A RE-INTERPRETATION OF THE LINEAR-QUADRATIC MODEL WHEN INVENTORIES AND SALES ARE POLYNOMIALLY COINTEGRATED

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SUMMARY

Estimation of the linear quadratic model, the workhorse of the inventory literature, traditionally takes inventories and sales to be first-difference stationary series, and the ratio of the two variables to be stationary. However, these assumptions do not always match the properties of the data for the last two decades in the United States. We propose a model that allows for the non-stationary characteristics of the data, using polynomial cointegration. We show that the closed-form solution has other recent models as special cases. The resulting model performs well on aggregate and disaggregated data. Copyright © 2006 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The linear-quadratic (L-Q) model, developed by Holt et al. (1960) and Lovell (1961), has been the workhorse of the literature on inventories and stock adjustment (see Blinder and Maccini, 1991; West, 1995; Ramey and West, 1999). The ‘production smoothing’ class of inventory models that arises has been estimated using time-series methods by, inter alia, Kashyap and Wilcox (1993), West (1995) and Ramey and West (1999), with the common feature that inventories and sales should each have a single unit root (follow an I(1) process), and a linear combination (i.e., a cointegrating relationship), for which the ratio of inventories to sales is a special case, should be a stationary series.

A recent paper by Hamilton (2002) notes that the traditional formulation has some unappealing characteristics, since marginal production and inventory management costs tend to infinity while profits tend to minus infinity under standard assumptions about the driving variables. Hamilton’s solution is to consider productivity shocks as an I(1) process and to reconfigure the system in order to allow cointegration between sales and productivity shocks, which preserves a dynamic structure identical to that of the original model.

Both the traditional model and Hamilton’s reconfiguration require that a linear combination of inventories-to-sales be stationary. However, examination of the data in Figure 1 illustrates that during the last two decades there has been a pronounced downward trend in the ratio of inventories.
Figure 1. Left-hand panels illustrate the downward sloping ratio of aggregate stocks to sales, and the more volatile disaggregated stock-to-sales ratios for manufacturing raw materials, work-in-progress and final goods inventories relative to manufacturing sales. Right-hand panels illustrate the underlying time series for the disaggregated relationships, reporting manufacturing sales in the top panel, raw materials in the second panel, work-in-progress in the third panel and finally the final goods inventories. In three of the four panels data are separated further into durable and non-durable categories. Exact definitions are given in the text for the aggregate ratio (aggratio) and disaggregated ratios (RMratio, WIPratio, FGmratio). Categories include manufacturers' raw materials and supplies (RMmanuf), work-in-progress (WIPmanuf) and finished goods (FGmanuf). These are further disaggregated into durable (.duram) and non-durable (.nonduram) real inventories at each stage of fabrication. Data source: BEA
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1 Two potential explanations for this phenomenon are (a) improvements to stock management processes such as computerized stock control and just-in-time delivery, and (b) the relative decline in the manufacturing sector as a share of GDP (a major contributor to total stockholding) may have reduced inventories; cf. Cuthbertson and Gasparo (1993), Sensier, (1997) and Mizen (2003).

2 It is well known that allowance for this form of ‘non-stationarity’ cannot be made by the straightforward incorporation of deterministic time polynomials (see, for example, Phillips 1986, 1987a, 1987b). Nor is there evidence of a cointegrating relationship (which reduces the order of integration of the linear combination to zero) for other linear combinations of inventories and sales, over the sample period from the early 1980s to the present. Rather the combination of inventories and sales appears to be an I(1) series for any linear combination.

3 Each of the series is individually I(1), while the cointegrating combination reduces order of integration by 1.

4 Investigations of cumulated series by Engle and Yoo (1991), Granger and Lee (1989) and the recent interest in multicointegration (Engsted and Haldrup, 1999a; Engsted and Johansen, 1999) show that I(2) properties are common in stock-flow models such as inventory–sales relationships.

to sales in the United States. This gives rise to the possibility that the linear combination consisting of the ratio is an integrated series. The downward trend in the inventories-to-sales ratio undermines the case for a simple reduction in the order of integration through a CI(1,1) error correction system. Inspection of the properties of the data suggest that the series for sales and inventories are in fact I(2) series, and the linear combination is I(1). This implies that a more generalized approach to the dynamic model is required in which we allow for cointegrating systems of the CI(2,1) variety. In this respect, this paper draws on earlier work by Dolado et al. (1991), where integrated variables are introduced to L-Q models in general form, with similarities to labour and money demand equations applications in Engsted and Haldrup (1994, 1997, 1999b). As far as we are aware, no application has yet been made to inventories, where the issue is pertinent to the current debate.

In this paper we consider a simple generalization of the cointegrating system offered by Hamilton (2002)—which is itself a representation of the model of Ramey and West (1999)—that can allow for more sophisticated dynamics involving polynomial cointegration (Dolado et al., 1991; Engsted and Haldrup, 1999a; Engsted and Johansen, 1999). In this case we do not require the variables to be I(1), nor the ratio to be stationary. In fact, the variables could have I(2) properties, and the error correction system may exhibit cointegration between the levels and the differences of the series.

We provide a closed-form solution for the dynamic vector error correction model under polynomial cointegration, with Hamilton (2002) as a special case. We illustrate our model using aggregate and disaggregated data for the United States during the decline in the inventories-to-sales ratio in recent years.

The next section generalizes the L-Q model to allow for polynomial cointegration, and Section 3 explains how it is implemented. Section 4 reports two illustrative examples using aggregate and disaggregated data sources drawn from the Bureau of Economic Analysis (BEA) in the United States. Section 5 concludes the paper.

