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Characterization of Pure Strategy Equilibria in Uniform Price IPO Auctions

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Abstract

We characterize pure strategy equilibria of common value multi-unit uniform price auctions under the framework of initial public offerings, where bidders have incomplete private information regarding the value of shares and submit discrete demand schedules. We show that there exists a continuum of equilibria where investors with a higher expectation about the value of shares bid for higher quantities at higher prices, and as a result the market price increases with the market value. The collusive equilibria, in which investors place bids regardless of their expectation about the value, are obtained under stricter conditions than in the continuous price case.

Keywords:

IPO, uniform price auction, divisible goods auction, share auction, tacit collusion

JEL Classification Codes:

D44, G12, D82

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1. Introduction

Uniform price auctions are one of the mechanisms for selling initial public offerings (IPOs). In such auctions, bidders submit quantity-price pairs indicating the price that they are willing to pay for obtaining the corresponding quantities. The goods are allocated from high bids to low bids and bidders pay the market-clearing price for all units won.

There is an extensive literature on uniform price auctions that explores one particular type of equilibrium that predicts low revenues for sellers, namely tacit collusion equilibrium (see e.g. Wilson 1979; Back and Zender 1993; Biais and Faugeron-Grouzet 2002). This equilibrium suggests that an increase in the number of bidders cannot improve sellers' revenues because bids are independent of bidders' expectations about the value of shares.

Nevertheless, the tacit collusion equilibrium is generally not supported by field observations and experimental outcomes (see e.g. Kandel et al. 99; Sade et al. 2006; Zhang, forthcoming). Uniform price auctions are widely used for selling multi-unit goods to multiple buyers in financial and commodity markets. [2] Although there are alternative IPO mechanisms available, uniform price auctions are still used in, e.g. Israel, U.S. and France.

The divergence between the theoretical prediction and empirical findings motivates the current study. We found that a common assumption of the theoretical research is that bidders submit continuous demand functions. However, in naturally occurring markets, not only is submitting full-demand schedules costly (Kastl 2008), but the number of bids, the minimum price increment and the quantity multiple are also often bounded by regulators.[3] Hence instead, bidders submit a limited number of price-quantity pairs. This issue is important because the equilibrium that holds with

[2] Other examples include Treasury bills, spectrum and electricity.

[3] For example, the online IPO auction company WR Hambrecht+ Co. used to require a minimum bid increment of 1/32 of a dollar, which has been changed to 1 cent in 2005. On the Singapore stock market, there are five tick size categories (i.e. price increments) ranging from 0.5 cents for stocks priced less than \$1.00 to 10 cents for stocks priced above \$10 (Comerton-Forde et al. 2003). In stock markets a minimum order quantity for a new issue is also required. For example, in the U.S the number is 100 shares. There are usually restrictions on the number of price-quantity pairs allowed. In the Spanish electricity market generators may submit up to twenty-five price-quantity pairs (Fabra et al. 2002), while a maximum of three bids are allowed in Italian treasury bills markets (see Scalia 1996, or Kremer and Nyborg 2004, p.858).

continuous demand functions does not necessarily work in the case with discrete bids. For example, Kremer and Nyborg (2004) show that the collusive equilibrium of the Wilson/Back and Zender model does not survive when bidders only make a finite number of bids.

However, Kremer and Nyborg (2004) assume that bidders are certain about the value of the good for sale. As they mentioned in their paper, this assumption of complete information is not applicable in many markets where bidders only have private information. To make the settings closer to markets such as IPOs, where investors have expectations about the value of shares when placing orders, we relax this assumption in the current study while keeping the discreteness of demand schedules. We take the simple binomial informational structure adopted by Benveniste and Wilhelm (1990) and Biais and Faugeron-Crouzet (2002) and frequently used in IPO modelling, in which each investor receives a signal that can be either high or low, revealing partial information about the market value of shares.

Our paper differs from previous research in the following respects. Unlike Wilson (1979) and Back and Zender (1993) who assume continuous demand functions, we focus on discrete bids and found that the collusive equilibrium, although remaining valid, is obtained under stricter conditions. In addition, we found a continuum of equilibria where investors with a higher expected valuation bid more aggressively and as a result the market price is positively related with the market value. Unlike Kremer and Nyborg (2004) who assume bidders are certain of the value of goods and find that the equilibrium price can be as high as the value, in this paper bidders only have partial information about the value and this leads to underpricing in most equilibria. The binomial informational structure adopted in this paper also implies that the Bertrand-like competition by bidding for the whole issue at one price is one of the equilibrium strategies in models of Kremer and Nyborg (2004) and Wang and Zender (2002; in which bidders receive private information from a set of signals distributed conditional on the value), but is generally not an equilibrium in ours.

The new set of equilibria characterized in this paper has some properties that are consistent with field observations and experimental evidences: an increase in the number of bidders improves competition and revenues (Kandel, et al. 1999); bidders place more bids as the number of bidders increases (Malvey et. al 1997); bidders with higher expected market values place higher bids for more shares than those with lower expected market values (Zhang, Forthcoming). The existence of the enlarged set of equilibria may explain why uniform price auctions are still widely used despite the low revenue prediction that is consistent with the tacit collusion equilibrium. The performance of IPO auctions depends significantly on bidders' equilibrium selection.

The rest of the paper is organized as follows. We discuss the related literature in Section 2 and then introduce the model in section 3. In section 4 we present the main result. Starting in section 4.1 with the tacit collusion equilibrium where all bidders behave symmetrically, we then examine cases where the behaviour of different types of bidders is asymmetric in section 4.2. We discuss our results in section 5.

2. Related Literatures

The equilibrium solution in multi-unit uniform price auctions is quite sensitive to alternative modelling settings. For example, it would depend on the choices between single-unit or multi-unit demand, private values or common values, discrete or continuous demand schedules; it would depend on the information that bidders hold, as well as whether supply uncertainty exists.

Although the truth-telling property of the second-price sealed-bid auction (Vickrey 1961) still holds for uniform price auctions in the context of *single-unit demand* (McCabe, et al. 1990; Pesendorfer and Swinkels 1997; Weber 1983), it does not extend to situations with *multi-unit demand* (Krishna 2002). In the studies carried out on uniform price auctions with multi-unit supply and demand, much research focuses on the two-unit demand model with *private values*. They show that sincerely bidding on the first unit and applying demand reduction, i.e. bidding at a level below the marginal unit value on the second unit is an equilibrium strategy. Bidders have an incentive to shade their bids after the first-unit demand in order to enjoy a lower market price in the case their bids are marginal (Engelbrecht-Wiggans and Kahn; 1998; Krishna 2002; Noussair 1995).[4]

The above mentioned equilibrium has some similarity with the tacit collusion equilibrium in uniform price *common value* auctions. In such auctions, if a bidder has a positive probability of influencing the price, in a situation where the bidder obtains

[4] Levin (2005) demonstrates that bidding well above value on the first-unit and zero on the second-unit (when the reservation price is zero) is also an equilibrium. The theoretical prediction of demand reduction behaviour is supported by experiment studies (Engelmann and Grimm 2003; Kagel and Levin 2001) and field experiment studies (List and Riley 2000). In these studies, overbidding on the first unit is also frequently observed.

some allocation, then she has an incentive to shade her bid (Ausubel and Cramton 2002). Many researchers (e.g. Wilson 1979; Back and Zender 1993; Maxwell,1983) demonstrate the existence of multiple equilibria which yield a sale price well below the competitive price, when bidders submit *continuous demand functions*. In such equilibria, bidders follow inelastic demand functions regardless of their expected market value and, therefore, the market price provides little information about the market value. With inelastic demand functions, it would take a big price increase to increase one's allocation and, consequently collusive strategies become self-enforcing in this non-cooperative game. Thus despite Friedman's (1960, 1990) argument that uniform price auctions have the advantage of increasing competition, theoretical results suggest that sellers would not benefit from increased competition because bidders can manipulate the market price.

However, the equilibrium that holds with continuous demand functions does not necessarily work in the case with discrete bids. Fabra et al. (2006) model an electricity market as an auction with *private values*, capacity constraint and demand uncertainty. They claim that a collusive equilibrium only exists when the demand function is continuous while in the *discrete* case there exists a unique, Bertrand-like, equilibrium. Kremer and Nyborg (2004), who assume that the *value of good is common knowledge*, show that the collusive equilibrium does not survive in the Wilson/Back and Zender model when bidders only make a finite number of bids. Instead Bertrand-like price competition is induced, and the equilibrium price can be as high as the market value.

