COMPLETING THE SURVIVOR

DERIVATIVES MARKET

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Abstract

Survivorship is a risk of considerable importance to developed economies. Survivor derivatives are in their early stages and manage a risk which is arguably more serious than that managed by credit derivatives. This paper takes the approach developed by Dowd et al. [2006] for pricing survivor swaps and shows its application to other forms of survivor derivatives, namely forwards, basis swaps, futures, forward swaps, swaptions, futures options and combined option products. It concludes by considering applications for these products by hedgers and financial engineers.

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1. INTRODUCTION

A new global capital market, the ‘life market’, is developing and ‘survivor pools’ (or ‘longevity pools’ or ‘mortality pools’ depending on how one views them) will become the new asset class of the twenty-first century. It began with the securitization of insurance company life and annuity books (see, e.g., Millette et al [2002], Cowley and Cummins [2005], and Lin and Cox [2005]). But with investment banks, such as Goldman Sachs, entering the growing market in pension plan buyouts in the UK, for example, it is only a matter of time before full trading of ‘survivor pools’ in the capital markets begins.\footnote{1}

There are, however, a number of issues to resolve before that happens and key amongst these are the design and pricing of the survivor derivatives contracts that will be needed to hedge the survivor risk in the survivor pools.

Dowd et al. [2006] discussed the pricing and applications of vanilla survivor swap contracts, in which the pay-fixed party pays, at specified intervals over a defined period, a series of fixed payments predicated on the expected survival rates of a predefined cohort and receives in return a series of variable payments, predicated on the actual survival rates of this cohort. The receive-fixed party would obviously be contracted to the same cashflows, but in the opposite direction.

In this paper, we first derive a generalized pricing model from the approach taken by Dowd et al [2006] and then we apply to it to a further range of survivor derivatives,
namely forwards, forward swaps, basis swaps and futures. This is followed by the presentation of a model for pricing and hedging options on such derivatives. The paper concludes with some examples of applications of these products by hedgers and financial engineers.

2. PRICING IN AN INCOMPLETE MARKET

2.1 The Dowd et al. [2006] pricing methodology

Whilst, as will be seen throughout this paper, there are many similarities between survivor swaps and interest-rate swaps, Dowd et al. [2006] identify an important difference between them. The latter can be priced in a risk-free environment given the zero-arbitrage condition implied by the spot term structure. Lack of market completeness means that survivor swaps cannot be priced in such a manner. Thus, with the fixed premiums based on the expected survival probabilities at the time of contract agreement, a survival premium, $p$, should be added to the payments of the party perceived to be at greater risk.

Determination of the size and sign of the premium, $p$, is a function of three factors:

1. the level of interest rates, or, more specifically, the discount factors applying to each cashflow,

2. the market price of risk – in this context, Lin and Cox [2005], using comparable mortality data, estimate this to be 0.1792 for male annuitants and 0.2312 for female annuitants – and Dowd et al. [2006] factor this into the pricing by means of a Wang transform, and
3. the degree of longevity risk facing the benchmark cohort, i.e., the distribution of longevity shocks which constitute the principal risk in such contracts. For example, the benchmark cohort might be 65-year-old males from the national population of the country in which a particular contract originates. Dowd et al. [2006] propose a transformed beta distribution, allowing the distribution to have a positive (longevity increases) or negative (longevity decreases) mean.

In the vanilla survivor swap structure analyzed in Dowd et al. [2006], on each of the payment dates, \( n \), the fixed-rate payer pays the notional principal multiplied by \((1+p)H(n)\) to the floating-rate payer and receives in return the notional principal multiplied by \(S(n)\). \(H(n)\) is the survival rate of members of the benchmark cohort at time (and age) \( n \) which was expected at the time of contract agreement and is thus certain at that time. \(S(n)\) is the benchmark cohort’s actual survival rate at time (and age) \( n \) and is thus uncertain at the time of contract agreement. \(p\) is set at the agreement of the contract and remains fixed for its duration.

As with interest-rate swaps, a fairly valued survivor swap has zero value at inception. This is achieved by setting the present value of the fixed-rate leg equal to the expected present value of the floating-rate leg. Denoting each Wang-transformed expected floating-rate payment at time \( n \) by \(E[S(n)]\), and letting \(D_n\) denote the price at time zero of a bond paying a value of 1 at time \( n \), the fair value for an \(k\)-period survivor swap requires:
From this structure, it becomes possible to price a range of related derivatives securities.