2. A GENERALIZATION OF THE COINTEGRATION INTERPRETATION

Following Ramey and West (1999) and Hamilton (2002) the decision problem for the representative firm is

\[
\max_{\{Q_t, H_t\}_{t=0}^{\infty}} E_0 \left\{ \sum_{t=0}^{\infty} \rho^t (P_t S_t - C_t) \right\}
\]

subject to

\[
C_t = \left( \frac{1}{2} \right) [a_0 (\Delta Q_t)^2 + a_1 (Q_t - U_{ct})^2 + a_2 (H_{t-1} - a_4 - a_3 S_t)^2]
\]

\[
Q_t = S_t + \Delta H_t
\]

\[
Q_t, S_t, H_t, \Delta Q_t, \Delta H_t \in \mathbb{R}
\]

\[
\rho \in (0, 1)
\]

\[
\rho^t = \rho^t \rho^{t-1} \cdots \rho^1
\]

\[
E_0 \left\{ \sum_{t=0}^{\infty} \rho^t (P_t S_t - C_t) \right\}
\]
where $P_t$ is the price of the good, $S_t$ is unit sales, $C_t$ is the cost of production and $Q_t$ is the level of production. $H_t$ is the level of inventories, and the change in inventories is defined as the difference between production and sales. $U_{ct}$ is a shock to marginal production, and $\rho$ is the discount rate. All variables are measured at the end of the period. If sales follow a random walk with drift, $S_t = S_{t-1} + a_5 + v_{st}$, and productivity shocks are stationary (i.e., $U_{ct} = v_{ct}$), the first-order condition in this case is

$$E_t[\Delta H_t + S_t - v_{ct} - \rho(\Delta H_{t+1} + S_t + a_5) + \rho a_2 (H_t - a_4 - a_3 S_t - a_3 a_5)] = 0.$$  

The resulting error correction system is given by

$$\Delta H_t = (\lambda_1 - 1)(H_{t-1} + \gamma_0 + \gamma_1 S_{t-1}) + \frac{\gamma_1 a_5 (\lambda_1 - 1)}{1 - \lambda_1 \rho} + \lambda_1 v_{ct} + (\lambda_1 - 1)\gamma_1 v_{st}$$

(2)

where $\lambda_1$ is the real root of the difference equation and

$$\gamma_0 = -(a_5/a_2) - a_4 - a_3 a_5, \quad \gamma_1 = \frac{1 - \rho}{\rho a_2} - a_3.$$  

In the derivation above $a_0 = 0$ for simplicity.\(^5\) Suppose now that $a_0 \neq 0$, and we specify the driving process for $S_t$ as a driftless I(2) process, $\Delta S_t = \Delta S_{t-1} + v_{st}$, which can also be written as

$$(1 - \theta_1 L)(1 - L)S_t = a_5 + v_{st}, \quad \text{where } a_5 = 0 \text{ and } \theta_1 = 1.$$ \(^6\)

All other aspects of the model are the same. The Euler equation now becomes

$$E_t \left[ a_0 (\Delta S_t + \Delta^2 H_t) - 2\rho (\Delta S_{t+1} + \Delta^2 H_{t+1}) + \rho^2 (\Delta S_{t+2} + \Delta^2 H_{t+2}) \right] = 0$$

(3)

A major difference between (3) and the Euler equation of Hamilton is the presence of $\Delta S_t$ terms in (3).

Defining

$$\gamma_0 = -(a_5/a_2) - a_4 - a_3 a_5, \quad \gamma_1 = \frac{1 - \rho}{\rho a_2} - a_3, \quad \gamma_2 = \frac{a_0 - \gamma_1 a_1 - 2\rho + \rho^2}{\rho a_1 a_2} + \frac{\gamma_1 - 1 - a_2 a_3}{a_2}$$

and the polynomial cointegrating relationship as

$$w_t = H_t + \gamma_0 + \gamma_1 S_t + \gamma_2 \Delta S_t$$

\(^5\)This has important drawbacks. If adjustment of production were costless (as it would be if $a_0 = 0$) then the reasons for a firm to hold inventories are considerably undermined since an unexpected shock to demand could be met by costlessly adjusting production. The restriction would also imply, if inventories were reduced to zero, that the firm would set marginal revenue (prices) equal to marginal costs. Practically, Ramey and West (1999) note that, although a model in which $a_0 = 0$ is useful for illustrative purposes it is ‘not a good [assumption] empirically’ (p. 889). Recent data suggest that manufacturing firms hold between 20 and 30 weeks of output in the form of inventories.

\(^6\)This assumption is an integral part of the polynomial cointegration approach that the generalization of the cost function allows, but recovery of Hamilton’s original specification can be achieved if we set $a_0 = 0$, $a_5 \neq 0$ and $\theta_1 = 0$; therefore Hamilton’s version is a special case of our model.
we can then make use of this equation to define

$$\Delta^j H_{t+i} = \Delta^j w_{t+i} - \gamma_1 \Delta^j S_{t+i} - \gamma_2 \Delta^{j+1} S_{t+i}$$ for $i = 0, 1, 2, \ldots, j = 1, 2, \ldots$

Substituting out terms in $\Delta^j H_{t+i}$, we can then make use of this equation to define

$$E_t [a_0 (\rho^2 a_2^2 \Delta^2 w_{t+1} + \rho^2 \Delta^2 w_{t+2}) - \rho a_1 \Delta w_{t+1} + \rho a_2 \Delta w_t + a_1 \Delta w_t] = \psi_t$$

(5)

where $\psi_t = a_1 v_{c,t} + A v_{s,t} + B \Delta v_{s,t} + C \Delta^2 v_{s,t}, A = -\gamma_1 (\rho^2 - 2\rho + \alpha_0) - \gamma_2 (\rho^2 - \rho^2 + 1 + \alpha_1), B = A + \alpha_0 \gamma_1 + (2\rho + a_0 + a_1) \gamma_2$, and $C = \rho \gamma_2 (2 - \rho)$. This can be solved in the same way as Hamilton’s problem using the methods for higher-order difference equations in Sargent (1987, p. 199ff.) since the equation is identical to Hamilton’s except that it includes the second-differenced terms:

$$E_t [a_0 (\rho - L)^2 (1 - L)^2 - \alpha_1 (\rho - L)(1 - L)L + \rho a_1 a_2 L^2]w_{t+2}] = \psi_t$$

(6)

The solution to this fourth-order equation will be subject to two boundary conditions provided by the initial value for $w_t$ and the transversality condition. By satisfying the transversality condition we ensure that the path for $w_t$ is not explosive. Since there are four roots to equation (6), we calculate the solution by first factorizing the above expression to give

$$E_t \left[ (z^2 - a_1 a_2) \frac{a_1}{a_0} z - \frac{\rho a_1 a_2}{a_0} \right] = \psi_t$$

(7)

where $z = (1 - \rho^{-1} L)(1 - L)$.

