate pattern of inter-firm interactions that characterize a modern economy. Fortunately, these data are now becoming available, as well as the tools (theoretical, algorithmic, and computational) needed to analyze them. As a case in point, we refer to the work in progress that we are currently undertaking with panel data from the Spanish Tax Agency. The data include all bilateral transactions among Spanish firms (and any other economically relevant entities, public or private) that amount annually to more than 3000 during the decade 2004-2013. The analysis conducted so far, still preliminary, has delivered promising results. In particular, we have found a very strong and significant relationship between the profitability of firms and their respective centrality, which is one of the basic predictions of our model (cf. Proposition 2). Subsequent research will test some of its other predictions.

Appendix A: Proofs of the main results

We start this Appendix by providing a formal definition of Walrasian Equilibrium. The we proceed by providing detailed proofs of all the results in the in text.

Definition 2 (Walrasian Equilibrium)**.** *A WE is an array* $[(\boldsymbol{p}^*, w^*), (\boldsymbol{c}^*, \boldsymbol{y}^*, \boldsymbol{Z}^*, \boldsymbol{l}^*)]$ *such that:*

1. *The representative consumer chooses* $\boldsymbol{c}^*$ *by solving:*

$$\begin{aligned} \max_{\boldsymbol{c}} \quad & \prod_{i \ inM} c_i^{\gamma} \\ s.t. \quad & \sum_{i \in M} p_i c_i = w + \sum_{k=1}^{n} \pi_k \end{aligned}$$

2. *Firms choose* $\boldsymbol{y}^*, \boldsymbol{Z}^*, \boldsymbol{l}^*$ *by solving:*

$$\begin{aligned} \max_{(z_{ji})_j, l_i} \quad & p_i y_i - \sum_{j \in r_i^+} p_j z_{ji} g_{ji} - w l_i \\ s.t. \quad & z_i = A_i l_i^{\beta} \left(\prod_{j \in N_i^+} z_{ji}^{g_{ji}} \right)^{\alpha} \end{aligned}$$

3. *Markets for labour and intermediate goods clear:*

$$y_i = \sum_j z_{ij} + c_i \ \ \forall (i \in N)$$

$$\sum_i l_i = 1$$

Proof of Proposition 1: The production function (2) implies that in order to be active a firm has to have at least one active in-neigbor. As this is true for all active firms, we can consider the upstream walk from i choosing only active firms. As number of firms is finite and each active firm must have at least one active in-neighbor, this means that there will an upstream walk from i that visits the same active node more than once. This proves part (a).

To prove part note that the demand for produced good in the model is generated by the consumer. If there is no path from a firm to consumer there is no direct or indirect demand for the good a firm is producing. Therefore a firm will choose not to produce. □

Proof of Corolary 2. Omitted □

Proof of Proposition 2. We omit the asterisk (*) to simplify notation, but it will be clear from the context when we talk about the equilibrium amounts. The first order conditions for the maximization problem of firm i, with respect to z_{ji} are given with $(\forall j)$

$$A_i p_i \alpha g_{ji} l_i^{\beta} z_{ji}^{\alpha g_{ji}-1} \prod_{k,k\neq j} z_{ki}^{\alpha g_{ki}} = p_j \Rightarrow z_{ji} = \frac{p_i \alpha g_{ji}}{p_j} y_i \tag{28}$$

and with respect to labour l_i:

$$A_i p_i \beta l_i^{\beta-1} \prod_{k} z_{ki}^{\alpha g_{ki}} = w \Rightarrow l_i = \frac{p_i \beta}{w} y_i \tag{29}$$

Using (28) and (29) we write the profit of firm i as:

$$\pi_i = p_i y_i - \sum_j \left(p_j \frac{p_i \alpha g_{ji}}{p_j} y_i \right) - w \frac{p_i \beta}{w} y_i = (1-\alpha-\beta) p_i y_i \tag{30}$$

Using (29), from market clearing condition from labor we get $w = \beta \sum_i p_i y_i$ which together with expression from profit (30) and FOC for consumption gives:

$$c_i = \frac{\sum_{j=1}^{n} (1-\alpha-\beta) p_j y_j + w}{p_i} \gamma_i = \frac{(1-\alpha)w}{\beta p_i} \gamma_i \tag{31}$$

We can now write the market clearing condition for good i (plugging in (28) and (31)) as: $s_i = \frac{(1-\alpha)w}{\beta m} \gamma_i + \alpha \sum_j g_{ij} s_j$, where $s_i = p_i y_i$ is the revenue of firm i. Using vector notation, we write the system of market clearing conditions for all intermediate goods as:

$$\boldsymbol{s} = \frac{(1-\alpha)w}{\beta} \boldsymbol{\gamma} + \alpha G \boldsymbol{s} \Rightarrow \boldsymbol{s} = \frac{(1-\alpha)w}{\beta} (I - \alpha G)^{-1} \boldsymbol{\gamma} \tag{32}$$

It follows directly from (30) that: $\boldsymbol{\pi} = (1-\alpha-\beta)\boldsymbol{s}$ The system of equations 32 has the unique solution as the matrix G is column stochastic and $\alpha < 1$. □

Proof of Corollary 2. Suppose now that all goods are consumed. Let us define:

$$\boldsymbol{v} := \frac{\beta}{w} \boldsymbol{s} = (1-\alpha)(I-\alpha G)^{-1} \boldsymbol{\gamma} \tag{33}$$

Expanding (33) we can write $\boldsymbol{v} = (1-\alpha)\left(\sum_{k=0}^{\infty} \alpha^k G^k\right) \boldsymbol{\gamma}$. This sum converges as $0 \leq \alpha < 1$ and G is column stochastic. Note also that $\mathbf{1}' G \boldsymbol{\gamma} = 1$, as G is column stochastic and $\sum_{i=1}^{n} \gamma_i = 1$. Since the product of two stochastic matrices is stochastic, we have that $\mathbf{1}' G^k \boldsymbol{\gamma} = 1 \ \forall k \in N$. This implies:

$$\mathbf{1}' \boldsymbol{v} = (1-\alpha) \sum_{k=0}^{\infty} \alpha^k = \frac{1-\alpha}{1-\alpha} = 1 \tag{34}$$

