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A Complete Characterization of Pure Strategy Equilibrium in Uniform Price IPO Auctions

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Collusive equilibria in share auctions despite being the focus of previous theoretical research, have received little empirical or experimental support. We develop a theoretical model of uniform price initial public offering (IPO) auctions and show that there exists a continuum of pure strategy equilibria where investors with a higher expected valuation bid more aggressively and as a result the market price increases with the market value. The collusive equilibria lie in fact on the boundary of this set, which is obtained under stricter conditions when demand is discrete than in the continuous format. Our results have important implications for the design of IPO auctions.

Keywords:
IPO, uniform price auction, divisible goods, share auctions, tacit collusion

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1 Introduction

Uniform price auctions are widely used for selling multiple units of goods to many buyers. Examples include Treasury bills, spectrum, electricity and initial public offerings (IPOs). In uniform price auctions, each bidder submits quantity-price combinations indicating the price that she is willing to pay for obtaining the corresponding quantities. The market-clearing price is determined according to the accumulative quantity-price schedule; all bids exceeding the market-clearing price are accepted and bidders pay the market-clearing price for all units won.

Some countries use this type of auction for IPOs (e.g. Israel, U.K. and U.S.). In many countries, other IPO mechanisms such as fixed price offerings and Bookbuilding are also available. The choice of the IPO mechanism is important for both issuers and market regulators. From the point of view of issuers a proper IPO mechanism generates high revenues and achieves full subscription, while from the point of view of regulators it can help to maintain a stable stock market. In evaluating different IPO mechanisms, the ability to generate revenues is ranked highly by both practitioners and academics.

Most theoretical research so far has focused on one particular type of equilibrium in uniform price auctions, namely the tacit collusion equilibrium (e.g. Biais and Faugeron-Grouzet, 2002; Back and Zender, 1993; Wang and Zender, 2002; Wilson, 1979). This equilibrium predicts low revenues for sellers. It also suggests that an increase in the number of bidders cannot improve sellers' revenues because bidders' bids are independent of their expectations about the value of shares. According to this equilibrium prediction, uniform price auctions are inferior to other IPO mechanisms in terms of raising revenue.

However, field observations and experimental evidence suggest that collusion is not very common in practice (e.g. Kandel, Sarig and Wohl, 1997; Sade, Schnitzlein and Zender, 2006; Zhang, 2006). This motivates us to look for other equilibrium solutions for uniform price auctions. Of particular interest is whether performance comparisons of different mechanisms are sensitive to equilibrium selection.

In this paper, we fully characterize the pure strategy equilibrium set of the Biais and Faugeron-Grouzet (2002) model for the discrete strategy space case. The reason that we focus on the discrete case is that in real IPO markets or other share auctions it is impossible to submit continuous demand functions. This is either because submitting full-demand schedules is costly (in preparation or submission), or because there exists a minimum increment requirement in prices.[1] However, the equilibrium that holds in the continuous case does not necessarily work in the same way with discrete bids. We find a continuum of equilibria in which the tacit collusion equilibrium is only on the boundary of the whole equilibrium set. In the new set of equilibria demand reduction is generally inevitable and the market price is positively related with the market value. On the other boundary of the equilibrium set the market price is equal to the market value.

The new set of equilibria has some properties that are consistent with field observations and experimental evidences: more competition, i.e. an increase in the number of bidders, improves revenues (Kandel, Sarig and Wohl, 1999); bidders place more bids as the number of bidders increases (Malvey, Archibald and Flynn, 1997); bidders with higher expected market values bid more aggressively than those with lower expected market values (Zhang, 2006). The existence of the new set of equilibria may explain why uniform price auctions are still widely used despite the low revenue prediction of the tacit collusion equilibrium.

The rest of the paper is organized as follows. We discuss the related literature in Section 2. The model is introduced in section 3. In section 4, we demonstrate that submitting flat demand functions, i.e. bidding for all shares at one price, is not an equilibrium strategy. This leads us to a list of necessary conditions that an equilibrium should satisfy. In section 5, we first derive the tacit collusion equilibrium and then complete the characterization of the set of pure strategy equilibria by examining cases where the behaviour of different types of bidders is asymmetric. We sum up our results and conclude in section 6.

2. Related literatures

Uniform price auctions used in IPOs are close to share auctions in the sense that there are a large number of identical stocks for sale. In share auctions, if a bidder has a

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[1] For example, the online IPO auction company WR Hambrecht+ Co. used to require a minimum bid increment of 1/32 of a dollar, which has been changed to 1 cent in 2005. On, the Singapore stock market, there are five tick size categories (i.e. price increments) ranging from 0.5 cents for stocks priced less than \$1.00 to 10 cents for stocks priced above \$10 (Comerton-Forde, Lau and McInish, 2003). On stock markets the minimum order quantity for a new issue varies; in the U.S. it is often 100 shares. There are also usually restrictions on the number of price-quantity bids allowed; for example, in the Spanish electricity market generators may submit up to twenty-five price-quantity pairs (Harbord, Fabra and von der Fehr, 2002), while a maximum of three bids are allowed in Italian treasury bills markets (see Scalia, 1996, or Kremer and Nyborg, 2004, p.858).

positive probability of influencing the price, in a situation where the bidder obtains some allocation, then she has an incentive to shade her bid (Ausubel and Cramton, 2002). Wilson (1979), Ausubel and Cramton (2002), Back and Zender (1993) and Maxwell (1983) demonstrate the existence of multiple equilibria which yield a sale price well below the competitive price. In such tacit collusion equilibria, bidders place bids regardless of their expected market value and, therefore, the market price provides little information about the market value. Submitting steep demand schedules implies that it would take a big price increase to increase one's allocation and, consequently collusive strategies become self-enforcing in this non-cooperative game. Despite the uniform price auction's advantage of increased competition (Friedman, 1961, 1990), existing theoretical results suggest that because bidders can manipulate the market price, this type of auction will generate low revenues for sellers who would not benefit from the increase in competition.

Harbord, Fabra and von der Fehr (2002) claim that a collusive equilibrium only exists when the demand is continuous while in the discrete case there exists a unique, Bertrand-like equilibrium. Kremer and Nyborg (2004) show that the collusive equilibrium of the share auction models of Wilson (1979) and Back and Zender (1993) does not survive when bidders only make a finite number of bids. Instead, a Bertrand-like price competition is induced. In the discrete version of the Wilson (1979) and Back and Zender (1993) model, the equilibrium market price can be as high as the market value even when investors face an uncertain supply. They suggest that this may explain why uniform price auctions are still popular in spite of the severe theoretical warnings. [2]

In fact, the empirical evidence generally does not support the tacit collusion equilibrium. Although collusive behaviour is observed among large dealers in the market for Treasury bills when uniform price auctions are used, collusion is less severe than under the discriminatory setting (e.g. see Malvey, Archibald and Flynn (1997) for the U.S. [3] and Umlauf (1993) for Mexico). Keloharju, Nyborg and Rydqvist (2003) using individual bidder data from Finnish treasury auctions, find little evidence of collusion. An empirical study on the Zambian foreign exchange

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[2] Biais, Bossaerts, and Rochet (2002) show that an optimal IPO mechanism, which maximizes the issuer's proceeds, can be implemented through a uniform price rule (however, the mechanism does not work the same as the uniform price auction in this paper).

[3] After an experiment with uniform price auctions on two-year and five-year notes that started on September 1992, the U.S. Treasury switched entirely to the uniform-price auction in November 1998 (Ausubel, 2002).

market, where a large number of relatively small bidders are involved, provides no evidence of collusive behaviour, though demand reduction is evident (Tenorio, 1993). In all these markets, higher participation rates are reported, which indicates that the market is widen and competition is encouraged. Though it has been argued that the existence of collusive equilibria in the uniform price auction is one reason for the Britain's decision of adopting a discriminatory auction format in markets for electricity (Klemperer, 2001), a report for the California power exchange concludes that a shift from a uniform to a discriminatory auction is unlikely to result in lower electricity prices (Kahn et. al, 2001; see Harbord, Fabra and von der Fehr, 2002). In Israel's IPO market, contrary to the steep collusive demand function derived by Biais and Faugeron-Grouzet (2002), the demand schedule is flat and elastic (Kandel, Sarig and Wohl, 1997). A common feature of the auctions where collusion is observed is that a relatively small number of bidders compete on a relatively large number of items; for example in a spectrum auction (Engelbrecht-Wiggans and Kahn, 2005).[4]

There are also plenty of experimental evidences. Engelbrecht-Wiggans, List, and Reiley (2006) suggest that it is relatively difficult to find statistically significant evidence of demand reduction when there are more than two bidders. Moreover, Porter and Vragov (2003), in an experiment with two bidders who each has two units demand and private information, report that though demand reduction is observed, bids for low valued units is higher than the equilibrium prediction of zero. Sade, Schnitzlein and Zender (2006) find little evidence of collusive behaviour in uniform price auctions even when communication is allowed and when financial professionals participate.[5] In markets for IPO where there are many investors, including a large number of usually inexperienced retail investors, a collusive equilibrium would be more difficult to achieve. Zhang (2006) compares the performances of uniform price auctions and another IPO mechanism called fixed price offerings. Given the tacit collusion equilibrium that earlier theoretical work has focused on, uniform price auctions should generate lower revenues between the two mechanisms. However, the results of the experiments are contrary to this prediction, because the tacit collusion equilibrium was not achieved in the experiment. Instead, subjects place bids according

[4] Other collusive behaviour is reported in electricity auctions in England and Wales where the same bidders compete repeatedly (Wolfram, 1998).

[5] Goswami, Noe and Rebello (1996) report that subjects are able to reach the collusive equilibrium when nonbinding preplay communication between bidders is introduced. However, the small number of experimental sessions used precludes strong conclusions from being drawn from their investigation.

to their expected values and, as a consequence, the market price varied in the same direction with the market value. The above evidence suggests that in order to choose an appropriate mechanism we need to understand better the behaviour of participants. The present theoretical investigation is a step in this direction.

3. The model

The basic model in this paper follows Biais and Faugeron-Crouzet (2002).

The volume of shares offered in the IPO auction is normalized to 1. There are $N \geq 2$ large institutional investors and a fringe of small retail investors as potential buyers. All investors are risk-neutral. Each institutional investor has private information about the valuation of the shares by the market as well as a large bidding capacity. The retail investors, however, are uninformed and cannot absorb the whole issue.

The private information that an institutional investor has is represented by the private signal s_i, which is identically and independently distributed and can be *high* with probability π or *low* with the complementary probability. However, each signal only reveals part of the information. The value of shares on the secondary market increases with the number of *high* signals n. Denote by v_n the market value of a share when there are n *high* signals. Each informed investor can buy the whole issue.[6] Uninformed investors do not observe signals and all together can purchase up to $1-k$ units shares, with $k \in [0,1]$.[7] All investors have the same constant marginal value for shares.