If we take

$$\left( z^2 - \frac{a_1}{a_0} z - \frac{\rho a_1 a_2}{a_0} \right) = \frac{1}{\lambda_1 \lambda_2} (1 - \lambda_1 z)(1 - \lambda_2 z)$$

where $0 < \lambda_1 < 1$ and $\lambda_2 > 1/\rho$, for $[(\lambda_1 + \lambda_2)/(\lambda_1 \lambda_2)] = -a_1/a_0$ and $[1/\lambda_1 \lambda_2] = -\rho a_1 a_2/a_0$, the solution to equation (7) is

$$w_t = \lambda_1 \Delta w_t - \rho^{-1} \lambda_1 (1 - L)w_{t-1} + \eta_t$$

(8)

where $\eta_t = -\lambda_1 \sum_{i=0}^{\infty} \left( \frac{1}{\lambda_2} \right)^i \psi_{t+i}$.

We then solve this equation for the remaining two roots, $\mu_1$ and $\mu_2$, by rewriting (8) as

$$\left(1 - \lambda_1 (1 - L)(1 - \rho^{-1} L) \right)w_t = \eta_t$$

(9)

and solving for the roots of the polynomial

$$\left[ \frac{1}{\mu_1 \mu_2} (1 - \mu_1 z)(1 - \mu_2 z) \right]$$

(10)

7 The transversality condition for the optimization problem (specified in terms of $w_t$) is given by

$$\lim_{T \to \infty} \rho^T [a_0 a_2^2 w_{T+T} + D w_{T+T-1} + F w_{T+T-2} + G w_{T+T-3} + a_0 w_{T+T-4}] w_{T+T} = 0$$

where $D = (a_0 - \rho a_1) - 2a_0 \rho (1 + \rho)$, $F = a_0 \rho (\rho + 4) + \rho a_1 (1 + a_2) + a_1$ and $G = -2a_0 (1 + \rho) - a_1$. © 2006 John Wiley & Sons, Ltd. J. Appl. Econ. 21: 1249–1264 (2006) DOI: 10.1002/jae
which are given by
\[ -\lambda_1(1 + \rho^{-1}) \pm \sqrt{\lambda_1^2(1 + \rho^{-1})^2 + 4\rho^{-1}(1 - \lambda_1)\lambda_1} \over 2(1 - \lambda_1). \]

Depending on the value of \( \rho \) the process for \( w_t \), which is the polynomially cointegrating relation, could be explosive. We could consider mapping the range of values for \( \rho \) and other parameters from the cost function to determine where the process becomes explosive. Except for \( \rho \), which is a discount factor, we have no priors about the values that can be taken by the remaining parameters of the cost function apart from requiring them to be positive. We therefore need to rely on our empirical results to show that for our empirical application a stationary polynomially cointegrating relation exists for most of the categories of sales and inventories studied.

3. IMPLEMENTING A POLYNOMIAL COINTEGRATION APPROACH

In order to describe the formal general framework briefly, consider a \( k \)th-order vector autoregression of the core variables, \( x_t \), of dimension \( n \times 1 \):
\[
\Delta x_t = \Pi x_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta x_{t-i} + \Phi D_t + \mu + \varepsilon_t
\]
where \( \Pi = \alpha \beta' \), \( \mu \) is a constant term that may be unrestricted and \( D_t \) is a vector of trends and dummy variables. Equation (10) may be rewritten
\[
\Delta^2 x_t = \sum_{i=1}^{k-2} \Psi_i \Delta^2 x_{t-i} + \Pi x_{t-1} - \Gamma \Delta x_{t-1} + \Phi D_t + \mu + \varepsilon_t
\]
where \( \Psi_i = -\sum_{j=i+1}^{k-1} \Gamma_j \), \( i = 1, \ldots, k-2 \), \( \Pi = \alpha \beta' \) and \( \Gamma = I - \sum_{i=1}^{k-1} \Gamma_i \). The variable \( \varepsilon_t \) is an \( n \)-dimensional vector of errors assumed to be Gaussian with mean vector 0 and variance–covariance matrix \( \Sigma \). The parameters (\( \Psi_i, \Pi, \Gamma, \Phi, \mu, \Sigma \)) are assumed to be variation free.

For a system to be I(2) requires not only that the long-run matrix \( \Pi = \alpha \beta' \) is of reduced rank but that \( \alpha \perp \Gamma \beta \perp \) is also of reduced rank \( s \). \( \alpha \perp \Gamma \beta \perp \) is therefore expressible as \( \alpha \perp \Gamma \beta \perp = [\xi \eta]' \), where \( \xi \) and \( \eta \) are matrices of order \( (n - r) \times s \) with \( s < n - r \). The I(2) system can then be decomposed into I(0), I(1) and I(2) directions with dimensions \( r, s \) and \( n - r - s \) respectively. Moreover, the \( r \) cointegrating relationships are further decomposable into \( r_0 \) directly cointegrating relationships where the levels of the I(2) variables cointegrate directly to an I(0) variable and \( r_1 \) polynomially cointegrating relationships where the levels cointegrate with the differences of the levels to give an I(0) variable. Thus:
\[
\beta_0' x_t \sim I(0) \quad \text{where} \quad \beta_0 \text{ is } n \times r_0 \text{ with rank } r_0;
\]
\[
\beta_1' x_t + \kappa' \Delta x_t \sim I(0) \quad \text{where} \quad \beta_1 \text{ and } \kappa \text{ are } n \times r_1;
\]
\[
r_0 + r_1 = r.
\]

8 See Haldrup (1998) for a detailed account of this framework.
It is possible of course for either $r_0$, $r_1$ or both to be zero. In general, the number of polynomially cointegrating relationships may equal the number of I(2) common trends in the system such that $r = r_0 + r_1$ and $r_1 = n - r - s = s_2$. If $s_2$ equals zero, or equivalently $n - r = s$, the I(2) system collapses to the I(1) case.

