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Optimal Delegation with a Finite Number of States*

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Abstract

This paper contributes to the literature on optimal delegation, dating back to Holmstrom's (1984) seminal work. In contrast to models in the Holmstrom tradition, we assume that the set of states is finite. We provide a full characterization of the class of optimal delegation sets under this assumption, and show that they have a different structure from that in the continuous-state model. As the number of states tends to infinity, however, every optimal delegation set converges to that of Holmstrom (1984). We also show that, for intermediate bias, the Ally Principal fails for small changes in bias, the Uncertainty Principle may fail, and the principal prefers to appoint an amateur agent.

Keywords: Optimal delegation, finite states, Ally Principle, Uncertainty Principle, expertise

1 Introduction

This paper studies delegation without monetary transfers in finite environments. Consider the following example. It is widely believed that juries are unduly biased towards reaching some verdict (rather than hanging), and that they may therefore compromise by convicting on some lesser charge when evidence on a more serious charge is unclear.[1] Accordingly, juries are instructed to convict a defendant on a charge only if they agree that the offense was committed.[2] For instance, if jurors believe that the defendant may have committed murder but definitely did not commit manslaughter then they may not convict on the lesser charge. The judge's instruction addresses the jury's bias, relative to the presumption of

*We thank Claudio Mezzetti and Eyal Winter for their comments.

[1] See Hannaford-Agor et al (2002) pp 42-43 for field evidence in support of this possibility.

[2] See *Stein v. NY* 346 US (1953) on jury instructions and compromise verdicts. According to *Beck v. Alabama* 447 US (1980), juries need only be instructed to consider a lesser offense if there is evidence that it might have been committed. Hoffheimer (2006) discusses these issues.

innocence, by restricting the verdicts which the jury may reach, inducing the jury to pool on acquittal unless the evidence proves guilt on some charge beyond a reasonable doubt.

One can think of this situation as an optimal delegation problem without monetary transfers in which the principal is the judge and the jury is the expert agent: expert, in the sense that its role is fact-finding. The uninformed principal (he) offers a delegation set – namely, a collection of scalar decisions (the permissible verdicts) – to the biased agent (she), who takes one of these decisions after observing the state (the evidence).[3] All of these features appear in the literature on optimal delegation initiated by Holmstrom (1984). Almost all of this literature also supposes that the set of possible states is an interval and, therefore, an infinite set.[4] In the jury example, there may be many feasible decisions (lesser charges), but there is only a finite number of possible states: the evidence either clearly exonerates, or clearly inculpates the defendant of murder, or identifies the crime as murder but leaves the defendant's factual guilt in doubt.

Our approach encompasses such examples because we consider optimal delegation problems in which there is a finite number of possible states. One might conjecture that the distinction between finite and continuous sets of states is only a mathematical detail, and that all the results from continuous-state models can be applied to situations with a finite number of states. Our results reveal that such an impression is wrong. Analysis of the finite-state case is important not only because it describes situations that are clearly realistic, but also because the optimal delegation set (ODS, for short) has strikingly different properties when there are finitely many as opposed to an interval of possible states. Such dissimilarities also generate significantly different implications when we come to comparative statics exercises.

We demonstrate the differences between finite and continuous cases by analyzing Holmstrom's (1984) seminal model of optimal delegation in an environment with a finite number of states. Holmstrom assumes that the principal's and the agent's preferences are each represented by loss functions which are quadratic around the player's ideal decision. The principal's ideal decision is simply the realization of the state, while the agent's ideal decision is b higher: where $b > 0$ is the agent's *bias*. The agent observes the actual realization of the state; the principal does not, but believes that it is uniformly distributed on $[0, 1]$.[5] The ODS has a particularly simple structure. The agent takes her ideal decision in low enough states, and otherwise pools: so the ODS is an interval.

We extend Holmstrom's model by retaining all these assumptions except for the uniformly distributed state. Instead, we assume that the state is equally likely to take each of a finite number of equi-distanced numbers; so our model approximates Holmstrom's when we let the number of states tend to infinity. Staying close to this benchmark model will facilitate comparisons between the finite and continuum cases when we come to study the implications of the model, as elaborated on below. Our first step is to characterize ODSs

[3] Melumad and Shibano (1991) demonstrate that the principal cannot improve on the optimal delegation set via any other mechanism.

[4] We discuss the exceptions below.

[5] This is the benchmark example in Crawford and Sobel's (1982) cheap talk model. The optimal delegation problem is equivalent to a cheap talk game in which the Receiver moves first by committing to the decisions he will take in response to any message.

in this finite environment. The ODS is again simple, albeit with a different structure to Holmstrom's model:

No discretion. Generically, no agent takes her ideal decision in any state, irrespective of the bias. This stands in radical contrast to Holmstrom's (1984) result: in the continuum-state model, the agent takes her ideal decision in every low enough state.

The chain property and top loading. In common with Holmstrom's model, the agent takes the same decision in all high enough states: a property which we call *top loading.* If the bias is sufficiently large then the ODS consists of a single decision; otherwise, and in contrast to Holmstrom (1984), the agent is indifferent between the decision she takes and the next highest decision in the ODS in low states (the *chain property*).

The chain property implies that the principal must exclude some intermediate decisions to prevent the agent from compromising (as in our example of judicial instructions) as well as some decisions above the ODS. Gailmard (2009) p26 notes that the FTC, like other agencies, is typically limited to up-down choices; Szalay (2005) p1174 mentions several other examples. By contrast, the ODS in Holmstrom's model need only exclude decisions above the ODS (cf. Melumad and Shibano (1991)).[6] In contrast, Alonso and Matouschek (2008) show that the principal may exclude some intermediate decisions when the support of states is an interval and the agent's ideal decisions are insufficiently state-sensitive. This motive is absent in our model because a state-independent bias precludes any conflict over state-sensitivity.[7]

Decisions outside the support of the state's distribution. The ODS characterized by Holmstrom (1984) and Melumad and Shibano (1991) for the benchmark model is a strict subset of $[0, 1]$ for all positive values of the bias. By contrast, we show that there are values of the bias for which the principal allows the agent to take decisions outside [0,1] — that is, decisions that are never ideal for him — in the highest state.[8] This result suggests a possible explanation for the tenure system: the university/principal forces the dean/agent to offer unduly generous terms to good candidates to motivate her to dismiss bad candidates.

Convergence. Despite these contrasts, the ODS in our model converges to that in Holmstrom's model as we add equiprobable and equi-distant states, holding the location of the lowest and the highest states fixed.

Since Holmstrom (1984), the interval support model of optimal delegation has been a prominent analytical tool in the literature on legislative control of agencies.[9] Two standard

[6]Martimort and Semenov (2006), Alonso and Matouschek (2008) and Gailmard (2009) provide other general conditions for this result when the support of states is an interval; Kovac and Mylovanov (2009) and Goltsman et al (2009) prove that principal does not gain from offering mixtures of decisions in Holmstrom's benchmark model.

[7]In Szalay (2005), the principal may exclude intermediate decisions to induce costly information acquisition.

[8]We provide an example without equiprobable states in which the agent also takes a decision above 1 in lower states.

[9]Aside from delegation, the model applies to self-control problems for individuals (cf. Amador et al (2006)), time-inconsistency for central banks (cf. Athey et al (2005)), paternalism (with an informed child/agent), and to corporate structure (cf. Harris and Raviv (1996)).

predictions of this literature are the *Ally Principle* (i.e., the principal prefers to appoint less biased agents) and the *Uncertainty Principle* (i.e., delegation is more valuable, the more risky is the state).[10] Our next step is to investigate these principles with the finite-state model.

A change in the agent's bias has two important effects in Holmstrom's model: (i) the principal's equilibrium loss increases when the agent becomes more biased, and (ii) the principal offers more discretion to less biased agents: the delegation interval shrinks as the bias increases, and becomes a singleton (delegation is not valuable) for sufficiently high bias. We show that the former property – the Ally Principle – also holds in our model, but that variations in bias have a more subtle effect on the structure of the ODS. With a finite number of states, variations in bias typically do not change discretion (according to the set inclusion ordering). Generically, a marginal increase in bias leaves the number of decisions in the ODS unchanged, and shifts all decisions shifts to the right. Nevertheless, discretion *does* decrease at some critical levels of bias in the sense that, at these values of b, a marginal increase in bias leads the principal to drop the largest decision from the ODS and to retain all of the other decisions. More generally, a sufficiently large increase in bias reduces the number of decisions in the ODS. This does not reduce discretion; but we show that the principal then raises the minimal and reduces the maximal decision in the ODS.

According to the Uncertainty Principle, a mean-preserving increase in the risk of states raises the value of delegation, defined as the difference between the principal's loss when he takes the decision himself and when he delegates the decision to the agent. The Uncertainty Principle obviously holds in models of full delegation (such as Dessein (2002)), where the agent takes her ideal decision in every state. It also holds in Holmstrom's model if the support of states increases symmetrically round its expectation. We show that the latter result carries over to our finite-state model. However, the Uncertainty Principle relies on equi-probable states: we provide a three-state example in which a mean-preserving increase in risk reduces the value of delegation.

Finally, we turn to a property which we dub the *Expertise Principle.* It asserts that the principal prefers to appoint the more expert of two agents with a common bias. This property underlies a recent literature which studies a trade-off between loyalty and competence (cf. Gailmard and Patty (2007) and Huber and McCarty (2004) Proposition 5). The Expertise Principal holds trivially in existing models (such as Bendor and Meirowitz (2004)), which treat an amateur as an agent who does not observe the state with positive probability. By contrast, we treat an amateur as an agent who is unable to distinguish between a subset of states.[11] The principal cannot gain by appointing an amateur if the bias is low enough because the expert would take almost first-best decisions in every state, whereas an amateur must be less state-sensitive. We also show that the principal cannot gain from appointing an amateur if the bias is large enough that delegation to an expert has no value. Surprisingly, however, the Expertise Principle fails for every intermediate bias, in the sense that there is an amateur (i.e. an event which the agent cannot distin-

[10] We refer the reader to Huber and Shipan (2006) for a survey of this literature.

[11] An agent's expertise is fixed in our model and in Bendor and Meirowitz (2004); whereas agents exert effort in Aghion and Tirole (1997) and Szalay (2005) inter alia.

guish) who outperforms the expert. This result holds whenever there are at least four states and, therefore, also holds in Holmstrom's model. Consequently, evidence that political appointees are less expert (cf. Lewis (2007)) does not imply that there is a trade-off between loyalty and competence.

Related Work

Although the literature on optimal delegation is now too large to survey exhaustively, we briefly explain the paper's relationship to a few of the most closely related game-theoretic contributions.