The empirical evidence generally does not support the tacit collusion equilibrium in uniform price auctions. Although collusive behaviour is observed among large dealers in the market for Treasury bills when uniform price auctions are used, collusion is less severe than under the discriminatory setting.[5] Using individual bidder data from Finnish treasury auctions, Keloharju et al. (2005) find little evidence of collusion. An empirical study on the Zambian foreign exchange market, where a large number of relatively small bidders are involved, provides no evidence of collusive behaviour, though demand reduction is evident (Tenorio 1993). In all these markets, higher participation rates are reported under the uniform price setting, which indicates that the market is widened and competition is encouraged. In Israel's IPO market, contrary to the steep collusive demand function derived by Biais and Faugeron-Grouzet (2002), the demand schedule is flat and elastic (Kandel, et al. 1999). However, it is vulnerable to collusion in markets where a relatively small number of bidders compete on a relatively large number of items (for example spectrum auctions, see Engelbrecht-Wiggans and Kahn 2005), or where the same bidders compete in a frequently repeated auction market (for example electricity markets, see Klemperer 2002; Sweeting 2007). [6]

Empirical evidence from experiments does not generally support the collusive equilibrium either. Engelbrecht-Wiggans et al. (2006) suggest that it is difficult to find statistically significant evidence of demand reduction in multi-unit auctions when there are more than two bidders. Porter and Vragov (2006), in an experiment with two bidders who each has two units demand and private information, report that though demand reduction is observed, bids for low valued units is higher than the equilibrium prediction of zero. Sade et al. (2006) find little evidence of collusive behaviour in uniform price auctions even when communication is allowed and when financial professionals participate.[7] Some experimental literature on collusion in laboratory multi-unit auctions suggests that collusion is only achievable if there are two bidders (Sherstyuk, forthcoming), subjects have a coordination device (Brown et al, forthcoming), or anonymity is abandoned (Füllbrunn and Neugebauer 2007). These results suggests that in markets for IPO where there are many investors, including a large number of usually inexperienced retail investors, a collusive equilibrium would

[5] For examples see Malvey et al. (1997) for the U.S. and Umlauf (1993) for Mexico. In fact, after an experiment with uniform price auctions on two-year and five-year notes that started on September 1992, the U.S. Treasury switched entirely to the uniform-price auction in November 1998 (Ausubel 2002).

[6] Klemperer (2001) argues that collusion is one reason for the UK's decision to adopt a discriminatory auction format in electricity markets, because "the repeated interaction among bidders expands the set of signalling and punishment strategies available to them and allows them to learn to cooperate". Sweeting (2007) found evidence of collusion by the two largest generators of the England and Wales wholesale electricity market in the late 1990s: both could have increased their profits by lowering their bids and significantly increasing their outputs. Nevertheless, he also mentions that this may have occurred because of the generators' attempt to raise the prices that they could negotiate in future hedging contracts.

[7] Goswami et al. (1996) find collusive behaviour in uniform price auctions when communication is allowed. One key difference between the two experiments is that bidders are allowed to place bids at four fixed price levels in Sade et. al (2006) versus three price levels in Goswami et al. (1996). Thus

be difficult to achieve. Zhang (Forthcoming) compares the performances of uniform price auctions and another IPO mechanism fixed price offerings. Given the tacit collusion equilibrium, uniform price auctions should generate lower revenues between the two mechanisms. However, the results of the experiments are contrary to this prediction, because the tacit collusion equilibrium was not achieved. Instead, subjects place bids according to their expectation regarding the market values and, as a consequence, the market price varied in the same direction with the value. This property, as shown later, is consistent with the property of the new set of equilibria characterized in this paper.

3. The Model

The basic model in this paper follows Biais and Faugeron-Crouzet (2002). The volume of shares offered in the auction is normalized to 1. There are $N \geq 2$ large institutional investors and a fringe of small retail investors as potential buyers. All investors are risk-neutral. Each institutional investor has private information, represented by the private signal s_i, about the valuation of shares. Signals are identically and independently distributed and can be *high* with probability π or *low* with the complementary probability. The value of shares increases with the number of *high* signals n.[8] Denote by v_n the market value of a share when there are n *high* signals. The realized value of shares is the same for all bidders on all units. Each informed investor has a large bidding capacity thus can buy the whole issue.[9] The retail investors, however, are uninformed and all together can purchase up to 1- k units shares, with $k \in [0,1]$.[10]

there are fewer alternatives to coordinate on in the latter case (Kagel and Levin 2008).

[8] The market value can be regarded as the price of shares on the first trading day in the secondary stock market. Though there exists equilibrium in which bidders, strategically, do not reveal their private information regarding the value of shares in the primary stock market, according to the *Efficient Markets Hypothesis*, in the secondary stock market prices reveal all available information including bidders' private information.

[9] This assumption is reasonable "given the bidding power of the large financial institutions participating regularly to IPOs, compared to the relatively small size of most of these operations. In addition this assumption simplifies the analysis." (Biais and Faugeron-Crouzet 2002, p15).

[10] In practice, either because retail investors have small demand capacity or because their demand is difficult to predict, firms who go public always try to attract large institutional investors to guarantee full subscription. It is rare to rely on small retail investors to absorb all shares of an IPO, even if the

The price rule and the allocation rule are as follows. If the total demand exceeds the supply, the market price p_m is the highest bid price p where $D(p)$, the total demand at p exceeds supply.[11] Otherwise if the total demand is less than or equal to the supply, the market price is zero.[12] Formally:

$$p_m = \begin{cases} \max(p \mid D(p) > 1) & if \quad D(0) > 1 \\ 0 & otherwise \end{cases} \qquad \textbf{[1]}$$

Thus if there is excess demand, the aggregate demand above p_m is no more than 1, and at p_m exceeds 1. Denote $d_i(p)$ as bidder i's aggregate demand at prices *higher than or equal to p* and $d_i^a(p)$ as her aggregate demand at prices *higher than* p. Notice that differing from Biais and Faugeron-Crouzet (2002), we do not require $d_i(p)$ or $d_i^a(p)$ to be continuous functions.[13] Bidder i's allocation a_i is expressed by the following formula:

$$a_i = \begin{cases} d_i^a(p_m) + [1 - \sum_{i=0}^{N} d_i^a(p_m)] \dfrac{d_i(p_m) - d_i^a(p_m)}{\sum_{i=0}^{N} [d_i(p_m) - d_i^a(p_m)]} & if\ D(0) > 1 \\ d_i(0) & otherwise \end{cases} \qquad \textbf{[2]}$$

This is simply the rule of pro-rata on the margin. Where $i \in (0,1,\ldots, \mathrm{N})$ and i = 0 represents the group of uninformed investors as a whole.[14] If the aggregate demand at

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resulting market price is low. In some issues there is a maximum subscription amount for a retail investor. Hence the assumption that the retail investors can purchase up to 1-k units shares is reasonable. Moreover, k is allowed to take a value as low as zero. In that case, the retail investors as a whole can buy the whole issue.

[11] Following the convention of auction theory, we use the highest losing price rather than the lowest winning price as the market-clearing price. This simplifies our description of bidders' strategies. For example, suppose the quantity multiple is w, an equilibrium in which each bidder bids (1-w)/N at a price of v_N and 1- (1-w)/N at zero when using the lowest winning price as the market price results the same price and allocation as an equilibrium in which each bids 1/N at a price of v_N and 1-1/N at zero when applying the highest losing price as the market price. The results of this paper remain valid if using the lowest winning price and substitute 1 with 1-w where appropriate.

[12] In other words, the market price is the maximum price between the highest bid price where demand exceeds supply and the reservation price, which, without loss of generality, is assumed to be zero.

[13] As mentioned in the Introduction, in practice, including markets for IPOs, either because submitting full-demand schedules is costly (Kastl 2008), or because the number of bids, the price tick or the quantity multiple are bounded by regulators, bidders submit only a limited number of bids, which is not continuous in price.

[14] Following Biais and Faugeron-Crouzet (2002), the group of uninformed investors is regarded as a single player.

the reservation price exceeds the supply then, after allocating to each bidder the amount she bids for at prices higher than the market price, i.e. $d_i^a(p_m)$, the rest of shares (the multiplicand of the second term) are prorated among the bidders. In that case each bidder obtains a proportion equal to the ratio of her bids at the market price over the total bids placed at the market price (the multiplier of the second term). The bids below the market price do not receive any allocation. Otherwise, if the total demand at 0 is less than or equal to the supply, each bidder obtains the amount she bids for, i.e. $d_i(0)$.

A strategy of bidder i in this game is defined as a demand-price schedule $d_i(p, s_i)$ indicating how many shares bidder i would like to bid for at price p, given the observed signal s_i (s_i is either H, L or U representing high, low or uninformed). The allocation rule implies that if a bidder indicates the willingness to buy any amount at some price, then she would also like to buy at least the same amount at a lower price. Thus the function $d_i(p,s_i)$ is nonincreasing in p.

When the realization of the market value is v_n, bidder i's payoff Π_i equals the per unit payoff, $v_n - p_m$, multiplied by the number of units allocated:

$$\Pi_i = (v_n - p_m) \times a_i \qquad [3]$$

As both the market price and the allocation are determined by bidders' demand schedules, bidder i's payoff can be written as a function of bidder i's and all the other bidders'(-i) demand schedules, given the profile of signals $s = (s_0, s_1,\dots, s_N)$:

$$\Pi_i(v_n) = \Pi_i(d_i(p,s_i), d_{-i}(p,s_{-i}), s) \qquad [4]$$

An investor obtains a zero profit by demanding zero:

$$\Pi_i(0, d_{-i}(p,s_{-i}), s) = 0 \qquad [5]$$

Bidder i's problem is to maximize her payoff by choosing a demand schedule $d_i(p,s_i)$. The solution concept is *ex post* Nash equilibrium in bidding strategies, which is a set of strategies $d_i(p,s_i)$ for all investors such that for each bidder i, $d_i(p,s_i)$ maximizes her expected payoff. In addition, given the other bidders' demand schedules, for each realization of market value, bidders would not change their bids even though they were allowed to do so *ex post*. [15] Thus for each bidder $i \in (0,1,\ldots, N)$, for all $d_i(p,s_i)$ and every profile of signals $s = (s_0, s_1,\ldots, s_N)$**:**

$$\Pi_i(d^*(p,s),s) \geq \Pi_i(d_i(p,s_i),d^*_{-i}(p,s_{-i}),s) \qquad \textbf{[6]}$$

We assume that the same types of investors, i.e., informed investors with an H signal (H investors) or with an L signal (L investors) and uninformed investors (U investors) have symmetric beliefs and behaviour. We call an equilibrium that satisfies this assumption a *symmetric-in-type equilibrium*. We express the demand of an H investor at any price p >0 as the sum of the demand of an L investor and an amount $c(p)$ at p, where $c(p)$is also a function of price:

$$d(p,H) = d(p,L) + c(p) \qquad \textbf{[7]}$$

We will show later that in equilibrium $c(p)$ is nonnegative at any price above v_0.