4. ILLUSTRATIVE EXAMPLES FROM AGGREGATE AND DISAGGREGATED INVENTORIES AND SALES

4.1. Data Characteristics

To illustrate our point we use quarterly inventory and sales data at the aggregate and the disaggregated level for the United States. Aggregate real private inventories are recorded in billions of chained (2000) dollars, seasonally adjusted, end-of-period quarterly totals drawn from the BEA database, and are denoted $H_t$. Real private domestic final sales recorded in billions of chained (2000) dollars, seasonally adjusted end-of-period quarterly totals are obtained from the same source and are denoted $S_t$. The time span for all these series is 1982:q1 to 2000:q2.

The disaggregated data comprise real manufacturing inventories by stage of fabrication and real manufacturing sales, seasonally adjusted, recorded at the end of each period in millions of chained (2000) dollars, denoted $S^M_t$. By isolating the real inventories into components at different stages of fabrication we can examine separately the behaviour of raw materials (RM), work-in-progress (WIP), and finished goods (FG) for manufactured goods producers to investigate the impact of shocks on the different types of inventories held by manufacturing firms. These are denoted $H^{M(RM)}_t$, $H^{M(WIP)}_t$, $H^{M(FG)}_t$ and respectively. The time span for all these series is also 1982:q1 to 2000:q2.

Inspection of the time series over the sample 1983q1–2000q2 for these data in Figure 1 reveal that the inventory sales ratio shows that the aggregate series moves persistently downwards over the sample (upper left panel), while the ratios for the disaggregated components (lower left panel) also move downwards but with more noticeable variation over the cycle. Observation of the real sales and real inventory series (right panels) reveal that the downward trend in the ratio results from the faster growth of sales (top right panel) versus inventories (remaining right panels). The strongly cyclical response of the WIP inventories suggests that this type of inventory behaves differently to stocks of raw materials and finished goods, and the majority of the variation stems from the variation in the inventory series rather than from sales. The downward trends suggest some ratios are not stationary.

4.2. Estimating a Polynomially Cointegrating System with Aggregate Data

We begin our analysis by exploring the properties of an I(2) system, which is estimated without any restrictions except for the exclusion of quadratic trends, and is reported with a constant but without a trend in the cointegrating space.

The data reveal that aggregate real inventory and real final sales series are polynomially cointegrated. We specify our system initially to include three lags of the levels of stocks and sales and a constant that is restricted to lie in the cointegration space. Joint trace test statistics reported in Table I show, with the relevant block shaded, that the hypothesis that $r = 1$, $s = 0$, and $n - r - s = 1$ cannot be rejected. Since the number of I(2) trends equals the number of polynomially cointegrating relationships, the only cointegrating relationship detected above must
Table I. Polynomial cointegration in an I(2) system with aggregate data

<table>
<thead>
<tr>
<th>Constant, no trend</th>
<th>Vector diagnostics</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n - r )</td>
<td>( r )</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( n - r - s )</td>
<td>2</td>
</tr>
</tbody>
</table>

Notes: Statistics are computed with 3 lags and the estimation sample is 1982q4–2000q2 (71 observations). \( p \)-values for the Joint Trace Test \( S_\delta(s, r) \) (without trend) from Paruolo (1996), reported in square brackets; \( p \)-values for diagnostics are in square brackets.

be of the polynomially cointegrating variety. There is no I(1) common trend \( (s = 0) \) and one I(2) common trend \( (n - r - s = 1) \), suggesting that there ought to be two unit roots in the companion matrix. The four largest roots in the modulus of the characteristic polynomial are 1.018, 0.9010, 0.659 and 0.659, which suggests that there are two unit roots in the system.

The results of the I(2) system estimation deliver normalized cointegrating vectors represented as \( \beta'_1 \). The polynomial component based on the parameter estimates, \( \kappa' \), gives the following representation:

\[
\text{Aggregate : } H_t = 2S_t - 2.597\Delta H_t - 1.299\Delta S_t
\]

Estimation of the I(2) system shows that the I(0) direction of the data (with the polynomially cointegrating terms included) is given by the vector \( \beta'_1 \equiv (1, -a) \); the vector \( \beta'_3 \equiv (1, 1/a) \) provides the I(2) direction; and these vectors are orthogonal to each other. A basis for the space orthogonal to \( \beta'_3 \) is given by the matrix \( F = \begin{pmatrix} 1 \\ -a \end{pmatrix} \). Thus \( \begin{pmatrix} F'x_t \\ b'\Delta x_t \end{pmatrix} \), where \( b \) is any \( 2 \times 1 \) vector that satisfies the restriction that \( b'\beta'_3 \neq 0 \), provides the transformation to I(1), which keeps all the cointegrating and polynomially cointegrating information. Hence if we take \( b \) to be \( (0, 1)' \), then the bivariate system given by the I(1) representation \( \begin{pmatrix} H_t - a S_t \\ \Delta S_t \end{pmatrix} \) is a valid full reduction. We define \( x_t = (H_t, S_t)' \) in this notation.

Further exploration of the system reveals that all the coefficients are significant and cannot be excluded from the system, leaving a relationship in levels between stocks and final sales of \( H_t - 1.983S_t - 267.61 \), which can be used in the second stage of the analysis when investigating potential cointegrating relationships between levels and differences of the variables in our system.

To investigate the properties of the system in an I(1) framework we re-estimate the model using the levels relationship and the first difference of aggregate final sales. Initially we took three lags of each variable and included an unrestricted constant in the system, but subsequent testing revealed that the constant was insignificant and could be dropped. The results in the final model without a constant are reported in Table II. A single cointegrating relationship can be found, which is confirmed by a single unit root in the first four roots in modulus of the companion matrix being given by 1.012, 0.7204, 0.7204, 0.7136. The error correction term in this system is given by \( ECM_t \equiv (H_t - 1.983S_t - 267.61) - 9.808\Delta S_t \).

