Optimal Growth Through Product Innovation*

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Abstract

In Lentz and Mortensen (2008), we formulate and estimate a market equilibrium model of endogenous growth through product innovation. In this paper, we provide quantitative equilibrium solutions to the model based on our parameter estimates, and compare them with a social planner’s solution. We find that the socially optimal growth rate is double that of the market equilibrium growth rate when firm differences in the ability to create productive products are highly persistent as a consequence of the “business stealing” externality present in the model. The welfare loss of the decentralized economy relative to that of the planner is equivalent to a 20% tax on the planner consumption path. The planner’s solution differs from the decentralized economy in that it discourages innovation by low ability innovators as well as entry.

We introduce two mechanisms that temper the strength of the negative externality: Transitory firm types and the possibility of buyouts. We show that both sharply reduce the inefficiency due to innovation by low ability innovators. But with caveats: If firm types are completely transitory, then the notion of firm heterogeneity is for practical purposes lost. Buyouts improve efficiency, but the efficiency gain depends significantly on the strength of the innovator’s ability to extract rents from incumbents through the buyout.

Keywords: Optimal growth, planner’s problem, product innovation, innovation spill overs, creative-destruction externality.


1 Introduction

In Lentz and Mortensen (2008), we formulate and estimate a structural market equilibrium model of growth through product innovation. The model is an extended version of that proposed by Klette and Kortum (2004) originally designed to explain the link between innovation investment and the size distribution of firms. Their framework in turn is an elaboration of the Aghion and Howitt (1992) and Grossman and Helpman (1991) model of endogenous growth through creative-destruction combined with the firm dynamics insights in Hopenhayn (1992).

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In our version of the model, firms differ with respect to the quality of the intermediate products they create. We find that heterogeneity in this sense is needed to explain the cross firm correlations between valued added, labor force size and labor productivity. One implication of this form of heterogeneity is that labor reallocation from slower growing firms to faster growing firms that create more profitable higher quality products plays an important role in determining the aggregate growth rate.

The purpose of this paper is to explore the quantitative welfare implications of our estimated model. Namely, we formulate and compute the socially optimal R&D strategy for the modeled environment and compare its implications for the performance with that implied by the market equilibrium.

The market equilibrium solution to the model need not be socially optimal for three different reasons spelled out in Grossman and Helpman (1991). First, the most recent product innovator has monopoly power and use it to set prices above the marginal cost of production. Second, every innovation replaces an older version of some product and by doing so truncates the stream of quasi rents accruing to its previous innovator. Finally, because each new improvement builds on those of previous innovators, innovation has a positive “spill-over” effect on future productivity which is not fully captured by the innovator in a market equilibrium. The net deviation of the equilibrium growth rate from that which is socially optimal is unclear. One of the contributions of a quantitative equilibrium model is its ability to reflect light on the relative magnitudes of these effects.

We begin by characterizing the market and planner’s solution. We then use the parameter estimates obtained by Lentz and Mortensen (2008) to compute the implications of both for aggregate growth and economic welfare. If firm differences with respect to creative ability are highly persistent, then the planner’s optimal innovation investment strategy yields a growth rate which is much larger than that obtained in market equilibrium. However, if the differences between market equilibria and optimal growth rates are much less if firm heterogeneity is not very persistent.

2 The Model

Firms come in an amazing range of shapes and sizes. This fact cannot be ignored in any analysis of the relationship between firm size and factor productivity. Furthermore, an adequate theory must account for entry, exit, and firm evolution in order to explain observed size distributions. Klette and Kortum (2004) construct a stochastic model of firm product innovation and growth that is consistent with stylized facts regarding the firm size evolution and distribution. In Lentz and Mortensen (2008),
we find that an extension of their model that allows for cross firm heterogeneity in the quality of innovations is needed to explain our Danish data.

2.1 Preferences and Technology

The utility of the representative household at time $t$ is given by

$$U_t = \int_t^\infty \ln C_s e^{-\rho(s-t)} \, ds \quad (1)$$

where $\ln C_t$ denotes the instantaneous utility of the single consumption good at date $t$ and $\rho$ represents the pure rate of time discount. Each household is free to borrow or lend at interest rate $r_t$. Nominal household expenditure at date $t$ is $E_t = P_t C_t$. Optimal consumption expenditure must solve the differential equation $\dot{E}/E = r_t - \rho$. Following Grossman and Helpman (1991), we choose the numeraire so that $E_t = 1$ for all $t$ without loss of generality, which implies $r_t = r = \rho$ for all $t$. Note that this choice of the numeraire also implies that price of the consumption good, $P_t$, falls over time at a rate equal to the rate of growth in consumption.

The quantity of the consumption produced is determined by the quantity and quality of the economy’s intermediate inputs. Specifically, there is a unit continuum of intermediate good inputs and consumption is determined by the production function

$$\ln C_t = \int_0^1 \ln(A_t(j)x_t(j)) \, dj = \ln A_t + \int_0^1 \ln x_t(j) \, dj \quad (2)$$

where $x_t(j)$ is the quantity of input $j \in [0,1]$ at time $t$, $A_t(j)$ is the productivity of input $j$ at time $t$, and $A_t$ represent aggregate productivity. The level of productivity of each input and aggregate productivity are determined by the number of technical improvements made in the past. Specifically,

$$A_t(j) = \prod_{i=1}^{J_t(j)} q_i(j) \text{ and } \ln A_t = \int_0^1 \ln A_t(j) \, dj. \quad (3)$$

where $J_t(j)$ is the number of innovations made in intermediate input $j$ up to date $t$ and $q_i(j) > 1$ denotes the quantitative improvement (step size) in productivity attributable to the $i^{th}$ innovation in product $j$. Innovations arrive at rate $\delta$ which is endogenous in market equilibrium and the same for all intermediate products.

2.2 The Behavior of a Firm

Because intermediate goods of the same type are perfect substitutes, the creator of the latest most productive version is the sole supplier at any point in time. The price charged is limited by the ability of suppliers of previous versions to supply one of equal value. In Nash-Bertrand equilibrium,
any successful innovator takes over the market for its good type by setting the price just below that at which final good producers are indifferent between the new more productive product supplied by the innovator and the alternative supplied by the previous provider. The price charged is the product of the relative productivity of the innovation and the previous producer’s marginal cost of production. Given the symmetry of demands for the different good types and the assumption that future innovation likelihoods are independent of the type of good, one can drop the good subscript without confusion. Given stationary of quantities along the equilibrium growth path, the time subscript can be dropped as well.