In equilibrium one investor has to give up an amount of shares in order to lower the market price, or raise the market price to a higher level in order to absorb more residual supply. Equilibrium requires that the gain from either kind of deviation is not enough to compensate for the corresponding loss. Equation [5] implies that all bidders obtain a nonnegative payoff in equilibrium, otherwise one could improve the payoff by not participating.

Next, we will conduct the equilibrium analysis on this model of common value multi-unit uniform price auctions where bidders have *private information* regarding the value of shares and submit *discrete* demand schedules.

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[15] We are looking at the set of Nash equilibria that are also *ex post* Nash equilibria. There may be other equilibria that are not examined in this paper. Concentrating on the *ex post* equilibrium simplify the optimization problem with discrete functions because the equilibrium analysis is then free from the distribution of bidders' signals. The use of *ex post* equilibrium appears to originate in the work of Cremer and McLean (1985). Maskin (1992) refers to an *ex post* equilibrium as a "robust" Bayesian Nash equilibrium, because every *ex post* equilibrium is a Bayesian Nash equilibrium, under which bidders' expected payoffs are also maximized *ex ante* (see Appendix F in Krishna 2002 for a discussion). Malakhov and Vohra (2008) derive an ex post optimal mechanism under an environment with a single divisible good and two asymmetrically budget constrained bidders. They claim that the mechanism has the practical advantage over the Bayesian approach in that it is ex-post individually rational, and is completely transparent to the bidders without requiring them to calculate the odds of winning, since incentive compatibility holds for each of the realized profiles of types (p.246).

4. Characterization of Pure Strategy Equilibria

4.1 Tacit Collusion Equilibria

We start with the symmetric tacit collusion equilibria (TCE) which have been examined by much previous research. In such equilibria, *all* investors behave symmetrically regardless of their signals by submitting an inelastic demand function, so that the residual supply faced by a bidder increases only by a small amount when the price is raised by a large amount**.** Thus, the gain from the increase in the allocation cannot offset the loss from the increase in price. The equilibrium market price can be as low as the reservation price, and each investor obtains the same quota of the entire shares. One feature of such equilibria is that the total demand would remain unchanged regardless of the number of high signals, and hence the market price and each investor's allocation would be constant at any possible market value.

The above analysis is based on the assumption that the demand function is continuous. When bidders submit discrete demand schedules, we find that the equilibrium still exists but under stricter conditions. Denote p_n as the market price when there are n high signals. Before describing the equilibrium we introduce the following Lemma:

Lemma 1: *If there exists a symmetric-in-type equilibrium in which $p_{n+1}>p_n$ for all $n\in[0,N)$ then there must exist an equilibrium resulting the same price and allocation where the demand curve between p_n and p_{n+1} is vertical.*
If there exists asymmetric tacit collusion equilibrium where the highest bid of each bidder is $\overline{p}$ and the market price is p_m, there must exist an equilibrium resulting the same price and allocation where the demand curve between p_m and $\overline{p}$ is vertical.
Proof: see Appendix A.

Thus for simplicity we can concentrate on the equilibrium where the demand curve between p_m and $\overline{p}$ is vertical. [16] The same type of equilibria has been studied by Wilson (1979), Back and Zender (1993) and Kremer and Nyborg (2004).

Proposition 1: *There exists a continuum of equilibria where all types of bidders*

[16] See Zhang (2006) for a discussion on equilibria where the demand curve between p_m and $\overline{p}$ is not vertical.

behave symmetrically, the equilibrium price $p_m \in [0, v_0]$*, and for each* $i \in [0, N]$*:*

(i) $d_i(\overline{p}, s) = d_i^a(p_m, s) = \frac{1}{N+1}$*, where* $\overline{p} = \frac{Nv_N + p_m}{N+1}$ and $s \in \{H, L, U\}$*;*

(ii) $d_i(p_m, s) \geq \frac{1}{N+1} + \frac{p_m}{N(N+1)v_0}$.

Proof: see Appendix A.

An *ex post* equilibrium requires the price to be in the range $[0, v_0]$. Part (i) of Proposition 1 states that the demand of each bidder at a price $\overline{p}$ is $\frac{1}{N+1}$, and the demand curve remains vertical between $\overline{p}$ and the market price p_m. This strategy makes it unprofitable for a bidder to increase the allocation by bidding higher. In the case of a continuous demand function where the demand at p_m is $\frac{1}{N+1}$, the total demand above p_m is 1 minus a negligible amount. In contrast, in the discrete case the probability that no bid is placed between p_m and a price (slightly) higher than p_m, say p', is positive. If the total demand at p' is *m* units less than 1, rather than sharing *m* with other bidders by rationing, an investor can absorb all *m* units by bidding *m* units more above p_m without raising the market price. As long as p_m is lower than the realized market value, this deviation increases her payoff. Hence to prevent a profitable deviation the aggregate demand *above* the market price must be 1 and thus no share is left for prorating *at* the market price. In the symmetric case each bidder should demand $\frac{1}{N+1}$ above p_m.[17] Assume that the highest bid of each investors is $\overline{p}$. If one investor tries to absorb all the shares by outbidding the other bidders then the market price will be raised to, at least, $\overline{p}$. Then if $\overline{p}$ is sufficiently high (as described in Proposition 1), no investor will find it profitable to do so. Part (ii) of the proposition states that each bidder bids for at least an additional $\frac{p_m}{N(N+1)v_0}$ at p_m. This requirement makes it unprofitable for a bidder to lower the market price by bidding for less. A simple example of an equilibrium in Proposition 1 is that each bidder bids for

[17] In Biais and Faugeron-Crouzet (2002), when discussing TCE they assume that uninformed investors as a whole can behave in the same way as an informed investor. For simplicity we follow this assumption in Proposition 1. In Appendix A we also give the solution when the capacity of all the uninformed investors is less than 1/N.

$\frac{1}{N+1}$ at v_N and obtains $\frac{1}{N+1}$ allocation at price zero. This equilibrium weakly dominates the other TCE that lead to higher market prices.

4.2. Symmetric-in-type Equilibria

TCE are pooling equilibria where all types of investors follow the same strategy. If H and L investors behave asymmetrically, the market price will change with the market value. For simplicity and without loss of generality, based on Lemma 1, in the rest of paper we will restrict our equilibrium analysis to the case where the demand curve between two possible realization prices p_n and p_{n+1} is vertical.

In equilibrium it is possible that one type of investor is excluded from the market. By "exclude" we mean that this type of investors places no bids above v_0.

Lemma 2: *Unless U investors can buy the whole issue, there is no equilibrium where H investors are excluded from the market given that at least one H investor exists.*
Proof: see Appendix A.

Recall that U investors could hardly buy the whole issue (see footnote 9). Because of lemma 2, in this section we start with checking if there exists an equilibrium in which both L and U investors are excluded and H investors absorb all shares, when there is at least one type-H investor. We denote such an equilibrium as EH.

4.2.1 EH

In an *ex post* equilibrium the equilibrium price is bounded by v_n. The strategy of an H investor is described in the following lemma:

Lemma 3: *When each H investor follows the following strategy, L and U investors are excluded from the market, and an H investor does not have incentive to deviate.*

$$\begin{cases} d(v_n, H) = \frac{1}{n} \quad if\ p_n < v_n\ or\ n = 1; \quad n \in [1, N] \\ d(p_n, H) = \frac{1}{n-1}, \quad n \in (1, N]\ and\ p_n \in [v_{n-1}, v_n]. \end{cases}$$

Proof: see Appendix A.

Examples of this strategy are described in Figure 1, which shows the case when there are four informed investors. In the figure the shaded area is the strategy space described in Lemma 3. If L and U bidders are excluded the strategy described by the upper-right bound leads to a market price that equals the market value (the corresponding demand schedule of an H bidder is shown in Table 1.a). Table 1.b provides the demand schedule that results in the price at v_{n-1}, described by the lower-left bound of the shaded area. Any prices in between are possible market prices, and the bold step-bids demand curve represents one of them. Under such strategies, since the total demand at v_n is at least 1, an H investor cannot increase her allocation without raising the market price above v_n. Furthermore, because for any H bidder the demand of all the other H investors at p_n is 1, given that there are more than one high signals (n-1 investors each bids for $\frac{1}{n-1}$ at p_n), an H investor can only lower the market price below p_n by giving up the whole allocation. Thus an H investor cannot profitably deviate by either overbidding or underbidding.

[Figure 1 about here] [Table 1.a about here] [Table 1.b about here]

If all type-H investors follow the strategies described in Lemma 3, the demand of H investors is at least 1 at v_n. If L or U investors place bids above v_0, they would either get no share (if they bid below v_n), or get shares but earn a non-positive profit (getting a zero profit if the bid is at v_n; if the bid is above v_n, the market price would be raised to at least v_n), in neither case they could do better than by placing bids only at prices no higher than v_0. The following lemma describes the best response strategies of type-L and type-U investors.