Table III reports the results from estimating the inventories and sales equations in a system, using the dynamic vector error correction mechanisms (VECMs) given above. Excluding the
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Table II. Cointegration in an I(1) system with aggregate data

<table>
<thead>
<tr>
<th>Null $H_0 : r$</th>
<th>Eigen-values</th>
<th>Estimated trace statistic</th>
<th>$n - r$</th>
<th>Vector diagnostics</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.307</td>
<td>26.72** [0.000]</td>
<td>2</td>
<td>Portmanteau (8) 15.54</td>
</tr>
<tr>
<td>1</td>
<td>0.009</td>
<td>0.674 [0.474]</td>
<td>1</td>
<td>Normality $\chi^2 (4)$ 4.40 [0.355] Hetero $F(48,134)$ 1.21 [0.177]</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$H_t - aS_t$</th>
<th>$\Delta S_t$</th>
<th>$\alpha$</th>
<th>$\Delta (H_t - aS_t)$</th>
<th>$\Delta^2 S_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(normalized coefficients)</td>
<td>1.000</td>
<td>-9.81</td>
<td>Coefficient</td>
<td>-0.191</td>
<td>0.053</td>
</tr>
<tr>
<td>SE</td>
<td>(1.079)</td>
<td>SE</td>
<td>(0.035)</td>
<td>(0.0112)</td>
<td></td>
</tr>
</tbody>
</table>

Notes: Statistics are computed with 3 lags and the estimation sample is 1982q4–2000q2. $p$-values are in square brackets.

Table III. The dynamic system for aggregate data

$$\Delta^2 H_t = -0.295 \Delta^2 H_{t-1} - 0.098 \text{ECM}_{t-1} + 0.828 \Delta^2 S_{t-2}$$

ECM$_t \equiv (H_t - 1.983 S_t - 267.61) - 9.808 \Delta S_t$

Sample: 1982(4) to 2000(2)
Sigma 5.43 RSS 2003.04744
Log-likelihood −219.306 DW 1.95
No. of observations 71 No. of parameters 3

System diagnostics
| AR(1–4) | $F(5, 63) = 0.503$ [0.773] |
| ARCH(1–4) | $F(4, 60) = 1.215$ [0.314] |
| Jarque–Bera normality (2) | $\chi^2(2) = 0.961$ [0.618] |
| Heteroscedasticity test | $F(6, 61) = 0.916$ [0.490] |
| RESET test | $F(1, 67) = 0.341$ [0.561] |

Note: $p$-values are in square brackets.

insignificant variables on a 5% $t$-criterion the final form is

$$\Delta^2 H_t = \mu_1 - \alpha_1 (H + \theta_1 S + \theta_2 \Delta S)_{t-1} + \sum_{i=1}^{4} \tau_{1i} \Delta^2 S_{t-i} + \sum_{i=1}^{4} \tau_{2i} \Delta^2 H_{t-i} + \varepsilon_t$$

(12)

A parsimonious VECM has satisfactory diagnostics, and gives a significant negative coefficient on the polynomial error correction term.

4.3. Estimating a Polynomially Cointegrating System with Disaggregated Data

The declining trend in the ratio of aggregate inventories relative to aggregate sales could be due to the fact that inventory-holding industries have experienced declining sales relative to non-inventory-holding industries such as services. Since services do not hold inventories the proportion of aggregate inventories to aggregate sales may have fallen due to the changing composition of economic activity. Exploration of disaggregated data allows us to explore whether this is the explanation of the underlying downward trend of aggregated inventory and sales data. We should find

that we get quite different relationships between inventories and sales for aggregate data versus the
disaggregated data if the declining ratio of aggregate inventories to sales is a compositional effect.

In this section we compare manufacturing inventories at different stage of fabrication (raw
materials (RM), work-in-progress (WIP) and final goods (FG)) in relation to manufacturing sales.
Visual inspection of the data in the previous section revealed that the various components of
inventories at certain stages of fabrication behave differently from each other. Some components
such as RM and FG inventories show similar trends that broadly match the upward trend of sales,
but WIP inventories seem to have a more evident cyclical pattern with a slight upward trend.
These characteristics result from the diverse shocks that affect the components of real aggregate
inventories in different ways. 9

Unsurprisingly, the analysis of these components in an I(2) system demonstrates similar
differences in behaviour. Using the same sample period as we used for aggregate data reveals
evidence of polynomial cointegration for RM, WIP and FG, but with different properties in the
equilibrium relations. Our initial specification of the system includes three lags of the levels of
each variable and a constant that is restricted to lie in the cointegrating space. The relationship at
each stage of fabrication is estimated separately.

When we examine the joint trace test statistics in Table IV for each stage of fabrication we find
support for polynomial cointegrating relationships at all stages. We find that the hypothesis, with
the relevant blocks in each panel of Table IV shaded, that \( r = 1, s = 0, \) and \( n - r - s = 1 \) cannot
be rejected, and as before in the aggregate data we find that the number of I(2) trends in the model
here equals the number of polynomially cointegrating relationships, with no I(1) common trend
\( (s = 0). \) Further investigation of the levels relationships shows that in two cases the restricted
constants are insignificant and these are excluded for raw materials inventories and finished goods
inventories. The relationships between inventories and manufacturing sales without the constants
are then re-estimated. The resulting levels relationships at each stage of fabrication are as follows:

\[
\text{Raw materials : } (H_t^{M(RM)} - 0.438S_t^{M});
\]
\[
\text{Work-in-progress : } (H_t^{M(WIP)} - 0.219S_t^{M} - 77.19);
\]
\[
\text{Finished goods : } (H_t^{M(FG)} - 0.541S_t^{M}).
\]

In general, the responsiveness to sales in the manufacturing sector is lower than for the aggregate
series with coefficients below one, and the intercepts are generally smaller. 10

Taking the I(1) format of the system as before, we re-estimate the model. Again we begin with
a system of three lags of each variable and include an unrestricted constant in the system. The
results correspond to the aggregate results since subsequent testing reveals that the constant is
insignificant in all cases in the I(1) system and can be excluded. The results in the final model
without a constant are reported in Table V. There is a single cointegrating relationship reported
between the levels relation and the first difference of manufacturing sales in each case.