Labor and capital, in fixed proportions, are used in the production of intermediate inputs to the final goods production process. Labor productivity is the same across all intermediate products and is set equal to unity. The required capital expressed in units of output, a constant $\kappa$, is also the same for all products. The operating profit per unit obtained from supplying an intermediate product is $p(1 - \kappa) - w$ which implies that the lowest price that the previous supplier is willing to charge, that which yields no profit, is $w/(1 - \kappa)$. The quality leader will charge $p =qw/(1 - \kappa)$ because consumers are exactly indifferent between buying from the quality leader at this price and the zero profit price of the previous supplier. As profit maximizing by the competitive suppliers of the consumption good requires

$$P_t \frac{\partial C_t}{\partial x_t(j)} = P_t C_t x_t(j) = p_t(j)$$

and $P_t C_t = E_t = 1$ by choice of the numeraire, the demand function for any product and the labor requirement for its production is

$$x = \frac{1}{p} = \frac{1 - \kappa}{wq}.$$ 

(4)

Hence, the gross profit associated with supplying the good of quality $q$ is

$$\pi(q) = p(1 - \kappa)x - wx = (1 - \kappa) \left(1 - q^{-1}\right).$$

(5)

Following Klette and Kortum (2004), the discrete number of products supplied by a firm, denoted as $k$, is defined on the integers and its value evolves over time as a birth-death process reflecting product creation and destruction. In their interpretation, $k$ reflects the firm’s past successes in the product innovation process as well as current firm size. New products are generated by R&D investment. The firm’s R&D investment flow generates new product arrivals at frequency $\gamma k$. The total R&D investment cost is $wc(\gamma)k$ where $c(\gamma)k$ represents the labor input required in the research and development process. The function $c(\gamma)$ is assumed to be strictly increasing and convex. According to the creators of the model, the assumption that the total cost of R&D investment is
linear in the number of existing product, given the innovation rate per product line, “captures the idea that a firm’s knowledge capital facilitates innovation.” In other words, the current state of innovation capital for a continuing firm is equal to the number of its current products that are at the technological frontier. In any case, this cost structure is needed to obtain firm growth rates that are independent of size, as typically observed in the data.

The market for any version of an intermediate product currently supplied is destroyed, as well as a unit of it creation capital, by the creation of a new more productive alternative by some other firm. This event occurs with frequency \( \delta \) independent of the product type. Below we refer to \( \gamma \) as the firm’s product innovation rate and to \( \delta \) as the common creative-destruction rate faced by all firms. The firm chooses \( \gamma \) to maximize the expected present value of its future net profit flow. Of course, the R&D strategies of all incumbent firms and potential entrants determine the equilibrium value of \( \delta \).

There are persistent but generally temporary differences with respect to the improvement in productivity (the quality of an innovation \( q \)) offered by the products that a firm creates. For simplicity, we assume that the firm’s identity changes from time to type. Specifically, identity is a Markov chain defined on two states, denoted as \( j = 0 \) and \( 1 \). Without loss of generality, the quality of a product created by a firm in state \( j = 1 \) is larger. Formally, a firm creates higher quality product is state 1 in the sense that \( q_1 > q_0 \). Finally, the given exogenous transition rate out of type state \( j \) is \( \lambda_j \). Because a firm’s creativity type is transitory, the firm may be supplying products of both qualities at any point in time.

### 2.3 The Value Function of the Firm

Given that the innovator always takes over the market for the new version of the product it has created, the value of a firm that currently supplies \( k_0 \) products of low quality and \( k_1 \) high quality products is the solution to the asset pricing equation

\[
r V_0(k_0, k_1) = \max_{\gamma \geq 0} \left\{ k_0 \pi_0 + k_1 \pi_1 - w c(\gamma) k + \gamma k \left[ V_0(k_0 + 1, k_1) - V_0(k_0, k_1) \right] + \delta \left[ k_0 V_0(k_0 - 1, k_1) + k_1 V_0(k_0, k_1 - 1) - k V_0(k_0, k_1) \right] + \lambda_0 \left[ V_1(k_0, k_1) - V_0(k_0, k_1) \right] \right\}
\]

\[
r V_1(k_0, k_1) = \max_{\gamma \geq 0} \left\{ k_0 \pi_0 + k_1 \pi_1 - w c(\gamma) k + \gamma k \left[ V_1(k_0, k_1 + 1) - V_1(k_0, k_1) \right] + \delta \left[ k_0 V_1(k_0 - 1, k_1) + k_1 V_1(k_0, k_1 - 1) - k V_1(k_0, k_1) \right] + \lambda_1 \left[ V_0(k_0, k_1) - V_1(k_0, k_1) \right] \right\}
\]
where \( k \equiv k_0 + k_1 \) is the total number supplied and the subscript represents the firm’s current type. In each case, the first three terms on the right sides of (6) and (7) represent the cash flow obtained by supplying the portfolio of current products net of the current expenditure on R&D. The next three terms are respectively the expected charge in value attributable to the arrival of a new product line, the destruction of the existing product line, and a transition to the other creativity state.

The unique solution to (6) and (7) takes the form

\[
V_i (k_0, k_1) = \frac{k_0 \pi_0 + k_1 \pi_1}{r + \delta} + k \Psi_i, \quad i = 0, 1 \tag{8}
\]

\[
\Psi_i = \max_{\gamma \geq 0} \left\{ \gamma \left( \frac{\pi_i}{r + \delta} + \Psi_i \right) - wc (\gamma) + \lambda_i \Psi_j \right\}, \quad j = 1 - i, \tag{9}
\]

as one can verify by substitution. Obviously, the first term on the right side of (8) represents the expected present value of the future profit streams generated by the firm’s current products. The second term is the value of the firm R&D operation. One can think of there being \( k \) research operations, each associated with an existing product line, and regard \( \Psi_j \) as the asset price assigned to each operation when in creativity state \( j \). That price is the expected present value of the net profit generated by the choice of the current innovation frequency \( \gamma \).