From market clearing condition for labor, (29) and (34) it directly follows that $\forall (j \in N)$, $v_j = \frac{s_j}{\sum_{i=1}^{n} s_i}$. Recall that $\pi_i = (1-\alpha-\beta)s_i$ and the proof is complete. □

Proof of Proposition 3. To calculate utility of the consumer we proceed as follows. Plug in (28), and (29) into the production function (2):

$$y_i = A_i \left(\frac{p_i \beta}{w} y_i \right)^{\beta} \prod_j \left(\frac{p_i \alpha g_{ji}}{p_j} y_i \right)^{\alpha g_{ji}} \Rightarrow$$

$$\prod_{j \in N_i^+} \left(p_j^{\alpha g_{ji}} \right) y_i = A_i \left(\frac{p_i \beta}{w} y_i \right)^{\beta} \prod_j \left(p_i \alpha g_{ji} y_i \right)^{\alpha g_{ji}} \Rightarrow$$

$$\frac{\prod_{j \in N_i^+} p_j^{\alpha g_{ji}}}{p_i} s_i = A_i \left(\frac{\beta}{w} s_i \right)^{\beta} \prod_j \left(\alpha g_{ji} s_i \right)^{\alpha g_{ji}}$$

Taking natural logs and simplifying using (33) and (3) we get that for every firm i:

$$\alpha \sum_{j \in N_i} g_{ji} \log p_j - \log p_i = \tag{35}$$
$$\log A_i + \beta \log \beta - (1 - \alpha - \beta) \log s_i - \beta \log w + \alpha \log \alpha + \alpha \sum_j g_{ji} \log g_{ji}$$

Let us define $u := \alpha \log \alpha + \beta \log \beta + (1 - \alpha - \beta) \log \beta$ and write system of (35) in vector notation ($\forall i \in N$).

$$\alpha \left(I - \alpha G' \right) \log \boldsymbol{p} = (1 - \alpha) \mathbf{1} \log w - (\alpha + \nu) \log \boldsymbol{n} + (1 - \alpha - \beta) \log \boldsymbol{v} - \alpha H' \mathbf{1} - u \mathbf{1} \tag{36}$$

Premultiplying (36) with $\boldsymbol{v}' = (1 - \alpha) \boldsymbol{\gamma}' (I - \alpha G')^{-1}$ we get:

$$(1 - \alpha) \boldsymbol{v}' \mathbf{1} \log w = (\alpha + \nu) \boldsymbol{v}' \log \boldsymbol{n} + \alpha \frac{1 - \alpha}{n} \mathbf{1}' \log \boldsymbol{p} - (1 - \alpha - \beta) \boldsymbol{v}' \log \boldsymbol{v} + \boldsymbol{v}' \mathbf{1} u + \alpha \boldsymbol{v}' H' \mathbf{1} \Rightarrow$$
$$(1 - \alpha) \log w = (\alpha + \nu) \sum_i v_i \log n_i - (1 - \alpha - \beta) \sum_i v_i \log v_i + \alpha \sum_i \sum_j v_i g_{ji} \log g_{ji} + u \tag{37}$$

where we have normalized $\sum_i \log p_i = 0$. Recall that: $c_i = \frac{(1-\alpha)w}{\beta p_i} \gamma_i$. So we can write the utility of the representative consumer as:

$$U(\boldsymbol{c}) = \prod_{i=1}^{n} \left(\frac{(1 - \alpha) w}{\beta p_i} \gamma_i \right)^{\gamma_i} = \prod_{i=1}^{n} \gamma_i^{\gamma_i} \frac{1 - \alpha}{\beta} w$$

From this and (8) we get:

$$\log U = \sum_{i=1}^{n} \gamma_i \log \gamma_i + \log (1 - \alpha) \alpha^{\frac{\alpha}{1 - \alpha}} + \frac{1}{1 - \alpha}$$
$$\left((\nu + \alpha) \sum_i v_i \log n_i - (1 - \alpha - \beta) \sum_{i \in N} v_i \log v_i + \alpha \sum_{i \in N} \sum_{j \in N_i^+} v_i g_{ji} \log g_{ji} \right)$$

□

Proof. In the symmetric case $g_{ji} = g_{ki} = 1/n_i\ \forall(j,k \in N_i^+)$ and $\gamma_i = \frac{1}{n}\ \forall i$, we have: $\sum_i \sum_j v_i g_{ji} \log g_{ji} = \sum_i v_i \log \frac{1}{n_i} = -\sum_i v_i \log n_i$ and $\sum_{i=1}^n \gamma_i \log \gamma_i = -\log n$. From here and Proposition 3 it directly follows that:

$$\log U(\boldsymbol{c}) = -logn + \log(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} + \frac{1}{1-\alpha}\left(\nu \sum_{i=1}^{n} v_i \log n_i - (1-\alpha-\beta)\sum_{i=1}^{n} v_i \log v_i\right)$$

□

Definition 3 (WE with distortions)**.** *For an exogeneus vector of price distortions $\boldsymbol{\tau} = (\tau_i)_{i=1}^n$. A WE is an array $[(\boldsymbol{p}^*, w^*), (\boldsymbol{c}^*, \boldsymbol{y}^*, \boldsymbol{Z}^*, \boldsymbol{l}^*)]$ such that:*

1. *The representative consumer chooses $\boldsymbol{c}^*$ by solving:*

$$\begin{aligned} &\underset{\boldsymbol{c}}{max} && \prod_{i\ inM} c_i^{\gamma} \\ &s.t. && \sum_{i\in M} p_i c_i = w + \sum_{k=1}^{n} \pi_k \end{aligned}$$