We have already mentioned the main results from the continuous-state model as benchmarks against which to gauge the relevance of our results. The only previous models with a finite number of states are Green (1982), Huber and Gordon (2007) and Amador et al (2006). Green shows that the optimal stochastic delegation set is the solution to a linear programming problem if there is a finite number of feasible decisions. Huber and Gordon characterize the ODS for an example with three feasible decisions. Amador et al show that the principal might achieve first best in a two-state model if preferences are sufficiently aligned and any scalar decision is feasible (as in our model); that the ODS features separation with intermediate alignment; and that delegation is otherwise not valuable.

Various other papers have characterized the ODS in variants on Holmstrom's model (with a continuum of states). Decisions are two-dimensional in Koessler and Martimort (2009). Unlike the principal, the agent has different biases in each dimension, which allows the principal to screen the agent by distorting each dimension away from the agent's ideal decision. They show that the agent does not take her ideal decision on either dimension in any state (as in our model, but in contrast to Holmstrom's). Unlike Holmstrom's and our results, the agent takes a different decision-pair in each state at the ODS, for every bias. Other notable, but less related contributions include Ambrus and Egorov (2009), Armstrong and Vickers (forthcoming), Krishna and Morgan (2008) and Mylovanov (2008).

We present the model and characterize the ODS in Section 2. Sections 3, 4 and 5 respectively study comparative statics with respect to the bias, the risk of states, and the agent's expertise. We summarize in Section 6, and provide lengthier proofs in the Appendix and an online Appendix.

2 Optimal Delegation Sets

This section is divided into three subsections. We begin by presenting the model of optimal delegation with a finite number of states. We then provide a full characterization of the class of optimal delegation sets for every value of the agent's bias. We end the section with a discussion of our proof strategy.

2.1 Model

Players and Preferences

There are two players in the model: a principal (he) and an agent (she). The preferences of both players depend on a decision variable, d, and on a random state of the world, t, in which the decision is taken. We assume that there is only a finite set of conceivable states, which we denote $\mathbf{T} \equiv \{0, 1, \ldots, T-1\}$, but that d can take any real value. The principal's evaluation of the decision d is represented by the following loss function:

$$\lambda(d,t) \equiv \left(d - \frac{t}{T-1}\right)^2 .$$

A principal who knew the state t for sure would therefore take decision $d = t/(T-1)$. Accordingly, we refer to $t/(T-1)$ as the principal's ideal decision in state t. We normalize by dividing the state by $T-1$ in order to facilitate comparison with Holmstrom's (1984) model.

The agent's evaluation of decision d is given by loss function

$$\lambda^A(d,t) \equiv \left(d - b - \frac{t}{T-1}\right)^2$$

for some $b > 0$. For any value of b, the agent's ideal decision in state t is thus $b + t/(T-1)$, which exceeds the principal's ideal decision in state t by b, which we call the agent's *bias*. By construction, the bias is state-independent.

An obvious, but important, implication of these loss functions is that the principal strictly prefers the agent to take decision d over decision e in those states where the agent weakly prefers d over e. This property will play a decisive role in the analysis below.

Information and Timing

Events unfold as follows: 1) Nature chooses a state t in $\mathbf{T}$ and – unless otherwise stated – reveals the true value of t privately to the agent; 2) The principal offers the agent a delegation set (to be defined shortly); 3) The agent takes a decision d in the delegation set.

The agent is asymmetrically informed about the state relative to the principal, who only knows that every possible state is equally likely. Both players' loss functions are common knowledge.

Delegation Sets

Broadly defined, a (deterministic) delegation set is any collection of decisions: that is, any nonempty subset of $\mathbb{R}$. A delegation set (say, Δ) could include decisions which the agent would never take. We follow the literature by focusing on minimal delegation sets: that is, delegation sets with the property that a loss-minimizing agent with bias b would take each decision in Δ in some state. This is the right focus because an optimal delegation set minimizes the principal's expected loss over all incentive compatible mechanisms (cf.

Melumad and Shibano (1991)); so exactly the decisions in a minimal delegation set are taken according to every optimal mechanism. Accordingly, we define $\mathcal{D}(b)$ as the class of minimal delegation sets when the agent's bias is b. We will henceforth drop the qualifier 'minimal'.

A brief inspection of λ^A reveals that an agent who is offered $\Delta \in \mathcal{D}(b)$ can only be indifferent between two distinct decisions in any state; so a delegation set must contain a finite number of decisions. We will suppose that the agent takes the lowest decision in Δ which minimizes her loss in each state.[12] Formally, for every state $t \in \mathbf{T}$, define the decision function $d_t : \mathcal{D}(b) \times \mathbb{R}_+ \to \mathbb{R}$ as

$$d_t(\Delta, b) \equiv \min \left\{ \arg\min \left\{ \left(d - \frac{t}{T-1} - b \right)^2 : d \in \Delta \right\} \right\} .$$

Note that we implicitly assume here that the agent never mixes over decisions. This is without loss of generality because it is never optimal for the principal to induce the agent to randomize.

We say that $\Delta \in \mathcal{D}(b)$ is an *optimal delegation set* (or *ODS*) if and only if

$$\Delta \in \arg\min \left\{ \sum_{\tau \in \mathbf{T}} \left(d_\tau(\Phi, b) - \frac{\tau}{T-1} \right)^2 : \Phi \in \mathcal{D}(b) \right\} .$$

The class of optimal delegation sets when the agent's bias is b is denoted by $\mathcal{D}_T^*(b)$. A generic element of $\mathcal{D}_T^*(b)$ is of the form $\{\delta_0, \ldots, \delta_K\}$, where we order decisions such that $i < j$ implies $\delta_i < \delta_j$. In the next subsection, we characterize the class of ODSs for every bias.

Some more terminology will prove useful. A principal who offers a single-decision ODS must offer the decision which he would take if he did not delegate. Accordingly, we will say that *delegation is valuable* if and only if every ODS contains more than one decision.

It will also be useful to benchmark our results against an alternative model in which the principal places no restrictions on the agent (as in Dessein (2002)), implicitly offering the delegation set consisting of decisions $\left\{ b + \frac{t}{T-1} \right\}$ for $t \in \mathbf{T}$. We will say that such a principal *fully delegates*.

2.2 Results

2.2.1 Characterization of the ODS

It is important for both theoretical and practical purposes that ODSs have a simple structure. It turns out that every ODS satisfies two simple conditions. We need some more notation to specify these conditions. For every $\Delta = \{\delta_0, \ldots, \delta_K\} \in \mathcal{D}(b)$, define $t_k(\Delta, b)$ and $T_k(\Delta, b)$ as follows:

$$T_k(\Delta, b) \equiv \{\tau \in \mathbf{T} : d_\tau(\Delta, b) = \delta_k\} ,$$
$$t_k(\Delta, b) \equiv \max \{\tau : \tau \in T_k(\Delta, b)\} .$$

[12] Single crossing implies that this supposition is without loss of generality.

The elements of $T_k(\Delta, b)$ are the states in which the agent takes decision δ_k from Δ, and $t_k(\Delta, b)$ is the maximal such state.

Definition 1 *A delegation set* $\Delta = \{\delta_0, \ldots, \delta_K\} \in \mathcal{D}(b)$ *satisfies top loading if and only if* $|T_k(\Delta, b)| > 1$ *implies that* $k = K$.

Definition 2 *A delegation set* $\Delta \in \mathcal{D}(b)$ *satisfies the chain property if and only if* $\tau_1 = t_k(\Delta, b)$ *and* $\tau_2 = t_{k+1}(\Delta, b)$ *imply that*

$$\lambda^A \left(d_{\tau_1}(\Delta, b), \tau_1\right) = \lambda^A \left(d_{\tau_2}(\Delta, b), \tau_1\right)$$

for every $k = 0, \ldots, K-1$.

In words: Δ is top loaded when only the largest decision in Δ is taken in more than one state; and Δ satisfies the chain property if an agent who takes different decisions in states t and $t+1$ is indifferent between them in state t. If Δ satisfies the chain property then it must exclude any decisions between those taken in states t and $t+1$ because convexity of the agent's loss function implies that she would prefer this compromise in state t.[13] Our example of jury instructions in the Introduction illustrates a delegation set which excludes compromises.

Theorem 1 below asserts that every ODS satisfies the chain property and top loading. The ODS in Holmstrom's interval support model is top loaded, but does not satisfy the chain property.[14] Cheap talk models with interval support satisfy an analog of the chain property: the support is partitioned into a finite number of intervals in every equilibrium; and, at the maximal state in all but the largest element of the partition, the agent is indifferent between the decision it induces and the next highest decision taken by the principal.[15] However, these equilibria do not satisfy top loading.

An important implication of Definitions 1 and 2 is that a delegation set, say Δ, which is top loaded and satisfies the chain property is defined by two parameters: the lowest decision in Δ (namely, δ_0) and the number of decisions it contains (namely, $K+1$). Theorem 1 establishes that the number of decisions in an ODS for an agent with bias b is determined by the values of K which satisfy

$$b \in \left[b^{\min}(K, T), b^{\max}(K, T)\right]$$

where, for all integers $s \leq T$ and $K \leq s-1$:[16]

$$b^{\min}(K, s) \equiv \frac{s-K}{2(T-1)} \text{ and } b^{\max}(K, s) \equiv \begin{cases} \frac{s-K+1}{2(T-1)} & \text{if } K > 0 \,, \\ +\infty & \text{otherwise.} \end{cases}$$

[13] This argument implies that a non-minimal delegation set which satisfied the chain property would also have to exclude compromises.

[14] The ODS fails top loading in Koessler and Martimort (2009).

[15] This analog of the chain property does not hold in cheap talk games with a finite support.

[16] We introduce slightly more general notation for use in the next subsection.

This result immediately implies that: (i) there is a unique ODS if $b \neq \frac{k}{2(T-1)}$ for any $k \in \{1, ..., T-1\}$, and only two ODSs otherwise; and (ii) delegation is not valuable if and only if $b > b^{\min}(0, T) = \frac{T}{2(T-1)}$.

Having determined K, the lowest decision in the ODS is then chosen to minimize the principal's expected loss, subject to the agent being indifferent between δ_t and δ_{t+1} in every state $t \in \{0, \ldots, K-1\}$, and taking the maximal decision in every higher state.