The price rule and the allocation rule of the auction are as follows. The seller sets a reservation price p^0 ($p^0 \geq 0$). If the total demand at the reservation price $D(p^0)$ exceeds the supply, the market price p_m is set at the market-clearing price, i.e. the highest bid price where demand exceeds supply.[8] Otherwise if the cumulated demand

[6] Biais and Faugeron-Crouzet (2002, p15) think this assumption is reasonable "given the bidding power of the large financial institutions participating regularly to IPOs, compared to the relatively small size of most of these operations. In addition this assumption simplifies the analysis."
[7] In the real world, either because retail investors have small demand capacity or because their demand is difficult to predict, firms who go public always try to attract large institutional investors to guarantee full subscription. It is rare to rely on small retail investors to absorb all shares of an IPO, even if the resulting market price is low. In some issues there is a maximum subscription amount for a retail investor. Hence the assumption that the retail investors can purchase up to 1-*k* units shares is reasonable. Moreover, *k* is allowed to take a value as low as zero. In that case, the retail investors as a whole can buy the whole issue.
[8] Following the convention of auction theory, we use the highest losing price rather than the lowest winning price as the market-clearing price. This simplifies our description of bidders' strategies. Since there are numerous bids in real markets, the highest losing and the lowest winning prices are usually the same.

at p^0 is less than or equal to the supply, the market price is set as p^0. Formally:

$$p_m = \begin{cases} \max(p \mid D(p) > 1) & \text{if} \quad D(p^0) > 1 \\ p^0 & \text{otherwise} \end{cases} \qquad [\ 1]$$

Denote $d_i(p)$ as bidder i's cumulated demand at prices *greater than or equal to p* and $d_i^a(p)$ as her cumulated demand at prices *greater than* p, then bidder i's allocation a_i can be expressed by the following formula:

$$a_i = \begin{cases} d_i^a(p_m) + [1 - \sum_{i=0}^{N} d_i^a(p_m)] \dfrac{d_i(p_m) - d_i^a(p_m)}{\sum_{i=0}^{N} [d_i(p_m) - d_i^a(p_m)]} & \text{if } D(p^0) > 1 \\ d_i(p^0) & \text{otherwise} \end{cases} \qquad [\ 2]$$

Where $i = 0$ represents the group of uninformed investors as a whole. If the cumulated demand at the reservation price exceeds the supply then, after allocating to each bidder the amount she bids for at prices higher than the market price, i.e. $d_i^a(p_m)$, the rest of shares (the multiplicand of the second term) are prorated among the bidders. In that case each bidder obtains a proportion equal to the ratio of *her* bids at the market price over the *total* bids placed at the market price (the multiplier of the second term). The bids below the market price do not receive any allocation. Otherwise, if the total demand at p^0 is less than or equal to the supply, each bidder obtains the amount she bids for, i.e. $d_i(p^0)$.

When the realization of the market value is v_n, bidder i's payoff Π_i equals the per unit payoff, $v_n - p_m$, multiplied by the number of units allocated:

$$\Pi_i = (v_n - p_m) \times a_i \qquad [\ 3]$$

A strategy S_i in this game is defined as a demand-price schedule $d_i(p,\ s_i)$ indicating how many shares bidder i would like to bid for at price p, under the observed signal s_i (s is either H, L or U representing high, low or no signal). As both the market price and the allocation are determined by bidders' demand schedules, bidder i's payoff can be written as a function of bidder i's and all the other bidders'(-i) demand functions:

$$\Pi_i(S_i, S_{-i}) = \Pi_i(d_i(p, s_i), d_{-i}(p, s_{-i})) \qquad [\ 4]$$

An investor obtains a zero profit by demanding zero:

$$\Pi_i(0, d_{-i}(p, s_{-i})) = 0 .$$

Bidder i's problem is to maximize her expected payoff by choosing a demand schedule $d_i(p,s_i)$ conditional on the signal she observes, given the other bidders' demand schedules. The function $d_i(p,s_i)$ is nonincreasing in p. We assume that the same types of investors, i.e., the investors with high or low signals and uninformed investors (called H, L and U investors respectively hereafter) are symmetric in both beliefs and behaviour.

Since H and L investors in fact are the same investors with different signals, we assume that the demand is nondecreasing in the expected value and the demand schedule of informed investors is additive: an H investor bids for no less than an L investor at any given price as they have higher expected market value:

$$d(p,H) = d(p,L) + c(p)$$

where $c(p)$ is nonnegative. The equation states that when the price is p the demand of an H investor exceeds the demand of an L investor by the amount $c(p)$.

Equilibrium strategies must satisfy the following four conditions. [9]

The first condition is that each investor gains a nonnegative expected payoff in equilibrium:

$$E\Pi_i \geq 0 \text{ for each } i \in [0, N].$$

Let p_n denote the market price when there are n high signals and $a(p_n, s)$ the allocation of an investor with signal s. Then the first condition can be formally stated as:

Condition 1:

$$E(\Pi \mid H) = \sum_{h=0}^{N-1} \pi_h a(p_{h+1}, H)(v_{h+1} - p_{h+1}) \geq 0 \textit{ for each H investor,}$$

$$E(\Pi \mid L) = \sum_{h=0}^{N-1} \pi_h a(p_h, L)(v_h - p_h) \geq 0 \textit{ for each L investor,}$$

$$E(\Pi \mid U) = \sum_{h=0}^{N} \mu_h a(p_h, U)(v_h - p_h) \geq 0 \textit{ for the U investor.}$$

where π_h is the probability of h out of N-1 other investors observing high signals, and μ_h is the probability of h out of N investors observing high signals.

The second condition states that a set of strategies S^* under which the payoff of investor i is Π_i is an equilibrium only if no investor can improve her expected payoff by changing her strategy S_i^*. Formally,

Condition 2:

$$E\Pi_i(S_i^*, S_{-i}^*) \geq E\Pi_i(S_i, S_{-i}^*) \text{ for any } i \in [0, N].$$

This is simply a condition for a (Bayesian) Nash Equilibrium stating that no player could profitably deviate. In this particular model, the intuition is as follows. Recall that a strategy is a demand schedule in this game. If one investor tries to lower the market price, she has to give up a sufficient large amount of shares. Condition 2 requires that the gain from a decrease in price is not enough to compensate for the corresponding loss in allocation, and vice versa. In the case where the market price is the reservation price, the price is bounded in one direction and condition 2 is simplified to the following: if one investor raises the market price to a higher level in order to absorb more shares from the other players' demand reduction, the gain from the allocation increase must not be sufficient to compensate for the loss from the increase in price.

The third condition follows the price rule: the market price is the highest price where the total demand exceeds supply.

Condition 3 (market clearing condition): *The cumulated demand above p_m is no more than 1, and that at p_m exceeds 1 if there is excess demand at the reservation price*:

$$\sum_{i=0}^{N} d_i^a(p_m) \leq 1; \ \sum_{i=0}^{N} d_i(p_m) > 1, \text{ if } D(p^0) > 1$$

The final condition states that if an investor indicates that she would like to buy any amount at some price then she would also like to buy at least the same amount at a lower price.

Condition 4 (nonincreasing demand function): *Each investor's demand does not increase in price* (*either downward sloping or vertical*)*:*

$$d_i(p, s_i) \leq d_i(p', s_i) \text{ if } p > p', \text{ for any } i \in [0,N] \text{ and any signal } s.$$

[9] Equilibrium in this paper refers to Bayesian Nash equilibrium as the game is one with incomplete information.

There are two differences between this model and other models of share auctions. The first difference is about the signals or expected values. In other models for share auctions (e.g. Wilson, 1979; Back and Zender, 1993 and Wang and Zender, 2002), bidders either observe (different) signals from a set of signals revealing all the possible states of the world, or have the same expected value (Wilson, 1979). Wilson also provides a case where the market value of shares is common knowledge to all investors. In this model, however, the type of signals is either high or low. The expected value can take three possible values that depend on whether or not a signal has been observed and in the former case on the type of the signal denoted by $E(v \mid H)$, $E(v \mid L)$ and $E(v)$. The second difference is that in this model the demand functions are discrete which is a format closer to the real market. The equilibrium in the discrete case still survives with continuous demand functions, but the reverse does not hold.

4. Preliminaries

We begin the analysis of our model by demonstrating that submitting flat demand functions cannot be an equilibrium strategy. Then, we establish a series of necessary conditions, implied by conditions 1 to 4, which we will be using in the derivation of the main results of the paper.

4.1 Flat demand functions

Suppose each bidder submits a flat demand function at price p. If each of them obtains a positive expected payoff, then one bidder could be better off by bidding for the same amount at a higher price in order to obtain the whole allocation (1-k units for the U bidder) without raising the market price. Like in the Bertrand competition, bidders compete to raise their bids until further overbidding will not increase their expected payoff, i.e. until the expected payoff from deviating drops to zero. Thus bidders with lower expected valuation of shares drop out first. So L and U investors can only obtain allocations if there is no H investor in the market. Knowing this an L investor should not bid higher than v_0. In the case where all L investors place flat demand functions, they would compete to outbid the others until the price reaches v_0 in order to absorb the most allocation when there is no high signal observed. Hence we assume that L investors participate by bidding for 1 at v_0. Then the U investor is indifferent between placing 1-k at any price below the price at which H investors place bids, as all are generating zero expected payoff for them. Without lose of generality, we assume that the U investor also places bids at v_0. The market price and the allocation for each type of investor under the strategies so far are listed in Table 1.[10]

Table 1: Outcomes under Flat Demand Function Strategies

Number of high signals (n)	Market price	Market value	Allocation
>1	p	v_n	Each H investor: 1/n
1	v_0	v_1	The H investor: 1
0	v_0	v_0	U investor: (1-k)/(1-k+N) Each L investor: 1/(1-k+N)

The following lemma follows from the above discussion:

Lemma 1: *In equilibrium type L investors will not submit flat demand functions except at price* v_0.

In order to complete the proof of the main result of this section we need the following two lemmas.[11]

Lemma 2: *If H investors submit flat demand functions they will compete by raising the price until it is unprofitable to overbid.*

Lemma 3: *At the price where it is unprofitable for an H investor to overbid, it is profitable for the same type of investor to underbid.*

Hence we cannot find a price that makes both overbidding and underbidding unprofitable. Thus submitting flat demand functions cannot be an equilibrium in pure strategies for H investors.[12]

Proposition 1: *All investors submitting flat demand functions at any price is not an equilibrium in uniform price IPO auctions.*

[10] For simplicity, we assume that the reservation price is equal to zero.
[11] All proofs can be found in Appendix A.
[12] In fact, submitting flat demand functions cannot be an equilibrium in mixed strategies as well. Because if there exists a price vector of possible mixed strategies of flat demand functions where each bidder i bids p_i for the entire shares, such that the bidders' ex ante expected profits are positive, one investor can always increase her profit by bidding at a price above all the possible prices in the mixed strategy set to absorb all the shares at the same price. By bidding at a higher price, an H investor captures the payoffs of all H investors. Then investors face a similar scenario as in pure strategies: at the price level where no one can be better off by overbidding, one can improve the expected payoff by underbidding. Submitting flat demand functions at a price that is equal to the expected market value can be an equilibrium strategy if all investors have the same expected market value, or if investors all have different market values (see, e.g., Harbord, Fabra and von der Fehr, 2001; Kremer and Nyborg, 2004).