2. *Firms choose $\boldsymbol{y}^*, \boldsymbol{Z}^*, \boldsymbol{l}^*$ by solving:*

$$\begin{aligned} &\underset{(z_{ji})_j, l_i}{max} && (1-\tau_i)p_i y_i - \sum_{j\in r_i^+} p_j z_{ji} g_{ji} - w l_i \\ &s.t. && z_i = A_i l_i^{\beta}\left(\prod_{j\in N_i^+} z_{ji}^{g_{ji}}\right)^{\alpha} \end{aligned}$$

3. *Markets for labour and intermediate goods clear:*

$$y_i = \sum_j z_{ij} + c_i \ \ \forall(i \in N)$$

$$\sum_i l_i = 1$$

Lemma 1. *There exists unique WE with uniform price distortions as long as $(1-\tau)\alpha < 1$. The vector of equilibrium revenues is $\boldsymbol{s}^*(\tau) = (s_i^*(\tau))_{i=1}^n \equiv (p_i^* \cdot y_i^*)_{i=1}^n$ is given by*

$$\boldsymbol{s}^*(\tau) = \frac{w(1-\alpha)}{\beta}(I-\alpha G)^{-1}\boldsymbol{\gamma} \qquad (i = 1, 2, ..., n),$$

while the corresponding equilibrium profits $\boldsymbol{\pi}^*(\tau) = (\pi_i^*)_{i=1}^n = (1-\tau)(1-\alpha-\beta)\boldsymbol{s}^*(\tau)$.

Proof. The proof in analogous to the proof of Proposition 2. In the case of uniform price distortions $\tau_i = \tau_j = \tau \ \forall (i, j \in N)$. The equations (28) and (29) become:

$$z_{ji} = (1-\tau)\frac{p_i \alpha g_{ji}}{p_j} y_i$$
$$l_i = (1-\tau)\frac{p_i \beta}{w} y_i$$

So we can write the equilibrium profit of firm i as:

$$\pi_i(\tau) = (1-\tau)p_i y_i - \sum_{j \in N_i^+} p_j (1-\tau)\frac{p_i \alpha g_{ji}}{p_j} y_i - w(1-\tau)\frac{p_i \beta}{w} y_i = (1-\tau)(1-\alpha-\beta)p_i y_i \quad (38)$$

Analogously to the derivation in Proposition 2 we get: $\boldsymbol{s}(\tau) = \frac{(1-\alpha)w}{\beta}\left(I - \alpha(1-\tau)G\right)^{-1}\boldsymbol{\gamma}$ which together with (38) gives $\boldsymbol{\pi}(\tau) = (1-\tau)(1-\alpha-\beta)\boldsymbol{s}(\tau)$.

The uniqueness follows directly from the fact that G is column stochastic and $(1-\tau)\alpha < 1$. □

Proposition 13. *In the WE with uniform price distortion τ the following holds:*

$$\frac{d\boldsymbol{\pi}^*}{d\tau}(\tau) < 0$$

Proof. To see the detrimental effect of the distortion to profit, it is useful to recall that $\boldsymbol{s}(\tau) = \frac{1-\alpha}{\beta}\boldsymbol{\gamma} + \alpha(1-\tau)G\boldsymbol{s}(\tau)$. Differentiating both sides with respect to τ we get:

$$\frac{d\boldsymbol{s}}{d\tau}(\tau) = -\alpha G\boldsymbol{s}(\tau) + \alpha(1-\tau)G\frac{d\boldsymbol{s}}{d\tau}(\tau) \Rightarrow \frac{d\boldsymbol{s}}{d\tau}(\tau) - \alpha(I - \alpha(1-\tau)G)^{-1}G\boldsymbol{s}(\tau) \quad (39)$$

From (38) and (39) it easily follows that:

$$\begin{aligned}\frac{d\boldsymbol{\pi}}{d\tau}(\tau) &= -(1-\alpha-\beta)\boldsymbol{s}(\tau) - \alpha(1-\tau)(1-\alpha-\beta)(I-\alpha(1-\tau)G)^{-1}G\boldsymbol{s}(\tau) \\ &= -(1-\alpha-\beta)\boldsymbol{s}(\tau) - \alpha(I-\alpha(1-\tau)G)^{-1}G\boldsymbol{\pi}(\tau) << 0\end{aligned}$$

□

Proof of Proposition 4. From (38) it follows that we can write: $\boldsymbol{\pi}(\tau) = (1-\tau)(1-\alpha-\beta)\boldsymbol{s}(\tau) = (1-\tau)(1-\alpha-\beta)\frac{1-\alpha}{\beta}w\left(I-(1-\tau)\alpha G\right)^{-1}\boldsymbol{\gamma}$. Normalizing $w = \frac{1-(1-\tau)\alpha}{(1-\tau)(1-\alpha)(1-\alpha-\beta)}\beta$ we get: $\hat{\boldsymbol{\pi}}(\tau) = (1-\alpha(1-\tau)((I-\alpha(1-\tau)G)^{-1}\boldsymbol{\gamma}$ and $\mathbf{1}'\hat{\boldsymbol{\pi}}(\tau) = 1$. Let us now write the normalized profits as: $\hat{\boldsymbol{\pi}}(\tau) = (1-\alpha(1-\tau))\boldsymbol{\gamma} + \alpha(1-\tau)G\hat{\boldsymbol{\pi}}(\tau)$ and take the derivative from both sides with respect to τ. We get:

$$\frac{d\hat{\boldsymbol{\pi}}}{d\tau}(\tau) = \alpha\boldsymbol{\gamma} - \alpha G\hat{\boldsymbol{\pi}}(\tau) + \alpha(1-\tau)G\frac{d\hat{\boldsymbol{\pi}}}{d\tau}(\tau) \Rightarrow$$
$$\frac{d\hat{\boldsymbol{\pi}}}{d\tau}(\tau) = \alpha(I - \alpha(1-\tau)G)^{-1}(\boldsymbol{\gamma} - G\hat{\boldsymbol{\pi}}(\tau)) \Rightarrow$$
$$\frac{d\hat{\boldsymbol{\pi}}}{d\tau}(\tau) = \alpha(I - \alpha(1-\tau)G)^{-1}\boldsymbol{\gamma} - \frac{1}{1-\tau}\left((I - \alpha(1-\tau)G)^{-1}\hat{\boldsymbol{\pi}}(\tau) - (1-\alpha(1-\tau)(I-\alpha(1-\tau)G)^{-1}\boldsymbol{\gamma}\right) \Rightarrow$$
$$\frac{d\hat{\boldsymbol{\pi}}}{d\tau}(\tau) = \frac{\alpha}{1-\alpha(1-\tau)}\hat{\boldsymbol{\pi}}(\tau) - \frac{1}{1-\tau}\left((I - \alpha(1-\tau)G)^{-1}\hat{\boldsymbol{\pi}}(\tau) - \hat{\boldsymbol{\pi}}(\tau)\right) \Rightarrow$$
$$\frac{d\hat{\boldsymbol{\pi}}}{d\tau}(\tau) = \frac{1}{1-\tau}\left(\frac{1}{1-\alpha(1-\tau)}\hat{\boldsymbol{\pi}}(\tau) - (I - \alpha(1-\tau)G)^{-1}\hat{\boldsymbol{\pi}}(\tau)\right) \Rightarrow$$
$$\frac{d\hat{\boldsymbol{\pi}}}{d\tau}(\tau) = \frac{1}{1-\tau}(I - \alpha(1-\tau)G)^{-1}\left(\boldsymbol{\gamma} - \hat{\boldsymbol{\pi}}(\tau)\right)$$

For particular firm i we have:

$$\frac{d\hat{\boldsymbol{\pi}}}{d\tau}(\tau) = \frac{1}{1-\tau}\sum_{j=1}^{n}\tilde{m}_{ij}(\tau)\left(\gamma_j - \hat{\pi}_j(\tau)\right) \tag{40}$$

When all goods are symmetric consumption goods, (40) becomes:

$$\frac{d\hat{\boldsymbol{\pi}}}{d\tau}(\tau) = \frac{1}{1-\tau}\sum_{j=1}^{n}\tilde{m}_{ij}(\tau)\left(\frac{1}{n} - \hat{\pi}_j(\tau)\right)$$

At $\tau = 0$, (40) becomes:

$$\left.\frac{d\hat{\boldsymbol{\pi}}}{d\tau}\right|_{\tau=0} = \sum_{j=1}^{n} m_{ij}\left(\gamma_j - \hat{\pi}_j(0)\right)$$

□

Proof of Proposition 5. Analogously to the case of uniform price distortion we arrive to the expression for vector of revenues in the equilibrium $\boldsymbol{s}(\tau_k) = \frac{1-\alpha}{\beta}w\left(I - (\alpha G - \alpha\tau_k\boldsymbol{g}_k\boldsymbol{e}_k'))\right)^{-1}\boldsymbol{\gamma}$ Using Sherman-Morrison formula we get:

$$\boldsymbol{s}(\tau_k) = \frac{1-\alpha}{\beta}w\left((I-\alpha G)^{-1}\boldsymbol{\gamma} - \frac{(I-\alpha G)^{-1}\alpha\tau_k\boldsymbol{g}_k\boldsymbol{e}_k'(I-\alpha G)^{-1}}{1+\alpha\tau_k\boldsymbol{e}_k'(I-\alpha G)^{-1}\boldsymbol{g}_k}\boldsymbol{\gamma}\right) \tag{41}$$

Note first that $\frac{1-\alpha}{\beta}w(I-\alpha G)^{-1}\boldsymbol{\gamma} = \boldsymbol{s}(0)$ and $\frac{(1-\alpha)w}{\beta}(I-\alpha G)^{-1}\alpha\tau_k\boldsymbol{g}_k\boldsymbol{e}_k'(I-\alpha G)^{-1}\boldsymbol{\gamma} = -\alpha\tau_k s_k(0)(I-\alpha G)^{-1}\boldsymbol{g}_k$. Using this to simplify (41) we get:

$$\boldsymbol{s}(\tau_k) = \boldsymbol{s}(0) - s_k(0)\frac{\alpha\tau_k(I-\alpha G)^{-1}\boldsymbol{g}_k}{1+\alpha\tau_k\boldsymbol{e}_k'(I-\alpha G)^{-1}\boldsymbol{g}_k} = \boldsymbol{s}(0) - s_k(0)\frac{\tau_k((I-\alpha G)^{-1} - I)\boldsymbol{e}_k}{1-\tau_k+\tau_k m_{kk}} \tag{42}$$

where the last equality follows directly from Lemma2. For a specific firm $i \neq k$ we have that: $\pi_i(\tau_k) = (1-\alpha-\beta)\left(s_i(0) - s_k(0)\frac{\tau_k m_{ki}}{1-\tau+\tau m_{kk}}\right)$, and from here

$$\frac{d\pi_i}{d\tau_k}(\tau_k) = -(1-\alpha-\beta)s_k(0)\frac{m_{ki}}{(1-\tau_k(1-m_{kk}))^2}$$

For the firm k, $\pi_k(\tau_k) = (1-\tau_k)(1-\alpha-\beta)\left(s_k(0) - s_k(0)\frac{\tau_k(m_{kk}-1)}{1-\tau+\tau m_{kk}}\right) = (1-\tau_k)(1-\alpha-\beta)s_k(0)\frac{1-\tau_k}{1-\tau_k+\tau_k m_{kk}}$, and again

$$\frac{d\pi_k}{d\tau_k}(\tau_k) = -(1-\alpha-\beta)s_k(0)\frac{m_{kk}}{(1-\tau_k(1-m_{kk}))^2}$$