Theorem 1 *If $b \leq \frac{1}{2(T-1)}$ then the ODS consists of the T decisions which are ideal for the principal in each state: $\Delta^* = \left\{\frac{t}{T-1}\right\}_{t \in \mathbf{T}}$.*

If $b > \frac{1}{2(T-1)}$ then

(i) Every ODS satisfies the chain property and top loading;

(ii) If $b \neq \frac{k}{2(T-1)}$ for any $k \in \{2, ..., T-1\}$ then there is a unique ODS, $\Delta^* \equiv \{\delta_0^*, \ldots, \delta_K^*\}$. The number of decisions in the ODS, $K+1$, is implicitly determined by $b \in \left(b^{\min}(K,T), b^{\max}(K,T)\right)$; and

$$\delta_0^* = \begin{cases} \frac{K}{T}b - \frac{1}{2(T-1)} + \frac{(T-K)^2}{2T(T-1)} & \text{if } K \text{ is even,} \\ \frac{2T-K-1}{T}b - \frac{T-1}{2T(T-1)} - \frac{(T-K)^2}{2T(T-1)} & \text{if } K \text{ is odd.} \end{cases}$$

In particular, if $b > \frac{T}{2(T-1)}$ then delegation is not valuable: $\Delta^* = \left\{\frac{1}{2}\right\}$.

(iii) If $b = \frac{k}{2(T-1)}$ for some $k \in \{2, ..., T-1\}$ then there are two ODSs: the ODS with $T-k$ and with $T-k+1$ decisions defined in part (ii) above.

We prove Theorem 1 in the Appendix. We will explain how we prove the result in Subsection 2.3.

A few remarks are in order. First, the principal's problem could be solved by finding the loss-minimizing delegation set with K decisions, and then minimizing again over K. While Theorem 1 asserts that every ODS satisfies top loading, this is not true of the best delegation sets containing a fixed number of decisions. For example, if there are four states and $\frac{1}{6} < b < \frac{1}{3}$ then the best two-decision delegation set is $\{b, b + \frac{2}{3}\}$, which fails top loading because the agent takes decision b in states 0 and 1. However, the ODS contains three decisions and satisfies top loading when $\frac{1}{6} < b < \frac{1}{3}$.

Second, the chain property relies on the distribution of states. To see this, suppose that there are four states of the world (i.e., $T = 4$) and that the probability of state t is p_t where $p_0 = p_3 = \frac{1}{2}(1-\varepsilon)$ and $p_1 = p_2 = \frac{1}{2}\varepsilon$ for ε arbitrarily small. If b is close to but exceeds $1/6$ then the delegation set cannot be first best, but must contain four decisions. If the ODS satisfied the chain property then the i^{th} decision would be $b + \frac{2i-1}{6}$, and the principal would lose $(b - \frac{1}{6})^2$. Now consider a delegation set

$$\Delta^+ \equiv \{\delta_0^+, \delta_1^+, \delta_2^+, \delta_3^+\} \equiv \left\{\varepsilon\left(2b - \frac{1}{3}\right), 2(1-\varepsilon)b + \frac{\varepsilon}{3}, 2(1-\varepsilon)b + \frac{1+\varepsilon}{3}, 2\varepsilon b + \frac{3-\varepsilon}{3}\right\},$$

which is constructed such that the agent is indifferent between δ_0^+ and δ_1^+ in state 0, and between δ_2^+ and δ_3^+ in state 2, and strictly prefers δ_1^+ over every other decision in Δ^+ in state 1 if ε is small enough. Hence, Δ^+ fails the chain property. A principal who offers this delegation set loses $4\varepsilon(1-\varepsilon)\left(b-\frac{1}{6}\right)^2$, which is less than $\left(b-\frac{1}{6}\right)^2$ for fixed b whenever ε is small enough. Indeed, it is easy to confirm that Δ^+ is optimal for b close enough to 1/6 and ε sufficiently small.[17]

Third, top loading could fail if the bias were state-dependent: for example, if there were three states and the agent's preferences in states 0 and 1 were sufficiently close then the agent may pool in the lower states alone.

2.2.2 Properties of the ODS

Let $\mathcal{D}^*_\infty(b)$ be the class of ODSs when the agent's bias is b and the state is uniformly distributed on $[0,1]$ (viz. Holmstrom's model). Melumad and Shibano (1991) prove that $\mathcal{D}^*_\infty(b)$ consists of the unique delegation set

$$\Delta^*_\infty(b) = \begin{cases} [b, 1-b] & \text{if } b \in [0, 1/2] \ , \\ \{1/2\} & \text{if } b > 1/2 \ . \end{cases}$$

In particular, the agent never takes a decision below her lowest ideal decision (i.e., b) or above the principal's highest ideal decision (i.e., 1). Furthermore, if $b < 1/2$ then the agent takes her ideal decision in every state below $1-2b$.[18] Our next result implies that none of these properties holds in the finite-state model.

Corollary 1 *If Δ is an ODS, then the following statements are true:*
(i) $d_t(\Delta, b) > \frac{t}{T-1}$ for every $t \in T$ if and only if $b > \frac{1}{2(T-1)}$;
(ii) $d_t(\Delta, b) < b + \frac{t}{T-1}$ for every $t \in T$;
(iii) If delegation is valuable then the agent takes a decision below b in state 0, but in no higher state;
(iv) The agent takes a decision above 1 if and only if $\frac{1}{2(T-1)} < b < \frac{1}{T-1}$ and the state is $T-1$.

We prove Corollary 1 in Appendix B.

Part (ii) asserts that the agent never takes her ideal decision, in stark contrast to Holmstrom's model.[19]

Part (iv) asserts that a sufficiently unbiased agent takes a decision above 1 — which never happens in Holmstrom's model — but only when the ODS contains T decisions, and then in the highest state. In Appendix C, we demonstrate by example that this property relies on all states being equiprobable. There are three states in Example; and states 0 and 2 are equiprobable. If state 1 is unlikely enough and $1/2 < b < 1$ then the ODS consists of two decisions, the larger of which is taken in states 1 and 2, and exceeds 1.

[17] Examples of this sort require at least four states.

[18] Alonso and Matouschek (2008) Proposition 3 provides more general sufficient conditions for the ODS to be in $[0, 1]$.

[19] The agent never takes her ideal decision in Koessler and Martimort (2009) because decisions are multi-dimensional.

More generally, parts (iii) and (iv) are reminiscent of Proposition 2(iv) in Alonso and Matouschek (2008), which asserts that the ODS contains at most one decision above and one decision below $\widehat{D}$: namely, the set of decisions which are ideal for the principal in some state and ideal for the agent in some state (so $\widehat{D} = [b, 1]$ in Holmstrom's model).

If T is sufficiently large then the support of states is close to $[0, 1]$, and the probability distribution is almost uniform. Our next result asserts that the ODS in our model approaches the ODS in Holmstrom's model as T increases:

Proposition 1 *For all* $b \geq 0$, $\lim_{T\to\infty} \mathcal{D}^*_T(b) = \mathcal{D}^*_\infty(b)$.

We prove Proposition 1 in Appendix B.

Let K_T denote the (generic) number of decisions in the ODS when there are T states. Proposition 1 implies that the agent takes almost her ideal decision in states $t < K_T$. The reason is straightforward. If $b < T/2(T-1)$ then K_T/T is bounded away from 0 as T increases. Now suppose there is state t such that $b + \frac{t}{T-1} - \delta_t > \varepsilon$ for some $\varepsilon > 0$ and every T. The agent is indifferent between δ_τ and $\delta_{\tau+1}$ in states $\tau < K_T$ (by the chain property), so $\delta_{t+1} > b + \frac{t+1}{T-1}$ for T large enough: a contradiction. Thus, the agent takes approximately the same decisions as in Holmstrom's model for states $t < K_T$; so she pools on approximately $1 - b$ in higher states, again as in Holmstrom's model.

2.3 Proof strategy

The technical problem in determining the ODS in models with interval support is how to convert it into a finite-dimensional problem. On the other hand, Melumad and Shibano (1991) Lemma 1 demonstrates that incentive constraints are very restrictive in models with interval support: the agent takes her ideal decision if the decisions taken are locally continuous and strictly monotonic in the state. Alonso and Matouschek (2008) use this property to provide conditions for the principal to reduce his loss by adding or removing decisions from a given delegation set. In particular, adding an intermediate decision to any finite delegation set induces the agent to take a less extreme decision in some intermediate states. Alonso and Matouschek use such local patches (adding or removing intermediate decisions) to characterize the ODS.

Models with finite support are automatically finite-dimensional. However, the incentive constraints are no longer as strong because the agent's ideal decisions are $1/(T-1)$ apart; so the effect of adding or removing intermediate decisions depends on the original delegation set. In particular, adding a decision to those taken in states t and $t+1$ can induce the agent to take this compromise decision in states $s \leq t$ (if the delegation set satisfies the chain property) or in states $s \geq t+1$ (if the agent strictly prefers her decision in state t) or in no states. In sum, we cannot use Alonso and Matouschek's local patches; so we use entirely different arguments to characterize the ODS.

One might conjecture that Theorem 1 could be proved by induction on the number of states, but this turns out not to be fruitful. Furthermore, as we noted above, the best K-decision delegation sets do not necessarily satisfy the chain property or top loading; so induction on K is uninstructive. Instead, we define ODSs on connected subsets of states, and use induction arguments on the size of these events.

We need some additional notation for this purpose. Let $\mathbf{T}_{s,t} \equiv \{t, t+1, \ldots, t+s-1\} \subseteq \mathbf{T}$, and define the probability distribution $p_{s,t}$ as

$$p_{s,t}(\{\tau\}) \equiv \begin{cases} 1/s & \text{if } \tau \in \mathbf{T}_{s,t} \\ 0 & \text{otherwise,} \end{cases}$$

for every $\tau \in \mathbf{T}$, where $\mathbf{T}_{T,0} = \mathbf{T}$. Thus, assuming that the state is distributed according to $p_{s,t}$ amounts to assuming that only the states in $\mathbf{T}_{s,t}$ can occur, each with the same probability.

We now generalize the definitions of Subsection 2.1, using this interpretation. Let $\mathcal{D}_{s,t}(b)$ be the class of delegation sets when the agent's bias is b and the state is in $\mathbf{T}_{s,t}$, and say that $\Delta \in \mathcal{D}_{s,t}(b)$ is an *ODS for* $\mathbf{T}_{s,t}$ if and only if

$$\Delta \in \arg\min \left\{ \sum_{\tau \in \mathbf{T}_{s,t}} p_{s,t}(\{\tau\}) \left(d_\tau(\Phi, b) - \frac{\tau}{T-1} \right)^2 : \Phi \in \mathcal{D}_{s,t}(b) \right\} .$$

The class of ODSs for $\mathbf{T}_{s,t}$ when the agent's bias is b is denoted by $\mathcal{D}^*_{s,t}(b)$.

We are now in a position to provide an outline of the construction on which the proof of Theorem 1 is based. The following existence result proves necessary as a first step to doing so because we will use negative arguments to characterize ODSs.