The fact that a type 1 innovation yields a higher profit rate implies that the value of R&D is higher when the firm is high creativity state.

**Proposition 1.** \( \pi_1 > \pi_0 \) implies \( \Psi_1 > \Psi_0 \) if \( c(\gamma) \) is strictly convex and \( c(0) = c'(0) = 0 \).

**Proof.** To the contrary, suppose that \( \Psi_0 - \Psi_1 \geq 0 \). Under the supposition, the equations of (8) and \( \pi_1 > \pi_0 \) imply the following contradiction,

\[
\Psi_1 = \max_{\gamma \geq 0} \left\{ \gamma \left( \frac{\pi_1}{r + \delta} - wc (\gamma) + \lambda_1 (\Psi_0 - \Psi_1) \right) \right\} \\
\geq \max_{\gamma \geq 0} \left\{ \gamma \left( \frac{\pi_0}{r + \delta} - wc (\gamma) \right) \right\} > \max_{\gamma \geq 0} \left\{ \gamma \left( \frac{\pi_0}{r + \delta} - wc (\gamma) \right) \right\} \\
\geq \max_{\gamma \geq 0} \left\{ \gamma \left( \frac{\pi_0}{r + \delta} - wc (\gamma) + \lambda_0 (\Psi_1 - \Psi_0) \right) \right\} = \Psi_0.
\]

As the value of doing so is always positive, the choice of the state contingent innovation rate satisfies

\[
wc' (\gamma_i) = v_i \equiv \frac{\pi_i}{r + \delta} + \Psi_i, \tag{10}
\]

under the hypothesis. The sufficient second order condition \( c''(\gamma) > 0 \) and Proposition 1 imply that firm’s innovate more frequently while in the high creativity state. In other words, \( \gamma_1 > \gamma_0 \).
2.4 Entry and Labor Market Clearing

The entry of a new firm requires a successful innovation by a potential entrant. Suppose that there is a constant measure $m$ of identical potential entrants. The rate at which any one of them generates a new product is $\gamma$ and the total cost is $wc(\gamma)$ where the cost function is the same as that faced by an incumbent. The new firm’s type is unknown ex ante but is realized immediately after the arrival of an innovation. Since the aggregate entry rate is $\eta = m\gamma$, the entry rate satisfies the following free entry condition

$$wc' \left( \frac{\eta}{m} \right) = V_0(1,0)\phi_0 + V_1(0,1)\phi_1 = v_0\phi_0 + v_1\phi_1$$

(11)

where $\phi_i$ is the probability that the entrant will turn out to be of creativity type $i$ initially. Obviously, in this formulation learning one’s type takes no time, which is unrealistic but a useful abstraction for the purposes of this paper.

There is a fixed measure of available workers, denoted by $\ell$, seeking employment at any positive wage. In equilibrium, these are allocated across production and R&D activities, those performed by both incumbent firms and potential entrants. The number of workers required per product of type $i = 0,1$ is $x_i = 1/p_i = (1 - \kappa)/q_i w$ from equation (4). The number of R&D workers employed per product by incumbent firms of type $j = 0,1$ is $\ell_R(j) = c(\gamma_j)$. Because each potential entrant innovates at frequency $\eta/m$, the aggregate number of workers engaged by all $m$ in R&D is $\ell_E = mc(\eta/m)$. Hence, the equilibrium wage satisfies the labor market clearing condition

$$\ell = \frac{1 - \kappa}{w} \left( \frac{K_0}{q_0} + \frac{K_1}{q_1} \right) + K_0c(\gamma_0) + K_1c(\gamma_1) + mc(\eta/m)$$

(12)

where $K^i = K_{0i} + K_{1i}$ represents the fraction of products of productivity $q_i$ and $K_j = K_{0j} + K_{1j}$ denotes the fraction supplied by firms in state $j$.

2.5 Aggregate Dynamics

The overall state of the aggregate economy can be represented by the joint distribution of products over product types and firm states. Recall that $K_{ij}$ represent the number of products of type $i \in \{0,1\}$ supplied by firms in creativity state $j \in \{0,1\}$. Because every firm leaves state $i$ at rate $\lambda_i$, the following laws of motion hold.
\[ K_{11} = \lambda_0 K_{10} + \eta \phi_1 + \gamma_1 (K_{11} + K_{01}) - (\delta + \lambda_1) K_{11} \quad (13) \]
\[ K_{01} = \lambda_0 K_{00} - (\delta + \lambda_1) K_{01} \quad (14) \]
\[ K_{10} = \lambda_1 K_{11} - (\delta + \lambda_0) K_{10} \quad (15) \]
\[ K_{00} = \lambda_1 K_{01} + \eta \phi_0 + \gamma_0 (K_{00} + K_{10}) - (\delta + \lambda_0) K_{00} \quad (16) \]

where \( \phi_j \) is the fraction of firms that enter in creativity state \( j = 0, 1 \). By implication,
\[
\begin{align*}
\dot{K}_1 &= K_{01} + K_{11} = \lambda_0 K_0 + \eta \phi_1 + \gamma_1 K_1 - (\delta + \lambda_1) K_1 \\
\dot{K}_0 &= K_{00} + K_{10} = \lambda_1 K_1 + \eta \phi_0 + \gamma_0 K_0 - (\delta + \lambda_0) K_0. \quad (18)
\end{align*}
\]

Since every new product induces the destruction of an existing one, \( K_0 + K_1 = 0 \), the aggregate creative-destruction rate is
\[ \delta = \eta + \gamma_1 K_1 + \gamma_0 K_0. \quad (19) \]

To find the steady state solution, use equation (19) to substitute out \( \delta \) in equation of (18). Then use the result
\[ \dot{K}_0 = \lambda_1 K_1 - K_0 [\gamma_1 K_1 + \eta \phi_1 + \lambda_0] = 0 \]

As this expression defines a strictly increasing relationship between \( K_0 \) and \( K_1 \) that passes through the origin, the fact that \( K_0 + K_1 = 1 \) implies that a unique positive steady state solution for the pair \((K_0, K_1)\) exists for any set of innovation and transitions rates.