Nevertheless, submitting a flat demand function at v_0 is an equilibrium strategy for L investors should they only obtain an allocation when there is no high signal in the market. Because both the market value (v_0) and the number of investors who share the allocation (N) are fixed, they each get zero payoffs and cannot do better given that the other bidders follow the same strategy. Because of the existence of asymmetry and the negative relationship between market value and the allocation of type H investors, lower market values that correspond to lower payoffs add more weight in the expected payoff than higher values. Thus a flat demand function cannot be an equilibrium strategy in our model.

4.2 Necessary conditions for an equilibrium

We have shown that submitting flat demand functions is not an equilibrium strategy. The following lemmas will be useful for the derivation of the main results of the paper**.**

Lemma 4: *If L investors share the market with H investors, the market price p_n must be lower than the corresponding market value v_n.*

Lemma 5: *If in an equilibrium the market price is lower than the market value, all shares should be allocated above the market price, i.e. no share is left for prorating at the market price.*

Lemma 6: *If in an equilibrium the market price is lower than the market value, and at least one type of investor is excluded from the market then the total demand at the market value has to be equal to* 1.

Lemma 6 implies that, when there is an equilibrium where not all investors participate, the demand function between v_n and p_n is vertical.[13]

Lemma 7: *An equilibrium where H investors are excluded from the market does not exist.*

Lemma 8: *If H and L investors do not behave symmetrically then, given that not all types of investors participate, the market price cannot be below v_{n-1}.*

Lemma 9: *If there exists an equilibrium where $p_n > p_{n-1}$ for all $n > 0$ then there must also exist an equilibrium where the demand curve between p_{n-1} and p_n is vertical.*

[13] A type of investors is called to be excluded from the market if they place no bids above v_0.

Figure 1: Vertical Demand Curve

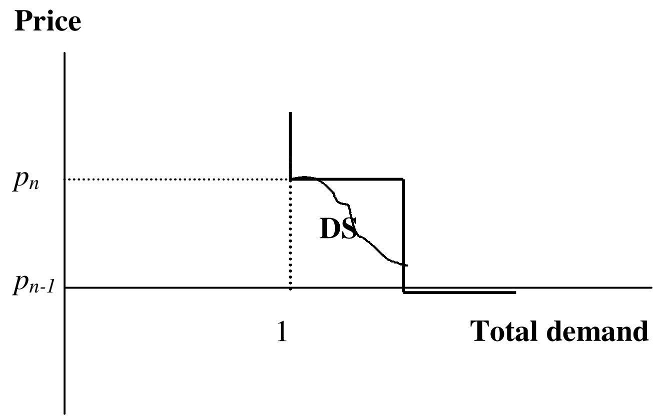


Figure 1 illustrates lemma 9 where DS denotes an arbitrary demand schedule.

Next, we shall utilize these lemmas to characterise the equilibria of this model. Given that H investors cannot be excluded from the market (Lemma 7), there are four kinds of equilibria that we need to consider according to the types of investors that participate; namely H investors alone (EH), H and L investors (EHL), H and U investors (EHU) and all types of investors (EHLU). We will examine the four possibilities in turn. Lemma 9 implies that, without loss of generality, we can restrict our equilibrium analysis to the case where the demand schedule is piecewise linear.

5. Characterization of pure strategy equilibria

5.1 Tacit collusion equilibrium

We start with the tacit collusion equilibrium which has been the focus of past research. In such an equilibrium, all investors behave symmetrically and submit a steep demand function regardless of their signals. The total demand at a given price would remain unchanged regardless of the number of high signals. Hence the market price and each investor's allocation would be constant at any possible market value. Biais and Faugeron-Crouzet (2002) provide a tacit collusion demand function solution for this model, according to which an investor's demand at price p is given by:[14]

$$d(p) = \frac{1}{N+1} - \sigma(p - p_0) \text{ ,with } \sigma \leq \frac{1}{N(N+1)} \frac{1}{E(v \mid H) - p_0} \qquad \textbf{[5]}$$

[14] Proof see Biais and Faugeron-Crouzet (2002, p19). Without any loss of generality, the authors have assumed that the U investor behaves the same way as an informed investor.

The condition for σ implies that the demand curve should be steep enough so that the residual supply faced by a bidder increases only by a small amount when the price is raised by a large amount**.** Thus, the gain from the increase in the allocation cannot compensate the loss from the increase in price. Hence the collusion is self-enforcing in this non-cooperative game. The equilibrium market price in this case is equal to the reservation price, and each investor obtains the same quota of the entire shares.

The above analysis is based on the assumption that the demand function is continuous. As we mentioned before, in reality, investors only indicate the amount they would like to buy at a limited number of prices. So we need to check if the tacit collusion equilibrium still holds in the discrete case. We find that the equilibrium still exists in the discrete case, under stricter conditions.

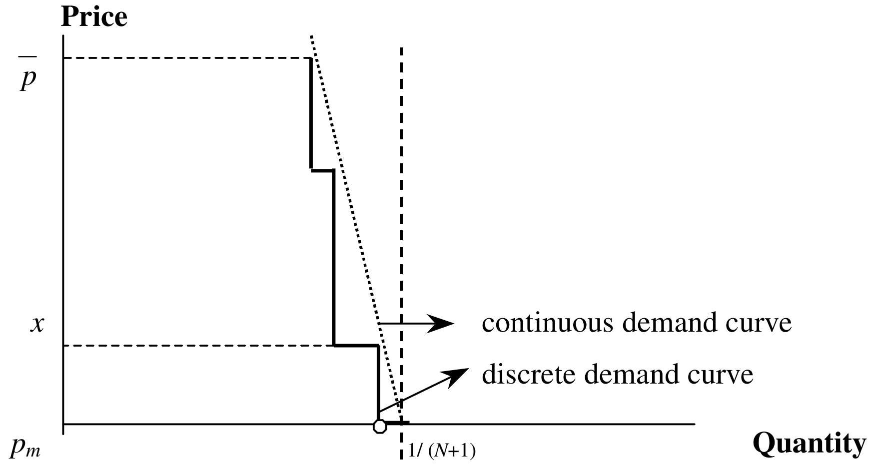
Proposition 2: *If all types of investors behave symmetrically then, in the discrete case,*

(i) *there exist a continuum of equilibria,* $p_m \in [p^0, E(v|L)]$, *where each investor demands* $\frac{1}{N+1}$ *of shares above* p_m*, and bidding for no less than* $\frac{1}{N}$ *of shares at* p_m if $p_m > p^0$*; and*

(ii) *the price* $\bar{p}$ *from which an investor starts bidding is no lower than* $\frac{NE(v \mid H) + p_m}{N+1}$.

Figure 2 offers some intuition for the above result. In the case of a continuous demand function [5], each investor faces a residual supply of $\frac{1}{N+1}$ above p_m. In contrast, in the discrete case since there is a gap between any two discrete prices, the expected residual supply above p_m is now equal to $\frac{1}{N+1}$ plus the expected residual demand gap from the other N bidders (see the bold curve in Figure 2). Now, the probability that no bid is placed between p_m and a price (slightly) higher than p_m, say, x, is positive. Then a bidder can exploit this demand gap at a negligible cost (a slight increase in price). Only a vertical demand function ($\sigma = 0$) can guarantee that each investor obtains $\frac{1}{N+1}$ of the total allocation and no shares are left for allocation for bids placed at p_m, and hence there is no room for a profitable deviation.

Figure 2 Residual Supply under A Discrete Demand Curve



Now, assume that all investors start bidding at a price at least as high as $\bar{p}$. If one investor tries to absorb all the shares by outbidding the other players then the market price will be raised to, at least, $\bar{p}$. Then if $\bar{p}$ is sufficiently high, even an H investor will find it unprofitable to deviate in this way.

Among the set of tacit collusion strategies, the one that leads to the market price as low as the reservation price weakly dominates the other strategies.[15] Note that, when p_m is equal to $E(v| L)$, the lowest possible price from which investors must start bidding, $\frac{NE(v \mid H) + p_m}{N+1}$, is higher than $E(v|L)$. So this equilibrium is risky especially for L investors. Because the demand function is vertical, a small amount of deviation may cause a loss. In addition, it would be reasonable not to follow this strategy without knowing the exact number of investors in the market.

5.2 EH equilibria

In the tacit collusion equilibrium all types of investors follow the same strategy. If different types of investors behave differently, i.e. if H and L investors have different demand functions, the market price will change with the market value. In this section, we characterize the equilibria where H investors absorb all the shares. Without loss of generality, from now on we assume that the reservation price is equal to zero. The following lemma imposes an upper bound on equilibrium prices.

[15] However, we don't eliminate the equilibria in weakly dominated strategies since in some experiments with a two-unit demand, it has been observed that investors overbid on the first unit frequently (e.g. Engelbrecht-Wiggans, List and Reiley, 2005). So we keep the whole equilibrium set for our future experimental examination.

Lemma 10: *Suppose that there is at least one type H investor, i.e.* $n \geq 1$*. If an EH equilibrium exists then the equilibrium price will be bounded by* v_n*.*

To prevent L and U investors from participating either the market price must be equal to the market value or the total demand of H investors at the market value must be equal to 1 when the market price is lower than the market value (Lemma 6). We examine these two cases separately.

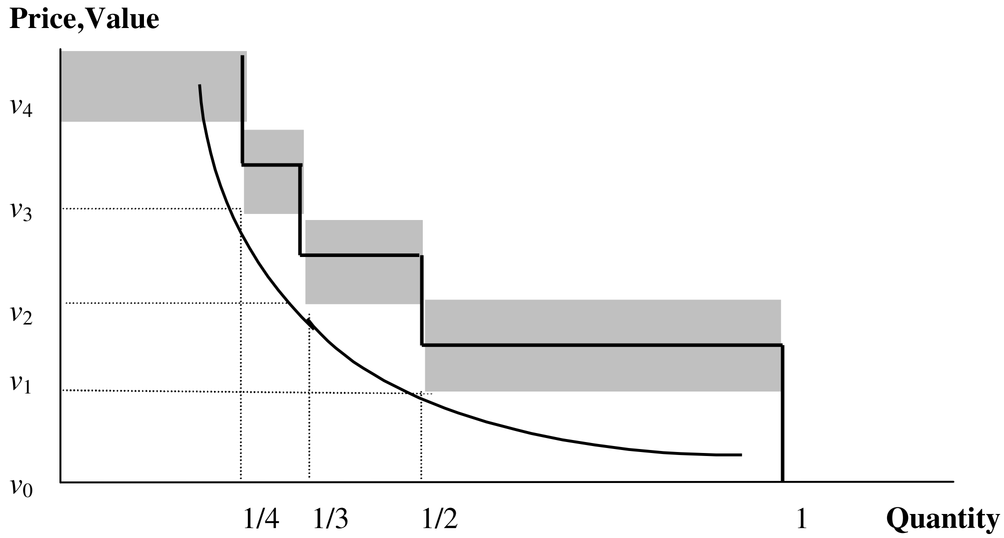
Lemma 11: *Suppose that there is at least one type H investor, i.e.* $n \geq 1$*. If an EH equilibrium exists and the market price equals the market value, then each H investor will bid for* $\frac{1}{n}$ *above* v_n *and* $\frac{1}{n-1}$ *at* v_n *for any* $n \in (1, N]$.