Lemma 1 *$\mathcal{D}^*_{s,t}(b) \neq \emptyset$ for every $t \in \mathbf{T}$, every integer $s \leq T - t$, and any $b \geq 0$.*

Lemma 1 establishes existence. The characterization argument is rather long, and involves three intermediate steps. These steps, however, are of some interest in their own right, as they expose interesting relationships between the diverse properties of ODSs. The first of them relies on the following notation:

$$\ell^e(b, s, K) \equiv \frac{K}{s} b - \frac{1}{2(T-1)} + \frac{(s-K)^2}{2s(T-1)} , \tag{1}$$

$$\ell^o(b, s, K) \equiv \frac{2s - K - 1}{s} b - \frac{s-1}{2s(T-1)} - \frac{(s-K)^2}{2s(T-1)} , \tag{2}$$

$$\text{and } \ell(b, s, K) \equiv \begin{cases} \ell^e(b, s, K) & \text{if } K \text{ is even,} \\ \ell^o(b, s, K) & \text{if } K \text{ is odd.} \end{cases} \tag{3}$$

Lemma 2 *Let $b > \frac{1}{2(T-1)}$. If $\Delta \equiv \{\delta_t, \ldots, \delta_{t+K}\}$ is in $\mathcal{D}^*_{s,t}(b)$ and satisfies the chain property and top loading then*

$$\delta_{t+\tau} = \begin{cases} \ell(b, s, K) + \frac{t+\tau}{T-1} & \text{if } \tau \text{ is even,} \\ 2b - \ell(b, s, K) + \frac{t+\tau-1}{T-1} & \text{if } \tau \text{ is odd,} \end{cases}$$

and K is implicitly determined by $b \in \left[b^{\min}(K, s), b^{\max}(K, s) \right]$.

Note that substituting $s = T$ into Lemma 2 yields the ODS identified in Theorem 1. The next step proceeds by induction on the number of states. We start from the following hypothesis:

($\mathbf{H_s}$) For every $z \leq s$ and every $t \leq T - z$, every $\Delta \in \mathcal{D}^*_{z,t}(b)$ satisfies the chain property and top loading.

($\mathbf{H_s}$) says that ODSs satisfy the chain property and top loading when the number of possible states does not exceed s. It is easy to confirm that, for every $t \in \mathbf{T} \setminus \{T-1\}$, an ODS for $\mathbf{T}_{2,t}$ satisfies the chain property and top loading: that is, ($\mathbf{H_2}$) is true.

($\mathbf{H_s}$) implies that every ODS for $s+1$ states has also the chain property:

Lemma 3 *If* ($\mathbf{H_s}$) *then, for every* $t \leq T - s - 1$, $\Delta \in \mathcal{D}^*_{s+1,t}(b)$ *implies that* Δ *satisfies the chain property.*

The last step asserts that the chain property and optimality jointly imply top loading. Formally:

Lemma 4 *If* $\Delta \in \mathcal{D}^*_{s,t}(b)$ *satisfies the chain property then* Δ *is top loaded.*

Lemmas 2, 3, and 4 jointly allow us to provide the full characterization of ODSs in Theorem 1.

3 The Ally Principle

There are two versions of the Ally Principle (AP) in the literature. One version states that the principal prefers to appoint an agent with a lower bias; the other version states that the principal gives more discretion to a less biased agent, where discretion is defined in the set-inclusion sense. Both versions of AP hold in Holmstrom's (1984) continuous-state model. In this section, we consider whether each version of AP holds in the finite-state version of the model.

3.1 AP: appointments

The literature suggests various reasons why the appointment version of AP might fail: for example, the principal may optimally appoint a more biased agent if her decision is itself a move in a larger game, as in Vickers (1985), where other players best-respond to the agent; or to motivate an agent to become better informed, as in Callander (2008). We now use Corollary 1 to show that this version of AP holds in our model:

Proposition 2 *The principal strictly prefers to appoint a less biased agent.*

Proof: Let $\Delta_1 = \{\delta_0, \ldots, \delta_K\} \in \mathcal{D}^*_T(b_1)$. Theorem 1 implies that an agent with bias b_1 is indifferent between δ_t and δ_{t+1} in states $t < K$. An agent with bias $b_2 < b_1$ must strictly prefer δ_t over δ_{t+1} in states $t < K$; so the principal loses no more by offering Δ_1 to agent b_2 than he loses when offering (the optimal) Δ_1 to agent b_1.

Corollary 1 implies that $\delta_0 = d_0(\Delta_1, b) > 0$. The principal can then improve upon Δ_1 when transacting with the agent b_2 by reducing δ_0 towards 0. □

3.2 AP: discretion

We now turn to the second version of AP: that the principal gives more discretion to a less biased agent. We follow the literature by defining discretion in terms of set inclusion. Specifically, the principal gives less discretion to agent 0 than to agent 1 if he offers Δ_i to agent i and $\Delta_0 \subset \Delta_1$. Although set-inclusion is only a partial ordering of delegation sets, this version of AP holds in Holmstrom's (1984) model.

AP would evidently fail if the principal fully delegated to the agent, as in Dessein (2002): if the agent's ideal decision in every state increases then the support of the decisions taken by the agent shifts to the right. The literature suggests various theoretical reasons why it might also fail if the principal offers an ODS to each agent: for example, Alonso and Matouschek (2008) show that the principal may give less discretion to a more aligned agent to induce more state-sensitive decisions when bias is state-dependent.[20]

For expositional convenience, define

$$\Delta : \mathbb{R}_+ \setminus \left\{ \frac{T-K}{2(T-1)} \right\}_{K=0}^{K=T-1} \rightrightarrows \mathcal{D}_T^*(b)$$

as the mapping that assigns the unique ODS to every non-critical value of the bias. The comparative statics of $\Delta(b)$ with respect to b only depend on the set of ideal decisions for the agent when the principal fully delegates and in Holmstrom's model. This is not true in our model, where the agent is indifferent between successive decisions in states where the chain property holds. In further contrast, we need to distinguish between variations in b which leave the number of decisions in the ODS fixed, and those that change its size.

We start with small changes. Theorem 1 implies that the agent is indifferent between δ_t and δ_{t+1} in state $t < K$. If the initial bias satisfies $b_0 \in \left(\frac{T-K_0}{2(T-1)}, \frac{T-K_0+1}{2(T-1)} \right)$ then a small enough increase in b leaves K fixed at K_0. However, the agent now strictly prefers δ_{t+1} over δ_t; so the principal must increase the t'th and/or the next highest decision in the ODS to restore the agent's indifference in state t. Theorem 1 implies that the ODS varies continuously for increases in bias in this range.

In sum, if $K_0 > 0$ (so delegation is valuable) then a small enough increase in b from a generic starting point shifts the ODS to the right, just as in models of full delegation; so AP again fails. However, in contrast to those models, the driving force for this result is the change in shape of the loss function away from its minimal value.

Now consider the principal's loss at a critical value of b, say $b^{\max}(K,T)$. The chain property implies that the principal then makes the same loss in states $t \geq K$ if the agent takes δ_{K-1} or δ_K; so $\{\delta_0, \ldots, \delta_{K-1}\}$ is also an ODS at this critical bias. Thus, a marginal increase in b, starting from $b_0 \in \left\{ \frac{t}{2(T-1)} \right\}_{t>1}$, induces the principal to drop δ_K from the ODS, and to change the smaller decisions continuously. In particular, if B is a critical bias then

$$\lim_{b \searrow B} \Delta(b) \subset \lim_{b \nearrow B} \Delta(b).$$

This property mimics the reduction in discretion in Holmstrom's model.

[20]They define alignment in terms of the difference between ideal decisions.

Consider an increase in bias from b_0 to b_1 which is large enough that $\Delta(b_1)$ contains fewer decisions than $\Delta(b_0)$. Although $\Delta(b_0)$ and $\Delta(b_1)$ are not ordered by discretion, the principal offers more decisions to a less biased agent. We will now argue that a large enough increase in b reduces the maximal decision and raises the minimal decision in the ODS.

We start with the first claim. It is sufficient to show that an increase in b which reduces the number of decisions from $K_0 + 1$ to K_0 reduces the maximal decision.

If K_0 is even then the requisite condition is equivalent to

$$\ell^e(b_0, T, K_0) + \frac{K_0}{T-1} > 2b_1 - \ell^o(b_1, T, K_0 - 1) + \frac{K_0 - 2}{T-1}.$$

Substituting out for ℓ^e and ℓ^o, this condition is equivalent to $b_1 - b_0 < \frac{T+K_0}{2(T-1)}$. Now

$$\frac{T-K_0}{2(T-1)} < b_0 < b_1 < \frac{T-K_0+2}{2(T-1)},$$

which implies that the condition holds.

If K_0 is odd and $K_0 > 1$ then the requisite condition is equivalent to

$$2b_0 - \ell^o(b_0, T, K_0) > \ell^e(b_0, T, K_0 - 1)$$

which, on rearrangement, yields $b_1 - b_0 < 1/(T-1)$. The bounds on b_0 and b_1 above imply that this condition holds. If $K_0 = 1$ then Theorem 1 directly implies that $\delta_1 > 1/2$. These observations imply our claim that the maximal decision in $\Delta(b_1)$ exceeds the maximal decision in $\Delta(b_0)$ whenever $b_1 - b_0$ is large enough that the principal reduces the number of decisions in the ODS.

We will now argue that the minimal decision in $\Delta(b_1)$ exceeds the minimal decision in $\Delta(b_0)$ by considering an increase in b which reduces the number of decisions from $K_0 + 1$ to K_0. The argument requires us to distinguish between cases in which K_0 is odd and even. In the latter case, Lemma 2 implies that the minimal decision in $\Delta(b_1)$ exceeds the minimal decision in $\Delta(b_0)$ if and only if

$$K_0 b_0 - \frac{T}{2(T-1)} + \frac{(T-K_0)^2}{2(T-1)} < (2T - K_0) b_1 - \frac{1}{2} - \frac{(T-K_0+1)^2}{2(T-1)}.$$

Now

$$K_0 b_0 \leq \frac{K_0(T-K_0+1)}{2(T-1)} \text{ and } (2T-K_0)b_1 \geq \frac{(2T-K_0)(T-K_0+1)}{2(T-1)}.$$

Substituting above yields the requisite inequality.

If K_0 is odd then Lemma 2 implies that the minimal decision in $\Delta(b_1)$ exceeds the minimal decision in $\Delta(b_0)$ if and only if

$$(2T - K_0 - 1)b_0 - (K_0 - 1)b_1 < \frac{(T-K_0)^2}{T-1} + \frac{T-K_0}{T-1} + \frac{1}{T-1}.$$

Now

$$(2T-K_0-1)b_0 \leq \frac{(2T-K_0-1)(T-K_0+1)}{2(T-1)} \text{ and } (K_0-1)b_1 \geq \frac{(K_0-1)(T-K_0+1)}{2(T-1)}.$$

Substituting above yields the requisite inequality.