For equations (2) and (3), the contribution of any innovation to the aggregate rate of consumption growth in the natural log of its improvement in productivity over the previous version. Hence, the growth rate is,
\[ g = \frac{\dot{C}}{C} = m \gamma_e (\phi_0 \ln q_0 + \phi_1 \ln q_1) + \gamma_0 K_0 \ln q_0 + \gamma_1 K_1 \ln q_1. \quad (20) \]

### 2.6 Market Equilibrium

A steady state market equilibrium is a triple composed of a labor market clearing wage \( w \), entry rate \( \eta \), and creative destruction rate \( \delta \) together with an optimal creation rate for each firm type, \( \gamma_j \), \( j \in (0, 1) \) and a distribution of products across product and firm types, \( K_{ij}, (i,j) \in \{0,1\}^2 \), that satisfy equations (8), (11), (12), (19), and the steady state conditions implied by equations (13)-(16).
3 The Social Planner’s Problem

3.1 Formulation

The social planner chooses a non-negative time paths for the production rate of both product types, \(x_i\), the rate of new product creation for each firm creativity state, \(\gamma_j\), and the rate of product innovation by potential entrants, \(\gamma_e\) to maximize the present discounted utility of the representative household’s consumption subject to the fact that there are a fixed number of intermediate products, a labor resource constraint, and laws of motion for the state variables. Under complete and symmetric information, both the firm and the planner observes the realized productivity of any innovation.

The planner’s strategy, the choice of \((x_0, x_1, \gamma_0, \gamma_1, \gamma_e)\) that maximizes the expected present value of the representative consumer’s utility stream subject to a set of constraints. Formally, the criterion is

\[
\int_0^\infty \ln C_t e^{-rt} dt = \int_0^\infty \ln A_t + \sum_{i=0}^1 \sum_{j=0}^1 \ln (x_i(t)) K_{ij}(t) \bigg] e^{-rt} dt
\]

where \(A\) represents aggregate productivity and \(K_{ij}\) is the fraction of products of type \(i\) currently supplied by incumbent firms in creativity state \(j\). The constraints follow: Employment cannot exceed the available labor supply,

\[
l \geq \sum_{i=0}^1 \sum_{j=0}^1 \left[ \frac{x_i}{1-\kappa} + c(\gamma_j) \right] K_{ij} + mc(\gamma_e).
\]

The entry rate is \(\eta = m\gamma_e\). As a consequence, the law of motion for aggregate productivity can be written

\[
\frac{d \ln A}{dt} = g = m\gamma_e [\phi_0 \ln q_0 + \phi_1 \ln q_1] + \gamma_0 K_0 \ln q_0 + \gamma_1 K_1 \ln q_1
\]

where \(K_j = K_{0j} + K_{1j}\) is the fraction of products currently supplied by firms in creative state \(j\). The distribution of products across firm state evolves according to

\[
\dot{K}_0 = \lambda_1 K_1 + m\gamma_e \phi_0 + \gamma_0 K_0 - (m\gamma_e \phi_0 + \gamma_0 K_0 + m\gamma_e \phi_1 + \gamma_1 K_1 + \lambda_0) K_0
\]

\[
\dot{K}_1 = \lambda_0 K_0 + m\gamma_e \phi_1 + \gamma_1 K_1 - (m\gamma_e \phi_0 + \gamma_0 K_0 + m\gamma_e \phi_1 + \gamma_1 K_1 + \lambda_1) K_1.
\]

Although the complete planner’s problem includes laws of motion for every \(K_{ij}\), only \(K_0\) and \(K_1\) are decision relevant state variables for reasons that will become apparent.

The planner’s problem is a relatively standard one in dynamic control. The constraint augmented
present value Hamiltonian for the problem can be written as,

$$ H = \ln A + \sum_i \sum_j \ln (x_i(t)) K_{ij}(t) $$

$$ + \omega \left( \ell - \sum_i \sum_j \left[ x_i + c(\gamma_j) \right] K_{ij} - mc(\gamma_e) \right) $$

$$ + \Lambda \left[ m\gamma_e (\phi_0 + \phi_1 \ln q) + \gamma_0 K_0 \ln q_0 + \gamma_1 K_1 \ln q_1 \right] $$

$$ + \nu_0 \left[ \frac{\lambda_1 K_1 + m\gamma_e \phi_0 + \gamma_0 K_0}{\lambda_0 K_0 + m\gamma_e \phi_1 + \gamma_1 K_1} \right] $$

$$ + \nu_1 \left[ \frac{-m\gamma_e \phi_0 + \gamma_0 K_0 + m\gamma_e \phi_1 + \gamma_1 K_1 + \lambda_0 K_0}{\lambda_0 K_0 + m\gamma_e \phi_1 + \gamma_1 K_1} \right] $$

where $\omega$ is the labor supply constraint multiplier, $\Lambda$ is the shadow price of the state variable $\ln A$ and $\nu_j$ is the shadow price associated with $K_j$, $j \in \{0, 1\}$.

As the optimal controls maximize the Hamiltonian given state and co-state variables, the first order necessary conditions for all the continuous choice variables are

$$ \frac{\partial H}{\partial x_i} = \left( \frac{1}{x_i} - \omega \right) K_i = 0, \text{ for all } i \in \{0, 1\}. $$

$$ \frac{\partial H}{\partial \gamma_0} = \left[ \Lambda \ln q_0 + \nu_0 K_0 - \nu_0 K_1 - \omega c'(\gamma_0) \right] K_0 \leq 0 \text{ (with = if } \gamma_0 > 0) $$

$$ \frac{\partial H}{\partial \gamma_1} = \left[ \Lambda \ln q_1 + \nu_1 K_1 - \nu_0 K_0 - \omega c'(\gamma_1) \right] K_1 \leq 0 \text{ (with = if } \gamma_1 > 0) $$

$$ \frac{\partial H}{\partial \gamma_e} = \left[ \Lambda \left[ \phi_0 \ln q_0 + \phi_1 \ln q_1 \right] + \nu_0 \left[ (1 - K_0) \phi_0 - K_0 \phi_1 \right] + \nu_1 \left[ (1 - K_1) \phi_1 - K_1 \phi_0 \right] - \omega c'(\gamma_e) \right] m \leq 0 \text{ (with = if } \gamma_e > 0). $$

The assumption that the cost of R&D is convex in the innovation rate is sufficient to guarantee that the second order necessary conditions are satisfied.