Because the resulting market price equals the market value for any realization of $n \in [1, N]$, the strategy described in Lemma 11, given that multiple equilibria exist, imposes an upper bound of the equilibrium set of bids. This strategy is described by the upper-right bound of the shaded area in Figure 3, which shows the case where there are four informed investors.

Lemma 12: *Suppose that there is at least one type H investor, i.e.* $n \geq 1$*. If an EH equilibrium exists and the market price is lower than the market value, then each H investor can bid for* $\frac{1}{n}$ *at* v_n *for any* $n \in [1, N]$*, and* $\frac{1}{n-1}$ *at* p_n *for any* $n \in (1, N]$*, where* $p_n \in [v_{n-1}, v_n)$.

The bold step-bids demand curve in Figure 3 is an example of the strategy stated in Lemma 12. Under this strategy, since the demand at v_n is already equal to 1, and all demand above the market price is fully allocated, an H investor cannot increase her allocation without raising the market price above v_n. Furthermore, as the demand of the other H investors at p_n is already 1 (n-1 investors each bids for $\frac{1}{n-1}$ at p_n), an H investor can only lower the market price below p_n by giving up the whole allocation. Thus an H investor cannot profitably deviate by either overbidding or underbidding.

Figure 3 Demand curve of EH



The last lemma of this section, describes the optimal strategies of type L and U investors.

Lemma 13: *Suppose that there is at least one type H investor, i.e.* $n \geq 1$. *If an EH equilibrium exists L and U investors will only place bids at prices no higher than* v_0.

L and U investors cannot obtain a positive expected payoff by placing bids above v_0. Since the two types of investors can only get an allocation when the market value is v_0, and the market value of shares is at least v_0, either submitting a flat demand function at v_0 or following the tacit collusion strategy in the range $[0, v_0]$ can be an equilibrium strategy for L and U investors.

The following Proposition follows immediately from Lemmas 10 to13.

Proposition 3: *There exists a continuum of equilibria where*

(i) *for any* $n \in (1, N]$ *each H investors places* $\frac{1}{n}$ *above* v_n , *places* $\frac{1}{n-1}$ *at price* $p_n \in [v_{n-1}, v_n]$, *places* 1 *at* v_1 *for* $n = 1$, *and places no bid at other prices;*

(ii) *L and U investors place no bids above* v_0 *and only obtain an allocation if there is no high signals in the market; and*

(iii) *any price between* v_{n-1} *and* v_n *for* $n \in [1, N]$ *and between* 0 *and* v_0 *for* $n = 0$, *can be an equilibrium market price in this set.*

If bidders have the same expected value, the equilibrium set degenerates to the tacit collusion equilibrium.

From Lemma 8 we know that the lowest possible market price is v_{n-1} in EH (described by the lower-left bound of the shading area in Figure 3). The strategy of the lower bound of the equilibrium set weakly dominates the others. This is not only because it is the most profitable strategy in this set, but also because it is a safe play. By choosing this strategy, an H investor obtains as large an allocation as the other H investors independently of the equilibrium strategy the other investors choose from the EH set. Moreover, with a positive probability, the other H investors do the same as her and then they each enjoy the lowest possible market price in this equilibrium set.

It has been argued that in multiunit uniform price auctions large bidders often make rooms for smaller ones by reducing demand to avoid competition, especially if the smaller bidders have the ability to increase prices (Tenorio, 1997). We will check if this can be an equilibrium in the following sections. The derivation of the following three sets of equilibria includes two steps. First we use Conditions 3, 4 and Lemmas 4 to 9 to specify what an equilibrium should look like supposing that such an equilibrium exists. Then we check if the proposed strategy satisfies Conditions 1 and 2.

5.3 EHL equilibria

Condition 4 requires that the demand functions of both L and H investors be downward sloping; formally

$$d(p_n,L) \le d(p_{n-1},L);\ \ d(p_n,H) \le d(p_{n-1},H) \text{ for any } n\in[1,N] \qquad \textbf{[6]}$$

Lemmas 4 and 6 require that the total demand at the market value of all informed investors is equal to 1:

$$Nd(v_n,L)+nc(v_n)=1 \text{ for any } n\in[1,N] \qquad \textbf{[7]}$$

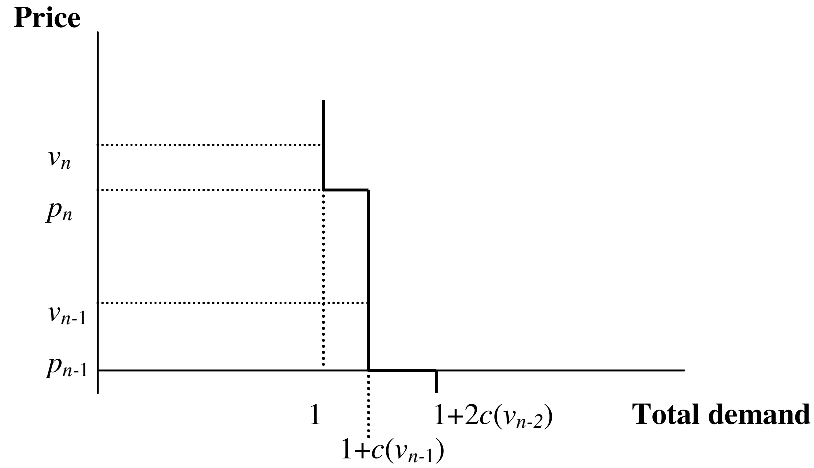
So $c(v_n) = \dfrac{1-Nd(v_n,L)}{n}$. Note that when [7] is satisfied Condition 1 is also satisfied.

Next, Condition 2 requires that no type of investor is able to profitably deviate. Suppose that demand functions satisfy equation [7] for every $n\in[1,N]$. Then we must have $Nd(p_n,L)+(n-1)c(p_n)=1$ (Lemma 5). When the realization of the number of high signals is n, the demand of all the investors above p_n is 1 and is equal to $1+c(v_{n-1})$ at p_n (see Figure 4). According to equation [7], an allocation can only be increased if the price is raised above v_n, which leads to a negative payoff. Thus we only need to ensure that it is unprofitable to lower the equilibrium price by reducing the bids:

$$(v_n - p_n)\, d(v_n , s) \geq (v_n - p_i) r_n (p_i , s) \quad i \in [0, n\text{-}1] \qquad \textbf{[8]}$$

Where s represents signals, that can be either H or L; $r_n(p_i,s)$ is the residual supply at price p_i when there are n high signals, given that a bidder who has received signal s has deviated.[16]

Figure 4 Demand Curve (part) in EHL



Condition [8] simply states that neither an H nor an L investor can increase her payoff by lowering the market price and taking all the residual supply. Together Lemma 8 and Condition [8] imply that the market price p_n lies in the interval $[v_{n\text{-}1}, v_n - \frac{r_n(p_i^*, s^*)}{d(v_n, s^*)}(v_n - p_i^*)]$ for every $n \in [1, N]$. [17]

Thus, we can have a continuum set of equilibria described in Proposition 4:

Proposition 4: *There exists a continuum of equilibria where*

(i) *both L and H investors place bids above* v_0 *where the bids of H investors are higher than those of L investors, while their total demand at* v_n *is* 1*;*

(ii) *the U investor places no bids above* v_0 *and only obtains an allocation when there are no high signals; and*

(iii) *any price between* $v_{n\text{-}1}$ *and* $v_n - \frac{r_n(p_i^*, s^*)}{d(v_n, s^*)}(v_n - p_i^*)$ for $n \in [1, N]$, *and between* 0 *and* v_0 *for n*=0 *can be an equilibrium market price in this set.*

[16] There exists a condition that is weaker than [8] such that if an H investor is unprofitable to lower the market price to $p_{n\,\text{-}1}$ by reducing the bids, it is also unprofitable for her to lower the market price further by reducing the bids further. If this condition holds then in an EHL equilibrium, it is also unprofitable for an L investor to deviate (See Appendix C).

[17] The inequality of [8] indicates that $p_n \leq v_n - r_n(p_i^*, s^*)(v_n - p_i^*)/d(v_n, s^*)$ for any $n \in [1, N]$, where p*

Notice that consistent with Lemma 4, p_n can only be equal to v_n if $d(p\ ,L)$ is zero (in this case we obtain an EH type of equilibrium).[18] Two examples of an EHL equilibrium is provided in Appendix B.

5.4 EHU equilibria

In equilibria where L investors do not place bids higher than v_0 we have $d(p,L) = 0$ for $p > v_0$.

Proposition 5: *There exists a continuum of equilibria where*

(i) *the total demand of H and U investors above v_n is no more than* 1*;*

(ii) *L investors place no bids above v_0 and only obtain an allocation when there are no high signals; and*

(iii) *any price between v_{n-1} and v_n for $n\in[1,N]$, and between* 0 *and v_0 for n=0 can be an equilibrium market price in this set.*

Because in this set of equilibria, L investors do not get any allocation when there is at least one H investor, the market price can be as high as the market value.[19] The derivative of this equilibrium is very similar as that of EHL. In Appendix B we provide two examples of this type of equilibrium. In one of these examples the U investor behaves the same way as an H investor, resulting in an equilibrium that is similar to an EH equilibrium with one more H investor.

5.5 EHLU equilibria

When all investors participate we know that the market price should be lower than the market value. According to Lemma 5, we should also have the following relationship in equilibrium:

$$Nd^{a}(p_{n},L)+nc^{a}(p_{n})+d^{a}(p_{n},U)=1 \quad \text{where } p_n < v_n \qquad \textbf{[9]}$$

The equilibria that we have proved so far can all be nested in this equation. In the tacit collusion equilibrium, H and L investors are symmetric in their bidding behaviour, so $c^{a}(p_{n})$ is zero, and the left hand side of the equation does not depend on n. Hence the

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and s* jointly maximize the value of $v_n - r_n(p_i\ ,s)(v_n - p_i\ )/d(v_n,s)$.

[18] For n =0, L and U investors' strategies are the same as their strategies in EH. If $d(p,L)$ is zero in an equilibrium, as in EH, the residual supply from deviation for an H investor, $r_n(p_i\ ,H)$, is at most zero.