Lemma 4: *If H investors follow the strategies described in Lemma 3, then the best response strategy of L and U investors is given by:*

$$\begin{cases} \mathrm{d(p,s)} = 0 & if \quad p > v_0 \\ \mathrm{d(p,s)} = \dfrac{1}{N+1} & if \quad p = v_0 \; and \; p_m < v_0 \qquad where \; s \in \{L,U\} \\ \mathrm{d(p,s)} \geq \dfrac{1}{N+1} + \dfrac{p_m}{N(N+1)v_0} & if \quad p = p_m \in [0, v_0] \end{cases}$$

L and U investors can only get an allocation when there are no H signals and thus the market value is v_0. When the market price is lower than v_0, the strategy is like the tacit collusion strategy described in Proposition 1. Since the sum of demand at v_0 is 1, one cannot increase the allocation without raising the market price above v_0. If the market price is equal to v_0, the strategy guarantees that underbidding is not profitable for any bidder. [18]

The following Proposition follows immediately from Lemma 3 and Lemma 4.

Proposition 2: *If H bidders follow the strategy described in Lemma 3, L and U bidders follow the strategies described in Lemma 4, there exist a continuum of equilibria in which the equilibrium price* p_m *can be any price between* v_{n-1} *and* v_n *for* $n \in [1, N]$ *and between 0 and* v_0 *for n=0, and L and U bidders only obtain an allocation if the market value is* v_0*.*

The equilibrium where $p_m=(v_{n-1}|v>v_0)$ and $p_m=(0|v=v_0)$ leads to the lowest market price among EH for any realization of the market value. By following the strategy of this equilibrium, an investor obtains at least the same payoff as playing other strategies in EH; and gets the highest payoff if the other H investors follow the same strategy. Hence this equilibrium weakly dominants the other EH equilibria.

It has been argued that in multi-unit uniform price auctions large bidders often make room for smaller ones by reducing demand to avoid competition, especially if the smaller bidders have the ability to increase prices (Tenorio 1997). In the following section, we will check if an equilibrium where H investors share the market with the other bidders exists. We denote such equilibrium as EHLU.

4.2.2 EHLU

Lemma 5 and Lemma 6 describe some necessary conditions for an EHLU equilibrium to exist.

In EH, though weakly dominated, there is an *ex post* equilibrium where the market price equals the market value. When L investors participate in the market, however, underpricing becomes inevitable.

Lemma 5: *If L investors share the market with H investors, the market price* p_n *must be lower than the corresponding market value* v_n*.*

Proof: see Appendix A.

H and L investors are informed investors with different signals. The following Lemma implies that the strategy of informed investors is monotone, i.e. an H investor bids no less than L investors, at prices above v_0.

[18] As in section 4.1, we assume that U investors behave symmetrically as an L investor. If their capacity is below 1/N, the solution will correspond to the solution given for Proposition 1 in Appendix A.

Lemma 6: *If there is at least one H investor, in equilibrium* $c(p)$ *is nonnegative at any price* above v_0.

Proof: see Appendix A.

The following Lemma requires the lowest price in EH to be v_{n-1}:

Lemma 7: *If either L or U investors are excluded from the market, and H and L investors do not behave symmetrically, then* $p_n \geq v_{n-1}$.

Proof: see Appendix A.

The lemma is developed from the restriction that the total demand at v_n must be at least 1 to keep some type of investors out of the market (see [A1]). When all investors participate, this restriction no longer applies and thus it is possible to have a lower market price than v_{n-1}.

An equilibrium must also satisfy the following conditions.

Firstly, as we explained in section 4.1 for the case of TCE, when the market price is below the market value, to prevent a profitable deviation the sum of demand above the market price must be 1 and thus no share is left for prorating *at* the market price:

$$Nd^a(p_n, L) + nc^a(p_n) + d^a(p_n, U) = 1 \qquad \text{if } p_n < v_n\text{, where } n \in [0, N] \qquad \textbf{[8]}$$

Secondly, since demand functions of all types of investors are nonincreasing in price, for every $n \in [1, N]$ and $i \in [0, N]$:

$$d(p_n, s_i) \leq d(p_{n-1}, s_i)\text{, where } n \in [1, N],\ s_i \in \{H, L, U\} \qquad \textbf{[9]}$$

Thirdly, no bidder should be able to improve the payoff by any kind of deviation. Thus in equilibrium all investors receive nonnegative payoff. The most profitable deviation given n high signals would be to raise or to lower the price to p_d ($d \in [0, N]$ and $d \neq n$) and absorb all the residual supply above that price(denote by $r_n(p_d, s_i)$). The following condition rules out such deviations. For every bidder $i \in [0, N]$:

$$(v_n - p_n)d^a(p_n, s_i) \geq \max(0, (v_n - p_d)r_n(p_d, s_i)) \ \forall\ d \neq n;\ d, n \in [0, N] \qquad \textbf{[10]}$$

Where s_i represents investor i's signal, which can be H, or L or U. Condition [10] indicates that:

$$p_n \leq \min\left(v_n, v_n - \frac{r_n(p_{d*}, s^*)}{d^a(p_n, s^*)}(v_n - p_{d*})\right)$$

where d^* and s^* is the combination that maximize the deviation profit, and when the profit is nonnegative d^* and s^* support the minimum level of underpricing that prevents a profitable deviation: $\frac{r_n(p_{d*}, s^*)}{d^a(p_n, s^*)}(v_n - p_{d*})$. Thus the minimum underpricing level is positively related with $(v_n - p_{d*})$. Since this holds for any realization of n, if the price discount is large for some market value, it should also be large for other possible market values. The upper bound of price p_n decreases with the ratio $\frac{r_n(p_{d*}, s^*)}{d^a(p_n, s^*)}$, so the lower (or higher) the improvement in allocation as a result of a deviation compared with the equilibrium allocation, the higher the market price that can sustain (or the larger the underpricing must be).

The inequality [10] also guarantees that the equilibrium price is bounded by v_n. No other restrictions are needed. All strategies that satisfy requirements [8] to [10] form an equilibrium, which have the properties implied by Lemma 5, Lemma 6 and Lemma 7.

Proposition 3: *There exists a continuum of equilibria where all investors can participate and the equilibrium price lies in the range of [0, v_n] for any* $n \in [0, N]$.

Here we provide an example of EHLU:

For any $p > 0$, (i) $d(p,L) = K$, where $K \in [0, \frac{1}{N}]$ is a constant; (ii) $d(p,H) = d(p,U) = K + c^a(p)$, where $c^a(p_n) = \frac{1-NK}{n+1}$; (iii) $\frac{v_n - p_{n+1}}{(n+2)(p_{n+1} - p_n)} \le \frac{K}{1-NK} \le \frac{v_n - p_n}{(n+1)(p_n - p_{n-1})}$.

In the example (i) and (ii) make it satisfy restrictions [8] and [9], and (iii) follows from the restriction [10]:

$$(v_n - p_n)K \ge \max[0, (v_n - p_{n+1})(K + \frac{1-NK}{n+2})],\ n \in [0, N-1], \text{ and}$$

$$(v_n - p_n)(K + \frac{1-NK}{n+1}) \ge (v_n - p_{n-1})K,\ n \in [1, N]$$

The first inequality implies that an L investor cannot improve her payoff by increasing the allocation at a cost of raising the market price. It can be shown that an L investor benefits the most from overbidding compared with either an H or U investors. If this condition is satisfied, it is also unprofitable to increase the allocation further by raising the market price to an even higher level. The second inequality ensures that it is unprofitable for an H investor to deviate by lowering the market price to p_{n-1} or further, at a cost of reducing demand. An H investor benefits more than an L or U investors when deviating by underbidding. Rearranging the two inequalities we get (iii). [19]

EHLU nests all other sets of equilibria including both TCE and EH.

In TCE, H and L investors are symmetric in their bidding behaviour, so $c^a(p_n)$ is zero, thus the left hand side of the equation [8] does not depend on n. Hence the market price is constant regardless of the market value. If assuming the uninformed investors as a whole behave in the same way as an informed bidder, then each of them would demand 1/(N+1) above the market price. The requirement from [10] completes the rest of Proposition 1.

In EH, both $d^a(p_n, L)$ and $d^a(p_n, U)$ are zero, so $c^a(p_n) = d^a(p_n, H)$ must be equal to $\frac{1}{n}$. To satisfy the inequality [10], for any $d>n$, since $r_n(p_d, s_i)$ is larger than $d^a(p_n, H)$, $(v_n - p_d)$ has to be nonpositive, and thus p_{n+1} must be at least v_n (Lemma 7). For $d < n$, since we only consider the cases when the demand curve between p_n and p_{n-1} is vertical, and thus $r_n(p_i, s)$ is zero, the inequality [10] is naturally satisfied and p_n can be as high as v_n.

There are also sets of equilibria where only L investors are excluded (EHU, $d^a(p_n, L)$=0 for $p_n>v_0$), or only informed investors participate (EHL, $d^a(p_n, U)$=0 for $p_n>v_0$). The construction of these equilibria is similar as that of EHLU and thus we put them in Appendix B. In both cases the minimum market price is v_{n-1} (Lemma 7).