Because they enter symmetrically in the production of the consumption good and they cost the same at the margin, the first equation of (21) implies that all intermediate products should be supplied at the same rate. Formally,

$$ x_i = x = \frac{1}{\omega}, i \in (0, 1). $$

(22)
Given this result, the co-state (Euler) equations are

\[
\begin{align*}
\frac{\partial H}{\partial \ln A} &= 1 = r \Lambda - \dot{\Lambda} \\
\frac{\partial H}{\partial K_0} &= \ln x - \omega [x + c(\gamma_0)] + (\Lambda \ln q_0 + \nu_0 (1 - K_0) - \nu_1 K_1) \gamma_0 \\
&\quad + (\nu_1 - \nu_0) \lambda_0 - \delta \nu_0 = rv_0 - \dot{v}_0 \\
\frac{\partial H}{\partial K_1} &= \ln x - \omega [x + c(\gamma_1)] + (\Lambda \ln q_1 + \nu_1 (1 - K_1) - \nu_0 K_0) \gamma_1 \\
&\quad + (\nu_0 - \nu_1) \lambda_1 - \delta \nu_1 = rv_1 - \dot{v}_1
\end{align*}
\]

Of course, \( \nu_i \) represents the social value of creating a quality \( i \) product. Finally, the necessary transversality conditions require that \( \Lambda e^{-rt} \) and \( \nu_j e^{-rt} \) converge to zero as \( t \to \infty \).

The transversality condition requires that the shadow price of log productivity equal the inverse of the discount rate at all dates,

\[
\Lambda = \frac{1}{r}.
\]

The remaining equations of (21) can be rewritten as

\[
\begin{align*}
\omega c'(\gamma_0) &= \frac{\ln q_0}{r} + \nu_0 - \nu_0 K_0 - \nu_1 K_1 \text{ if } \gamma_0 > 0 \\
\omega c'(\gamma_1) &= \frac{\ln q_1}{r} + \nu_1 - \nu_0 K_0 - \nu_1 K_1 \text{ if } \gamma_1 > 0 \\
\omega c'(\gamma_e) &= \sum_{i \in \{0, 1\}} \left[ \frac{\ln q_i}{r} + \nu_i \right] \phi_i - \nu_0 K_0 - \nu_1 K_1 \text{ if } \gamma_e > 0.
\end{align*}
\]

In all cases, the marginal cost of innovation is equal to the expected social return. For a firm in the more creative state (\( j = 1 \)), the social return is the sum of the value of the spill-over plus the difference between the value of a high quality product and the expected value of the product it will replace. The deviation between the socially optimal and private incentive to invest in R&D is apparent by comparing these with the market equilibrium first order conditions represented in equation (10). The first term on the right side of all the equation in (25) is the present value (expressed in utility terms) of an innovation’s contribution to future productivity. The second term is the difference between the expected present value of the innovation and the product that it will replace. The first term, representing the “spill over” economy, and the last term, the “business stealing” external effect, are not present in the private calculation.

### 3.2 Steady State Planner Solution

To simplify the analysis, we compare welfare across steady states. This exercise is the equivalent of solving the full planner problem subject to a zero discount rate. In steady state, \( d \ln A_j / dt = g \) for
all $t$. Consequently, $\ln A_t = gt + \ln A_0$, where $\ln A_0$ is the initial quality level. Furthermore, because the optimal production choice is the same for both product type; $x_i(t) = x$. Thus, the steady state social planner criterion is,

$$\max \int_0^\infty \ln C_t e^{-rt} dt = \max \int_0^\infty \left[ \ln A_t + \sum_i \sum_j \ln \left( x_{ij}(t) \right) K_{ij}(t) \right] e^{-rt} dt$$

$$= \max \int_0^\infty \left[ gt + \ln A_0 + \ln (x) \right] e^{-rt} dt$$

$$= \max \left[ \ln A_0 \int_0^\infty e^{(g-r)t} dt + \ln (x) \int_0^\infty e^{-rt} dt \right]$$

$$= \max \left[ g \int_0^\infty te^{-rt} dt + \frac{\ln (x) + \ln A_0}{r} \right]$$

$$= \max \left[ \frac{g}{r^2} + \frac{\ln (x) + \ln A_0}{r} \right]$$

$$= \frac{\ln A_0}{r} + r \max \left[ \ln (x) + \frac{g}{r} \right].$$

The full steady state planner problem reduces to a choice of the real vector $(\gamma_0, \gamma_1, \gamma_e)$ that solves

$$\max \{r \ln (x) + g\} \quad (26)$$

subject to 

$$x = \ell - c (\gamma_0) (1 - K_1) - c (\gamma_1) K_1 - mc (\gamma_e)$$

$$g = m \gamma_e (\phi_0 \ln q_0 + \phi_1 \ln q_1) + \gamma_0 (1 - K_1) \ln q_0 + \gamma_1 K_1 \ln q_1$$

$$K_0 = 0 = \lambda_1 K_1 + m \gamma_e \phi_0 - (m \gamma_e + \lambda_0 + (\gamma_1 - \gamma_0) K_1) (1 - K_1).$$

In the following section we will compare the market equilibrium outcomes with the social planner solution. Note that welfare in the market solution takes a similar form to the social planner except here products are not supplied at the same rates. Normalizing the initial quality level at unity, welfare in the steady state market equilibrium is given by

$$U = r \sum_i \ln \left( \frac{1 - \kappa}{q_i w} \right) K^i + g$$

where $K^i$ is the aggregate supply of type $i$ products in steady state.