[19] Lemma 4 is not relevant in this case.

market price is constant regardless of the market value. In the set of EH equilibria, both $d^a(p_n, L)$ and $d^a(p_n, U)$ are zero, so $c^a(p_n) = d^a(p_n, H)$ must be equal to $\frac{1}{n}$. Condition 1 requires that p_n is no higher than v_n. Lemma 8 implies that in order to prevent the other types of investors from entering the market the lowest possible price in equilibrium must be v_{n-1}. EHL and EHU are the equilibria obtained when $d^a(p_n, U)$ and $d^a(p_n, L)$ equal to zero, respectively. When all investors participate, $d^a(p_n, L)$, $c^a(p_n)$ and $d^a(p_n, U)$ are all positive.

Lemma 6 states that, when some type of investor is excluded from the market, the total demand at v_n must be equal to 1. However, when all the investors participate, we do not need this restriction anymore and so it is possible to have a lower market price than v_{n-1}. We proceed with the characterization of the set of EHLU equilibria.

Firstly, Condition 4 requires that the demand functions of all types of investors do not increase with price. So for every $n \in [1, N]$:

$d(p_n, L) \leq d(p_{n-1}, L)$,

$d(p_n, U) \leq d(p_{n-1}, U)$ and

$d(p_n, L) + c(p_n) \leq d(p_{n-1}, L) + c(p_{n-1})$ i.e.

$$\frac{1 - d(p_n, U) - (N - n)d(p_n, L)}{n} \leq \frac{1 - d(p_{n-1}, U) - (N - (n-1))d(p_{n-1}, L)}{n-1} \qquad \textbf{[10]}$$

Secondly, no bidder should be able to improve the payoff by any kind of deviation. The most profitable deviation would be to raise or to lower the price to p_i ($i \neq n$) and absorb all the residual supply at that price. The following condition rules out such deviations:

$$(v_n - p_n)d^a(p_n, s) \geq (v_n - p_i)r_n(p_i, s) \text{ for any } i \neq \text{ n, } i \in [0, N] \qquad \textbf{[11]}$$

Where s represents signals, which can be H or L or U. According to [11], the highest price, conditional on the number of high signals, is:

$$p_n \leq v_n - \frac{r_n(p_i^*, s^*)}{d(v_n, s^*)}(v_n - p_i^*)$$

where $i \in [0, N]$ and $i \neq n$, p* and s* maximize the value of $v_n - \frac{r_n(p_i, s)}{d(v_n, s)}(v_n - p_i)$.

No other restrictions are needed for an equilibrium. In Appendix B we produce an example of an equilibrium in this set.

Proposition 6: *There exists a continuum of equilibria where all investors participate and with market prices in the range from zero to v_n, for every $n \in [0, N]$.*

The equilibrium market price is no higher than the corresponding market value for any realization of the number of high signals. Except for the upper bound of the equilibrium set where the price equals the market value, underpricing always takes place. The underpricing level when there are n high signals, $v_n - p_n$, is positively related with $v_n - p_i^*$. Since this holds for any realization of n, if the price discount is large for some market value, it should also be large in all other possible market values. The price p_n decreases with the ratio $\frac{r_n(p_i^*, s)}{d(v_n, s)}$, so the lower (higher) the equilibrium allocation an investor obtains, the lower (higher) the market price must (could) be; and the lower (higher) an allocation an investor could gain by deviating, the lower (higher) the possible underpricing. When $r_n(p_i^*, s)$ is zero, p_n can be as high as v_n (e.g., EH equilibrium).

5.6 Discussion

We have demonstrated the existence of a continuum of equilibria in uniform IPO auctions that can be classified according to the types of investors that participate. This is in contrast to the existing literature that has exclusively focused on the set of tacit collusion equilibria. In all these additional equilibria the market price increases with the market value and thus, in general, the level of underpricing is lower. The new sets of equilibria have the following properties.

As N increases, the distances among the "steps" in the demand curve get smaller and thus the demand curve becomes smoother. As N approaches infinity the demand curve of each H investor becomes convex (see the curve in Figure 3). The average demand schedule in IPO auctions in Israel provided by Kandel, Sarig and Wohl (1999) appears to have this shape. Also, when N goes to infinity, the difference between v_n and v_{n-1} vanishes and therefore the market price gets closer to the market value. This implies that unlike the case of the tacit collusion equilibrium, competition increases revenues.

Another prediction of the new set of equilibria is that bidders would place more bids as the number of bidders increases. This is because they have to include bids for every possible market value in their demand schedules. This is consistent with the evidence from the U.S. Treasury bill market. Under the uniform price auction format, the number of investors participating in the market is higher and large dealers split bids into numerous smaller bids (Malvey, Archibald and Flynn, 1997).

Our theoretical results also suggest that, in contrast to the tacit collusion equilibrium case, type H investors place higher bids than L investors. This prediction has been confirmed experimentally (Zhang, 2006).

Lastly, recall that one difference between this model and other models for share auctions is that the expected value can take three possible values $E(v \mid H)$, $E(v \mid L)$ and $E(v)$. If bidders have the same expected values (like in the model by Wilson, 1979), the set of equilibrium reduces to the tacit collusion equilibrium. If each bidder observes a different signal thus has a different expected market value (like in the model by Back and Zender, 1993 and Wang and Zender, 2002), submitting a flat demand function can be an equilibrium strategy (however, in that case, the winner's curse is likely to happen).

6. Conclusions

In the previous sections, we have characterized the full set of pure strategies equilibria for discrete uniform price IPO auctions. The tacit collusion equilibria that much previous research has focused on are still equilibria in the discrete case but under stricter conditions: each investor's demand curve must be vertical above the market price. In the new set of equilibria investors with higher expected values bid for higher quantities at higher prices than those with lower expected values. The equilibrium market price can be any price between zero and the realized market value. Underpricing takes place in all equilibria but the upper bound of the equilibrium set.

Not only there exist multiple equlibria resulting in different market prices, but also a certain market price can result from different equilibrium strategies. The volatility observed in actual uniform price auctions may be explained by the existence of a large set of equilibria. Among these equilibria, the tacit collusion equilibrium is the most profitable one for investors. However, it is also risky for bidders who follow this strategy. Since the demand function is generally steep in a tacit collusion equilibrium, if some investors bid for a higher quantity, or intend to follow some other strategies in the equilibrium set, the market price may be raised by a significant amount. Choosing strategies that are consistent with this equilibrium requires that each bidder believes that her rivals will not raise their demand at a price higher than her expected market value. This may happen if a small number of bidders play repeatedly (e.g. the electricity market in England and Wale; see Wolfram, 1998). However, since many institutional and retail investors are involved in buying initial public offerings, the collusion of a small number of parties is not a likely scenario. If the tacit collusion equilibrium is not likely to happen in practice, uniform price auctions may generate higher revenues than other IPO mechanisms. In the Zhang (2006) experiment, uniform price auctions outperform fixed price offerings because bidders follow other strategies rather than the tacit collusion equilibrium.

In general, in auctions where either a small number of bidders participate, or some bidders are significant in size relative to the auction volume, the competition is limited and the auctioneer needs to address the potential exercise of market power in the auction design (Ausubel and Cramton, 2004). The set of equilibria characterized in this paper has the property that the underpricing level is lower when the number of investors increases, i.e. competition increases revenues for sellers. Thus the sellers' payoff may be improved under a design that encourages investors to enter the market. In that case even if a small number of institutional investors were capable of cooperating, the competition from a large number of bidders outside the cartel would offset the advantages of collusion. Such policies include allowing small investors to purchase a smaller block of shares and lowering the threshold for opening accounts.

APPENDIX A: PROOFS

Lemma 2: *If H investors submit flat demand functions they will compete by raising the price until it is unprofitable to overbid.*

Proof: Given L and U investors' strategies described in Lemma 1, if all H investors submit flat demand functions at price p, then an H investor's expected payoff under this strategy is:[20]

$$E(\Pi \mid H) = \sum_{h=1}^{N-1} \pi_h \frac{1}{h+1}[v_{h+1} - p] + \pi_0 [v_1 - v_0] \qquad \textbf{[A1]}$$

By bidding for 1 at a price higher than p, an H investor can absorb all the shares without raising the market price. The expected payoff of such an H investor is:

$$E(\Pi_d \mid H) = \sum_{h=1}^{N-1} \pi_h [v_{h+1} - p] + \pi_0 [v_1 - v_0] \qquad \textbf{[A2]}$$

We need the following three steps to proof Lemma 2.

Step* 1: The first term of* [A1] *is lower than that of* [A2] *when the market price is no higher than E (v |H).*

[A1] and [A2] can be rewritten as:

$$E(\Pi \mid H) = \sum_{h=0}^{N-1} \pi_h \frac{1}{h+1}[v_{h+1} - p] + \pi_0 [p - v_0] \qquad \textbf{[A1]}'$$

$$E(\Pi_d \mid H) = \sum_{h=0}^{N-1} \pi_h [v_{h+1} - p] + \pi_0 [p - v_0] \qquad \textbf{[A2]}'$$

The second terms of both equations are identical. The first term of [A2]′ is in fact

$E((v - p) \mid H)$, which is equal to:

$$\sum_{h=0}^{N-1} \pi_h v_{h+1} - \sum_{h=0}^{N-1} \pi_h p = E(v \mid H) - p \sum_{h=0}^{N-1} \pi_h = E(v \mid H) - p$$

When p is $E(v \mid H)$ it is equal to zero.

The first term of [A1]′equals

$$E(\tau(v-p) \mid H) = E(v - p \mid H) \cdot E(\tau \mid H) + \operatorname{cov}(\tau, v - p \mid H)$$

---

[20] The price p should be no lower than $E(v)$, otherwise the U investor would bid over p. The market price when there is only one H investor is the maximum price among L and U investors' bid prices. Without any lose of generality, we assume the price is v_0 .

Where τ, the proportion of the offering that is allocated to an H investor, is equal to $\frac{1}{n}$.

Unless n equals 1, τ is less than 1, so $E(\tau \mid H)$ is positive and smaller than 1.

Because the more high signals there are, the higher the market value is, and the lower the proportion of the offering that will be allocated to each H investor, the covariance between τ and v is negative, so $\mathrm{cov}(\tau, v - p \mid H)$ is negative.

Hence when $E((v - p) \mid H)$ is nonnegative, i.e., when p is no higher than $E(v \mid H)$, the first term of [A1]′ is always lower than that of [A2]′.

Since the second terms of [A1] and [A2] as well as [A1]′and [A2]′ are identical, the first term of [A1] is lower than that of [A2] when the market price is no higher than $E(v \mid H)$.