2.2 A generalized pricing model

The pricing model presented in Dowd et al [2006] can be generalized to a wide range of related securities. For ease of presentation, assume first that all members of the cohort whose survivorship is the underlying asset in these derivatives are of the same age and second that payments due under the derivatives are made annually. We denote this age over the evolution of the derivatives by the following subscripts:

- $t$: their age at the time of the contract agreement
- $s$: their age at the time of the first payment
- $f$: their age at the time of the final payment
- $n$: their age at the time of any given anniversary ($t \leq n \leq f$)

Let us also denote

- $N$: the size of the cohort at age $t$
- $D_n$: the discount factor from age $t$ to age $n$
- $Y_n$: the payment per survivor due at age $n$ ($= 0$ for $n < s$)

Life tables observed at age $t$ give us the probability of survival from age $n$ to age $n+1$, conditional on having survived to age $n$. Let us denote this conditional probability by $p_n$. 

\[
(1 + p) \sum_{n=1}^{k} a_n D_n H(n) = \sum_{n=1}^{k} D_n E \hat{S}(n) \hat{A}_n
\]

\[
\hat{p}_n = \frac{\sum_{n=1}^{k} D_n E \hat{S}(n) \hat{A}_n}{\sum_{n=1}^{k} D_n H(n)} - 1
\]
The present value (at time $t$) of a fixed payment due at time $n$ is therefore

\[ (1 + p) \sum_{i=t}^{n-1} p_i \]

From which it follows that the present value, at time $t$, of the payments contracted by the pay-fixed party to the swap is

\[ (1 + p) \sum_{i=t}^{f} Y_n D_n \sum_{i=t}^{n-1} p_i \]

which (assuming, at this stage, $p = 0$) can be determined with certainty at time $t$.

The present value of the floating-rate leg is dependent on realized survivorship and cannot be determined with certainty at age $t$. Instead, we need to determine the expected value after accounting for longevity shocks and the Wang transform described above. We denote the longevity shock at age $n$ as $e_n$, and it will be recalled that Dowd et al. [2006] model this as a transformed beta distribution, allowing $e_n$ to take both positive and negative values. The present value of the floating rate leg can therefore be modeled as

\[ N.E \sum_{i=t}^{f} Y_n D_n \sum_{i=t}^{n-1} p_i \]

Since a fairly valued swap requires the present value of the two legs to be equal at time $t$, we can combine (4) and (5) to calculate a value, $p_{s,f}$, for any swap-type contract, valued at time $t$, whose payments start at age $s$ and finish at age $f$ in the following equation
Assuming constant interest rates, the only random variable in equation (6) is $e_i$.

3. PRICING SURVIVOR DERIVATIVES

We can use the pricing methodology outlined in the previous section to price some key survivor derivatives.

3.1 Survivor forwards

Just as an interest-rate swap is essentially a portfolio of FRA contracts, so a survivor swap can be decomposed into a portfolio of survivor forward contracts. Consider two parties, each seeking to fix payments on the same cohort of 65-year-old annuitants. The first enters into a $k$-year, annual-payment, pay-fixed swap as described above. The second enters into a portfolio of $k$ annual survivor forward contracts, each of which requires payment of the notional principal multiplied by $(1 + p_n)H(n)$ and the receipt of the notional principal multiplied by $S(n)$, $n = 1, 2, \ldots, k$. $p_n$ differs for each $n$. The present value of the commitments faced by the two investors must be equal at the outset. Thus

\[
(1 + p) \sum_{n=1}^{k} D_n H(n) = \sum_{n=1}^{k} D_n H(n)(1 + p_n) \tag{7}
\]

\[
p = \frac{\sum_{n=1}^{k} D_n H(n)p_n}{\sum_{n=1}^{k} D_n H(n)} \tag{8}
\]
Hence it follows that $p$ in the survivor swap must be equal to the weighted average of the individual $p_n$'s in the portfolio of forward contracts, in the same way that the fixed rate in an interest-rate swap is equal to the weighted average of the forward rates.