# 4 Numerical Solutions

In this section, we use model parameter estimates reported by Lentz and Mortensen (2008) to compare the quantitative properties of the market and the planner’s solutions. We conclude the section by computing the welfare gain that could be achieved by a switch to the optimal solution when the market steady state solution characterizes the initial conditions.
Table 1: Model Parameter Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost scale parameter $c_0$</td>
<td>175.800</td>
</tr>
<tr>
<td>Cost curvature parameter $c_1$</td>
<td>3.728</td>
</tr>
<tr>
<td>Capital cost per product $\kappa$</td>
<td>0.410</td>
</tr>
<tr>
<td>Interest rate $r$</td>
<td>0.050</td>
</tr>
<tr>
<td>Labor supply $L$</td>
<td>45,734</td>
</tr>
<tr>
<td>Value added $Z$</td>
<td>16,859,000</td>
</tr>
<tr>
<td>Entrant mass $m$</td>
<td>1.341</td>
</tr>
<tr>
<td>Low type quality improvement $q_0$</td>
<td>1.001</td>
</tr>
<tr>
<td>High type quality improvement $q_1$</td>
<td>1.600</td>
</tr>
<tr>
<td>Low type entrant probability $\phi_0$</td>
<td>0.850</td>
</tr>
<tr>
<td>High type entrant probability $\phi_1$</td>
<td>0.150</td>
</tr>
</tbody>
</table>

4.1 Model parameters

The parameter estimates obtained by Lentz and Mortensen (2008) derived from Danish firm data using a simulated methods of moments method are reported in Table 1. In the empirical version of the model estimated, the R&D cost function is assumed to take the power form $c(\gamma) = c_0\gamma^{1+c_1}$. The parameter $Z$ is the average real value added per product line and $L$ is the total labor force. Since real value added per product is normalized at unity and the total measure of products is set to unity as well in the model, the total labor supply per product line is $\ell = L/Z$. The estimated model is based on 3 types of firms where the type conditional quality realization is based on Weibull distributions. The model in this paper is simplified to two types and the type conditional quality realization distribution is degenerate. The 3 types in Lentz and Mortensen (2008) can be grouped into one high type and two low types. The parameter values in Table 1 are based on such a grouping. Specifically, the type conditional quality improvement is chosen so as to match $E\ln(q_i)$ across the two models. Finally, the model in Lentz and Mortensen (2008) is estimated under the assumption that a firm’s initial type realization is permanent ($\lambda_0 = \lambda_1 = 0$).

The extreme right skew in the distribution of firm types at entry reflects the fact that very few innovations by potential entrants are of high quality. Indeed, around 85% of the entrants create innovation of only marginal value. These might be interpreted as simply “imitators.”

4.2 Comparison of planner and market outcomes

This section will present a comparison of the steady state equilibrium outcomes of economies that differ in the degree of type permanence. The market outcomes are compared to the steady state planner solution.
Table 2: Social planner and market outcomes for high and low type persistence.

<table>
<thead>
<tr>
<th>Planner</th>
<th>Market</th>
<th>Planner</th>
<th>Market</th>
<th>Constrained Planner ((L_R = 1.582))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(w)</td>
<td>189.552</td>
<td>184.555</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\delta)</td>
<td>0.060</td>
<td>0.069</td>
<td>0.076</td>
<td>0.084</td>
</tr>
<tr>
<td>(\eta)</td>
<td>0.000</td>
<td>0.043</td>
<td>0.045</td>
<td>0.046</td>
</tr>
<tr>
<td>(\gamma_0)</td>
<td>0.000</td>
<td>0.005</td>
<td>0.000</td>
<td>0.028</td>
</tr>
<tr>
<td>(\gamma_1)</td>
<td>0.060</td>
<td>0.054</td>
<td>0.061</td>
<td>0.049</td>
</tr>
<tr>
<td>(K_0)</td>
<td>0.000</td>
<td>0.577</td>
<td>0.501</td>
<td>0.524</td>
</tr>
<tr>
<td>(K_1)</td>
<td>1.000</td>
<td>0.423</td>
<td>0.499</td>
<td>0.476</td>
</tr>
<tr>
<td>(K^0)</td>
<td>0.000</td>
<td>0.577</td>
<td>0.507</td>
<td>0.547</td>
</tr>
<tr>
<td>(K^1)</td>
<td>1.000</td>
<td>0.423</td>
<td>0.493</td>
<td>0.439</td>
</tr>
<tr>
<td>(v_0)</td>
<td>0.001</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(v_1)</td>
<td>2.885</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\Psi_0)</td>
<td>0.000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\Psi_1)</td>
<td>1.025</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(L_X)</td>
<td>40.879</td>
<td>44.152</td>
<td>42.578</td>
<td>44.285</td>
</tr>
<tr>
<td>(L_R)</td>
<td>4.856</td>
<td>1.582</td>
<td>3.156</td>
<td>1.449</td>
</tr>
<tr>
<td>(g)</td>
<td>0.028</td>
<td>0.014</td>
<td>0.018</td>
<td>0.014</td>
</tr>
<tr>
<td>(U)</td>
<td>0.214</td>
<td>0.202</td>
<td>0.205</td>
<td>0.203</td>
</tr>
<tr>
<td>(\tau)</td>
<td>0.211</td>
<td>0.036</td>
<td>0.036</td>
<td>0.036</td>
</tr>
</tbody>
</table>

In Table 2, \(L_X\) and \(L_R\) are respectively the amount labor engaged in product manufacturing and innovation. Welfare, \(U\), in the market equilibria outcomes are compared to the planner problem by means of determining a tax, \(\tau\), on planner consumption such that agents are indifferent between the market outcome and the planner’s solution,

\[
r \ln (x^p (1 - \tau)) + g^p = \hat{U},
\]

where \(\hat{U}\) is the welfare in the market equilibrium in question and \(x^p\) and \(g^p\) are the planner consumption and growth levels, respectively.

First consider the case of high type persistence. A comparison of the planner’s solution to the market equilibrium reveals the fact that the optimal growth rate is twice that of the equilibrium growth rate. Although selection takes place in the sense that more creative firms innovate more frequently in equilibrium as well as in the planner’s solution, more creative firms accounts for only 42% of the products supplied in steady state equilibrium as compared with almost 100% in the planner’s solution. The planner obtains this outcome by decreasing the innovation rates of the less creative firm type and of the potential entrants to zero. In the high persistence case an increase in steady state \(K_1\) is sufficiently valuable that it outweighs any gains from innovations through \(\gamma_0\) and \(\gamma_e\). In other words, the expected contribution to future aggregate productivity of an innovation by
a low creativity firm or a potential entrant does not compensate for transfer of R&D capacity from
the more to the less creative firms. Consequently, innovation is done only by firms in the more
creative state and selection into the more creativity firms is almost perfect. The large difference
between private and social values of innovation by less creative firms and entrants translates into a
welfare difference between the planner and market solutions equivalent to a tax of 21% of planner
consumption.