Unlike the aggregate system we have evidence of quite different responses to differences in sales
at different stages of fabrication. Manufacturing firms’ RM inventories respond to sales growth in a
similar fashion as aggregate inventories respond to aggregate final sales: an increase in sales growth

---

9 We thank an anonymous referee for making this point.
10 Aggregate inventories and final sales include data from farm, wholesale and retail sectors besides the manufacturing
sector and for this reason might be expected to behave differently. Disaggregated data by stage of fabrication are not
available for these other sectors.
Table IV. Polynomial cointegration with disaggregated data

<table>
<thead>
<tr>
<th></th>
<th>Manufactured goods</th>
<th>Raw materials</th>
<th></th>
<th>Vector diagnostics</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Constant, no trend</td>
<td></td>
<td></td>
<td>Portmanteau (8)</td>
<td>30.7309</td>
</tr>
<tr>
<td></td>
<td>Q(r)</td>
<td></td>
<td></td>
<td>AR 1–5 F(20, 106)</td>
<td>1.5136 [0.092]</td>
</tr>
<tr>
<td>n – r</td>
<td>2</td>
<td>0</td>
<td>69.89 [0.000]</td>
<td>26.21 [0.052]</td>
<td>21.05 [0.037]</td>
</tr>
<tr>
<td></td>
<td>r</td>
<td></td>
<td>2</td>
<td>0.000</td>
<td>26.21 [0.052]</td>
</tr>
<tr>
<td></td>
<td>Q(r)</td>
<td></td>
<td></td>
<td>Normality $\chi^2(4)$</td>
<td>6.3817 [0.172]</td>
</tr>
<tr>
<td>n – r – s</td>
<td>2</td>
<td>1</td>
<td>7.96 [0.299]</td>
<td>6.95 [0.132]</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Hetero F(48, 134)</td>
<td>0.69677 [0.923]</td>
</tr>
<tr>
<td></td>
<td>Work-in-progress</td>
<td></td>
<td></td>
<td>Portmanteau (8)</td>
<td>15.4476</td>
</tr>
<tr>
<td></td>
<td>Constant, no trend</td>
<td></td>
<td></td>
<td>AR 1–5 F(20, 106)</td>
<td>0.6284 [0.883]</td>
</tr>
<tr>
<td></td>
<td>Q(r)</td>
<td></td>
<td></td>
<td>Normality $\chi^2(4)$</td>
<td>0.9787 [0.913]</td>
</tr>
<tr>
<td>n – r</td>
<td>2</td>
<td>0</td>
<td>75.09 [0.000]</td>
<td>49.76 [0.000]</td>
<td>38.879 [0.000]</td>
</tr>
<tr>
<td></td>
<td>r</td>
<td></td>
<td>2</td>
<td>0.000</td>
<td>49.76 [0.000]</td>
</tr>
<tr>
<td></td>
<td>Q(r)</td>
<td></td>
<td></td>
<td>Hetero F(48, 134)</td>
<td>1.2711 [0.144]</td>
</tr>
<tr>
<td>n – r – s</td>
<td>2</td>
<td>0</td>
<td>9.07 [0.221]</td>
<td>7.54 [0.103]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Finished goods</td>
<td></td>
<td></td>
<td>Portmanteau (8):</td>
<td>21.659</td>
</tr>
<tr>
<td></td>
<td>Constant, no trend</td>
<td></td>
<td></td>
<td>AR 1–5 F(20, 106)</td>
<td>1.2309 [0.244]</td>
</tr>
<tr>
<td></td>
<td>Q(r)</td>
<td></td>
<td></td>
<td>Normality $\chi^2(4)$</td>
<td>6.3475 [0.175]</td>
</tr>
<tr>
<td>n – r</td>
<td>2</td>
<td>0</td>
<td>73.39 [0.000]</td>
<td>36.97 [0.002]</td>
<td>27.78 [0.003]</td>
</tr>
<tr>
<td></td>
<td>r</td>
<td></td>
<td>2</td>
<td>0.000</td>
<td>36.97 [0.002]</td>
</tr>
<tr>
<td></td>
<td>Q(r)</td>
<td></td>
<td></td>
<td>Hetero F(48, 134)</td>
<td>0.76058 [0.861]</td>
</tr>
<tr>
<td>n – r – s</td>
<td>2</td>
<td>0</td>
<td>8.41 [0.265]</td>
<td>8.32 [0.072]</td>
<td></td>
</tr>
</tbody>
</table>

**Notes:** Statistics are computed with 3 lags and the estimation sample is 1982q4–2000q2 (71 observations). $p$-values for the Joint Trace Test $S(s, r)$ (without trend) from Paruolo (1996), reported in square brackets. $p$-values for diagnostics are in square brackets.

has a positive influence on the level of inventories relative to the level of sales. Manufacturing firms’ WIP and FG inventories, on the other hand, respond negatively to sales growth. This implies that when manufacturing sales increase, the level of inventories at the intermediate and final stages of production fall in relation to the level of sales. This makes sense from an economic perspective since a manufacturing firm faced with an increase in sales might very well implement a first response that reduces the level of inventories (rather than seek an instantaneous increase in production which would be more costly), while at the same time increasing the inputs (raw materials) to the production process to allow for a gradual increase in production over time.