### 3.2 Forward survivor swaps

Given the existence of the individual $p_n$'s for the portfolio of forward contracts, it becomes possible to price forward survivor swaps in a risk-free environment. In such a contract, the parties would agree at time zero, the terms of a survivor swap contract which would commence at some point in the future. Not only would such a contract meet the needs of those who are committed to providing pensions in the future, but this instrument could also serve as the hedging vehicle for survivor swaptions – see later.

The pricing of such a contract would be quite straightforward. As shown above, the position could be replicated by entering into an appropriate portfolio of forward contracts. Thus $p_{\text{forward}}$ – the risk premium for the forward swap contract - must equal the weighted average of the individual $p_n$'s used in the replication strategy. More straightforwardly, $p_{\text{forward}}$ can be derived directly from equation (6).

### 3.3 Basis swaps

Dowd et al. [2006] also discuss, but do not price, a floating-for-floating swap, in which the two counterparties exchange payments based on the actual survivorships of two different cohorts. Following practice in the interest-rate swaps market, such contracts should be called basis swaps. The analysis above shows how such contracts could be
priced. First, consider two parties wishing to exchange the notional principal\(^{ii}\) multiplied by the actual survivorship of cohorts \(j\) and \(k\). Assume equal notional principals and denote the risk premia and untransformed expected survival rates for such cohorts by \(p_j\) and \(p_k\) and \(H_j(n)\) and \(H_k(n)\), respectively. Given the existence of vanilla swap contracts on each cohort, the present values of the fixed leg of each such contract will be

\[
(1 + p_j) \sum_{n=1}^{\infty} D_n H_j(n) \quad \text{and} \quad (1 + p_k) \sum_{n=1}^{\infty} D_n H_k(n),
\]

respectively, and the zero value argument shows that these must also be the present values of the expected floating rate legs. Since there is no uncertainty about these fixed rate payments, it is possible to calculate with certainty, an exchange factor, \(k\), such that

\[
(1 + p_j) \sum_{n=1}^{\infty} D_n H_j(n) = k (1 + p_k) \sum_{n=1}^{\infty} D_n H_k(n) \quad \text{(9)}
\]

\[
k = \frac{(1 + p_j) \sum_{n=1}^{\infty} D_n H_j(n)}{(1 + p_k) \sum_{n=1}^{\infty} D_n H_k(n)} \quad \text{(10)}
\]

from which it follows that the fair value in a floating-for-floating basis swap requires one party to make payments determined by the notional principal multiplied by \(S_j(n)\) and the other party to make payments determined by the notional principal multiplied by \(k S_k(n)\); \(k\) is determined at the outset of the basis swap and remains fixed for the duration of the contract. The same approach can be used to price forward basis swaps, in which case \(k\) is defined as.
3.4 Cross-currency basis swaps

The same approach can be used to price a cross-currency basis swap, in which the cohort-
j payments are made in one currency and the cohort-k payments in another.

The single currency floating-for-floating basis swap analyzed in the preceding sub-
section required the cohort-j payer to pay \( S_j(n) \) at each payment date and to receive
\( k S_k(n) \) and this contract has zero value at the outset. Now consider a similar contract in
which the cohort-j payments are made in currency j and the cohort-k payments made in
currency k. Assume the spot exchange rate between the two currencies is \( F \) units of
currency k for each unit of currency j.

From the arguments above, we can determine the present value of each stream. These are
\[
\frac{(1 + p_j)^f_{\alpha_{n-s}} D_n H_j(n)}{(1 + p_k)^f_{\alpha_{n-s}} D_n H_k(n)}
\]

respectively, expressed in their respective
currencies. Multiplying by \( F \) expresses the value of the cohort-k stream in units of
currency j. The standard requirement that the two streams have the same value at the time
of contract agreement is again achieved by determining an exchange factor, \( k_{FX} \), such
that
Thus, in the case of a floating-for-floating cross-currency basis swap, on each payment
date, \( n \), one party will make a payment of the notional principal multiplied by \( S_j(n) \) and
receive in return a payment of \( k_{FX} S_k(n) \). Each payment will be made in its own currency,
so that exchange rate risk is present. However, in contrast with a cross-currency interest-
rate swap, there is no exchange of principal at the termination of the contract, so the
exchange rate risk is mitigated. The same procedure is used for a forward cross-currency
basis swap, except that the summation in equation (13) above is from \( n = s \) to \( f \), rather
than from \( n = 1 \) to \( f \). Since the desire is to equate present values, it should be noted that
the spot exchange rate, \( F \), is applied in this equation, rather than the forward exchange
rate.\(^{iv}\)

\[
(1 + p_j)^{\hat{a}} D_n H_j(n) = k_{FX} F (1 + p_k)^{\hat{a}} D_n H_k(n)
\]  
\[
k_{FX} = \frac{(1 + p_j)^{\hat{a}} D_n H_j(n)}{F (1 + p_k)^{\hat{a}} D_n H_k(n)}
\]  