4.2.3 Flat Demand Functions

In this section we consider a special kind of strategy where each bidder places one

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[19] According to (iii), $\frac{K}{1-NK}$ increases with K. The higher K is, the larger $\frac{v_n - p_{n+1}}{p_{n+1} - p_n}$ could be and $\frac{v_n - p_n}{p_n - p_{n-1}}$ should be. This implies that the more shares that an L investor obtains, the price discount gets larger (higher v_n- p_{n+1} and v_n- p_n), and the neighbour prices gets closer (smaller p_{n+1} - p_n and p_n-p_{n-1}). In the extreme case, when K equals $\frac{1}{N}$, p_n is equal to p_{n-1} and thus the market price must be constant. This is consistent with TCE. In the other extreme case, when K equals zero, p_n should be no higher than v_n but no lower than v_{n-1} (as v_n-p_{n+1} is non-positive for every n when K is zero), which is consistent with the equilibria EH. For values of K in between, we expect that the market price can take any value between zero and $v_n - \frac{r_n(p_{d*}, s^*)}{d(v_n, s^*)}(v_n - p_{d*})$ ([10]).

bid for all shares given that the capacity constrained is satisfied. In equilibrium the market price should be either lower than or equal to the market value. Since we consider the symmetric-in-type equilibrium, there are at most three prices at which investors place their bids, one for each type of investors. This means there are at most three possible prices besides zero. Hence an equilibrium where p_n *equals* v_n does not exist for N>2 (when there are more than three possible realizations of v_n). For N=2, only when U investors can buy all the shares, which can rarely happen (see footnote 9), there exists an equilibrium in which the equilibrium price equals the market value:

$$d(v_2, H)=1, d(v_1, U)=1, d(v_0, L)=1$$

For an equilibrium where the price is *lower* than the value, there are two possibilities: every type of investor obtains an allocation, or one or two types of investors do not receive an allocation.

In the first case, since each informed investor bids for 1, rationing must exist among bidders. Thus one bidder could improve the payoff by bidding higher than the market price and absorb a higher allocation rather than share with other bidders. Because of such Bertrand-like competition, no equilibrium exists in this case.

In the second case, in order to make it unprofitable for a bidder who does not obtain an allocation to bid higher than the market price, it must be the case that all the shares have been allocated at a price higher than or equal to the market value. Suppose it is informed bidders who place bids at or above the value and obtain all the allocation. Since the number of H or L bidders changes with n, there must exist cases when either n or N-n is larger than 1, and thus the market price is higher than or equal to the value. This is a contradiction. Hence if there exists an equilibrium where price is lower than the value, it must be U investors who absorb all the allocation at or above the market value. This is only possible if U investors can buy the whole issue:

$$d(v_N, U)=1, d(p, H)= d(p, L)=0 \text{ for } p>v_0.$$

Under this equilibrium, an informed investor can only obtain an allocation by raising the price above v_N. Since the market price will be between 0 and v_0, U investors would not like to change the strategy *ex post*. However, deviating by bidding higher than v_0 but lower than v_N is no cost (no benefit as well) for an informed investor. Thus for U investors following the strategy consistent with such an equilibrium is risky.

Proposition 4: *Submitting flat demand functions by all bidders is not an equilibrium* *strategy in uniform price IPO auctions, unless uninformed bidders as a whole can buy the whole issue.*

5. Discussion and Conclusion

In this paper we have characterized pure strategy equilibria for discrete uniform price common value auctions with private information. Though we employ the framework of IPOs, the results are applicable to other markets with similar environments. Tacit collusion equilibria (TCE) exist, but under stricter conditions than for the continuous price case. In addition we identify a continuum of equilibria that can be classified according to the types of investors that participate. In all these additional equilibria investors with higher expected values bid for higher quantities at higher prices. The market price increases with the market value and thus, though underpricing happens in most equilibria, the level of underpricing is lower than that under TCE. The new sets of equilibria have the following properties.

As the number of investors N increases, the distances among the "steps" in the demand curve get smaller and thus the demand curve becomes smoother. As N approaches infinity, one can conjecture that the demand curve of each H investor becomes smooth (see the curve in Figure 1). The average demand schedule in IPO auctions in Israel provided by Kandel et al. (1999) appears to have this shape. Also, when N goes to infinity, the difference between v_n and v_{n-1} vanishes and therefore the market price gets closer to the market value. This implies that unlike TCE, competition increases revenues.[20]

Another prediction is that bidders place more bids as the number of bidders increases. The reason is that bidders should include bids for every possible market value in their demand schedules. This prediction is consistent with the evidence from the US Treasury bill market. Under the uniform price auction format, the number of investors participating in the market is higher and large dealers split bids into numerous smaller bids (Malvey et al. 1997).

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[20] Although in this paper we assume that the market value increases with the number of high signals, in fact it is the proportion of high signals that matters. Thus even if the number of investors participating changes, the underlining market value can still remain the same. The result may relate to the finding by Bierbaum and Grimm (2006), that in uniform price auctions where a perfectly divisible good is sold to a large number of bidders, bidders act as price takers and a single buyer's bid has no impact on aggregate revealed demand, thus truthful bidding is an equilibrium.

Our theoretical results also suggest that, in contrast to TCE, investors with high signals place higher bids than those with low signals. This is consistent with the experimental evidence in Zhang (Forthcoming).

Not only are there multiple equlibria resulting in different market prices, but also a certain market price can result from different equilibrium strategies. The volatility observed in actual uniform price auctions (Jagannathan and Sherman 2006) may be explained by the existence of a large set of equilibria.

TCE requires that bidders bid for a considerable amount of shares from a considerably high price. Playing this strategy may leave bidders ending up paying a high market price if other bidders deviate from the strategy. In other word, TCE strategy can be risky. It may be achieved if a small number of bidders play repeatedly. With many institutional and retail investors involved in IPOs, the collusion of a small number of parties is not a likely scenario. If TCE is not likely to happen in practice, uniform price auctions may generate higher revenues than other IPO mechanisms. In the experiment by Zhang (Forthcoming), uniform price auctions outperform fixed price offerings because bidders follow other strategies rather than TCE.

Though risky, TCE are payoff dominant. In addition, TCE are robust to changes in model variations. Thus it may be worth introducing some market rules in order to prevent such equilibria from being played. McAdams (2006) shows that all equilibria of a uniform price auction with adjustable supply generate strictly higher expected profit than any other equilibrium given any fixed quantity and reserve price. Other research also suggests that the seller could benefit by randomizing the quantity offered (e.g. Back and Zender 2001). Such rules include, for example, a Greenshoe Option employed in US which allows issuing firms to increase the amount of offered shares by up to 15% after the bids. In Italy, the amounts of shares offered to institutional and retail investors are separated and can be transferred in between depending on the demand in each market (Boreiko and Lombardo 2008).

Another approach for eliminating undesirable equilibria is using an alternative allocation rule. Kremer and Nyborg (2004) suggest two allocation rules. One is the uniform rationing rule, in which all winning bids are pro-rated, regardless whether they are placed *above* or placed *at* the market price. The other is the average of the uniform rationing rule and the standard allocation rule used in this paper.

In auctions where either a small number of bidders participate, or some bidders are significant in size relative to the auction volume, the competition is limited and the auctioneer needs to address the potential effect of market power in the auction design (Ausubel and Cramton 2004). As we have pointed out, the new set of equilibria characterized in this paper has the property that competition increases sellers' revenues. Thus the sellers' payoff may be improved under a design that encourages investors to enter the market. In that case even if a small number of institutional investors were capable of cooperating, the competition from a large number of bidders outside the cartel would offset the advantages of collusion. Such policies include allowing small investors to purchase a smaller block of shares and lowering the threshold for opening accounts.

The assumed relationship between the number of high signals and the market value makes it possible to develop the new set of equilibria. This assumption is reasonable for the stock market because in such a market the demand on a stock is directly related to its market value. An investor who has a high signal can be regarded as an investor who is willing to buy the stock. On the other hand, the equilibria imply that investors place bids not only according to their expected market value, but also according to their expectation of the number of investors who would like to place orders on a certain price. A useful topic for future research would be to analyze uniform price auctions under alternative information structures.

APPENDIX A: PROOFS

Proof of Lemma 1: Suppose that we have an arbitrary equilibrium demand schedule (DS) under which the market price is p_n (see Figure A1). By keeping the demand curve between p_n and p_{n+1} vertical for any $0\leq n<N$ (called VDS), the aggregate demand above p_n and that at p_n remain unchanged, so both the market price and the allocation remain the same as those under DS. Moreover, because the arbitrary DS is downward sloping and lies to the left of VDS, the residual supply as well as the profit when deviating are minimized under VDS. Thus if DS is an equilibrium demand schedule, so is VDS. The same logic applies for strategies in TCE. □

[Figure A1 about here]

***Proof of Proposition* 1:** The price range [0, v_0] guarantees that the equilibria are *ex post*. Otherwise if the market price is higher than v_0, then for some possible realization of v_n, investors would suffer negative payoffs and thus would rather not participate after knowing v_n.

(i) As described in the main text, because in the discrete case there is a positive probability that no bids are placed between p_m and a price higher than p_m, there should be no shares left for allocating at p_m. Otherwise a bidder could bid more at a price slightly higher than p_m in order to increase the allocation at a negligible cost. Thus the demand function should be vertical above p_m.[21] In the symmetric case this means that each bidder bids for $\frac{1}{N+1}$ at a price higher than p_m, which guarantees that each bidder obtains $\frac{1}{N+1}$ in the first place.