The error correction terms for the stages of fabrication calculated from the cointegrating matrix $\beta'$ in this system are as follows:

- **Raw materials:** $ECM_i = (H_i^{M(RM)} - 0.438S_i^{M}) - 4.203\Delta S_i^{M}$,
- **Work-in-progress:** $ECM_i = (H_i^{M(WIP)} - 0.2195S_i^{M} - 77.19) + 5.109\Delta S_i^{M}$,
- **Finished goods:** $ECM_i = (H_i^{M(FG)} - 0.541S_i^{M}) + 9.240\Delta S_i^{M}$,

and these reflect the economic pattern of responses described above.
Table V. Cointegration in an I(1) system with disaggregated data

Manufactured goods

<table>
<thead>
<tr>
<th>Null $H_0 : r$</th>
<th>Eigenvales</th>
<th>Estimated trace statistic</th>
<th>$n - r$</th>
<th>Vector diagnostics</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.185</td>
<td>16.29** [0.010]</td>
<td>2</td>
<td>Portmanteau (8) 25.07</td>
</tr>
<tr>
<td>1</td>
<td>0.025</td>
<td>1.791 [0.212]</td>
<td>1</td>
<td>Normality $\chi^2(4)$ 8.60 [0.072]</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Hetero $F(81, 105)$ 0.78 [0.876]</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$H_t - aS_t$</td>
<td>$\Delta S_t$</td>
<td>$\alpha$</td>
<td>$\Delta(H_t - aS_t)$</td>
</tr>
<tr>
<td>(Normalized coefficients)</td>
<td>1.000</td>
<td>-4.202</td>
<td>Coefficient</td>
<td>-0.058</td>
</tr>
<tr>
<td>SE</td>
<td>(1.124)</td>
<td></td>
<td>SE</td>
<td>(0.018)</td>
</tr>
</tbody>
</table>

Work-in-progress

<table>
<thead>
<tr>
<th>Null $H_0 : r$</th>
<th>Eigenvales</th>
<th>Estimated trace statistic</th>
<th>$n - r$</th>
<th>Vector diagnostics</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.331</td>
<td>29.36** [0.010]</td>
<td>2</td>
<td>Portmanteau (8) 25.68</td>
</tr>
<tr>
<td>1</td>
<td>0.011</td>
<td>0.815 [0.425]</td>
<td>1</td>
<td>Normality $\chi^2(4)$ 5.61 [0.372]</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Hetero $F(81, 105)$ 0.67 [0.968]</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$H_t - aS_t$</td>
<td>$\Delta S_t$</td>
<td>$\alpha$</td>
<td>$\Delta(H_t - aS_t)$</td>
</tr>
<tr>
<td>(Normalized coefficients)</td>
<td>1.000</td>
<td>5.109</td>
<td>Coefficient</td>
<td>0.055</td>
</tr>
<tr>
<td>SE</td>
<td>(0.496)</td>
<td></td>
<td>SE</td>
<td>(0.019)</td>
</tr>
</tbody>
</table>

Finished goods

<table>
<thead>
<tr>
<th>Null $H_0 : r$</th>
<th>Eigenvales</th>
<th>Estimated trace statistic</th>
<th>$n - r$</th>
<th>Vector diagnostics</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.288</td>
<td>24.23** [0.000]</td>
<td>2</td>
<td>Portmanteau (8) 26.03</td>
</tr>
<tr>
<td>1</td>
<td>0.0009</td>
<td>0.066 [0.857]</td>
<td>1</td>
<td>Normality $\chi^2(4)$ 4.13 [0.389]</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Hetero $F(81, 105)$ 0.94 [0.595]</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$H_t - aS_t$</td>
<td>$\Delta S_t$</td>
<td>$\alpha$</td>
<td>$\Delta(H_t - aS_t)$</td>
</tr>
<tr>
<td>(Normalized coefficients)</td>
<td>1.000</td>
<td>9.240</td>
<td>Coefficient</td>
<td>0.019</td>
</tr>
<tr>
<td>SE</td>
<td>(1.062)</td>
<td></td>
<td>SE</td>
<td>(0.019)</td>
</tr>
</tbody>
</table>

Notes: Statistics are computed with 3 lags and the estimation sample is 1982q4–2000q2. p-values are in square brackets.

Table VI gives the dynamic model. Excluding the insignificant variables on a 5% t-criterion the final form of the dynamic inventories and sales equations in the system VECM has an error correction term with a significant negative coefficient for RM and FG, but has an insignificant error correction term for WIP (unreported). The equations are well specified and have satisfactory diagnostics. The results suggest that while RM and FG inventories have dynamic adjustment processes that are consistent with feedback from a polynomially cointegrating relationship between I(2) variables, WIP inventories, which unit root tests reported in the Appendix suggest had I(1) rather than I(2) properties, does not.

Table VI. The dynamic systems for disaggregated data

### Raw materials

\[
\Delta^2 H_t^{M(RM)} = 0.874 \Delta^2 H_{t-1}^{M(RM)} - 0.302 \Delta^2 H_{t-2}^{M(RM)} - 0.048 \text{ECM}_{t-1} + 0.746 \\
(0.098) \quad (0.099) \quad (0.099) \quad (0.175)
\]

\[
\text{ECM}_t \equiv (H_t^{M(RM)} - 0.438 \delta^M_t) - 4.203 \Delta^2 \delta^M_t
\]

Sample: 1983(1) to 2000(2)

<table>
<thead>
<tr>
<th></th>
<th>1.18</th>
<th>RSS</th>
<th>91.90</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sigma</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Log-likelihood</td>
<td>-108.85</td>
<td>DW</td>
<td>2.31</td>
</tr>
<tr>
<td>No. of observations</td>
<td>70</td>
<td>No. of parameters</td>
<td>4</td>
</tr>
</tbody>
</table>

### System diagnostics

- **AR(1–4)**: \(F(5, 61) = 3.22 [0.012]^*\)
- **ARCH(1–4)**: \(F(4, 58) = 1.044 [0.393]\)
- **Jarque–Bera normality (2)**: \(\chi^2(2) = 7.956 [0.019]^*\)
- **Heteroscedasticity test**: \(F(6, 61) = 1.326 [0.260]\)
- **RESET test**: \(F(1, 67) = 0.556 [0.459]\)

\(p\)-values are in square brackets.