3.5 Futures contracts

The need to customize the specification of the cohort(s) in the derivative contracts
described above implies trading in the over-the-counter (OTC) market. However, an
exchange-traded instrument offers attractions to many. As shown above, the uncertainty
in survivor swaps is captured in the factor \( p \), and a futures contract with \( p \) as the
underlying asset would serve a useful function both as a hedging vehicle and for
speculators who wished to achieve exchange-traded exposure to survival risk.
Such contracts would be based on the value of $p$ for a specified cohort over a specified time frame at a specified time in the future, in much the same way as the Eurodollar futures contract is based on 3-month Eurodollar LIBOR. Thus, if the notional principal were $1$ million and the timeframe were 1 year, a long position in a December 2008 contract at a price of 3\% would notionally commit the holder to pay $1.03$ million multiplied by the expected size of the cohort surviving and to receive $1$ million multiplied by the actual size. This is a notional commitment only; in practice, the contracts would be cash-settled, so that if the spot value of $p$ at expiry, which we denote $p_{\text{expiry}}$, were 4\%, the investor would receive a cash payment of $10,000: \text{i.e. } 1\% \text{ of } $1$ million$'.

Cohort specification would need to be determined by research among likely users of the contracts. Too many cohorts would spread the liquidity too thinly across the contracts: too few cohorts would lead to excessive basis risk.

Determination of the settlement price at expiry would be achieved by dealer poll. Such futures contracts could be expected to serve as the principal driver of price discovery in the survivorship market, with dealers in the OTC market using the futures prices to inform their pricing of customized survivor swap contracts.
4. SURVIVOR SWAPTIONS

Where there is demand for linear payoff derivatives, such as swaps, forwards, and futures, there is generally also demand for option products. A survivor swaptions contract is an obvious extension of the products discussed above.

4.1 Specification of swaptions

The specification of such options is quite straightforward. Consider a forward survivor swap, described above. A pension provider might choose to lay off anticipated survival risk by entering into the pay-fixed side of such a contract at zero cost, thus committing itself to payments of \((1 + p_{\text{forwardswap}})H(n)\) and to receipts of \(S(n)\). Alternatively, the provider might choose to pay an option premium which would give the right but not the obligation to enter into such a swap. Clearly, the exercise decision would depend on whether the market rate of \(p\) at expiry for such a swap was greater or less than \(p_{\text{forwardswap}}\). Thus, in the case described, \(p_{\text{forwardswap}}\) is the strike price of the swaption and the option is a de facto at-the-money option on \(p_{\text{forwardswap}}\).

The strike price of the option does not have to be \(p_{\text{forwardswap}}\). It can be any value that the parties agree. However, using \(p_{\text{forwardswap}}\) as an example shows how put-call parity applies to such swaptions. An investor who purchases a payer swaption, at strike price \(p_{\text{forwardswap}}\), and writes a receiver swaption with an identical specification has synthesized a forward survivor swap. Since such a contract could be opened at zero cost, it follows that a synthetic replication must also be available at zero cost. Hence the premium paid for the payer swaption must equal the premium received for the receiver swaption.
Exercise of such swaptions could be settled either by delivery (i.e. the parties enter into opposite positions in the underlying swap) or by cash, in which case the writer pays the holder

\[ \text{Max} \left\{ fp_{\text{ expiry}} - fp_{\text{ strike}} \prod_{n=1}^{f} N(D_{\text{ expiry,n}}Y_{f}H(n)) \right\} \text{ with } f \text{ set as } +1 \text{ for payer swaptions and } -1 \text{ for receiver swaptions, } p_{\text{ expiry}} \text{ representing the market value of } p \text{ at the time of swaption expiry, } p_{\text{ strike}} \text{ representing the strike price of the swaption and } D_{\text{ expiry,n}} \text{ representing the price at option expiry of a bond paying } 1 \text{ at time } n^{vii}. \]