Suppose that the highest price at which bidders place bids is $\overline{p}$. Then if an H investor tries to absorb the entire supply by bidding at a higher price, the market price would be raised from p_m to $\overline{p}$. The following inequality should be satisfied to make it unprofitable to do so:

[21] According to Lemma 1, we concentrate on the case where the demand curve is vertical above p_m for simplicity. However, in fact it is unnecessary that the whole demand function above p_m is vertical. In other words, a step-like demand curve can exist at prices higher than p_m. See Zhang (2006) for details.

$$v_i - \bar{p} \leq [v_i - p_m]\frac{1}{N+1}\text{, which requires } \bar{p} \geq \frac{Nv_i + p_m}{N+1}$$

When v_i takes the value of v_N, the right hand side of the inequality is maximized and all investors do not have incentives to deviate by overbidding for any realization of n.

ii) A bidder may try to lower the market price to p' by reducing her demand. This deviation will not occur if it cannot increase the bidder's profit:

$$\frac{1}{N+1}(v_i - p_m) \geq (1 - Nd(p_m))(v_i - p')$$

Thus $d(p_m) \geq \dfrac{Nv_i - (N+1)p' + p_m}{N(N+1)(v_i - p')}$

The right hand side is maximized when p' is zero. For any possible realization of the value, if each bidder's demand at p_m is $\dfrac{Nv_0 + p_m}{N(N+1)v_0}$, which is $\dfrac{p_m}{N(N+1)v_0}$ units additional to $\dfrac{1}{N+1}$, the above deviation will not be profitable.

For simplicity, the above analysis assumes that the uninformed bidders can buy $\dfrac{1}{N}$ units of shares (the maximum of $\dfrac{1}{N+1} + \dfrac{p_m}{N(N+1)v_0}$) and behave symmetrically with informed bidders. If their capacity constraint, say, K, is less than $\dfrac{1}{N}$, then the solution should be:

$$d(\bar{p}, U) = d(0, U) = K, d_i(\bar{p}, s) = d_i^a(p_m, s) = \frac{1-K}{N}, \textit{ where } \bar{p} = v_N(1-K) + p_m K,$$

$$d_i(p_m, s) = \frac{1-K}{N} + \frac{(1-K)p_m}{N(N-1)v_0}, s \in \{H, L\}.$$

The proof follows the same method and procedure as above and thus is ignored. □

Proof of Lemma 2: Suppose one type of investors stays out of the market. If the market price p_n is lower than the market value v_n, the total demand at v_n is equal to or less than 1(the price rule [1]). If it is less than 1, an investor who is out of the market can improve the payoff by bidding a positive amount at v_n. Thus, *if in an equilibrium the market price is lower than the market value, and at least one type of investors is excluded from the market, then the total demand at the market value is equal to* 1:

$(N-n)d(v_n,L)+nd(v_n,H)+d(v_n,U)=1$, if $p_n < v_n$ for $n \in [0,N]$, and $d(p,s)=0$ $\forall$ $p>v_0$ and for at least one signal $s \in (H,L,U)$. **[A1]**

Thus if H investors are excluded, whether the market price is equal to or less than the market value, we should have:

$$(N-n)d(v_n,L)+d(v_n,U)\geq 1, \text{ with } d(v_n,U)\in[0,1-k] \text{ and } d(v_n,L)\geq 0 \quad \textbf{[A2]}$$

Hence,

$$(N-(n-1))d(v_n,L)+d(v_n,U)\geq 1, \; \boldsymbol{n} \in \mathbf{[1,N]} \quad \textbf{[A3]}$$

If $d(v_n,L)>0$, the left hand side of [A3] is larger than 1. This implies that p_{n-1} is at least v_n and thus cannot sustain in equilibrium.[22] If $d(v_n,L)=0$, to satisfy both [A2] and [A3], $d(v_n,U)$ must take a value of 1.[23] Thus H investors could be excluded from the market in equilibrium only if U investors can absorb the whole issue, and which could rarely happen in practice (see footnote 9). □

Proof of Lemma 3: When the market price is lower than the market value, H investors' total demand at v_n is 1 ([A1]):

$$nd(v_n,H)=1 \text{ for each } n \in [1, N]$$

Hence each H investor should bid for $\frac{1}{n}$ at v_n for any possible realization of $n \in [1, N]$ to prevent L or U investors from obtaining an allocation. As we are considering the case where the demand curve between p_{n-1} and p_n, thus between v_{n-1} and p_n, is vertical, the quantity that an investor bids for at p_n must be $\frac{1}{n-1}$, and no bids are placed between v_{n-1} and p_n, for any n $\in$ (1, N].

When the market price equals the market value, each H investor gains a zero expected payoff. If an H investor can lower the market price below the market value, while still keeping some allocation, she would enjoy a positive payoff. So to prevent

[22] Since the two inequalities should hold for every n>0, the expected profit of a participant is negative, thus it cannot sustain even the solution concept is relaxed to a Bayesian Nash Equilibrium.

[23] The equilibrium strategy in this case is described in Section 4.3.

an H investor from deviating, the H investor could only lower the price by giving up the whole allocation. This requires that the total demand of the other H investors is at least 1 at v_n:

$$(n - 1)\, d\,(v_n, H) \geq 1 \text{ for } n \in (1, N] \text{ thus } d\,(v_n, H) \geq \frac{1}{n-1}.$$

Moreover, in order to have nonnegative payoffs the total demand of H investors *above* v_n cannot be more than 1:

$$nd^a(v_n, H) \leq 1 \text{ for each } n \in [1, \text{N}] \text{ Hence } d^a\,(v_n, H) \leq \frac{1}{n}.$$

Because this relation should be satisfied for any realization of n, the demand *at* v_n (*above* $v_{n\text{-}1}$) is no more than 1 when there are n-1 high signals. So

$$d\,(v_n, H) \leq \frac{1}{n-1} \quad \forall\ n \in (1, \text{N}].$$

Thus an H investor's demand *at* v_n must be $\frac{1}{n-1}$ and the demand function between $v_{n\text{-}1}$ and v_n should be vertical. An H investor need not increase the demand at prices below v_2. Thus p_1 equals the highest bid placed by either an L or U investors (which is v_0 according to Lemma 4).

The development of these strategies imply that, given H bidders' strategy, H and L bidders will stay excluded; given that L and U bidders stay excluded, an H bidder has a nonnegative expected payoff and cannot benefit from deviating. □

Proof of Lemma 5: Suppose that the market price equals the market value. An H investor could raise her profit from zero to a positive level by lowering the price by bidding less, unless the demand of all the other investors at v_n is at least 1:

$$(n-1)d(v_n, H) + (N-n)d(v_n, L) + d(v_n, U) \geq 1$$

If this condition holds, we also have

$$(n-1)d(v_n, H) + (N-(n-1))d(v_n, L) + d(v_n, U) > 1 \text{ if } d(p,L) \text{ is larger than zero.}$$

The inequality implies that the total demand at price v_n is larger than 1 when there are n-1 H investors, thus $p_{n\text{-}1}$ equals v_n. Hence investors have zero profits when the realization of the market value is v_n for the given n and have a negative payoff if the market value is lower. This leads to a negative expected payoff. Hence in equilibrium the market price should be lower than the market value. □

***Proof of Lemma 6*:** Lemma 2 implies that there is no equilibrium where H investors are excluded but L investors are not. If L investors are excluded from participating but H investors are not, $c(p)$ must be nonnegative at any price above v_0. If both H and L investors participate, Lemma 5 implies that p_m is below v_n. Then [A1] requires that:

$$Nd(v_n, L) + nc(v_n) + d(v_n, U) = 1,\ n \in (1, \mathrm{N}]$$

Thus if $c(p) < 0$, $Nd(v_n, L) + (n-1)c(v_n) + d(v_n, U) > 1$, which means the market price p_{n-1} is at least v_n when the value is v_{n-1} and leads to a negative payoff for some realization of the value. Hence $c(p)$ should be nonnegative at v_n. Since when L investors participate the market price is lower than v_n, the demand curve between p_n and v_n is vertical and thus $c(p)$ is nonnegative at any price between p_n and v_n. As we only consider equilibria where the demand curve between p_n and p_{n+1} is vertical (Lemma 1), $c(p)$ is nonnegative at any prices above v_0. □

Proof of Lemma 7: According to [A1], to prevent type-L and type-U investors from participating, we have:

$$(n-1)c(v_{n-1}) \geq 1$$

So $c(p)$ is positive and $nc(v_{n-1}) > 1$, hence $p_n \geq v_{n-1}$.

In the case that only L investors stay out of the market (then $c(p)$ is nonnegative):

$$(n-1)c(v_{n-1}) + d(v_{n-1}, U) \geq 1,$$

Thus $nc(v_{n-1}) + d(v_{n-1}, U) \geq 1$, $p_n \geq v_{n-1}$.

In the case that only U investors stay out of the market:

$$Nd(v_{n-1}, L) + (n-1)c(v_{n-1}) \geq 1$$

In this case $c(v_{n-1})$ is nonnegative (Lemma 6). When H and L investors behave asymmetrically $c(v_{n-1})$ is positive. Thus $Nd(v_{n-1}, L) + nc(v_{n-1}) > 1$, and $p_n \geq v_{n-1}$.