This pair of observations implies our claim that the minimal decision in $\Delta(b_1)$ exceeds the minimal decision in $\Delta(b_0)$. We summarize this subsection's arguments in

Proposition 3 *Suppose that the initial bias is low enough that delegation is valuable.*

(i) Consider a marginal increase in bias from b_0 to b_1. If $b \notin \left\{\frac{T-K}{2(T-1)}\right\}_{K=0}^{K=T-1}$ then the principal raises all decisions in the ODS: so $\Delta(b_0)$ and $\Delta(b_1)$ are not ordered by set-inclusion. However, if $b \in \left\{\frac{T-K}{2(T-1)}\right\}_{K=0}^{K=T-1}$ then the principal drops the highest decision in the ODS, and gives the agent less discretion in the sense that $\lim_{b\searrow b_0}\Delta(b) \subset \lim_{b\nearrow b_0}\Delta(b)$.

(ii) Consider an increase in bias from b_0 to b_1 which is large enough that $\Delta(b_1)$ contains fewer decisions than $\Delta(b_0)$. The two delegation sets are not ordered by set-inclusion. However, the principal raises [resp. reduces] the smallest [resp. largest] decision in the ODS.

Proposition 3 implies that a version of AP holds in our model, provided that bias increases enough. However, the principal shifts the entire ODS to the right in response to smaller increases in bias.

The empirical evidence on AP has used the distinction between divided and unified government as a proxy for differences in bias. The evidence is mixed: Huber and Shipan's (2002) data is supportive of AP, but Volden's (2002) evidence is not.

4 The Uncertainty Principle

A long-standing political science literature has considered how a principal responds when the state becomes riskier, yielding two related hypotheses. According to one version, a principal who delegates gives the agent more discretion when the state is riskier. Holmstrom (1984) shows that this hypothesis holds in his model when losses are quadratic, but that it may otherwise fail. Alonso and Matouschek (2008) provide a related example which does not rely on Holmstrom's assumption that the ODS is an interval. These results rely on the supposition that the support of states is an interval. By contrast, variations in the ODS are typically not ordered by (set-inclusive) discretion in our model. Accordingly, we will focus on the second hypothesis, which we will refer to as the Uncertainty Principle (UP).

According to UP, the principal is more likely to delegate when the state is riskier. Specifically, suppose that a principal who does not delegate takes the decision himself — so that his expected loss equals the variance of the state — and that, if he chooses to delegate then he incurs a fixed cost in addition to his expected loss from delegating. This principal would delegate (rather than take the decision himself) if and only if the gain —

or the reduction in the expected loss — from doing so exceeds the fixed cost. Following this idea, we measure the principal's willingness to delegate by V, the value of delegation (cf. Alonso and Matouschek (2008)): namely, the difference between the principal's (minimal) expected loss when he takes the decision himself and his expected loss when he delegates. UP then holds if a mean-preserving increase in risk raises V.

UP obviously holds with full delegation: for the agent takes her ideal decision and the principal loses b^2 in every state, irrespective of its distribution; whereas the principal is worse off, absent delegation, when the state is riskier (cf. Bendor and Meirowitz (2004)). We will now show that UP also holds if a delegating principal offers the ODS, and the state is uniformly distributed on $[-a, 1+a]$ (with variance $\frac{1}{3}a^2 + \frac{1}{3}a + \frac{1}{12}$). An increase in a is then a mean-preserving increase in risk, and UP fails if $\partial V/\partial a < 0$.[21]

A delegating principal offers $\Delta(b) = [b - a, 1 + a - b]$ if $b < (1 + 2a)/2$ and loses $b^2 - \frac{4}{3(1+2a)}b^3$. Consequently,

$$V(a) = \frac{1}{3}a^2 + \frac{1}{3}a + \frac{1}{12} - b^2 + \frac{4}{3(1+2a)}b^3 \text{ and} \tag{4}$$

$$\frac{\partial V}{\partial a} = \frac{1+2a}{3} - \frac{8}{3(1+2a)^2}b^3 , \tag{5}$$

which is positive because of the upper bound on b. We therefore conclude that UP must hold in this model. Note that a delegating principal is worse off with a riskier state; so the direct (and conventional) effect of an increased variance dominates the indirect effect on the principal's loss.

We will now demonstrate that UP also holds in the finite-state model. We model an increase in risk as a change in the probability distribution from $p_{s,t}$ to $p_{s+2\upsilon,t-\upsilon}$, $\upsilon \in \mathbb{N}$. In other words, the support of the distribution is enlarged by adding υ additional states to the left- and right-hand sides of the original support of the distribution; but the states are still $1/(T-1)$ apart. The ensuing distribution of states is a mean-preserving spread of the original one, with the variance increasing by $\upsilon(s+\upsilon)/3$.[22]

Suppose that the state is distributed according to $p_{s,t}$, and let $\Delta \in \mathcal{D}^*_{s,t}(b)$. The principal's expected losses when he does not delegate and when he delegates are, respectively,

$$\lambda^n(s) \equiv \frac{1}{s}\sum_{\tau=t}^{t+s-1}\left(\frac{2t+s-1}{2(T-1)} - \frac{\tau}{T-1}\right)^2 = \frac{s^2-1}{12(T-1)^2}$$

and

$$\lambda(s) \equiv \frac{1}{s}\sum_{\tau=t}^{t+s-1}\left(d_\tau(\Delta) - \frac{\tau}{T-1}\right)^2 .$$

The value of delegation when the state is distributed according to $p_{s,t}$ is therefore given by $V(s) \equiv \lambda^n(s) - \lambda(s)$.

The next proposition shows that an increase in risk raises the value of delegation.

[21] Huber and McCarty (2004) Proposition 3 draw the same conclusion, but under the additional condition that agents are incompetent enough. We discuss their model in the next section.

[22] Theorem 1 implies that the ODS contains at least as many decisions when the risk of the state increases in this manner: a weaker form of the other version of UP.

Proposition 4 *Let $t \in T$ and $s < T - t$. For every integer $v \in [1, \min\{t, T-t-s\}]$ and any $b > 0$, $V(s+2v, b) \geq V(s, b)$. Furthermore, this inequality is strict if and only if $b < \frac{s+2v}{2(T-1)}$.*

The proof of Proposition 4, which relies extensively on Lemma 2, is relegated to Appendix B. The Proposition asserts that UP holds in our model if risk is increased by adding states to the support.

The distribution of states could, alternatively, become more risky if we fix the support of states and add weight to the tails. In Appendix C, we return to Example, where $T = \{0, 1, 2\}$ and states 0 and 2 are each realized with probability p. We show that a marginal increase in p reduces V if b is small enough that the ODS contains three decisions, and state 1 is likely enough: so UP fails. Formally:

Remark 4: *UP fails for some distributions of states with a finite support..*

Volden (2002) provides strong empirical support for UP, which also seems to be consistent with Acemoglu et. al's (2007) evidence that firms in riskier environments are more likely to decentralize decision-making (delegate). Nevertheless, Remark 4 seems to be novel in the literature. This result is striking because of the simplicity of Example.

5 The Expertise Principle and the Optimal Agent

According to many commentators, political appointments to bureaucracies have become increasingly prevalent in the US and the UK. Hurricane Katrina focused concerns that political appointees are, on average, less competent than career civil servants. Lewis (2007) and Kelman and Myers (2009) provide evidence which supports this conjecture.

Bendor and Meirowitz (2004) model an amateur as an agent who knows the state with probability $q \in [0, 1]$, and otherwise has the same prior beliefs about the state as the principal: so agents are ordered by competence (q). They note that the principal can trivially not improve on appointing the agent with the highest q from a population of agents with common bias.[23] On the other hand, civil servants may be more biased towards the status quo (cf. McCarty (2004)). A recent literature has built on these features by studying the ensuing trade-off between loyalty and competence: cf. Egorov and Sonin (forthcoming), Gailmard and Patty (2007) and Huber and McCarty (2004).

In this section, we will use a different model of competence to investigate whether the principal optimally appoints the expert in our model. In a general framework, a representative agent observes a signal correlated with the state. Like Ivanov (forthcoming), we follow the literature by treating knowledge as partitional. Specifically, we suppose that each agent has a partition of the T states into $n \leq T$ events, which we denote $E \equiv \{E_1, ...E_n\}$. The agent in previous sections has $n = T$; we refer to such an agent as the *expert.* In contrast to Ivanov, we do not require that the events are connected, in the

[23] This is also true in Egorov and Sonin's (forthcoming) two-state model, where the agent's signal is wrong with probability q.

sense that an agent may be unable to distinguish between states $t-1$ and $t+1$, but can distinguish state t. (States are ordered by their payoff implications.)

In contrast to Bendor and Meirowitz's version of amateurs, agents are only partially ordered by competence; though the expert is more competent than any other agent. We will exploit this fact by studying the Expertise Principle (or EP): the principal loses no more by appointing the expert than appointing any amateur if he offers each agent her loss-minimizing delegation set. If EP fails then we say that some amateur *strictly outperforms* the expert.

EP obviously holds in Bendor and Meirowitz's version.[24] We will argue that our version of an amateur has radically different implications for EP.

At first sight, it seems implausible that an amateur could outperform the expert because an amateur who cannot distinguish between states in some event must take the same decision in that event. The decision taken by an expert must therefore be at least as state-dependent as that taken by an amateur. As we will demonstrate, however, this does not imply that the principal is better off appointing an expert.

5.1 Preliminary intuitions

It is expositionally useful to postpone the statement of our general results and begin with an intuitive presentation of the key mechanisms at work. Specifically, there are two reasons why the principal may be better off appointing an amateur rather than the expert.

The first reason is that an agent's expertise may force the principal to offer her too many decisions. To see this, it is useful to focus on putative improvements which replace a subset of decisions in the ODS. (We reserve the term *ODS* for the expert throughout this section). Specifically, suppose that there are four states, and that $1/3 < b < 1/2$. The ODS, which we describe in Appendix C.2, is

$$\Delta = \left\{ \frac{1}{2}b, \frac{3}{2}b, \frac{1}{2}b + \frac{2}{3} \right\} ,$$

and the principal loses $\frac{5}{4}b^2 - \frac{1}{2}b + \frac{1}{18}$ in event $E \equiv \{0, 1\}$. Consider another delegation set $\Delta' = \{\frac{3}{2}b - \frac{1}{3}, \frac{1}{2}b + \frac{2}{3}\}$: where $\frac{3}{2}b - \frac{1}{3}$ replaces the lower two decisions in Δ. If the agent were always to take $\frac{3}{2}b - \frac{1}{3}$ in E then the principal would lose $\frac{9}{4}b^2 - \frac{3}{2}b + \frac{5}{18}$ in E, which is less than $\frac{5}{4}b^2 - \frac{1}{2}b + \frac{1}{18}$. In other words, the principal would lose less when he offers Δ', provided that the agent takes $\frac{3}{2}b - \frac{1}{3}$ in E. The latter condition fails when the agent is an expert because she would take $\frac{1}{2}b + \frac{2}{3}$ in state 1.