4.2 Pricing swaptions

It was noted earlier, in respect of equation (6), that the only random variable in the pricing of \( p \) (in the current case, \( p_{\text{ forwardswap}} \)) is \( e \). Dowd et al. [2006] model this as a transformed beta distribution. One consequence of such modeling is that, in accordance with the Central Limit Theorem, \( p_{\text{ forwardswap}} \) is approximately normally distributed, \( p_{\text{ forwardswap}} \sim N(p_{\text{ forwardswap}}, s^{2}) \) with \( s^{2} \) presented in annual terms in accordance with convention\(^{viii}\). Normally distributed prices are rare in financial assets, since such a distribution permits prices to become negative. In the case of \( p_{\text{ forwardswap}} \), negative values are perfectly feasible. Dawson et al [2007] derive and test a model for pricing options on assets with normally distributed prices and application of their model to survivor swaptions gives

\[ p_{\text{ payer}} = e^{-rt} \left( (p_{\text{ forwardswap}} - p_{\text{ strike}})N(d) + s \sqrt{t} N(d) \right) \]

(14)

\[ p_{\text{ receiver}} = e^{-rt} \left( (p_{\text{ strike}} - p_{\text{ forwardswap}})N(-d) + s \sqrt{t} N(d) \right) \]

(15)
This valuation model is predicated on the standard Black-Scholes [1973] assumptions, including, *inter alia*, continuous trading in the underlying asset. In practice, it is unlikely that a liquid market will be found in the specific forward swap underlying any given swaption. Furthermore, survivor swaption dealers will likely need to hedge positions in swaptions on different cohorts, which will be self-hedging to a certain extent, leaving only residual risk to hedged. A liquid market in the $p$ futures described above would mitigate these problems. Option portfolio software would translate the residual risk into futures contract equivalents, thus dictating (and possibly automatically submitting) the orders necessary for maintaining delta-neutrality. Given a variety of cohorts, basis risk could be a problem, but a diversified set of cohorts will act to diversify the basis risk and, as noted earlier, an important pre-condition of futures introduction is research among industry participants to optimize the number of cohorts for which $p$ futures would be introduced.

Most liquid futures markets create a demand for futures options, and this leads to the possibility of $p$ futures options. Pricing such contracts is also accomplished in (14) - (15) above. All that is necessary is to substitute $p_{futures}$ for $p_{forwardswap}$ as the value of the optioned asset.

4.3 *The Greeks*

Dawson *et al* [2007] derive the Greeks for their option model. Table 1 below consolidates the formulae for option values and the delta, gamma, rho, theta and vega for payer and receiver swaptions.
4.4 **Survivor caps and floors**

The parallels with the interest rate swaps market can be carried still further. In the interest rate derivatives market, caps and floors are traded, as well as swaptions. These offer more versatility than swaptions, since each individual payment is optioned with a caplet or a floorlet, whilst a swaption, if exercised, determines a single fixed rate for all payments. The extra optionality comes at the expense of a significantly increased option premium. Such procedures could be implemented in the market for survivor derivatives. Again, equations (14) – (15) can be used to price the caplets and floorlets. In place of \( p_{\text{forwardswap}} \), each individual caplet or floorlet will require the \( p \) value for the survivor forward contract which serves as the underlying asset.

5. **HEDGING APPLICATIONS**

Consider a pension fund with a liability to pay $10,000 annually to each survivor of a cohort of 1,000 65-year-old males. For this example, assume a flat yield curve of 6%. Using the same life tables as Dowd et al. [2006], the present value of this liability is approximately $106.1 million and the pension fund is exposed to survivor risk. We consider several strategies to mitigate this risk.

The first hedging strategy which the fund might undertake is to enter into a 50-year survivor swap. As shown in (6), the price of this swap depends on the parameters of the
beta distribution which are used to model $e_i$. Using, values of 5,050 and 4,950 for the beta distribution generating the $e_i$ values gives a swap rate of 3.1667%. Entering a pay-fixed swap at this price would remove the survivorship risk entirely from the pension fund, but increase the present value of its liabilities to approximately $109.5 million (i.e. $106.1 million ×1.031667).