Together the above arguments complete the proof. □

APPENDIX B: EHU and EHL

Proposition 5 (EHU): *There exists a continuum of equilibria in which L investors are* *excluded, the equilibrium price satisfies* $p_m \in [v_{n-1}, v_n]$ *for* $n \in [1,N]$, *and* $p_m \in [0, v_0]$ *for* $n=0$, *and L investors only obtain an allocation when the market value is* v_0.

Proof: The proof is by construction.

The demand functions of both an L and an H investor are non-increasing in price:

$$d(p_n, U) \le d(p_{n-1}, U) \text{ and } c(p_n) \le c(p_{n-1}) \qquad \textbf{[B1]}$$

As in equilibrium L investors are excluded when there is at least one H investor, the market price could be as high as the market value (Lemma 5 is irrelevant). If p_n is equal to v_n, to prevent an investor from deviating, the total demand of the other players must be at least 1 at price v_n:

$$(n\text{-}1)c(v_n) + d(v_n, U) \ge 1 \text{ and } nc(v_n) \ge 1 \;\; \forall \;\; n \in [1, N] \text{ if } p_n = v_n$$

In addition, the total demand at v_n when there are n-1 H investors should be no more than 1(otherwise the payoff is negative when the market value is v_{n-1}):

$$(n\text{-}1)c(v_n) + d(v_n, U) \le 1 \;\; \forall \;\; n \in [1, N] \text{ if } p_n = v_n$$

Put the above two inequalities together we have:

$$(n\text{-}1)c(v_n) + d(v_n, U) = 1 \text{ and } nc(v_n) \ge 1 \;\; \forall \;\; n \in [1, N] \text{ if } p_n = v_n \qquad \textbf{[B2a]}$$

It is unprofitable to overbid as well (otherwise the price would be raised higher than the value). The strategy that satisfies [B2a] and [B1] can be an equilibrium strategy. [B2a] also implies that $c(v_n) \ge d(v_n, U)$. Thus an H bidder bids no less than the U bidders.[24]

If $p_n < v_n$, [A1] requires that the demand at v_n must be 1:

$$nc(v_n) + d(v_n, U) = 1 \;\; \forall \;\; n \in [1, N] \;\; \text{if } p_n < v_n \qquad \textbf{[B2b]}$$

Thus if U investors bid for $d(v_n, U) \in [0, 1-k]$, each H investor bids for $\frac{1-d(v_n, U)}{n}$ in $(p_n, p_{n+1}]$ for any $n \in [1, N]$ (Lemma 1, Figure A1). Under this strategy, the total demand above v_n is 1, so no investor would like to increase the allocation by raising the

[24] A simple example of such an equilibrium is: $c(v_n) = d(v_n, U) = 1/n \;\; \forall$ n $\in$ [1,N] (assume U investors can bid for up to 1), and L investors do not place any bids. This is an equilibrium similar to the EH with an extra H investor, and with U investors as a whole behaving the same as an H investor.

market price. To make it unprofitable for an H investor to lower the market price by demanding less, the payoff under the above strategy should be no less than that when lowering price to p_{n-1} and taking all the residual supply above p_{n-1} (at v_{n-1}):

$$(v_n - p_n)\frac{1-d(v_n,U)}{n} \geq \max[0,(v_n - p_{n-1})(1-d(v_{n-1},U)-(n-1)c(v_{n-1}))]\ \forall\, n \in [1,N]$$

Because $d(v_{n-1},U)+(n-1)c(v_{n-1})=1$, the right hand side is zero, thus the equation is always satisfied. An H investor can only reduce the market price by giving up the whole allocation.

To make U investors unprofitable to deviate, when she gives up part of the allocation and lowers the market price to p_{n-1}, the profit from the deviation should be no more than that under the equilibrium strategy:

$$(v_n - p_n)d(v_n,U) \geq \max[0,(v_n - p_{n-1})(1-nc(v_{n-1}))]\ \forall\, n \in [1,N] \qquad \textbf{[B3]}$$

The condition is satisfied if $d(v_{n-1},U)$ is no more than $c(v_{n-1})$ (thus no residual share is left at p_{n-1}), otherwise it requires $\frac{d(v_n,U)}{d(v_{n-1},U)-c(v_{n-1})} \geq \frac{v_n - p_{n-1}}{v_n - p_n}$. It can be shown that if [B3] is satisfied and $c(p)$ is non-increasing in n, an even lower market price is also unprofitable (Appendix C).

As in EH, L investors place no bids above v_0 and only obtain an allocation when there is no H investors in the market. Specifically:

$$d(p,L) = \begin{cases} 0 & if \quad p > v_0 \\ \frac{1-d(v_0,U)}{N} & if \quad p = v_0\ and\ p_m < v_0 \\ \geq \max[\frac{1}{N} - \frac{(v_0-p_m)d(v_0,U)}{Nv_0}, \frac{1-d(v_0,U)}{N} + \frac{p_m(1-d(v_0,U))}{N(N-1)v_0}] & if \quad p = p_m \in [0,v_0] \end{cases}$$

The proof follows the same method as in Lemma 4 so is ignored. Together with strategies that satisfy [B1], [B2b] and [B3], we obtain a continuum of equilibria stated in Proposition 5. □ [25]

Example: (i) $d(p,U) = K\ \forall\, p>0$, where $K \in (0,1/N]$ is a constant;

[25] For n =0 in Proposition 4 and 5, L and U investors' strategies are the same as those described in Lemma 4.

(ii) $\begin{cases} d(v_n, H) = \dfrac{1-K}{n} \quad if\ p_n < v_n\ or\ n = 1; \quad n \in [1, N] \\ d(p_n, H) = \dfrac{1-K}{n-1}, \quad n \in (1, N]\ and\ p_n \in [v_{n-1}, v_n]; \end{cases}$

(iii) $d(p,L) = \begin{cases} 0 & if \quad p > v_0 \\ \dfrac{1-K}{N} & if \quad p = v_0\ and\ p_m < v_0 \\ \dfrac{1-K}{N-1} & if \quad p = p_m \in [0, v_0] \end{cases}$

H investors' strategy is similar as that in EH, except that now the supply they face is 1- $d(p,U)$ instead of 1. For the same reason as before, they are unable to profitably deviate. It's unprofitable for U bidders to overbid. To prevent U investors to underbid:

$$(v_n - p_n)d(p,U) \geq (v_n - p_{n-1})(1 - nd(v_{n-1}, H))$$

The right hand side equals $(v_n - p_{n-1})\dfrac{nd(p,U)-1}{n-1}$. Rearranging it we get:

$$d(p,U) \leq \frac{1}{1 + (n-1)\dfrac{p_n - p_{n-1}}{v_n - p_{n-1}}}$$

The right hand side decreases with p_n, so the condition holds if it is satisfied when $p_n = v_n$, in which case $d(p,U) \leq \dfrac{1}{n}$. Since U investors' demand curve is constant, the condition always holds if $d(p,U) \leq \dfrac{1}{N}$. It is easy to see that the example also satisfies restrictions [8] and [9].

Proposition 6 (EHL): *There exists a continuum of equilibria in which U investors are excluded; the equilibrium price satisfies* $p_m \in [v_{n-1},\ v_n - \dfrac{r_n(p_{d*}, s^*)}{d(v_n, s^*)}(v_n - p_{d*})]$ *for* $n \in [1, N]$*, and* $p_m \in [0,\ v_0]$ *for n=0; the bid of an H investor is higher than that of an L investor by* $c(v_n) > 0\ \forall\ n \in [1, N]$*; and U investors only obtain an allocation when the market value is* v_0*.*

Notice that consistent with Lemma 5, in EHL p_n can only be equal to v_n if $d(p\ ,L)$ is zero (in this case we obtain an EH type of equilibrium).

The construction of EHL is similar as that of EHU and EHLU and thus is ignored.

Below are two examples.

Example 1: (i) $d(v_0,L) = 1$ *and* $d(p,L) = K \; \forall \; p>v_0$, where $K \in (0,\frac{1}{N}]$; (ii) $d(p,H) = K + \frac{1-NK}{n}$ for $p \in (p_n,p_{n+1}]$ and $n \in [1,N]$; and (iii) U investors place no bids.

In this example p_n lies in the interval $[v_{n-1}, v_n - \frac{nK}{1-(N-n)K}(v_n - p_{n-1})] \; \forall \; n \in [1,N]$.

Example 2: (i) $d(v_0,L) = 1$ *and* $d(p,L) = \frac{K}{N(n+K)} \; \forall \; p>v_0$ and $n \in [1,N]$;[26] (ii) $d(p,H) = \frac{K}{N(n+K)} + \frac{1}{n+K} \; \forall \; p>v_0$, $p \in (p_n,p_{n+1}]$ and $n \in [1,N]$; and (iii)U investors place no bids.

In this example p_n lies in the interval of $[v_{n-1}, v_n - \frac{K(n+K)}{(n-1+K)(N+K)}(v_n - p_{n-1})]$.

The upper bound of the price range in both examples decreases with the allocation of an L investor, which means that the larger the allocation that H investors lose to L investors, the lower that market price should be. Also, the market price can be equal to the market value only if an L investor does not get any allocation (K=0).

APPENDIX C

Part 1*: *Deriving a condition such that it is unprofitable to lower the price to p_{n-2} by reducing the bids if it is unprofitable to lower the price to p_{n-1}.