It is obvious that:

$$\frac{d\pi_i}{d\tau_k}(\tau_k) = \pi_k m_{ki} \ \ (i = 1, 2, ..., n)$$

□

Proof of proposition 6. For a fixed n, let us find the network topology that maximizes expression (6). In order to do that is useful to define $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ with $\Phi(\boldsymbol{v}, \boldsymbol{n}) = \sum_{i=1}^n v_i \log n_i$. Let us also define function $\Psi : \mathbb{R} \to \mathbb{R}$ as:

$$\begin{aligned} \Psi(\bar{y}) = & \max_{(x_i)_{i=1}^n, (y_i)_{i=1}^n} \sum_{i=1}^n x_i logy_i \\ \text{s.t.} \ & \sum_{i=1}^n y_i = \bar{y} \\ & \sum_{i=1}^n x_i = 1 \\ & y_i \leq n-1 \wedge y_i \geq 1 \wedge x_i > 0, \ i = 1, ..., n \end{aligned}$$

One can easily show that function Ψ is strictly increasing in $\bar{y}$. For any graph with $\hat{n}$ nodes and $\hat{L}$ links ($\sum_i \hat{n}_i = \hat{L}$) we have $\Phi(\hat{v}_i, \hat{n}_i) \leq \Psi(\hat{L}) < \Psi(n(n-1))$. Note that for complete network $\forall(i, j \in N)(n_i = n_j) \Rightarrow \Phi(v_i^v, n_i^c) = \Psi(n(n-1))$ (since Ψ is strictly increasing). Thus, the complete network is the unique maximizer of Φ.

The value of expression $-\sum_i^n v_i \log v_i$ which is in fact the entropy measure of simplex vector $\boldsymbol{v}$ will be maximized when $v_i = v_j \ \forall(i, j \in N)$, which will incidentally be true in the case of the complete network and the ring network. It fallows now that utility will be maximized at the complete network when $\nu > 0$ and that complete network will be the unique network that maximizes consumers utility. When $\nu < 0$ the ring will maximize social welfare. When $\nu = 0$, then any network such that centrality of each node is equal will maximize social welfare. This will be the case for the subclass of the class of regular strongly connected networks. □

Proof of Proposition 7. Using Sherman-Morrison formula we write:

$$\begin{aligned}\tilde{\boldsymbol{s}} &= \frac{1-\alpha}{\beta} w (I - \alpha \tilde{G})^{-1} \boldsymbol{\gamma} = \frac{1-\alpha}{\beta} w (I - \alpha G - \alpha \boldsymbol{q} \boldsymbol{e}_i')^{-1} \boldsymbol{\gamma} \\ &= \frac{1-\alpha}{\beta} w \left((I - \alpha G)^{-1} \boldsymbol{\gamma} + \frac{(I-\alpha G)^{-1} \alpha \boldsymbol{q} \boldsymbol{e}_i' (I - \alpha G)^{-1}}{1 - \alpha \boldsymbol{e}_i' (I - \alpha G)^{-1} \boldsymbol{q}} \boldsymbol{\gamma} \right) \end{aligned} \tag{43}$$

Let us focus on the expression: Proceeding analogue to the analysis in Proposition 4, we get that the effect of adding a link (i, j) on the centrality of firm k is:

$$\tilde{s}_k - s_k = \alpha s_j \frac{q_i m_{ki} + \sum_{l \in N_j^+} q_l m_{kl}}{1 + q_i m_{ji} + \alpha \sum_{l \in N_j^+} q_l m_{jl}}$$

where $\tilde{s_k}$ is the centrality of k after adding link (i, j). Since $\boldsymbol{\pi} = (1-\alpha-\beta)\boldsymbol{s}$ this completes the proof. □

Proof of Proposition 8. We can write the profit

$$\begin{aligned}\pi_j(\tilde{G}) &= (1-\alpha-\beta)\frac{1-\alpha}{\beta} w \sum_{\substack{k=1 \\ k \neq i}}^{n} m_{jk}(\tilde{G}) \gamma_k \\ &= \frac{1-\alpha}{\beta} w \left(\sum_{\substack{k=1 \\ k \neq i}}^{n} m_{jk}(\tilde{G}) \gamma_k + \left(\frac{m_{ii}(G) m_{ji}(G) - m_{ji}(G) m_{ji}(G)}{m_{ii}(G)} \right) \gamma_i \right) \\ &= \frac{1-\alpha}{m_{ii}(G)\beta} w \sum_{k=1}^{n} \left(m_{ii}(G) m_{jk}(G) - m_{ji}(G) m_{ik}(G) \right) \gamma_k \end{aligned}$$

where the second line follows directly from Lemma 3. Furthermore:

$$\begin{aligned}&\pi_j(\tilde{G}) - \pi_j(G) = \\ &(1-\alpha-\beta)\frac{1-\alpha}{\beta} w \left(\frac{1}{m_{ii}(G)} \sum_{k=1}^{n} (m_{ii}(G) m_{jk}(G) - m_{ji}(G) m_{ik}(G)) \gamma_k - \sum_{k=1}^{n} m_{jk}(G) \gamma_k \right) = \\ &-(1-\alpha-\beta)\frac{1-\alpha}{m_{ii}(G)\beta} w \left(\sum_{k=1}^{n} m_{ji}(G) m_{ik}(G) \gamma_k \right) = \\ &-\frac{m_{ji}(G)}{m_{ii}(G)} (1-\alpha-\beta) \frac{1-\alpha}{\beta} w \sum_{k=1}^{n} m_{ik} \gamma_k = \\ &-\pi_i(G) \frac{m_{ji}(G)}{m_{ii}(G)}\end{aligned}$$

□

Proof of Proposition 9. From (35) by normalizing: $\alpha \log \alpha + \beta \log \beta - \beta \log w = 0$ and by writing $b_i := -(1-\alpha-\beta)\log s_i + \alpha\sum_{j=1}^n g_{ji}\log g_{ji}$ we get:

$$\log \boldsymbol{p} = (I-\alpha G')^{-1}(\log \boldsymbol{A} + \boldsymbol{b})$$