The second hedging strategy has the pension fund choosing to accept survivor risk for the next five years, and entering into a forward swap today to hedge survivor risk from age 71 onwards. The value of $p$ is now 5.0910%. However, since this premium will not be paid for the first five years of the liability, when by definition, survivorship will be highest, the impact of the higher premium is much reduced. The $106 million referred to above can be decomposed into two components. The present value of the liabilities for the first five years is approximately $40.7 million, with approximately $65.4 million for the remainder of the survivorship. Thus, accepting survivor risk for the first five years, but hedging it for the remainder of the survivorship would increase the present value of the fund’s liabilities to approximately $109.4 million ($40.7 million + $65.4 million × 1.05091).

The fact that both strategies considered so far appear to increase the present value of the fund’s liabilities to approximately $109.5 million implies that there is almost zero benefit, and therefore almost zero cost, to hedging for the first five years. This is borne out by analysis of the forward curve. An alternative to the 50 year swap is for the hedger to use a portfolio of 50 survivor forward contracts. The no-arbitrage arguments presented earlier
mean that the cost of hedging in this manner must equal the cost of hedging with the swap. In this case it amounts to about $3.4 million ($106.1 \times 0.031667). Figure 1 below shows the distribution of this cost across the evolving ages of the cohort. It can be seen the cost is minimal during the early years\(^x\) (since there has been little time for longevity shocks to make any impact) and during the late years (when the cohort size has declined to very small numbers). The maximum cost occurs during the middle years, when there has been sufficient time for longevity shocks to make a significant impact and the cohort is still sufficiently large for this impact to be economically significant.

*Insert Figure 1 about here*

These three hedging strategies fix the commitments of the pension fund, either for the entire period of survivorship or for all but the first five years. The fund would have no exposure to any financial benefits of decreasing survivorship. These benefits could be obtained by our fourth hedging strategy, namely a position in a five year payer swaption. Again, the fund would accept survivor risk for the first five years, but would then have the right, but not the obligation, to enter a pay fixed swap. Using the same beta parameters as above, the annual volatility of the forward contract is 0.51% and the premium for an at-the-money forward swaption is 0.3378%. It was seen above that the present value of these liabilities at age 65 was $65.4 million. Thus the pension fund would pay an option premium of approximately $221,000 to gain the right, but not the obligation to fix the payments at a \(p\) rate of 5.0910% thereafter. If survivorship declines,
the fund will not exercise the option and will have lost merely the $221,000 swaption premium.

Rather more optionality could be obtained through a survivor cap, rather than a survivor swaption. As described above, this is constructed as a portfolio of options on survivor forwards. The expiry value of each option is $NYH(n)\text{Max}[0, p_{\text{settlement}} - p_{\text{strike}}]$, in which $p_{\text{settlement}}$ refers to the value of $p$ prevailing for that particular payment at the time the settlement is due. Its value would need to be determined by dealer poll or any other market-rated mechanism. It is defined by equation (1) as being $(S(n) - H(n)) \div H(n)^{xi}$.

Thus, on each payment date, $n$, the pension fund holding a survivor cap effectively pays $NY(1+\text{Min}[p_{\text{settlement}}, p_{\text{strike}}])H(n)$ on each payment date $n$ and receives $NYS(n)$ in return, with this receipt designed to match their liability to their pensioners. This extra optionality comes at a price: it was noted above that the option premium for a five-year survivor swaption, at strike price 5.0910%, with our standard parameters, was approximately $220,000. The equivalent survivor cap (in which the pension fund again accepts survivor risk for the first five years, but hedges it with a survivor cap again struck at 5.0910% for the remaining 45 years) would carry a premium of $1.84 million.

Our next hedging strategy is a zero-premium collar. Again using the same beta parameters, the premium for a payer swaption with a strike price of 5.5% is 0.2068%. The same premium applies to a receiver swaption with a strike price of 4.682%. Thus a zero premium collar can be constructed with a long position in the payer swaption financed by a short position in the receiver swaption. With such a position, the pension
fund would be committed to a pay-fixed swap contract at cohort age 70, but at a rate of no more than 5.5% and no less than 4.682%.

One downside to such a zero-premium collar is that the pension fund puts a floor on its potential gains from falling survivorship. An alternative zero-premium structure which would give unlimited participation in falling survivorship, as well as capping the price of rising survivorship at 5.5%, is to finance the purchase of the payer swaption by the sale of a receiver swaption of the same strike price. Since the payer swaption is out of the money, its premium is considerably less than that of the in the money receiver swaption – 0.2068% versus 0.5124%. Thus to finance a payer swaption on the $65.4 million liabilities, it would be necessary to sell a receiver swaption on only $26.4 million (= $65.4 million × 0.002068 ÷ 0.005124) of liabilities. The pension fund would thus enjoy, at zero premium, complete protection against the survival premium rising above 5.5%, and unlimited participation, albeit at a little less than 60c on the dollar, on the survivorship premium being less than this.