Now consider an amateur who can't distinguish between the states in E, but knows the exact state otherwise. The first decision in Δ' has been calibrated such that the amateur is indifferent across Δ' when the state is in E, and strictly prefers $\frac{1}{2}b + \frac{2}{3}$ in states 2 and 3 (by the single-crossing property). Consequently, this amateur would always take $\frac{3}{2}b - \frac{1}{3}$ in E, and would take $\frac{1}{2}b + \frac{2}{3}$ in states 2 and 3. In sum, the principal is better off appointing

[24] By contrast, Huber and McCarty (2004) model an amateur as an agent who observes the state, but trembles. EP can't be studied in their model because they focus on agents who are likely enough to tremble.

this amateur and offering her Δ' than appointing the expert and offering her the ODS. In other words, EP fails.

Notice that we have constructed Δ' such that the principal only gains in those states that the amateur cannot distinguish. Most of our arguments below will rely on this property; but it is important to understand that the principal might also gain in states outside E: the second reason why EP might fail. To see this, consider another delegation set, where $\frac{3}{2}b - \frac{1}{3}$ is replaced by a slightly higher decision. An amateur who can't distinguish between states 0 and 1 would now strictly prefer the new decision over $\frac{1}{2}b + \frac{2}{3}$ in event E. This allows the principal to reduce his loss in states 2 and 3 by replacing $\frac{1}{2}b + \frac{2}{3}$ with a slightly lower decision.[25] In other words, EP might fail because the agent takes lower decisions in states which she can distinguish. We will use this property at the end of this section, when we calculate loss-minimizing delegation sets for every amateur, and thereby the best amateur.

The two reasons why an amateur might outperform the expert are also necessary. In particular, an amateur who can distinguish between all states bar $E \equiv \{t, ..., T-1\}$ can't outperform the expert: for the principal can induce the expert to take the same decisions as such an amateur.

5.2 Expert vs. amateur agents

We now turn to the more general setting of T states, first establishing two main results: (i) the expert cannot be outperformed by amateurs when b is high enough that delegation is not valuable or when b is low enough that the principal can achieve first best with the expert; and (ii) for intermediate bias, there is an amateur who strictly outperforms the expert.

Proposition 5 ***(i)*** *EP holds if* $b \leq \frac{1}{2(T-1)}$ *or* $b > \frac{T}{2(T-1)}$*;*
(ii) *EP fails if there are at least four states and* $\frac{1}{T-1} < b < \frac{T}{2(T-1)}$.

Proof: **(i)** Replacing the expert with an amateur must make the agent's decisions less state-sensitive; so EP must hold if the agent would take first-best decisions in every state: that is, when $b \leq \frac{1}{2(T-1)}$.

If $b > \frac{T}{2(T-1)}$ then the principal loses $\frac{T^2+T}{12(T-1)^2}$ if he delegates to an expert. The principal can improve on appointing an expert if and only if he can offer some amateur a two-decision delegation set.

Suppose that the principal offers $\{\delta_0, \delta_1\}$ to an amateur, who takes δ_0 if and only if the state is in $E \equiv \{0, ..., \tau - 1\}$. The principal is then best off if the amateur cannot distinguish between the states in E, and is indifferent between δ_0 and δ_1: so

$$\delta_1 = 2b + \frac{\tau - 1}{T - 1} - \delta_0 \ .$$

[25] Recall that the expert takes a decision above the principal's ideal in every state if $b > 1/2(T-1)$ by Corollary 1(i).

For fixed δ_0 and τ, the principal expects to lose

$$\lambda(\delta_0, \tau, b) \equiv \frac{1}{T-1}\left[\sum_{t=0}^{\tau-1}(\delta_0 - \frac{t}{T-1})^2 + \sum_{t=\tau}^{T-1}\left(\delta_0 - 2b - \frac{\tau-1}{T-1} + \frac{t}{T-1}\right)^2\right].$$

For fixed τ, the principal minimizes his loss by offering the amateur

$$\delta_0^*(\tau, b) \equiv \frac{2(T-\tau)}{T}b - \frac{1}{2} + \frac{\tau-1}{T-1} \text{ and } \delta_1^*(\tau, b) = \frac{2\tau}{T}b + \frac{1}{2}\ .$$

Now $\lambda[\delta_0^*(\tau, b), \tau, b]$ is increasing in b; so $\lambda[\delta_0^*(\tau, b), \tau; b] > \lambda\left[\delta_0^*(\tau, b), \tau, \frac{T}{2(T-1)}\right]$ in the relevant range. We will argue that the principal is indifferent between appointing the expert and an amateur when $b = T/2(T-1)$, which will then imply that the expert outperforms the amateur for larger bias. Substituting for b yields

$$\delta_0^*\left(\tau, \frac{T}{2(T-1)}\right) = \frac{1}{2} \text{ and } \delta_1^*\left(\tau, \frac{T}{2(T-1)}\right) = \frac{\tau}{T-1} + \frac{1}{2}\ .$$

The expert takes decision $1/2$ in all states; so the amateur outperforms the expert when $b = T/2(T-1)$ if and only if the principal loses less in states $t \geq \tau$: that is, if there is τ such that

$$L(\tau) \equiv \sum_{t=\tau}^{T-1}\left(\frac{1}{2} + \frac{\tau}{T-1} - \frac{t}{T-1}\right)^2 - \sum_{t=\tau}^{T-1}\left(\frac{1}{2} - \frac{t}{T-1}\right)^2 < 0\ .$$

Rearranging terms reveals that $L(\tau) = 0$ for every τ, proving this part.

(ii) We prove this part in Appendix B.□

Proposition 5 generalizes our arguments in the last subsection. In particular, it asserts that some amateur outperforms the expert for intermediate bias. Our construction of a superior amateur generalizes that used in the last subsection: such an amateur can't distinguish between states $K-2$ and $K-1$; and we again construct a delegation set such that the amateur and the expert take the same decisions in all states other than $K-2$ and $K-1$. As above, we prove the result by showing that the principal loses less by offering some delegation set to some amateur, rather than by characterizing the loss-minimizing delegation set for a given amateur.

Part (ii) is reminiscent of Case II in Postlewaite (1982), where an amateur (in our sense) outperforms the expert. However, preferences therein fail the single crossing property such that an amateur alone can coordinate with the principal; so Case II captures a different effect.

The premise of part (ii) excludes $b \in (\frac{1}{2(T-1)}, \frac{1}{T-1})$: when the ODS contains T decisions. It is easy to confirm that the expert outperforms an amateur who can't distinguish between states $T-2$ and $T-1$, and takes the same decisions as the expert in all other states.[26] This observation and part (ii) imply that no amateur outperforms the expert if there are

[26] We provide a proof of this claim, which we dub Observation 1, in the online Appendix.

fewer than four states. This is obvious when $T = 2$, and slightly more subtle when $T = 3$. Arguments used in the proof of part (ii) then imply that no amateur can outperform the expert if $b > 1/(T-1)$; the observation implies that an amateur who can't distinguish between states 0 and 1 cannot outperform the expert; and an argument at the end of Section 5.1 precludes an amateur who can't distinguish between states 1 and 2 outperforming the expert.[27]

As T increases, the interval $(\frac{1}{2(T-1)}, \frac{1}{T-1})$ shrinks. We can therefore construct an amateur akin to that used in part (ii) to prove

Proposition 6 *If the state is uniformly distributed on* $[0,1]$ *and delegation is valuable then EP fails.*

Proof: Consider an amateur who cannot distinguish between states in $[1-2b-\varepsilon, 1-2b]$ for some $\varepsilon > 0$, and write $1-2b-\varepsilon$ and $1-2b$ respectively as α and β. The decision(s) which leaves the principal indifferent between an agent and an amateur are the roots of the equation

$$f(d) = d^2 - (\alpha+\beta)d + \frac{1}{3}\left(\alpha^2 + \alpha\beta + \beta^2 - b^2\right) = 0:$$
$$\text{viz. } f^{-1}(0) = 1 - 2b \pm \sqrt{b^2 - \varepsilon^2/12}\ ,$$

which are well-defined if $b > \varepsilon/2\sqrt{3}$.

Now consider a delegation set consisting of the interval $[b, 1-b-\varepsilon]$, a decision $d \in (1-b-\varepsilon, 1-b)$, and decision $1-b$. The agent strictly prefers d over both $1-b-\varepsilon$ and $1-b$ if and only if $d \in (1-b-\varepsilon, 1-b)$. If $b > \varepsilon/2\sqrt{3}$ then this interval has a nonempty intersection with $(1-2b-\sqrt{b^2-\varepsilon^2/12}, 1-2b+\sqrt{b^2-\varepsilon^2/12})$ because

$$1 - b > 1 - 2b - \sqrt{b^2 - \varepsilon^2/12} \text{ and } 1 - b - \varepsilon < 1 - 2b + \sqrt{b^2 - \varepsilon^2/12}\ .$$

Consequently, this agent takes a decision in this intersection when the state is in $[1-2b-\varepsilon, 1-2b]$, and therefore strictly outperforms the expert.

The proof is completed by noting that there is ε (and therefore a possible agent) which satisfies $b > \varepsilon/2\sqrt{3}$ for every $b \in (0, 1/2)$. □

Ivanov (forthcoming) proves an analog of Proposition 6 by constructing an amateur with a finite partition of $(0,1)$ who outperforms the expert whenever delegation is valuable. Our proof relies on constructing an amateur with an infinite partition. This construct, like that in the finite state model, only discretizes for states close to $1-2b$: the highest state at which the expert takes her ideal decision.

Proposition 5(i) states that EP holds when $b \leq \frac{1}{2(T-1)}$; so it also holds whenever b just exceeds $\frac{1}{2(T-1)}$. On the other hand, if $T > 3$ and b is close to $1/(T-1)$ then the principal might improve on the expert by offering another delegation set to an amateur who can't distinguish between states $T-2$ and $T-1$ (as above) and/or another amateur might

[27]Delegation to an amateur who can't distinguish between states 0 and 2 is never valuable because her preferences over decisions are state-independent.

outperform the expert. It seems natural to consider this possibility by addressing another question, of independent interest: the identity of the 'best' amateur, whose loss-minimizing delegation set minimizes the principal's loss.