If it is unprofitable for an investor who has observed a signal $s \in \{H, L, U\}$, to lower the price from p_n to p_{n-1} by underbidding then the following relationship must hold:

$$(v_n - p_n)d^a(p_n,s) \geq (v_n - p_{n-1})r_n^a(p_{n-1},s)$$
$$\Rightarrow \frac{v_n - p_n}{v_n - p_{n-1}} \geq \frac{r_n^a(p_{n-1},s)}{d^a(p_n,s)} \quad , n \in [1,N] \qquad \textbf{[C1]}$$

where $r_n^a(p_{n-1},s)$ is the residual supply above price p_{n-1} when there are n high signals ($n \in [1,N]$), given that a bidder who has received signal s has deviated. If [C1] holds, we also have

[26] K is a positive constant which is no higher than $[(N-1)(v_n - v_{n-1})/(v_{n-1} - p_{n-1})] - 1$, where $n \in (1,N]$, such that the upper bound of the price range is no lower than v_{n-1}.

$$(v_{n-1} - p_{n-1}) d^a(p_{n-1}, s) \geq (v_{n-1} - p_{n-2}) r_{n-1}^a(p_{n-2}, s),\ n \in [2, N]$$

Adding $(v_n - v_{n-1}) d^a(p_{n-1}, s)$ to both sides of the above inequality:

$$(v_n - p_{n-1}) d^a(p_{n-1}, s) \geq (v_{n-1} - p_{n-2}) r_{n-1}^a(p_{n-2}, s) + (v_n - v_{n-1}) d^a(p_{n-1}, s)$$

Because $d^a(p_{n-1}, s) \geq r_{n-1}^a(p_{n-2}, s)$, if the second inequality is satisfied, then

$$(v_n - p_{n-1}) d^a(p_{n-1}, s) \geq (v_n - p_{n-2}) r_{n-1}^a(p_{n-2}, s)$$

Thus

$$\frac{v_n - p_{n-1}}{v_n - p_{n-2}} \geq \frac{r_{n-1}^a(p_{n-2}, s)}{d^a(p_{n-1}, s)} \qquad \textbf{[C2]}$$

The product of [C1] and [C2] implies that $\dfrac{v_n - p_n}{v_n - p_{n-2}} \geq \dfrac{r_n^a(p_{n-1}, s) r_{n-1}^a(p_{n-2}, s)}{d^a(p_n, s) d^a(p_{n-1}, s)}$. If

$$r_n^a(p_{n-1}, s) r_{n-1}^a(p_{n-2}, s) \geq d^a(p_{n-1}, s) r_n^a(p_{n-2}, s), \qquad \textbf{[C3]}$$

Then

$$(v_n - p_n) d^a(p_n, s) \geq (v_n - p_{n-2}) r_n^a(p_{n-2}, s)$$

which implies that it is unprofitable to lower the price further to p_{n-2}.

Hence if all investors' demand functions satisfy [C3], we only need to check if it is profitable for them to lower the price from p_n to p_{n-1} by reducing the bids.

We can derive the residual supplies from the market clearing condition [8]:

$$N d^a(p_n, L) + n c^a(p_n) + d^a(p_n, U) = 1$$

Then [C3] is reduced to

$$c^a(p_{n-1})[d^a(p_{n-2}, L) - c^a(p_{n-2})] \leq c^a(p_{n-2}) d^a(p_{n-1}, L)$$

for an H or an L investor; and

$$c^a(p_{n-1})[d^a(p_{n-2}, U) - c^a(p_{n-2})] \leq c^a(p_{n-2}) d^a(p_{n-1}, U)$$

for U investors.

Together the two inequalities we have:

$$\frac{c^a(p_{n-2})}{c^a(p_{n-1})} \geq \max(\frac{d^a(p_{n-2},L)-c^a(p_{n-2})}{d^a(p_{n-1},L)}, \frac{d^a(p_{n-2},U)-c^a(p_{n-2})}{d^a(p_{n-1},U)}) \qquad \textbf{[C3]}'$$

If $d^a(p_{n-2},L)$ and $d^a(p_{n-2},U)$ is small than or equal to $C^a(p_{n-2})$, then [C3]′are naturally satisfied.

Part 2: Deriving a condition such that it is unprofitable to raise the price to p_{n+2} by increasing the bids if it is unprofitable to raise the price to p_{n+1}.

Since in EH, EHL, EHU $p_n \geq v_{n-1}$, which means $p_{n+2} \geq v_{n+1} > v_n$, this condition is only relevant for EHLU.

If we have $(v_n - p_n)d^a(p_n,s) \geq (v_n - p_{n+1})r_n(p_{n+1},s)\ \forall\ n \in$ [0,*N*-1]

and $(v_{n+1} - p_{n+1})d^a(p_{n+1},s) \geq (v_{n+1} - p_{n+2})r_{n+1}(p_{n+2},s)\ \ \forall\ n \in$ [0,*N*-2],

by subtracting $(v_{n+1} - v_n)d^a(p_{n+1},s)$ from both sides of the second equation we have:

$$(v_n - p_{n+1})d^a(p_{n+1},s)\ \geq (v_n - p_{n+2})r^a_{n+1}(p_{n+2},s).$$

Thus

$$\frac{v_n - p_n}{v_n - p_{n+2}} \geq \frac{r^a_n(p_{n+1},s)r^a_{n-1}(p_{n+2},s)}{d^a(p_n,s)d^a(p_{n+1},s)}$$, and so if

$$r^a_n(p_{n+1},s)r^a_{n+1}(p_{n+2},s) \geq d^a(p_{n+1},s)r^a_n(p_{n+2},s) \qquad \textbf{[C4]}$$

If it is satisfied, it is unprofitable to raise the price further to p_{n+2} even if p_{n+2} is lower than $v_{n.}$. The condition is reduced to

$c^a(p_{n+1})[d^a(p_{n+2},L)+c^a(p_{n+2})] \geq c^a(p_{n+2})d^a(p_{n+1},L)$ for an H or an L investor

and

$c^a(p_{n+1})[d^a(p_{n+2},U)+c^a(p_{n+2})] \geq c^a(p_{n+2})d^a(p_{n+1},U)$ for U investors.

Together the two inequalities we have:

$$\frac{c^a(p_{n+1})}{c^a(p_{n+2})} \geq \max(\frac{d^a(p_{n+1},L)}{d^a(p_{n+2},L)+c^a(p_{n+2})}, \frac{d^a(p_{n+1},U)}{d^a(p_{n+2},U)+c^a(p_{n+2})}) \qquad \textbf{[C4]}'$$

Because the right hand sides of both [C3]′and [C4]′ are smaller than 1, they are both satisfied if $c(p)$ is non-increasing in *n* (then $\frac{c^a(p_{n+1})}{c^a(p_{n+2})} \geq 1$). For simplicity we require that $c(p)$ is non-increasing in *n*, and thus we only need to check if condition [B3] is satisfied.□

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LIST OF TABLES:

Table 1. Examples of the demand schedule of an H bidder in EH

Table1.a

	Price	Quantity	Aggregate quantity
bid 1	$> v_4$	1/4	1/4
bid 2	v_4	1/12	1/3
bid 3	v_3	1/6	1/2
bid 4	v_2	1/2	1

Table 1.b

	Price	Quantity	Aggregate quantity
bid 1	v_4	1/4	1/4
bid 2	v_3	1/12	1/3
bid 3	v_2	1/6	1/2
bid 4	v_1	1/2	1

LIST OF FIGURES:

Figure 1 Demand curve of EH

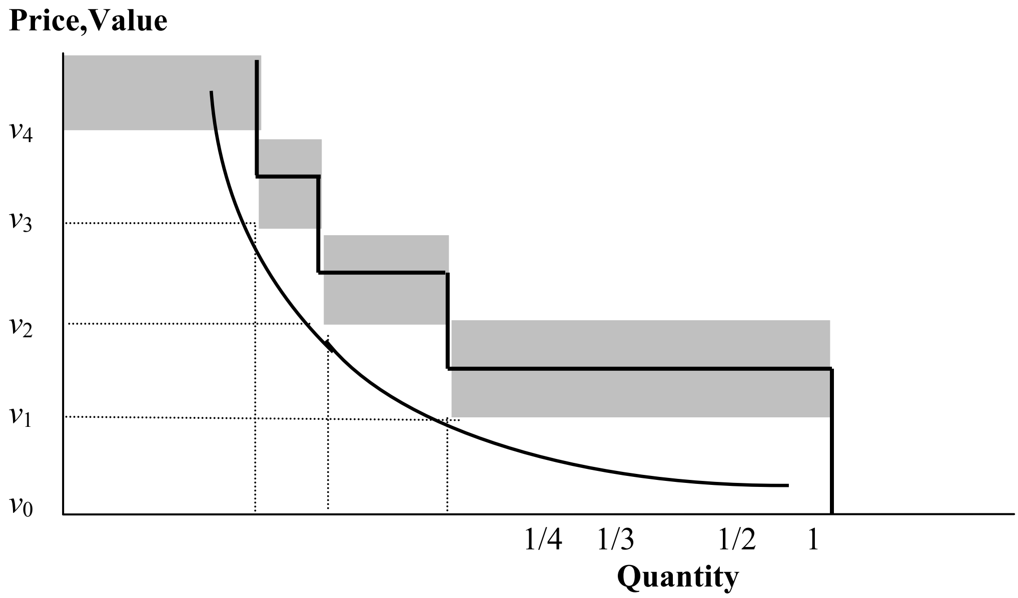

Figure A1: Vertical Demand Curve between p_n and p_{n+1}