The hedging strategies presented so far in this section serve to transfer the survival risk embedded in a pension fund to an outside party. In all cases, this is done either at a cost or, in the case of the zero-premium option structures, at the willingness to forego some of the financial benefits of falling survivorship. An alternative risk management technique would be for the pension fund to diversify its exposure. Using a basis swap or a cross-currency basis swap, the pension fund could swap some of its exposure to the existing cohort for an exposure to a different cohort (either in its domestic economy or overseas).
Hence, in return for receiving cashflows to match some of its obligations to its pensioners, it would assume liability for paying according to the actual survivorship of a different cohort. As the derivations of equations (11) and (13) show, this does not change the value of the pension fund’s liabilities, but, assuming less than perfect correlation between the survival rates of the two cohorts, the pension fund will enjoy benefits of diversification.

It would be wrong to leave this discussion of available hedging techniques without a brief discussion of structured products. The financial press has already begun to make comparisons between survivor derivatives and credit derivatives. The creative structuring which has been so prevalent in credit derivatives, can be expected in survivor derivatives as well. The simplest example would be for the pension fund to issue a 50-year survivor-linked note, paying a coupon of LIBOR + 3.1667% (i.e., the premium for a 50-year survivor swap). The holder of the note would enjoy a significant premium over a standard floating rate note, but the cashflows paid would be adjusted each year according to the evolution of the cohort survivorship. Thus the pension fund would be transferring the survivor risk, but in a funded, rather than an unfunded form.

The next step up in complexity would be for a market for tranched products on a defined cohort. The notional principal on the instruments would be linked to the evolving survivorship and the impact of mortality would first be borne by the equity tranche, until that is eliminated at which point the impact would be borne by successively higher tranches.
6. CONCLUSION

The products presented and priced in this paper go a long way towards completing the market for survivor derivatives. Further developments can be expected. Reference was made earlier to the incorporation of quanto features to eliminate currency risk. Knock-out and knock-in features might also be anticipated. For example, a pension fund might be quite willing to forego protection against increasing survivorship in the event of an avian flu pandemic. Alternatively, such a fund might seek protection which knocks-in in the event of a major breakthrough in the treatment of cancer. Survivorship is a risk of considerable importance to developed economies. It is surprising that the market has been so slow to develop derivative products to manage such risk. However, parallels with the interest rate and credit derivatives markets seem apposite: once the initial products were launched, the growth in these markets was rapid. Survivor derivatives are in their early stages and manage a risk which is arguably more serious than that managed by credit derivatives. Market completion is both important and inevitable.
REFERENCES


Figure 3: Hedging cost using individual forward contracts
<table>
<thead>
<tr>
<th></th>
<th>Payer swaptions</th>
<th>Puts or receiver swaptions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Option value</strong></td>
<td>( e^{-r t} \left( \frac{1}{2} p_{\text{forward swap}} - X \right) N(d) - \frac{s \sqrt{t}}{s} N\left(\frac{d}{s}\right) )</td>
<td>( e^{-r t} \left( \frac{1}{2} X - p_{\text{forward swap}} \right) N(-d) + \frac{s \sqrt{t}}{s} N\left(\frac{-d}{s}\right) )</td>
</tr>
<tr>
<td><strong>Delta</strong></td>
<td>( e^{-r t} N(d) )</td>
<td>(-e^{-r t} N(-d))</td>
</tr>
<tr>
<td><strong>Gamma</strong></td>
<td>( \frac{e^{-r t}}{s \sqrt{t}} N\left(\frac{d}{s}\right) )</td>
<td>( \frac{e^{-r t}}{s \sqrt{t}} N\left(\frac{-d}{s}\right) )</td>
</tr>
<tr>
<td><strong>Rho (per percentage point rise in rates)</strong></td>
<td>(- \frac{t P_{\text{payer}}}{100})</td>
<td>(- \frac{t P_{\text{receiver}}}{100})</td>
</tr>
<tr>
<td><strong>Theta (for 1 day passage of time)</strong></td>
<td>( \frac{2\sqrt{t} r P_{\text{payer}} - e^{-r t} s N\left(\frac{d}{s}\right)}{730\sqrt{t}} )</td>
<td>( \frac{2\sqrt{t} r P_{\text{receiver}} - e^{-r t} s N\left(\frac{-d}{s}\right) + 4\sqrt{t} re^{-r t}(F - \frac{1}{2})}{730\sqrt{t}} )</td>
</tr>
<tr>
<td><strong>Vega (per percentage point rise in volatility)</strong></td>
<td>( \frac{e^{-r t} \sqrt{t} N\left(\frac{d}{s}\right)}{100} )</td>
<td>( \frac{e^{-r t} \sqrt{t} N\left(\frac{-d}{s}\right)}{100} )</td>
</tr>
</tbody>
</table>
Notes