Unfortunately, this is a challenging problem in our model because the loss-minimizing delegation set for a given amateur need not satisfy the chain property (cf. the four-state example in Section 3.1). It is yet more demanding in Holmstrom's model because the problem there almost encompasses that in our model. To see this, consider an amateur in Holmstrom's model who can't distinguish between the states in events $\{E_t\}_{t=1}^{t=T-2}$, where $E_t \equiv [x_{t-1}, x_t)$, $x_0 \equiv \varepsilon > 0$, T is odd, and

$$x_t = \begin{cases} \frac{t}{T-1} + \varepsilon & \text{if } t \text{ is even,} \\ \frac{t+1}{T-1} - \varepsilon & \text{if } t \text{ is odd.} \end{cases}$$

Such an amateur behaves as if the state were sure to be $t/(T-1)$ in event E_t; so she behaves as if the state space were $\mathbf{T} \setminus \{0\}$, except in the event $[0, \varepsilon) \cup [1 - \varepsilon, 1]$. The expert in our model (with T states) yields almost the same loss as such an amateur in Holmstrom's model when ε is small enough; and any amateur in our model can be approximated analogously.

Nevertheless, our finite state structure allows us to obtain some insights by considering the special case of four states, where we can determine the optimal amateur for every bias (see Appendix C.2 for a detailed proof):

If $b \in (1/6, 3/8)$ then the best amateur can't distinguish between states 1 and 2; and if $b \in (3/8, 2/3)$ then there are two best amateurs: one can't distinguish between states 0 and 1; the other can also not distinguish between states 2 and 3.[28] Furthermore, EP fails if and only if $7/24 < b < 2/3$. In particular, if $b \in (\frac{7}{24}, \frac{1}{3}) \cup (\frac{3}{8}, \frac{1}{2})$ then the principal offers fewer decisions to the best amateur than to the expert.

Note that the expert can be outperformed when $b < 1/(T-1) = 1/3$. The best amateur can then not distinguish between states 1 and 2, as in our construction above. However, she takes different decisions than the expert in states 0 and 3 when offered her loss-minimizing delegation set. This property confirms the importance of the second reason for appointing an amateur, as detailed in Section 5.2 above: the principal may relax the incentive constraints of an amateur in states outside E by changing the decision which she takes in E. On the other hand, contrary to our construction in Proposition 5, the best amateur can't distinguish between states $K-1$ and K when $1/3 < b < 3/8$.[29]

Our results in this section imply that EP fails when the right sort of amateur is available and there are more than three states. The relevance of these conditions is, of course, an empirical question. Nevertheless, our results contradict a supposition in the related literature: evidence of incompetent political appointees does not imply that there is a trade-off between bias and expertise. Indeed, Proposition 5 implies that the principal could optimally appoint an agent who is less loyal than an expert.

Our demonstration that EP fails is reminiscent of a result in the signalling literature: the principal may prefer to take advice from an amateur: cf. Fischer and Stocken (2001) and Ivanov (forthcoming).

[28] The loss-minimizing delegation set for the first amateur induces her to pool in states 2 and 3.

[29] She again takes different decisions than the expert in states 0 and 3 when offered her loss-minimizing delegation set.

Remark 5 We have supposed that the principal appoints an amateur from a set of candidates with exogenously given competence. Suppose, per contra, that a single agent chooses her competence (the collection of events E) before observing the state. If the principal observes E before offering a delegation set then the agent may choose to be an amateur: for example, if there are four states and $\frac{1}{2} < b < \frac{2}{3}$ then she would lose $\frac{1}{4}b^2 + \frac{5}{36}$ as the expert and $\frac{5}{36}$ if $E = \{0, 1\}$.[30] Indeed, both parties are better off for b in this interval if the agent is an amateur because $E = \{0, 1\}$ is then an optimal amateur (see above), as in Postlewaite (1982) Case II. Postlewaite notes that the principal must observe E for the agent to be an amateur in a pure strategy SPE. This argument also applies in our model: for an amateur who is indifferent between decisions $\delta < \delta^*$ in event E must strictly prefer δ^* in the highest state in E. The agent could then profitably deviate to being an expert if the principal believes that she can't distinguish states in E.

6 Summary

The ODS in Holmstrom's model has various well-known properties: the agent takes her ideal decision in lower states, and only takes decisions which are ideal for the principal in some state; an increase in bias reduces discretion; and the principal is more willing to delegate if the state is riskier.

This paper addresses optimal delegation in empirically salient situations that cannot be studied within Holmstrom's framework, namely those in which the number of states is finite. We have characterized ODSs in a simple variant on Holmstrom's model, and shown that they exhibit two remarkable and extremely useful properties: the chain property — which implies that the agent is prevented from compromising — and top loading. We have then shown that all the above-mentioned properties of Holmstrom's ODS may fail in this finite environment: the agent never takes her ideal decision, but may take a decision which is not ideal for the agent or for the principal in any state; a small increase in bias shifts the ODS to the right. Finally, the Uncertainty Principle relies on equi-probable states: the principal may otherwise be less willing to delegate if the state is more risky.

We have also shown that the ideal agent for a principal is not an expert when the bias is intermediate: a property which also holds in Holmstrom's model.

[30] Kamenica and Gentzkow (2009) provide analogous results for a signalling game with interval support.

APPENDIX A. Proof of Theorem 1

Proof of Lemma 1

Without loss of generality, we normalize t to zero. Define the optimization problem $(\mathcal{P})$ as

$$\min_{(d_0,\ldots,d_{s-1})} \sum_{i=0}^{s-1} \left(d_i - \frac{i}{T-1}\right)^2 ,$$

$$\text{subject to } \left(d_i - b - \frac{i}{T-1}\right)^2 \leq \left(d_{i+1} - b - \frac{i}{T-1}\right)^2 , \forall i = 0, \ldots, s-1.$$

Our first step is to establish the existence of a solution to problem $(\mathcal{P})$ by using the Weierstrass Theorem. To do so, we must ensure that the objective function in $(\mathcal{P})$ is continuous, and that the collection of incentive-compatibility constraints defines a feasible set that is compact in $\mathbb{R}^s$. While continuity of the principal's loss function and closedness of the feasible set are trivial (inequalities are weak and the agent's loss function is always continuous), the feasible set of $(\mathcal{P})$ is obviously not bounded.

Nevertheless, any solution to $(\mathcal{P})$, say $\left(d_0^*, \ldots, d_{s-1}^*\right)$, must satisfy

$$0 \leq d_i^* \leq 2\left(b + \frac{s-2}{T-1}\right) , \tag{6}$$

for each $i = 0, \ldots, s-1$. To see this, suppose first that some of the d_i^*'s are negative. In such a situation, the principal can reduce her loss by replacing every $d_i^* < 0$ by $d_i = 0$. Indeed, the agent would take decision 0 in every state $i \leq I \equiv \max\{i : d_i^* < 0\}$, making the principal strictly better off. Moreover, the principal would also be better off in every state $i > I$ in which the agent would take 0 instead of $d_i^* > 0$. This proves that $d_i^* \geq 0$, for each $i = 0, \ldots, s-1$.

Furthermore, the incentive-compatibility constraint in state $i+1$, $i < s-1$, requires that $d_i^* \leq b + \frac{s-1}{T-1}$ whenever $d_i^* < d_{i+1}^*$. Now, to show that the above inequality is also true for $i = s-1$, let $\tau + 1 \equiv \min\left\{j : d_j^* = d_{s-1}^*\right\}$ and suppose that $d_{s-1}^* > b + \frac{s-1}{T-1}$. This implies that the incentive-compatibility constraint in state τ must be satisfied with equality, else the principal could reduce his expected loss by reducing $d_{\tau+1}^*$. Thus, either $d_\tau^* = d_{\tau+1}^*$ or

$$d_\tau^* + d_{\tau+1}^* = 2\left(b + \frac{\tau}{T-1}\right) .$$

Combined with $d_\tau^* \geq 0$, this implies that

$$d_i^* \leq d_{s-1} = d_{\tau+1}^* \leq \max\left\{b + \frac{s-1}{T-1}, 2\left(b + \frac{\tau}{T-1}\right)\right\} \leq 2\left(b + \frac{s-1}{T-1}\right)$$

for each $i = 0, \ldots, s-1$.

We have thus established that any solution to $(\mathcal{P})$, $\left(d_0^*, \ldots, d_{s-1}^*\right)$, must satisfy (6). Consequently, problem $(\mathcal{P}')$, obtained by adding the constraints (6) to $(\mathcal{P})$, has the same solution(s) as $(\mathcal{P})$. The conditions in (6) guarantee that the feasible set of $(\mathcal{P}')$ is bounded (and therefore compact). By Weierstrass Theorem, $(\mathcal{P}')$ – and therefore $(\mathcal{P})$ – has at least one solution.

Let $\left(d_0^*, \ldots, d_{s-1}^*\right)$ be a solution to $(\mathcal{P})$. Recursively define

$$\begin{aligned} \delta_0^* &\equiv d_0^* \ , \\ D_i &\equiv \left\{d \in \left\{d_0^*, \ldots, d_{s-1}^*\right\} : d > \delta_{i-1}^*\right\} \ , \\ \delta_i^* &\equiv \min_{d \in D_i} d \ . \end{aligned}$$

If K is the smallest integer such that $D_{K+1} = \emptyset$ then $\{\delta_0^*, \ldots, \delta_K^*\}$ is, by definition, an optimal delegation set.

Proof of Lemma 2

As Δ is top loaded, the agent takes decision $\delta_{t+\tau}$ in state $t+\tau$, for every $\tau < K$, and takes δ_{t+K} in states $K \leq t+\tau \leq s-1$.