In the low type persistence case reported in Table 2, the less creative type’s negative innovation
externality on the R&D capacity of the creative type firms is less dramatic. In the low persistence
case, the model is solved given \( \lambda_0 = \lambda_1 = 0.2 \), which implies that a firm switches type at a rate
more than double that by which it develops new products. In this case, the type switching by itself
remedies a substantial part of the skew toward imitator firms at birth. The fraction of products held
by low and high type firms, respectively, is not all that different from the high persistence solution.
However, this is now more a result of the intrinsic type shifting rather than pure selection. In the
planner solution, the selection effect is stronger and the fraction of high type products increases.
Firm types exist in roughly equal proportion with slightly more selection in the planner solution.
But even in the planner solution, the more creative firms only supply 49% of the products.

The imitator’s negative innovation externality is less important when firm type is less persistent
simply because firm type is now a less meaningful distinction. The value of moving R&D capacity
to firms in the more creative state is significantly lower in the low persistence case, because a
firm’s creativity is subject to rapid change. The planner solution increases productivity growth
from 1.4% annually to 1.8%. Product pricing is also more efficient. In all, the welfare loss in the
decentralized equilibrium corresponds to a 3.6% tax on planner consumption, a substantially more
moderate efficiency loss relative to the high type persistence case.

Finally, the last column in Table 2 shows a constrained planner solution for the permanent firm
types case, where the planner is constrained to maintain the decentralized solution’s aggregate level
of resources devoted to innovation. As can be seen, the planner does not obtain quite as much
productivity growth, but the loss relative to the unconstrained planner solution is only about 4%
of the unconstrained planner consumption. Hence, the inefficiency of the decentralized solution is
primarily a result of misallocation of innovation resources over firm types.
5 Buyout

Innovation in the Klette and Kortum (2004) and Lentz and Mortensen (2008) model is associated with a negative impact on the innovation technology of existing product leaders: Because innovation ability is embodied in product leadership, once a firm loses its leadership for a given product, it also loses the innovation ability embodied in it. Thus, once another firm moves the knowledge frontier forward, the previous leader’s innovation production technology associated with the market in question is rendered obsolete. The possible inefficiency associated with this effect can be very strong. When a low ability innovator takes over product leadership from a high ability innovator, innovation capacity is shifted away from a more efficient producer of innovations toward a less efficient innovator. As shown in this paper, the planner can increase the growth rate in a Lentz and Mortensen (2008) calibrated model from 1.4% to almost 3% annually with a welfare difference corresponding to a 20% tax on planner consumption. The planner achieves this by almost shutting down entry and in particular innovation by low innovation ability types.

The market can by itself attempt to address the inefficiency. One way is for the high innovation ability firms to defensively buy out an innovator’s innovation into its market so as to protect its leadership and thereby its innovation production scale. A defensive buyout is modeled as the purchase of the innovator’s technology which is then scrapped. This reflects the assumption that knowledge cannot be transferred to other firms. If it could, the most efficient innovators would simply purchase all product lines and there would be no firm type dispersion in equilibrium.

In the model above with two transitory types, denote by $v_{ij}$ the value of quality $i$ product line owned by a type $j$ firm, $i = 0, 1$ and $j = 0, 1$. The value of a new innovation to a low type firm is $v_{00}$. To keep the market, an incumbent of type $j$ with a quality $i$ product line is willing to pay $v_{ij}$. It is straightforward to show that $v_{00} \leq v_{ij}$ and $v_{11} \geq v_{ij}$ for $i, j = 0, 1$. Hence, assuming efficient bargaining between innovators and incumbents, a high ability innovator is never bought out, and a low ability innovator is always bought out except when innovation is into a low quality product owned by a low ability incumbent. The surplus of the buyout is split by a lottery: With probability $\alpha$ the incumbent makes the innovator a take-it-or-leave-it offer, and with probability $1-\alpha$ the innovator makes the take-it-or-leave-it offer. The value functions for the two types of firms

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for given product portfolios \((k_0, k_1)\) can be written as,

\[
 V_0(k_0, k_1) = \frac{k_0(\pi_0 - \delta \alpha \beta v_0) + k_1(\pi_1 - \delta \alpha \beta v_0)}{r + \delta (1 - \alpha \beta)} + k\Psi_0 \\
 V_1(k_0, k_1) = \frac{k_0(\pi_0 - \delta \alpha \beta v_0) + k_1(\pi_1 - \delta \alpha \beta v_0)}{r + \delta (1 - \alpha \beta)} + k\Psi_1,
\]

where the firm type conditional innovation ability value embodied in a product, \(\Psi_i\) is given by,

\[
\Psi_0 = \max_{\gamma_0} \frac{\gamma_0 v^* - wc(\gamma_0) + \lambda_0 \Psi_1}{r + \delta (1 - \alpha \beta) + \lambda_0}, \\
\Psi_1 = \max_{\gamma_1} \frac{\gamma_1 v_1 - wc(\gamma_1) + \lambda_1 \Psi_0}{r + \delta (1 - \alpha \beta) + \lambda_1},
\]

where \(\beta\) is the fraction of overall product creation by type 0 firms,

\[
\beta = \frac{\eta \phi_0 + K_0 \gamma_0}{\eta + K_0 \gamma_0 + K_1 \gamma_1},
\]

and \(v^* = \alpha v_{00} + (1 - \alpha) \sum_i \sum_j K_{ij} v_{ij}\) is the expected value of an innovation to a low ability firm. \(K_{ij}\) is the fraction of quality \(i\) products owned by type \(j\) firms in the economy.