i Dunbar [2006]. On 22 March 2007, the Institutional Life Markets Association (ILMA) was established in New York by Bear Stearns, Credit Suisse, Goldman Sachs, Mizuho International, UBS and West LB AG. The aim is to ‘encourage best practices and growth of the mortality and longevity related marketplace’.

ii Following practice in the interest rate swaps market, we avoid constant reference to the notional principal henceforth by quoting swap prices as percentages. The notional principal in survivor swaps can be expressed as the cohort size, $N$, multiplied by the payment per survivor at time $n$, $Y_n$.

iii In foreign exchange markets parlance, currency $j$ is the base currency and currency $k$ is the pricing currency.

iv The foreign exchange risk could be eliminated by use of a survivor swap contract in which the payments in one currency are translated into the second currency at a predetermined exchange rate, similar to the mechanics of a quanto option. Derivation of the pricing of such a contract is left for future research.

v Dowd et al. [2006] show that $p$ can take values between $-1$ and $1$. Since negative values are rare for traded assets, user systems might be unable to cope. To avoid problems, the market for $p$ futures contracts could either be quoted as $(1 + p)$, with $p$ as a decimal figure, or else follow interest rate futures practice and be quoted as $(100 - p)$ with $p$ expressed in percentage points.

vi In swaptions markets, usage of terms such as put and call can be confusing. Naming such options payer (i.e. the right to enter into a pay-fixed swap) and receiver (i.e. the right to enter into a receive-fixed swap) swaptions is preferable. We denote the options premia for such products as $P_{\text{payer}}$ and $P_{\text{receiver}}$ respectively.

vii Under Black-Scholes (1973) assumptions, interest rates are constant, so that $N \sum_{n=1}^{\infty} D_{\text{exp}}(n) Y_n H(n)$ is known from the outset. Let us call this the settlement sum. Following the approach in footnote ii, we can dispense with constant repetition of the settlement sum by expressing option values in percentages and recognising that these can be turned into a monetary amount by multiplying by the settlement sum.

viii We should report that large Monte Carlo simulations (250,000 trials) across a sample of different sets of input parameters for $p_{\text{forward}}$ reveal small but statistically significant non-zero skewness values. Furthermore, whilst excess kurtosis is insignificantly different from zero when drawing from beta distributions with relatively low standard deviations, the distribution of $p_{\text{forward}}$ is observed to become increasingly platykurtic as the standard deviation of the beta distribution is increased. Our option pricing model can deal with these effects in the same way as the Black-Scholes models deal with skewness and leptokurtosis. In this case of platykurtosis, a volatility frown, rather than a smile, is dictated.

$\times$ The $p$ value for a swap covering the just first five years of this exposure is a mere 0.0798%, representing a hedging cost of approximately $32,500.

ix Equation (1) is $(1 + p) \hat{a}_k \sum_{n=1}^{\infty} D_n H(n) = \hat{a}_k \sum_{n=1}^{k} D_n E \hat{S}(n) \hat{u}$. However, since the forward contract refers to just one specific maturity, $n$, the summation operator is redundant. Furthermore, since there is no time difference between this valuation and the payment, the discount factors are irrelevant. Finally, at expiry, $S(n)$ is known with certainty, so that the expectations operator is irrelevant. Thus

$$\left(1 + p\right) H(n) = S(n)$$

$$p = \frac{H(n) - S(n)}{H(n)}$$

xii See, for example, Wighton and Tett [2006].