Consider the technology shock that changes A_i to $\tilde{A}_i := \rho_i A_i$ (we shall use $\tilde{}$ to denote the variables after the technology shock). It is clear that this shock will not affect the relative demand of goods in the equilibrium which implies that (32) will not be affected by the technology shock. So, $s_i = \tilde{s}_i \Rightarrow b_i = \tilde{b}_i$. From here it directly follows:

$$\log \tilde{\boldsymbol{p}} - \log \boldsymbol{p} = (I-\alpha G')^{-1}\left(\log \tilde{\boldsymbol{A}} - \log \boldsymbol{A}\right) = (I-\alpha G')^{-1}\log \boldsymbol{\rho}$$

Where we have used the notation $\log \boldsymbol{x} := (\log x_i)_{i=1}^n$ □

Proof of proposition 11. Proceeding analogously to Proposition 2 one can see that the profit of a firm in the equilibrium will be given $\pi_i = (1-\alpha-\beta)s_i - wf$. Using this and the market clearing condition for labor 23 we get that the total income of the consumer is: $Y = w + \sum_i \pi_i = w + (1-\alpha-\beta)\sum_{i=1}^n s_i - nwf = (1-\alpha)\frac{1-nf}{\beta}w$. Then, the revenue vector in the equilibrium is defined with:

$$\boldsymbol{s} = \frac{(1-\alpha)(1-nf)}{\beta}w\boldsymbol{\gamma} + \alpha G\boldsymbol{s} \Rightarrow \boldsymbol{s} = \frac{(1-\alpha)(1-nf)}{\beta}w(I-\alpha G)^{-1}\boldsymbol{\gamma}$$

and the corresponding vector of equilibrium profits is $\boldsymbol{\pi} = (1-\alpha-\beta)\boldsymbol{s} - wf\boldsymbol{e}$ □

Proof to the Corolary 5. Directly follows from the Proposition 11 and (33). □

Proof of proposition 12. Analogue way to Proposition 8 □

Appendix B: General Framework: A Model with full inter-firm heterogeneity

Here we present the model and the main result relating centrality to profit allowing for the general heterogeneity. We still require that production function is Cobb-Douglas, but we allow general heterogeneity with respect to parameters of CD production function (α, β) and TFP A. Recall that before we have allowed for heterogeneity in link weights $(g_{ij})_{i=1,j=1}^n$ and in preference weights $(\gamma_i)_{i=1}^n$. We write the production function as: $y_i = A_i l_i^{\beta_i}\left(\prod_{j\in N_i} z_{ji}^{g_{ji}}\right)^{\alpha_i}$ with constraint that $\alpha_i + \beta_i \leq 1$. The utility function also takes a general form: $U(\boldsymbol{c}) = \prod_{i=1}^n c_i^{\gamma_i}$, with $\gamma_i \geq 0\ \forall i \in N$ and $\sum_{i=1}^n \gamma_i = 1$. Optimizing, we get that the firm j's demand for intermediate inputs and labor are:

$$z_{ij} = \frac{p_j \alpha_j g_{ij}}{p_i} y_j$$
$$l_j = \frac{p_j \beta_j}{w} y_j$$

The consumer's demand for consumption good i is:

$$c_i = \gamma_i \frac{\sum_{j=1}^n \pi_j + w}{p_i} = \gamma_i \frac{w + \sum_{j=1}^n (1 - \alpha_j - \beta_j) p_j y_j}{p_i}$$

The market clearing condition for good i can be written as:

$$s_i = \gamma_i w + \gamma_i \sum_{j=1}^n (1 - \alpha_j - \beta_j) s_j + \sum_{j=1}^n \alpha_j g_{ij} s_j \tag{44}$$

Before proceeding any further, let us introduce the following notation:

$$\boldsymbol{\rho} := \begin{pmatrix} 1 - \alpha_1 - \beta_1 \\ 1 - \alpha_2 - \beta_2 \\ \vdots \\ 1 - \alpha_n - \beta_n \end{pmatrix} \text{ and } \aleph := \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_n \end{pmatrix}$$

We can now write (44) for every firm i in the matrix notation as:

$$\boldsymbol{s} = w\boldsymbol{\gamma} + \boldsymbol{\gamma}\boldsymbol{\rho}'\boldsymbol{s} + G\aleph\boldsymbol{s}$$

and thus:

$$\boldsymbol{s} = w(I - \boldsymbol{\gamma}\boldsymbol{\rho}' - G\aleph)^{-1}\boldsymbol{\gamma} \tag{45}$$

The equation (45) relates the revenue of the firm i with it's centrality in the network which adjacency matrix is: $\boldsymbol{\gamma}\boldsymbol{\rho}' - G\aleph$. Let us show that inverse matrix in (45) exists. Note that the sum of column i of matrix $\boldsymbol{\gamma}\boldsymbol{\rho}'$ is $(1 - \alpha_i - \beta_i) \sum_{j=1}^n \gamma_j = (1 - \alpha_i - \beta_i) < 1$. The sum of column i of matrix $G\aleph$ is $\sum_k = g_{ki}\alpha_i = \alpha_i < 1$. Then the sum of column i of matrix $\boldsymbol{\gamma}\boldsymbol{\rho}' + G\aleph$ is $0 < 1 - \beta_i < 1$. Furthermore, it is clear that all elements of matrix $\boldsymbol{\gamma}\boldsymbol{\rho}' + G\aleph$ are positive. Thus, matrix $\boldsymbol{\gamma}\boldsymbol{\rho}' + G\aleph$ can be seen as a sub-matrix of a column stochastic matrix. From Perron-Frobenius theorem we know that the spectral radius of matrix $\boldsymbol{\gamma}\boldsymbol{\rho}' + G\aleph$ is smaller than 1. This implies that $I - \boldsymbol{\gamma}\boldsymbol{\rho}' - G\aleph$ is invertible (as $(I - \boldsymbol{\gamma}\boldsymbol{\rho}' - G\aleph)^{-1} = \sum_{i=0}^{\infty} (\boldsymbol{\gamma}\boldsymbol{\rho}' + G\aleph)^i$ converges).

Appendix C: Distortions under a Budget Constraint

We provide the basic results from Section 4 for the case when the payements of firms are redistributed to final consumers.