Step 2*: The first term of* [A1] *is lower than that of* [A2] *when the latter is zero.*

We want to show that $\sum_{h=1}^{N-1} \pi_h \frac{1}{h+1}[v_{h+1} - p]$ *is negative when* $\sum_{h=1}^{N-1} \pi_h [v_{h+1} - p]$ *is zero.* Because both items strictly decrease with p, the statement is true if $\sum_{h=1}^{N-1} \pi_h [v_{h+1} - p]$ is positive when $\sum_{h=1}^{N-1} \pi_h \frac{1}{h+1}[v_{h+1} - p]$ is zero. So it is equivalent to prove that $\sum_{h=1}^{N-1} \pi_h [v_{h+1} - \overline{p_d}]$is positive, i.e. $\frac{\sum_{h=1}^{N-1} \pi_h v_{h+1}}{\sum_{h=1}^{N-1} \pi_h} > \overline{p_d}$, where $\overline{p_d}$ denotes a price such that $\sum_{h=1}^{N-1} \pi_h \frac{1}{h+1}[v_{h+1} - p]$equals zero. Solving for $\overline{p_d}$ we get:

$$\overline{p_d} = \frac{\sum_{h=1}^{N-1} \frac{\pi_h v_{h+1}}{h+1}}{\sum_{h=1}^{N-1} \frac{\pi_h}{h+1}}$$

Hence we need to prove that $\frac{\sum_{h=1}^{N-1} \pi_h v_{h+1}}{\sum_{h=1}^{N-1} \pi_h} > \frac{\sum_{h=1}^{N-1} \frac{\pi_h v_{h+1}}{h+1}}{\sum_{h=1}^{N-1} \frac{\pi_h}{h+1}}$. By rearranging the inequality what we need to prove is that $\sum_{h=1}^{N-1} \pi_h v_{h+1} \sum_{h=1}^{N-1} \frac{\pi_h}{h+1} - \sum_{h=1}^{N-1} \pi_h \sum_{h=1}^{N-1} \frac{\pi_h v_{h+1}}{h+1} > 0$. The left hand side of this inequality equals

$$\sum_{h=1}^{N-1}\sum_{i=1}^{N-1}\pi_h v_{h+1}\frac{\pi_i}{i+1}-\sum_{h=1}^{N-1}\sum_{i=1}^{N-1}\frac{\pi_h v_{h+1}}{h+1}\pi_i=\sum_{h=1}^{N-1}\sum_{i=1}^{N-1}\pi_i\pi_h v_{h+1}\left(\frac{1}{i+1}-\frac{1}{h+1}\right) \quad \textbf{[A3]}$$

Both i and h take the value of each integer between 1 and N-1, and thus in the expanded expression there will be N-1 times h (for $h = i$), $\frac{(N-1)^2-(N-1)}{2}$ times h (for $h > i$) and $\frac{(N-1)^2-(N-1)}{2}$ times i (for i>h) terms, respectively. The terms corresponding to $h = i$ in the expanded expression are equal to zero. For each combination of $i = x< h = y$, we have an inverse combination of $i = y> h =x$. The sum of each pair of these combinations is $\pi_x\pi_y c(v_{y+1}-v_{x+1})$ where c equals $\frac{1}{x+1}-\frac{1}{y+1}$.

Because x< y, c is positive, and so is $v_{ly+1}-v_{x+1}$. Hence equation [A3] is positive and the statement is true.

Step* 3*: *(Completion of the proof of Lemma* 2). The first term of [A1] is lower than that of [A2] when the market price is no higher than E (v |H) (step 1) and at a level higher than E(v |H) (step 2: the market price p must be higher than E (v |H) when the first term of [A2] is zero[21]). Because both [A1] and [A2] strictly decrease in price p, the first term of [A1] is always at a lower level than that of [A2] when the latter is nonnegative. The second terms of both equations are identical. Thus the expected payoff by bidding the same as other H bidders is less than that by deviation: by bidding 1 at a price higher than p, an H investor can absorb all the shares without changing the market price, thus improve the expected payoff to $E(\prod_d |H)$. Like in Bertrand competition, every H investor would compete until it is unprofitable to do so. Since the second term of [A1] and [A2], the payoff when no other investor observes a high signal, is positive, H investors would compete to raise the price until the first term of [A2] equals zero (denote the price level as $\overline{p_d}$). In other words, any price below $\overline{p_d}$ cannot be an equilibrium price if each H investor submits a flat demand function. □

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[21] $\sum_{h=1}^{N-1}\pi_h[v_{h+1}-p]=\sum_{h=0}^{N-1}\pi_h[v_{h+1}-p]-\pi_0[v_1-p]$ is positive when $p = E$ (v |H) and N is higher than 1, so it is zero only when p is higher than E (v |H).

***Lemma* 3*:** *At the price where it is unprofitable for an H investor to overbid, it is profitable for the same type of investor to underbid.*

***Proof**:* Every H investor would compete by overbidding until the price climbs up to $\overline{p_d}$. But they cannot share the payoff with each other at this price because at $\overline{p_d}$ the first term of [A1] is negative (see step 2 in the proof for Lemma 2). Thus an H investor can be better off by bidding at a lower price and only obtaining an allocation when there is no other H investor in the market.

***Lemma* 4*:** *If L investors share the market with H investors, the market price p_n must be lower than the corresponding market value v_n.*

Proof: Suppose that the market price equals the market value. An H investor can raise her profit from zero to a positive level by lowering the price by bidding less, unless the demand of all the other investors at v_n is at least 1:

$$(n-1)d(v_n, H) + (N-n)d(v_n, L) + d(v_n, U) \geq 1$$

If this condition holds, we also have

$$(n-1)d(v_n, H) + (N-(n-1))d(v_n, L) + d(v_n, U) > 1 \text{ if } d(p,L) \text{ is larger than zero.}$$

The inequality implies that the total demand at price v_n is larger than 1 when there are n-1 H investors. In this case p_{n-1} equals v_n, so investors have zero profit when the realization of the market value is v_n and negative payoff if the market value is lower. This leads to a negative expected payoff and condition 1 is violated. Hence in an equilibrium Lemma 4 must hold. □

***Lemma* 5*:** *If in an equilibrium the market price is lower than the market value, all shares should be allocated above the market price, i.e. no share is left for prorating at the market price.*

Proof: According to the allocation rule, bids placed above the market price are fully allocated; then the rest of shares are prorated among the investors who place bids at the market price. So if the total demand above the market price is less than 1, an investor can raise it to 1 by bidding more above the market price. By doing this her allocation is increased but the market price still remains the same. Increasing the allocation can lead to an increase in profits since the market price is lower than the market value. Hence to prevent a profitable deviation the accumulative demand above the market price must be 1 and thus all shares must be absorbed above the market price. □

***Lemma* 6*:* *If in an equilibrium the market price is lower than the market value, and at least one type of investor is excluded from the market then the total demand at the market value has to be equal to* 1.

Proof: The fact that p_n is lower than v_n requires that the total demand at v_n is no more than 1. If it is less than 1, then an investor who stays out of the market can get a positive payoff by demanding at v_n. To prevent the other investors from entering the market, the total demand of the existing investors has to be 1 at v_n. □

***Lemma* 7*:* *An equilibrium where H investors are excluded from the market does not exist.*

Proof: To prevent H investors from participating, either the market price equals v_n, or the market price is lower than v_n but the total demand at v_n is 1 (Lemma 6). Hence we should have

$$(N-n)d(v_n, L) + d(v_n, U) \geq 1$$

Because U investors in total can buy at most 1-k, they cannot absorb the entire shares, $d(v_n, L)$ is larger than zero.

So $(N-(n-1))d(v_n, L) + d(v_n, U) > 1$

This means p_{n-1} is at least v_n. Since this should hold for every n>0, the expected profit for a participant is negative which violates condition 1. Thus unless the uninformed investors could absorb the whole shares, H investors cannot be excluded from the market. □

***Lemma* 8*:* *If H and L investors do not behave symmetrically then, given that not all types of investors participate, the market price cannot be below* $v_{n\text{-}1}$.

Proof: If H and L investors are not symmetric in their behaviour, because $d(p,H)=d(p,L)+c(p)$, $c(p)$ is positive. According to Lemma 6, to prevent any types of investors from participating, with n-1 H investors we should have:

$(n-1)c(v_{n-1}) \geq 1$ when both L and U investors are absent;

$(n-1)c(v_{n-1}) + d(v_{n-1}, U) \geq 1$ when L investors only stay out of the market,

$Nd(v_{n-1}, L) + (n-1)c(v_{n-1}) \geq 1$ when U investors only stay out of the market.

With n H investors, the total demand at v_{n-1} is larger than 1. Thus the market price cannot be lower than v_{n-1}. □

Lemma 9: *If there exists an equilibrium where $p_n > p_{n-1}$ for all $n > 0$ then there must also exist an equilibrium where the demand curve between p_{n-1} and p_n is vertical.*

Proof: Suppose we have an arbitrary equilibrium demand schedule (DS) under which the market price is p_n (see Figure 1). By keeping the demand curve between p_{n-1} and p_n vertical (for every $n>0$), when the realization of the number of high signals is n-1, the demand above p_{n-1} is still 1 and that at p_{n-1} is larger than 1, so both the market price and the allocation are the same as those under DS. Moreover, as DS is downward sloping, it lies to the left of the vertical demand curve. Thus the residual supply as well as the profit when deviating are minimized under this piecewise linear demand function. □

Proof of Proposition 2:

(i) According to Condition 1, in equilibrium the highest possible market price p_m is $E(v \mid L)$. Because the other bidders' demand is no less than 1 at p_m if $p_m > p^0$, a bidder can only lower the market price by giving up the whole allocation, or cannot lower the market price further by reducing her demand. Hence underbidding is unprofitable. According to Lemma 7, there should be no share left for allocating at p_m. Thus in discrete case the demand function should be vertical above p_m.[22] This guarantees that each bidder obtains $\frac{1}{N+1}$ allocation in the first place.

(ii) Suppose all the investors start bidding from price p (i.e., no bid is placed above p). Then if an H investor tries to absorb the entire supply by bidding at a higher price, the market price would be raised to p from p_m. To make it unprofitable to do so, the gain from deviation should be no higher than that from following the equilibrium strategy:

$$E(v \mid H) - p \leq \left[E(v \mid H) - p_m\right]\frac{1}{N+1}$$

[22] It is unnecessary that the whole demand function is vertical, In other word, the step-like demand curve can exist at prices higher than p_m. More proof about the step-like demand curve see Zhang, PhD thesis.

which requires $p \geq \frac{NE(v \mid H) + p_m}{N+1}$

If this condition is satisfied, the U investor and each L investor also do not have an incentive to deviate by absorbing the entire shares at p (which requires that p is no lower than $\frac{NE(v) + p_m}{N+1}$ and $\frac{NE(v \mid L) + p_m}{N+1}$, respectively). □

Lemma* 10*: Suppose that there is at least one type H investor, i.e. $n \geq 1$. If an EH equilibrium exists then the equilibrium price will be bounded by v_n.