### Finished goods

\[
\Delta^2 H_t^{M(FG)} = 0.958 \Delta^2 H_{t-1}^{M(FG)} - 0.669 \Delta^2 H_{t-2}^{M(FG)} + 0.019 \text{ECM}_{t-1} \\
(0.098) \quad (0.127) \quad (0.125) \quad (0.093)
\]

\[
\text{ECM}_t \equiv (H_t^{M(FG)} - 0.541 \delta^M_t) + 9.240 \Delta^2 \delta^M_t
\]

Sample: 1983(1) to 2000(2)

<table>
<thead>
<tr>
<th></th>
<th>1.30</th>
<th>RSS</th>
<th>109.29</th>
</tr>
</thead>
<tbody>
<tr>
<td>sigma</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Log-likelihood</td>
<td>-114.918</td>
<td>DW</td>
<td>1.98</td>
</tr>
<tr>
<td>No. of observations</td>
<td>70</td>
<td>No. of parameters</td>
<td>5</td>
</tr>
</tbody>
</table>

### System diagnostics

- **AR(1–4)**: \(F(5, 60) = 1.999 [0.092]\)
- **ARCH(1–4)**: \(F(4, 57) = 1.436 [0.234]\)
- **Jarque–Bera normality (2)**: \(\chi^2(2) = 0.826 [0.662]\)
- **Heteroscedasticity test**: \(F(10, 54) = 0.311 [0.975]\)
- **RESET test**: \(F(1, 64) = 0.002 [0.961]\)

\(p\)-values are in square brackets.

### 5. CONCLUSIONS

We have argued the case in this paper for a much richer modelling framework for stocks and sales. We offer a theoretical model that allows for the non-stationary characteristics of the data, using polynomial cointegration, and we show that the closed-form solution has other recent models as special cases. The resulting I(2) model performs well when put to the test on aggregate and disaggregated US data relating to inventories and sales.

Our empirical modelling strategy is derived directly from theoretical considerations. Indeed the possibility of casting the problem within an I(2) framework arises directly from the solution to the optimization problem and estimating the polynomially cointegrating relationship may be seen as the empirical analogue of the theoretical solution. Viewed in this light, our work offers well-specified dynamic models in accord with the time series properties of the data. Our approach fits in
with existing theoretical interpretations presented not only here but also in the general formulation reported by Dolado et al. (1991). It also marks the way forward for modelling inventory–sales relations during periods of transition in inventory management and changing industrial structure, extending Ramey and West’s and Hamilton’s excellent contributions in this area.

Appendix: Unit Root Tests

Aggregate Data

<table>
<thead>
<tr>
<th>Lag</th>
<th>$Y = \text{Stocks}$</th>
<th>$Y = \text{Final sales}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t$-adf $Y$</td>
<td>$t$-adf $DY$</td>
</tr>
<tr>
<td>8</td>
<td>2.503</td>
<td>–0.042</td>
</tr>
<tr>
<td>7</td>
<td>2.318</td>
<td>–0.612</td>
</tr>
<tr>
<td>6</td>
<td>2.615</td>
<td>–0.542</td>
</tr>
<tr>
<td>5</td>
<td>2.798</td>
<td>–0.847</td>
</tr>
<tr>
<td>4</td>
<td>3.131</td>
<td>–1.108</td>
</tr>
<tr>
<td>3</td>
<td>2.473</td>
<td>–1.503</td>
</tr>
<tr>
<td>2</td>
<td>2.768</td>
<td>–1.124</td>
</tr>
</tbody>
</table>

Notes: Critical values stocks: ADF tests ($T = 71$; (*) $5\% = -1.94$, (**) $1\% = -2.60$); sales: ADF tests ($T = 71$; (*) $5\% = -1.94$, (**) $1\% = -2.60$).

Disaggregated Data

<table>
<thead>
<tr>
<th>Lag</th>
<th>$Y = \text{Raw materials stocks}$</th>
<th>$Y = \text{Work-in-progress stocks}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t$-adf $Y$</td>
<td>$t$-adf $DY$</td>
</tr>
<tr>
<td>8</td>
<td>2.686</td>
<td>–0.6257</td>
</tr>
<tr>
<td>7</td>
<td>3.083</td>
<td>–1.103</td>
</tr>
<tr>
<td>6</td>
<td>3.358</td>
<td>–1.552</td>
</tr>
<tr>
<td>4</td>
<td>3.096</td>
<td>–2.268*</td>
</tr>
<tr>
<td>3</td>
<td>2.561</td>
<td>–2.347*</td>
</tr>
<tr>
<td>2</td>
<td>2.776</td>
<td>–1.955*</td>
</tr>
<tr>
<td>1</td>
<td>4.401</td>
<td>–2.536*</td>
</tr>
<tr>
<td>0</td>
<td>5.783</td>
<td>–5.158**</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Lag</th>
<th>$Y = \text{Final goods stocks}$</th>
<th>$Y = \text{Manufacturing sales}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$t$-adf $Y$</td>
<td>$t$-adf $DY$</td>
</tr>
<tr>
<td>8</td>
<td>3.176</td>
<td>–1.115</td>
</tr>
<tr>
<td>7</td>
<td>2.921</td>
<td>–1.296</td>
</tr>
<tr>
<td>5</td>
<td>3.677</td>
<td>–1.900</td>
</tr>
<tr>
<td>4</td>
<td>3.071</td>
<td>–2.431*</td>
</tr>
<tr>
<td>3</td>
<td>2.824</td>
<td>–2.264*</td>
</tr>
<tr>
<td>2</td>
<td>3.368</td>
<td>–2.243*</td>
</tr>
<tr>
<td>1</td>
<td>3.821</td>
<td>–3.239**</td>
</tr>
<tr>
<td>0</td>
<td>5.981</td>
<td>–4.645**</td>
</tr>
</tbody>
</table>

Notes: Sample: 1993 (4)–2000 (2). Critical values stocks: ADF tests ($T = 71$; (*) $5\% = -1.94$, (**) $1\% = -2.60$), sales: ADF tests ($T = 71$; (*) $5\% = -1.94$, (**) $1\% = -2.60$).
ACKNOWLEDGEMENTS

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REFERENCES