Uniform price distortion

In this case the income of the consumer changes, as now the consumer receives the collected *tax* from firms. The consumer's income reads:

$$Y = (1-\tau)(1-\alpha-\beta)\sum_{i\in N} p_i y_i + (1-\tau)\sum_{i\in N}\beta p_i y_i + \tau\sum_{i\in N} p_i y_i = \frac{w(1-(1-\tau)\alpha)}{\beta(1-\tau)} \quad (46)$$

From (46) it directly follows that the consumption demand in the equilibrium must satisfy:

$$c_i = \frac{Y}{p_i}\gamma_i = \frac{w(1-\alpha(1-\tau))}{p_i\beta(1-\tau)}\gamma_i$$

From here it is easy to see that equation (32) in this case becomes:

$$\boldsymbol{s}(\tau) = \frac{w(1-\alpha(1-\tau))}{\beta(1-\tau)}\boldsymbol{\gamma} + \alpha(1-\tau)G\boldsymbol{s} \Rightarrow$$
$$\boldsymbol{s}(\tau) = \frac{w(1-\alpha(1-\tau))}{\beta(1-\tau)}\left(I-\alpha(1-\tau)G\right)^{-1}\boldsymbol{\gamma}$$

It is not difficult to see that with this formulation the comparative static results from Subsection 4.1 will still hold.

Individual price distortion

Suppose firm k is affected by distortion. We can write income of the consumer in this case as:

$$\begin{aligned} Y &= \sum_{i\in N}(1-\alpha-\beta)\,p_i y_i - \tau\,(1-\alpha-\beta)\,p_k y_k + \beta\sum_{i\in N} p_i y_i - \tau\beta p_k y_k + \tau p_k y_k \\ &= \frac{1-\alpha}{\beta}w + \tau p_k y_k \end{aligned}$$

And the centrality equation becomes:

$$\boldsymbol{s}(\tau) = \left(\frac{1-\alpha}{\beta}w + \tau s_k(\tau)\right)\boldsymbol{\gamma} + \alpha(G+Q(\tau))\boldsymbol{s}(\tau) = \frac{1-\alpha}{\beta}w\boldsymbol{\gamma} + \alpha(G+Q(\tau)+U(\tau))\boldsymbol{s}(\tau)$$

where:

$$Q(\tau) := \begin{pmatrix} 0 & \dots & -g_{1k}\tau & \dots & 0 \\ 0 & \dots & -g_{2k}\tau & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & -g_{nk}\tau & \dots & 0 \end{pmatrix} \quad V(\tau) := \begin{pmatrix} 0 & \dots & \gamma_1\frac{\tau}{\alpha} & \dots & 0 \\ 0 & \dots & \gamma_2\frac{\tau}{\alpha} & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & \gamma_n\frac{\tau}{\alpha} & \dots & 0 \end{pmatrix}$$

Let us define $W(\tau) := V(\tau) + Q(\tau)$. The matrix $W(\tau)$ is nxn matrix with non-zero elements only in $k-th$ column, and with (i,k) element equal to $\frac{\tau}{\alpha}(\gamma_i - \alpha g_{ik})$. Let us denote with $\boldsymbol{x} = \boldsymbol{x}(\tau)$ a column vector with elements equal to $\frac{\tau}{\alpha}(\gamma_i - \alpha g_{ik})$. So we can write $W(\tau) = \boldsymbol{x}\boldsymbol{e}_k'$ and $\boldsymbol{s}(\tau) = \frac{(1-\alpha)w}{\beta}(I-(\alpha G+\alpha \boldsymbol{x}\boldsymbol{e}_k'))^{-1}\boldsymbol{\gamma}$. Using the Sherman-Morrison formula we get:

$$\boldsymbol{s}(\tau) = \frac{(1-\alpha)w}{\beta}\left((I-\alpha G)^{-1}\boldsymbol{\gamma} + \frac{(I-\alpha G)^{-1}\alpha \boldsymbol{x}\boldsymbol{e}_k'(I-\alpha G)^{-1}}{1-\alpha \boldsymbol{e}_k'(I-\alpha G)^{-1}\boldsymbol{x}}\boldsymbol{\gamma}\right) \tag{47}$$

Recall now that $\frac{(1-\alpha)w}{\beta}(I-\alpha G)^{-1}\boldsymbol{\gamma} = \boldsymbol{s}(0)$. Furthermore:

$$\begin{aligned}
&\frac{(1-\alpha)w}{\beta}(I-\alpha G)^{-1}\alpha \boldsymbol{x}\boldsymbol{e}_k'(I-\alpha G)^{-1}\boldsymbol{\gamma} = \\
&\alpha s_k(0)(I-\alpha G)^{-1}\boldsymbol{x} = \\
&\alpha s_k(0)\left((I-\alpha G)^{-1}\frac{\tau}{\alpha}\boldsymbol{\gamma} - \tau(I-\alpha G)^{-1}\boldsymbol{g}_k\right) = \\
&s_k(0)\left(\tau\frac{\beta}{w(1-\alpha)}\boldsymbol{s}(0) - \tau(I-\alpha G)^{-1} - I)\boldsymbol{e}_k\right)
\end{aligned}$$

On the other hand, the denominator of (47) becomes:

$$\begin{aligned}
&1-\alpha \boldsymbol{e}_k'(I-\alpha G)^{-1}\boldsymbol{x} = \\
&1-\left(\tau \boldsymbol{e}_k'(I-\alpha G)^{-1}\boldsymbol{\gamma} - \tau \boldsymbol{e}_k'((I-\alpha G)^{-1} - I)\boldsymbol{e}_k\right) = \\
&1-\tau\left(\frac{\beta}{(1-\alpha)w}s_k(0) - m_{kk} + 1\right)
\end{aligned}$$