Proof: In equilibrium, Condition 1 $E(\Pi \mid H) = \sum_{h=0}^{N-1} \pi_h a(p_{h+1}, H)(v_{h+1} - p_{h+1}) \geq 0$ must be satisfied by each H investor. Let $(p_1, p_2, \ldots p_N)$ be a price vector that satisfies this condition as an equality. If for example, market price p_n lies above v_n, then an H investor could lower her bidding price from v_n in her demand schedule. If the other investors' demand at p_n exceeds 1, market price is still p_n but by deviation this investor could avoid obtaining an allocation and paying at a price higher than the market value; if the other investors' demand at p_n is no more than 1, this investor's demand would be marginal and market price would be lowered, so a deviation could still benefit her. Hence if an EH equilibrium exists then the equilibrium price will be bounded by v_n. □

Lemma* 11*: Suppose that there is at least one type H investor, i.e. $n \geq 1$. If an EH equilibrium exists and the market price equals the market value, then each H investor will bid for $\frac{1}{n}$ above v_n and $\frac{1}{n-1}$ at v_n.

Proof: When the market price equals market value, each H investor gains a zero expected payoff. If an H investor can lower the market price below the market value, while still keeping some allocation, she would enjoy a positive payoff. So to prevent an H investor from deviating, in equilibrium the H investor could only lower the price by giving up the whole allocation. This requires that the total demand of the other H investors is at least 1 at v_n:

$$(n - 1)\, d(v_n, H) \geq 1 \text{ for } n \in [1, N] \text{ thus } d(v_n, H) \geq \frac{1}{n-1}.$$

Moreover, Condition 1 requires that the total demand of H investors *above* v_n is no more than 1:

$$nd^a(v_n, H) \le 1 \text{ for each } n \in [1, N] \qquad \text{Hence } d^a(v_n, H) \le \frac{1}{n}.$$

Because this relation should be satisfied for any realization of n, the demand *at* v_n (*above* v_{n-1}) is no more than 1 when there are n-1 high signals. So

$$d(v_n, H) \le \frac{1}{n-1} \text{ (for any } n>1).$$

Thus an H investor's demand *at* v_n must be $\frac{1}{n-1}$ and the demand function between v_{n-1} and v_n should be vertical. The development of this strategy implies that it also satisfies both Condition 1 and Condition 2. □

Lemma* 12*: *Suppose that there is at least one type H investor, i.e.* $n \geq 1$. *If an EH equilibrium exists and the market price is lower than the market value, then each H investor can bid for* $\frac{1}{n}$ *at* v_n *for any* $n \in [1, N]$, *and* $\frac{1}{n-1}$ *at* p_n *for any* $n \in (1, N]$, *where* $p_n \in [v_{n-1}, v_n)$.

Proof: When the market price is lower than the market value, Lemma 6 requires that H investors' total demand at v_n is 1:

$$nd(v_n, H) = 1 \text{ for each } n \in [1, N]$$

Hence each H investor should bid for $\frac{1}{n}$ at v_n for any possible realization of $n \in [1, N]$ to prevent L or U investors from obtaining an allocation. As we are considering the case where the demand curve between p_{n-1} and p_n, thus between v_{n-1} and p_n is vertical, the quantity that an investor bids for at p_n must be $\frac{1}{n-1}$, and no bids are placed between v_{n-1} and p_n, for any n $\in$ (1, N]. □

Lemma* 13*: *Suppose that there is at least one type H investor, i.e.* $n \geq 1$. *If an EH equilibrium exists L and U investors will only place bids at prices no higher than* v_0.

Proof: In an EH equilibrium exists, the demand of H investors must be at least 1 at v_n. If L or U investors place bids above v_0, they would either get no share (if they bid below v_n), or get shares but earn a nonpositive profit (if the bid is at v_n, they get a zero profit; if the bid is above v_n, the market price would be raised to at least v_n), in neither case they could improve their expected payoff than by only placing bids at prices no higher than v_0. □

***Proof of Proposition* 5:**

Condition 4 requires that the demand functions of both an L and an H investor be nonincreasing in price:

$d(p_n, U) \le d(p_{n-1}, U)$ and

$c(p_n) \le c(p_{n-1})$ **[A4]**

Also, if $p_n < v_n$, Lemma 6 requires that the demand at v_n must be 1:

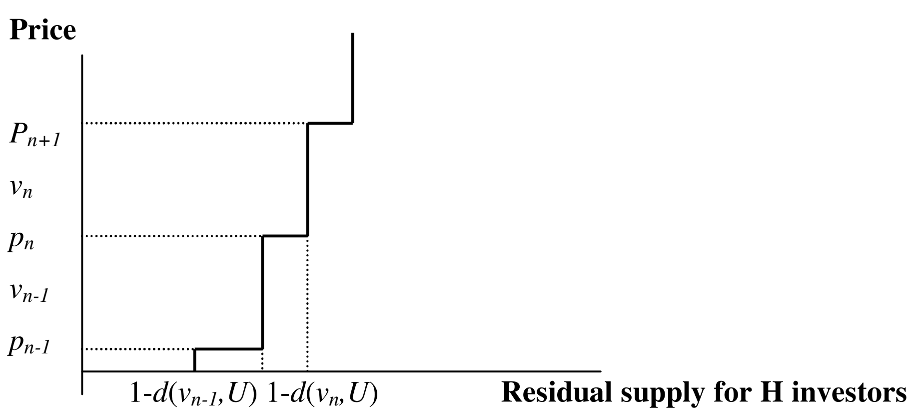
$nc(v_n) + d(v_n, U) = 1$ for any $n \in [1, N]$ if $p_n < v_n$ **[A5a]**

As in equilibrium L investors are excluded when there is at least one H investor, the market price can be as high as the market value (Lemma 4 is irrelevant). If p_n is equal to v_n, to prevent an L or U investor from deviating, the total demand of the other players must be at least 1 at price v_n; however, the total demand at v_n when there are n-1 H investors should be no more than 1(otherwise Condition 1 is violated), so the demand given n-1 H investors must be 1:

$(n-1)c(v_n) + d(v_n, U) = 1$ and $nc(v_n) = 1$ for any $n \in [1, N]$ if $p_n = v_n$ **[A5b]**

Finally, in equilibrium no player can profitably deviate. The residual supply faced by all the H investors, which equals 1 minus the demand of the U investor, is illustrated in Figure A (Here each H investor faces a similar scenario as that in EH, in the sense that they absorb all the residual supply above the market price p_n).

Figure A Residual Supply of H investors in EHU



Consider the following strategy: the U investor bids for $d(v_n, U) \in [0,1-k]$ and each H investor bids for $\frac{1-d(v_n,U)}{n}$ in $(p_n, p_{n+1}]$ for any $n \in [1,N]$. Under this strategy, the total demand above v_n is 1, so no investor would like to increase the allocation by raising the market price. To make it unprofitable for an H investor to lower the market price by demanding less, the payoff under the above strategy should be no less than that when lowering price to p_{n-1} and taking all the residual supply at that price:

$$(v_n - p_n)\frac{1-d(v_n,U)}{n} \geq (v_n - p_{n-1})(1-d(v_{n-1},U)-(n-1)c(v_{n-1}))$$

This is always satisfied, because $d(v_{n-1},U)+(n-1)c(v_{n-1})=1$ thus the right hand side is zero. An H investor can only reduce the market price by giving up the whole allocation.

To make the U investor unprofitable to deviate, when she gives up part of the allocation and lowers market price to p_{n-1}, the profit from the deviation should be no more than that under the equilibrium strategy:

$$(v_n - p_n)d(v_n,U) \geq (v_n - p_{n-1})(1-nc(v_{n-1})) \qquad \textbf{[A6]}$$

The condition is satisfied if $d(v_{n-1},U)$ is less than $c(v_{n-1})$ (thus no residual share is left at p_{n-1}), otherwise it requires $\frac{d(v_n,U)}{d(v_{n-1},U)-c(v_{n-1})} \geq \frac{v_n - p_{n-1}}{v_n - p_n}$. It can be shown that if [A6] is satisfied, an even lower market price is also unprofitable (Appendix C).

Like in EH, L investors place no bids above v_0 and only obtain an allocation when there is no H investor in the market. □

APPENDIX B: EXAMPLES

Examples of EHL equilibria:

***Example* 1:** Each L investor places bids according to a vertical demand function, i.e. $d(p,L)$ is a constant:[23] while H investors demand an additional amount of $\frac{1-Nd(p,L)}{n}$ in the price range $(p_n, p_{n+1}]$. For any $n \in [1,N]$, p_n is in the interval of $[v_{n-1}, v_n - \frac{nd(p,L)}{1-(N-n)d(,L)}(v_n - p_{n-1})]$.

***Example* 2:** Each L investor bids for $\frac{K}{N(n+K)}$ at p_n, and each H investor demands an extra amount $\frac{1}{n+K}$ at any $p \in (p_n, p_{n+1}]$, where K is a positive constant.[24] For any $n \in [1,N]$, p_n lies in the interval of $[v_{n-1}, v_n - \frac{K(n+K)}{(n-1+K)(N+K)}(v_n - p_{n-1})]$.

The upper bound of the price range in both examples decreases with the allocation of an L investor, which means that the larger the allocation that H investors lose to L investors, the lower that market price should be. Also, the market price can be equal to the market value only if an L investor does not get any allocation (K=0).

Examples of EHU equilibria:

***Example* 1:** Condition [A5b] implies that when the market price p_n equals v_n, $c(v_n) = d(v_n,U) = \frac{1}{n}$. Conditions [A4] and [A6] are also satisfied under this strategy. The demand of the other players is 1 at v_n, and the total demand above v_n is also 1, so no investor can be better off by either reducing price (by reducing bids) or increasing the allocation (by raising the price).

If the market price is lower than v_n, the total demand at v_n is 1 (Lemma 6). There is an equilibrium that is similar as the EH with one more H investor: the U investor

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[23] When $d(p,L)$=0, the equilibrium is EH; $d(p,L)$ is a positive constant smaller than min$[1/[N + n(v_{n-1} - v_{n-2})/(v_n - v_{n-1})]]$$(n \in [2, N-1])$ such that the upper bound of p_n is no less than v_{n-1}. If each high signal increases by the same amount the market value, i.e., if v_{n-1}-v_{n-2} = v_n-v_{n-1}, the maximum value of $d(p,L)$ reduces to1/(2N-1).

[24] K is no higher than $[(N-1)(v_n-v_{n-1})/(v_{n-1}-p_{n-1})]-1, n \in [1,N-1]$ such that the upper bound of the price range is no lower than v_{n-1}. If each additional high signal increases by the same amount the market value, the condition is satisfied if K is no higher than N-2.

behaves the same as an H investor by bidding for $\frac{1}{n}$ at a price $p_n \in [v_{n-1}, v_n]$while place no bids between p_{n-1} and p_n; and L investors place no bids above v_0. □

***Example* 2:** Suppose in equilibrium $d(p,U)$ is a constant between zero and 1-k. Then H investors' strategy is similar as that in EH, except that now the supply they face is 1- $d(p,U)$ instead of 1. Thus instead of bidding $\frac{1}{n}$, each H investor bids for $\frac{1-d(,U)}{n}$ in $(p_n, p_{n+1}]$, and for the same reason as before, they are unable to profitably deviate.