The product dynamics are as follows,

\[
\dot{K}_{11} = \lambda_0 K_{10} + \eta \phi_1 + \gamma_1 (K_{11} + K_{01}) - \left[ \gamma_1 (K_{11} + K_{01}) + \eta \phi_1 + \lambda_1 \right] K_{11} \\
\dot{K}_{01} = \lambda_0 K_{00} - \left[ \gamma_1 (K_{11} + K_{01}) + \eta \phi_1 + \lambda_1 \right] K_{01} \\
\dot{K}_{10} = \lambda_1 K_{11} - \left[ \gamma_1 (K_{11} + K_{01}) + \eta \phi_1 + \lambda_0 \right] K_{10} \\
\dot{K}_{00} = \lambda_1 K_{01} + (\eta \phi_0 + \gamma_0 K_{00}) K_{00} - (\delta + \lambda_0) K_{00}.
\]

In the permanent types case, \(\lambda_0 = \lambda_1 = 0\) the system becomes,

\[
\dot{K}_{11} = \eta \phi_1 + \gamma_1 K_{11} - \delta (1 - \beta) K_{11} \\
\dot{K}_{00} = -\delta (1 - \beta) K_{00}.
\]

The stable solution to this system has \(K_{11} = 1 - K_{00} = 1\). That is, with permanent types, the buyout regime eventually eliminates all low types. Given that the estimation in Lentz and Mortensen (2008) calls for a mix of types in steady state, this suggests that as an empirical model, this case is of limited interest. It is however notable how the decentralized steady state product distribution across types coincides with the desired planner solution in this case, although depending on \(\alpha\), the low ability firm may be doing too much innovation in the decentralized solution.

The transitory types case does deliver a steady state with coexistence of low and high type innovators. Suppose to the contrary that \(K_{00} + K_{10} = 1 - K_{11} - K_{01} = 0\). Steady state, \(\dot{K}_{10} + \dot{K}_{00} = 0\)
then implies, $\lambda_1 K_{11} + \lambda_1 K_{01} = 0 \iff \lambda_1 = 0$. But this is in violation of the assumption of transitory types. Hence, it must be that the steady state has $K_{00} + K_{10} > 0$.

In Table 3 we show the counterfactual outcome of the estimated Lentz and Mortensen (2008) economy if buyouts are allowed. We show the two bargaining extremes: $\alpha = 0$ where the innovator makes the incumbent a take-it-or-leave it offer, and $\alpha = 1$ where the bargaining power is fully with the incumbent who makes the innovator a take-it-or-leave-it offer. In addition we also include the planner solution in this case giving the planner the option as to whether an innovation should be implemented or discarded. The optimal planner behavior is the same as buyout behavior, that is, a low firm type’s innovation is only implemented if it is into the market of a low quality product held by a low type firm. As can be seen, the two buyout cases differ primarily in the incentives for a low type to innovate. In the case where the innovator has the full bargaining power ($\alpha = 1$), innovation incentives are identical across firm types. In steady state, there are no low quality products. A low quality product can only enter a market of another low quality product. Once all low quality products have been eliminated, a low quality product is always bought out. Hence, a low type innovator will always be bought out. Since, the innovation ability value embodied in a product is
the same regardless of whether the owner is a high or low type firm, the low type innovator will extract the same value from an innovation that a high type innovator does. Consequently, the low type innovator does much more innovation in this equilibrium than in the no buyout case. Relative to the social planner solution, the $\alpha = 0$ case is equivalent in welfare terms to a 5% tax on planner consumption in the case of permanent types and a 6% tax in the case with transitory firm types.

The other extreme where the bargaining power rests fully with the incumbent ($\alpha = 1$), is considerably more attractive in terms of efficiency. In this case, the low type innovator’s incentives to innovate are kept at a minimum. Furthermore, low type innovations never take away innovation ability from the high type incumbents. As a result, this equilibrium is very close to the social planner. In the permanent firm types case, the tax on planner consumption that makes the two equivalent is 0.1% and in the transitory case it is 0.7%.

Thus, a market that facilitates buyouts can go a long way to address the inefficiency due to innovation by low ability innovators, but with the caveat that a substantial fraction of the efficiency gains can be lost as a function of the innovator’s ability to extract the surplus of the buyout.

6 Concluding remarks

Optimal innovation policy in the Lentz and Mortensen (2008) model with permanent firm types is one that discourages innovation by low type innovators. To the extent that the policy can sufficiently discourage creative destruction by low types the policy will also want to discourage entry, because the incumbents will not need replacement. This is the design of the optimal planner solution. The paper thus provides the core insights into what an optimal policy design must try to obtain. An example of such a policy is a fixed patent fee combined with R&D subsidies. The fee must sit in between the low ability innovator’s valuation of an innovation and that of the high ability innovator. This will perfectly discourage low ability innovators from engaging in innovation since they will not find it optimal to implement innovations upon discovery. The R&D subsidy is then used to offset the diminished innovation incentives of the high ability innovators due to the patent fee. Monopolistic competition remains an inefficiency problem as incumbents will choose a different output level relative to that of the planner, but the results in the paper suggest that it is a minor efficiency loss relative to the issue of allocating innovation resources optimally across heterogeneous innovators.

The paper documents the substantial inefficiency in the decentralized economy due to innovation ability heterogeneity across firms. A similar mechanism arises in Acemoglu et al. (2013) where substantial inefficiencies related to heterogeneity are also documented. Thus, in both cases, the
analyses emphasize the importance of designing optimal policy with a view of targeting incentives conditional on firm heterogeneity.

Galaasen and Irarrazabal (2014) estimate the Lentz and Mortensen (2008) model on Norwegian data and come to much the same conclusions regarding the strength of the selection. If anything, once R&D expenditure at the firm level is included in the estimation, they find a stronger selection effect and therefore contribution to growth. They find that innovation policy design such as R&D subsidies can have very large impacts on growth rates depending on how they are targeted to the innovation ability heterogeneity across firms.

Atkeson and Burstein (2014) discuss optimal innovation policy in models related to Klette and Kortum (2004), leaving out the issue of heterogeneity in innovation ability across firms. In contrast to the results in this paper, they find that innovation policy changes must have very moderate impacts on productivity growth. In their paper, the impact is naturally bounded by the growth rate of the economy and the social depreciation of innovation expenditures. Thus, in the absence of firm heterogeneity, the design of optimal innovation policy in their analysis comes to depend crucially on the existence of social depreciation of innovation expenditures. Our paper highlights that firm heterogeneity may well be of first order importance for the design of optimal innovation policy.
References