So we can finally write:

$$\boldsymbol{s}(\tau) = \boldsymbol{s}(0) + \frac{s_k(0)}{1-\tau\left(\frac{\beta}{(1-\alpha)w}s_k(0) - m_{kk}+1\right)}\left(\tau\frac{\beta}{w(1-\alpha)}\boldsymbol{s}(0) - \tau(I-\alpha G)^{-1} - I)\boldsymbol{e}_k\right)$$

which for a firm i takes a form:

$$s_i(\tau) = s_i(0) + \frac{s_k(0)}{1-\tau\left(\frac{\beta}{(1-\alpha)w}s_k(0) - m_{kk}+1\right)}\left(\tau\frac{\beta}{w(1-\alpha)}s_i(0) - \tau m_{ki}\right) \tag{48}$$

The positive effect of the transfer to the consumer in (48) relative to what we had in Proposition 5 is captured by the expression in the numerator $s_k(0)\left(\tau\frac{\beta}{w(1-\alpha)}s_i(0)\right)$ which reflects the increase in centrality of firm i as demand for good i increases due to the income effect (the consumer receives additional income as the 'tax' is transferred to him). As in (42) this effect is discounted with properly adjusted expression that captures how much distorted firm contributes to it's own centrality. To conclude, the transfer will mild down detrimental consequences of the (negative) distortion, but including it into the model will not bring much more insights in the questions we are studying in the paper.

Appendix D: Useful Mathematical Results

Lemma 2. *Let G be a n-dimensional matrix, and $\alpha \in \mathbb{R}$ such that there exist $(I-\alpha G)^{-1}$. Then:* $(I-\alpha G)^{-1}G = \frac{1}{\alpha}\left((I-\alpha G)^{-1} - I\right)$

Proof of Lemma 2. $(I-\alpha G)^{-1}G = \left(\sum_{k=0}^{\infty}\alpha^k G^k\right)G = \sum_{k=0}^{\infty}\alpha^k G^{k+1} = \frac{1}{\alpha}\sum_{k=0}^{\infty}\alpha^{k+1}G^{k+1} = \frac{1}{\alpha}\sum_{k=1}^{\infty}\alpha^k G^k = \frac{1}{\alpha}\left(\sum_{k=0}^{\infty}\alpha^k G^k - \alpha^0 G^0\right) = \frac{1}{\alpha}\left((I-\alpha G)^{-1} - I\right)$ □

Lemma 3 (BZC)**.** $m_{ji}(G)m_{ik}(G) = m_{ii}(G)(m_{jk}(G) - m_{jk}(G_{-i}))$

Proof.

$$m_{ii}(G)(m_{jk}(G) - m_{jk}(G_{-i})) = \sum_{p=1}^{\infty}\alpha^p \sum_{\substack{r+s=p \\ r\geq 0, s\geq 1}} g_{ji}^{[r]}(g_{jk}^{[s]} - g_{j(-i)k}^{[s]}) = \sum_{p=1}^{\infty}\alpha^p \sum_{\substack{r+s=p \\ r\geq 0, s\geq 2}} g_{ii}^{[r]} g_{j(i)k}^{[s]}$$

$$= \sum_{p=1}^{\infty}\alpha^p \sum_{\substack{r'+s'=p \\ r'\geq 1, s'\geq 1}} g_{ji}^{[r']} g_{ik}^{[s']} = m_{ji}(G)m_{ik}(G)$$

where G_{-i} is the resulting network after elimination of node i from network G. □

Proposition 14. *Elimination of the node i with the highest value of $\frac{s_i m_i^I}{m_{ii}(G)}$ maximizes $\Delta \boldsymbol{s}(G_{-i})\mathbf{1}' = \sum_{k=1}^{n}(s_k(G) - s_k(G_{-i}))$.*

Proof. Suppose that i is eliminated. The following holds:

$$\Delta \boldsymbol{s}(G_{-i})\mathbf{1}' = -\frac{f}{1-nf}\sum_{k=1}^{n} s_k(G) + s_i(G)\sum_{k=1}^{n}\frac{m_{ki}(G)}{m_{ii}(G)} = -\frac{f}{1-nf}\sum_{k=1}^{n} s_k(G) + \frac{s_i(G)}{m_{ii}(G)}m_i(G)$$

using market clearing for labor, the above expression simplifies to:

$$\Delta \boldsymbol{s}(G_{-i})\mathbf{1}' = -\frac{wf}{\beta} + \frac{s_i(G)m_i(G)}{m_{ii}(G)}$$

Normalizing $w = 1$ we complete the proof. □

Matrix G in general has elements such that $\sum_j g_{ji} = 1\ \forall i$ so it is column stochastic matrix. For a column stochastic matrix P we have $P\boldsymbol{x} = \boldsymbol{x}$ for stationary distribution $\boldsymbol{x}$. Recall that $\sum_i p_i y_i = \mathbf{1}'\boldsymbol{s} = \frac{w}{\beta}$. Normalizing $w = \beta$ we can write:

$$\boldsymbol{s} = \frac{1-\alpha}{m}\boldsymbol{h}\mathbf{1}'\boldsymbol{s} + \alpha G\boldsymbol{s} = \left(\frac{1-\alpha}{m}\boldsymbol{h}\mathbf{1}' + \alpha G\right)\boldsymbol{s} \tag{49}$$

The Matrix $\frac{1-\alpha}{m}\boldsymbol{h}\mathbf{1}'$ in the benchmark case will be the matrix having sum of columns (and rows) equal to $1-\alpha$. The sum of elements of each column in matrix αG is equal to α. Thus, matrix $\frac{1-\alpha}{m}\boldsymbol{h}\mathbf{1}' + \alpha G$ is a column stochastic matrix. The vector of profits $\boldsymbol{\pi} = (1-\alpha-\beta)\boldsymbol{s}$ will be a scaled vector of the stationary distribution of (column) stochastic, irreducible (because we look only at the active firms) Markov chain with the transition matrix $\frac{1-\alpha}{m}\boldsymbol{h}\mathbf{1}' + \alpha G$. In general case, for an arbitrary $\boldsymbol{h}$ and weights attached to each consumption good this matrix will still be stochastic.

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