The residual supply to the U investors is $\frac{nd(p,U)-1}{n-1}$ above p_{n-1}. [A6] requires:

$$d(p,U) \le \frac{1}{1+(n-1)\frac{p_n - p_{n-1}}{v_n - p_{n-1}}}$$

The right hand side decreases with p_n, so the condition holds if it is satisfied when $p_n = v_n$, in that case $d(p,U) \le \frac{1}{n}$. Since the U investor's demand curve is constant, the condition always holds if $d(p,U) \le \frac{1}{N}$.

An example of EHLU equilibria:

Consider the following strategy:

Each L investor bids for a constant amount $K \in [0, \frac{1}{N}]$ at any prices. Each H investor demands an extra amount $c^a(p)$, and the U investor behaves the same as an H investor.

According to [9]:

$NK + (n+1)\, c^a(p_n) = 1$

So $c^a(p_n) = \frac{1-NK}{n+1}$, which satisfies the inequality [10].

Then [11] requires

$$(v_n - p_n)K \ge (v_n - p_{n+1})(K + \frac{1-NK}{n+2}),\ n \in [0, N-1]$$

The above inequality implies that an L investor cannot improve her payoff by increasing the allocation (at a cost of raising the market price). It can be shown that an

L investor benefits the most from this kind of deviation compared with either an H or the U investor. If this condition is satisfied, it is also unprofitable to increase the allocation further (by raising the market price to a even higher livel). Condition [11] also requires that

$$(v_n - p_n)(K + \frac{1-NK}{n+1}) \geq (v_n - p_{n-1})K \text{ , } n \in [1, N]$$

This condition ensures that it is unprofitable for an H investor to deviate by lowing the market price to p_{n-1} or further (at a cost of reducing demand). An H investor benefits the most than an L or U investor when deviating by lowering the price.

Rearranging the above two conditions we get:

$$\frac{v_n - p_{n+1}}{(n+2)(p_{n+1} - p_n)} \leq \frac{K}{1-NK} \leq \frac{v_n - p_n}{(n+1)(p_n - p_{n-1})} \qquad \textbf{[A7]}$$

Thus when [A7] is satisfied, the above strategy is an equilibrium strategy.

According to [A7], $\frac{K}{1-NK}$ increases with *K*. The higher *K* is, the larger $\frac{v_n - p_{n+1}}{p_{n+1} - p_n}$ could be and $\frac{v_n - p_n}{p_n - p_{n-1}}$ should be. This implies that the price discounted is larger (larger v_n-p_{n+1} and v_n-p_n), and the neighbour prices locate closer (smaller p_{n+1} - p_n and p_n-p_{n-1}) because the lowest market price p^0 is bounded. The higher the price discount H investors try to achieve, the more allocation they have to lose to L investors. In the extreme case, when *K* equals $\frac{1}{N}$, p_n is equal to p_{n-1} so the market price must be constant. This is consistent with the tacit collusion equilibrium. In the other extreme case, when *K* equals zero, p_n should be no higher than v_n but no lower than v_{n-1} (as v_n-p_{n+1} is nonpositive for every *n* when K is zero), this is consistent with the equilibria EH and EHU. For values of *K* in between, we expect that the market price can be any between p^0 and $v_n - \frac{r_n(p_i^*, s^*)}{d(v_n, s^*)}(v_n - p_i^*)$ ([11]). Because with higher *n* the larger $\frac{v_n - p_{n+1}}{p_{n+1} - p_n}$ could be and $\frac{v_n - p_n}{p_n - p_{n-1}}$ should be, higher market value corresponds to severe price discounts at least when each high signal contributes the same amount to the market value.

APPENDIX C

Part* 1: *Deriving a condition such that it is unprofitable to lower the price to p_{n-2} if it is unprofitable to lower the price to p_{n-1} by reducing the bids.

If it is unprofitable for an investor who has observed a signal *s*, which is H or L or U, to lower the price from p_n to p_{n-1} by reducing the bids then the following relationship must hold:

$$(v_n - p_n)d^a(p_n, s) \geq (v_n - p_{n-1})r_n^a(p_{n-1}, s)$$
$$\Rightarrow \frac{v_n - p_n}{v_n - p_{n-1}} \geq \frac{r_n^a(p_{n-1}, s)}{d^a(p_n, s)} \quad , n \in [1,N] \qquad \textbf{[A8]}$$

where $r_n^a(p_{n-1}, s)$ is the residual supply above price p_{n-1} when there are *n* high signals (n>0), given that a bidder who has received signal *s* has deviated. If [A8] holds, we also have

$$(v_{n-1} - p_{n-1})d^a(p_{n-1}, s) \geq (v_{n-1} - p_{n-2})r_{n-1}^a(p_{n-2}, s), \ n>1$$

Adding $(v_n - v_{n-1})d^a(p_{n-1}, s)$ to both sides of the second inequality:

$$(v_n - p_{n-1})d^a(p_{n-1}, s) \geq (v_{n-1} - p_{n-2})r_{n-1}^a(p_{n-2}, s) + (v_n - v_{n-1})d^a(p_{n-1}, s)$$

Because $d^a(p_{n-1}, s) \geq r_{n-1}^a(p_{n-2}, s)$, if the second inequality is satisfied, then

$$(v_n - p_{n-1})d^a(p_{n-1}, s) \geq (v_n - p_{n-2})r_{n-1}^a(p_{n-2}, s)$$

Thus

$$\frac{v_n - p_{n-1}}{v_n - p_{n-2}} \geq \frac{r_{n-1}^a(p_{n-2}, s)}{d^a(p_{n-1}, s)} \qquad \textbf{[A9]}$$

The product of [A8] and [A9] implies that $\frac{v_n - p_n}{v_n - p_{n-2}} \geq \frac{r_n^a(p_{n-1}, s)r_{n-1}^a(p_{n-2}, s)}{d^a(p_n, s)d^a(p_{n-1}, s)}$

If

$$r_n^a(p_{n-1}, s)r_{n-1}^a(p_{n-2}, s) \geq d^a(p_{n-1}, s)r_n^a(p_{n-2}, s), \qquad \textbf{[A10]}$$

then

$$(v_n - p_n)d^a(p_n, s) \geq (v_n - p_{n-2})r_n^a(p_{n-2}, s)$$

which implies that it is unprofitable to lower the price further to p_{n-2}.

Hence if all investors' demand functions satisfy [A10], we only need to check if it is profitable for them to lower the price p_n to p_{n-1} by reducing the bids.

We can derive the residual supplies from the general market clearing condition:

$$Nd^a(p_n, L) + nc^a(p_n) + d^a(p_n, U) = 1$$

Then [A10] is reduced to

$$c^a(p_{n-1})[d^a(p_{n-2}, L) - c^a(p_{n-2})] \leq c^a(p_{n-2})d^a(p_{n-1}, L) \qquad \textbf{[A10a]}$$

for an H or an L investor; and

$$c^a(p_{n-1})[d^a(p_{n-2}, U) - c^a(p_{n-2})] \leq c^a(p_{n-2})d^a(p_{n-1}, U) \qquad \textbf{[A10b]}$$

for the U investor.

In the first example of EHL, $d^a(p_{n-2}, L) = d^a(p_{n-1}, L) = K$ and $c^a(p_{n-1}) = \dfrac{1-NK}{n-1}$, so $\dfrac{(1-NK)^2}{(n-1)(n-2)} \geq K(\dfrac{1-NK}{n-1} - \dfrac{1-NK}{n-2})$ thus [A10a] is satisfied. So an investor will not try to reduce price further if it is unprofitable to reduce the price from p_n to p_{n-1}.

The same method and results apply for all the other examples we have provided for EH, EHL, EHU and EHLU. (The proof is trivial and thus is ignored).

Part 2*: *Deriving a condition such that it is unprofitable to raise the price to p_{n+2} if it is unprofitable to raise the price to p_{n+1} by increasing the bids.

If we have $(v_n - p_n)d^a(p_n, s) \geq (v_n - p_{n+1})r_n(p_{n+1}, s)$ for any $n \in [0, N-1]$

and $(v_{n+1} - p_{n+1})d^a(p_{n+1}, s) \geq (v_{n+1} - p_{n+2})r_{n+1}(p_{n+2}, s)$ for any $n \in [0, N-2]$,

by subtracting $(v_{n+1} - v_n)d^a(p_{n+1}, s)$ from both sides of the second equation we have:

$$(v_n - p_{n+1})d^a(p_{n+1}, s) \geq (v_n - p_{n+2})r^a_{n+1}(p_{n+2}, s).$$

Thus

$$\frac{v_n - p_n}{v_n - p_{n+2}} \geq \frac{r^a_n(p_{n+1}, s)r^a_{n-1}(p_{n+2}, s)}{d^a(p_n, s)d^a(p_{n+1}, s)}, \text{ and so if}$$

$$r^a_n(p_{n+1}, s)r^a_{n+1}(p_{n+2}, s) \geq d^a(p_{n+1}, s)r^a_n(p_{n+2}, s) \qquad \textbf{[A11]}$$

is satisfied, it is unprofitable to raise the price further to p_{n+2} even if p_{n+2} is lower than v_n. The condition is reduced to

$$c^a(p_{n+1})[d^a(p_{n+2},L)+c^a(p_{n+2})] \geq c^a(p_{n+2})d^a(p_{n+1},L)$$

for an H or an L investor; and

$$c^a(p_{n+1})[d^a(p_{n+2},U)+c^a(p_{n+2})] \geq c^a(p_{n+2})d^a(p_{n+1},U)$$

for the U investor.

All the examples that we have provided for EH, EHL, EHU and EHLU satisfy the above conditions.

Thus when [A10] and [A11] are satisfied, we only need to check if the following conditions, such that it is unprofitable for an investor to lower (raise) price to p_{n-1} for $n>0$ (p_{n+1} for $n<N$) are satisfied:

$$\frac{v_n - p_n}{v_n - p_{n-1}} \geq \frac{r_n(p_{n-1}, s^*)}{d^a(p_n, s)} \quad \textbf{[A12a]}$$

$$\frac{v_n - p_n}{v_n - p_{n+1}} \geq \frac{r_n(p_{n+1}, s^*)}{d^a(p_n, s)} \text{ if } p_{n+1} < v_n. \quad \textbf{[A12b]}$$

If $p_{n+1} \geq v_n$, investors will not try to increase price and we do not need [A12b]. The right hand sides in both inequalities represent the maximum ratio between each player's residual supply given a deviation and her allocation without a deviation, given signals H, L and U.

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