For each $\tau \leq K$, let $\kappa_{t+\tau} \equiv \delta_{t+\tau} - \frac{t+\tau}{T-1}$. By optimality, $(\kappa_{t+\tau})_{\tau=0\ldots,K}$ must solve

$$\min_{\{\kappa_{t+\tau}\}_{\tau=t,\ldots,K}} \left\{ \sum_{\tau=0}^{K-1} \kappa_{t+\tau}^2 + \sum_{\tau=K}^{s-1} \left(\kappa_{t+K} - \frac{\tau-K}{T-1}\right)^2 \right\}$$

subject to

$$\kappa_{t+\tau} + \kappa_{t+\tau+1} = 2b - \frac{1}{T-1} \ , \forall \tau = 0, \ldots, K-1 \ ; \tag{7}$$

where the equality in (7) must hold at Δ because it satisfies the chain property. It is easy to see that any solution to the above problem is also a solution to

$$\min_{\{\kappa_{t+\tau}\}_{\tau=t,\ldots,K}} \left\{ \sum_{\tau=0}^{K-1} \kappa_{t+\tau}^2 + (s-K)\left(\kappa_{t+K} - \frac{s-K-1}{T-1}\right)\kappa_{t+K} \right\} \ , \tag{8}$$

subject to (7). From the incentive constraints (7), we obtain by recursion that

$$\kappa_{t+\tau} = \begin{cases} \kappa_t & \text{if } \tau \text{ is even} \\ 2b - \frac{1}{T-1} - \kappa_t & \text{if } \tau \text{ is odd} \end{cases} \ , \forall \tau = 0, \ldots, K.$$

Substituting into (8), simple convex optimization reveals that, at an optimum:

$$\kappa_t = \begin{cases} \ell^e(b,s,K) & \text{if } K \text{ is even,} \\ \ell^o(b,s,K) & \text{if } K \text{ is odd,} \end{cases}$$

where $\ell^e(b,s,K)$ and $\ell^o(b,s,K)$ are defined before Lemma 2 in Section 2.3.

We end the proof by showing that the principal can reduce his expected loss whenever $b \notin \left[b^{\min}(K,s), b^{\max}(K,s)\right]$ — a contradiction to Δ being optimal. More precisely, for every $m \in \mathbb{N}$, let $D^*(m) \equiv \left\{\delta_t^*(m), \ldots, \delta_{t+m-1}^*(m)\right\}$ where

$$\delta_{t+\tau}^*(m) = \begin{cases} \ell(b,s,m-1) + \frac{t+\tau}{T-1} & \text{if } \tau \text{ is even,} \\ 2b - \ell(b,s,m-1) + \frac{t+\tau-1}{T-1} & \text{if } \tau \text{ is odd.} \end{cases}$$

We will show that, if $b \notin \left[b^{\min}(K,s), b^{\max}(K,s)\right]$ then there is $K' \neq K$ such that the principal can reduce his loss by offering either $D^*(K'-1)$ or $D^*(K'+1)$ to the agent.

Suppose that K is even. The change in the principal's loss following a deviation to $D^*(K+2)$ when $K < s-1$ can be decomposed as follows:

(i) In each state $t+\tau$, with τ even and no more than K, the change in the principal's loss is:

$$\ell^o(b,s,K+1)^2 - \ell^e(b,s,K)^2 = \varphi(b,s,K)\left[\varphi(b,s,K) + 2\ell^e(b,s,K)\right] \ ,$$

where

$$\varphi(b,s,K) \equiv \ell^o(b,s,K+1) - \ell^e(b,s,K) = \frac{s-K-1}{s}\left(2b - \frac{s-K}{T-1}\right) \ .$$

(ii) In each state $t+\tau$, with τ odd and no more than $K-1$, the change in the principal's loss is:

$$\left[2b - \frac{1}{T-1} - \ell^o(b,s,K+1)\right]^2 - \left[2b - \frac{1}{T-1} - \ell^e(b,s,K)\right]^2$$
$$= \varphi(b,s,K)\left[\varphi(b,s,K) - 2\left(2b - \frac{1}{T-1} - \ell^e(b,s,K)\right)\right] \ .$$

(iii) In each state $t+K+i$, $i = 1,\ldots,s-K-1$, the change in the principal's loss is:

$$\left[2b - \ell^o(b,s,K+1) - \frac{i}{T-1}\right]^2 - \left[\ell^e(b,s,K) - \frac{i}{T-1}\right]^2$$
$$= \xi(b,s,K)\left\{\xi(b,s,K) + 2\left[\ell^e(b,s,K) - \frac{i}{T-1}\right]\right\} \ ,$$

where

$$\xi(b,s,K) \equiv 2b - \ell^o(b,s,K+1) - \ell^e(b,s,K) = \frac{1}{s}\left(2b + \frac{K}{T-1}\right) \ .$$

Note that

$$\begin{aligned}\sum_{i=1}^{s-K-1}\left\{\xi(b,s,K) + 2\left[\ell^e(b,s,K) - \frac{i}{T-1}\right]\right\} &= (s-K-1)\left[\xi(b,s,K) + 2\ell^e(b,s,K) - \frac{s-K}{T-1}\right] \\ &= (K+1)\frac{s-K-1}{s}\left(2b - \frac{s-K}{T-1}\right) \\ &= (K+1)\varphi(b,s,K) \ .\end{aligned}$$

Thus, summing the changes in the principal's loss over all states, we obtain

$$(K+1)\varphi(b,s,K)\left[\varphi(b,s,K) + 2\ell^e(b,s,K) - \frac{K}{K+1}\left(2b - \frac{1}{T-1}\right) + \xi(b,s,K)\right]$$

$$= (K+1)\varphi(b,s,K)\left[\frac{1}{K+1}\left(2b + \frac{K}{T-1}\right)\right] = \frac{2(s-K-1)}{s}\left(2b + \frac{K}{T-1}\right)\left[b - b^{\min}(K,s)\right] \ .$$

This proves that the deviation to $D^*(K+2)$ is unprofitable only if $b \geq b^{\min}(K,s)$. The argument above also implies that the principal can profitably deviate from $D^*(K+2)$ to $D^*(K+1)$ unless $b \leq b^{\max}(K+1,s) = b^{\min}(K,s)$.

Analogous arguments imply (see the online Appendix for a detailed proof) that the change in the principal's expected loss if he deviates to $D^*(K)$ (when $K > 0$) equals

$$-\frac{2K}{T-1}\frac{s-K}{s}\left[b - b^{\max}(K,s)\right] \ .$$

This proves that the deviation to $D^*(K)$ is unprofitable only if $b \leq b^{\max}(K,s)$. The above argument also implies that the principal can profitably deviate from $D^*(K)$ to $D^*(K+1)$ unless $b \geq b^{\min}(K-1,s) = b^{\max}(K,s)$.

We have therefore proved that the principal can reduce his expected loss by choosing $D^*(K+2)$ instead of Δ when $b < b^{\min}(K,s)$, and by choosing $D^*(K)$ instead of Δ when $b > b^{\max}(K,s)$. [31] As $\Delta \in \mathcal{D}^*_{s,t}(b)$, this implies that $b \in \left[b^{\min}(K,s), b^{\max}(K,s)\right]$.

Proof of Lemma 3

Suppose that $(\mathbf{H_s})$ is true and that, contrary to the Lemma, some element of $\mathcal{D}^*_{s+1,t}(b)$, say Δ^*, does not satisfy the chain property.

$(\mathbf{H_s})$ and the optimality of Δ^* imply that the chain can only break once. To see this, note that there would otherwise be k_1 and k_2 such that the agent would strictly prefer $d_{t+\tau_i}(\Delta^*,b)$ to $d_{t+\tau_i+1}(\Delta^*,b)$, $i=1,2$ in states $t+\tau_i = t_{k_i}(\Delta^*,b)$, $t \leq \tau_1 < \tau_2 \leq t+s$. A brief inspection of problem of the principal's optimization problem (see above) reveals that this would in turn imply that $\Delta^* = \Delta_1 \cup \Delta_2 \cup \Delta_3$, where $\Delta_1 \in \mathcal{D}^*_{\tau_1+1,t}(b)$, $\Delta_2 \in \mathcal{D}^*_{\tau_2-\tau_1,t+\tau_1+1}(b)$, and $\Delta_3 \in \mathcal{D}^*_{s-1-\tau_2,\tau_2+1}(b)$. But this would contradict $\Delta^* \in \mathcal{D}^*_{s+1,t}(b)$: for $(\mathbf{H_{\tau_2+1}})$ implies that $(\Delta_1 \cup \Delta_2) \notin \mathcal{D}^*_{\tau_2+1,t}(b)$; so the principal could reduce his expected loss by replacing Δ^* with $(\Delta_1' \cup \Delta_3)$, where $\Delta_1' \in \mathcal{D}^*_{\tau_2+1,t}(b)$.

In sum, there is a unique k such that the agent strictly prefers $d_{t+\tau-1}(\Delta^*,b)$ over $d_{t+\tau}(\Delta^*,b)$ in state $t+\tau-1 = t_k(\Delta^*,b) \geq 1$. Consequently, there is $\Delta_1^* \in \mathcal{D}^*_{\tau,t}(b)$ and $\Delta_2^* \in \mathcal{D}^*_{s-\tau+1,t+\tau}(b)$ such that $\Delta^* = \Delta_1^* \cup \Delta_2^*$, and

$$d_{t+\tau-1}\left(\Delta_1^*,b\right) + d_{t+\tau}\left(\Delta_2^*,b\right) = d_{t+\tau-1}(\Delta^*,b) + d_{t+\tau}(\Delta^*,b) > 2\left(b + \frac{t+\tau-1}{T-1}\right) \ . \quad (9)$$

Lemma 2 implies that there are integers $K \leq \tau-1$ and $K' \leq s-\tau$ such that

$$d_{t+\tau-1}\left(\Delta_1^*,b\right) = \begin{cases} \ell^e(b,\tau,K) + \frac{t+K}{T-1} & \text{if } K \text{ is even,} \\ 2b - \frac{1}{T-1} - \ell^o(b,\tau,K) + \frac{t+K}{T-1} & \text{if } K \text{ is odd,} \end{cases} \ , \quad (10)$$

$$d_{t+\tau}\left(\Delta_2^*,b\right) = \begin{cases} \ell^e(b,s-\tau+1,K') + \frac{t+\tau}{T-1} & \text{if } K' \text{ is even} \\ \ell^o(b,s-\tau+1,K') + \frac{t+\tau}{T-1} & \text{if } K' \text{ is odd} \end{cases} \ , \quad (11)$$

and

$$b \geq \max\left\{\frac{\tau-K}{2(T-1)}, \frac{s-\tau+1-K'}{2(T-1)}\right\} \ . \quad (12)$$

[31] Note that we have ignored the case in which $K = s-1$ and $b < b^{\min}(K,s)$, since it implies $b < 1/2(T-1)$.

To obtain the desired contradiction, we will use (12) to prove that (10) and (11) are inconsistent with (9). Before we proceed any further, however, the following observations are worth making. First, $b \geq 1/2(T-1)$ implies that, for any s and any K,

$$2b - \frac{1}{T-1} - \ell^o(b,s,K) - \ell^e(b,s,K) = \frac{1}{s}\left(b - \frac{1}{2(T-1)}\right) \geq 0 \ .$$

Second, $b \geq (s-K)/2(T-1)$ implies that, for any s and any K,

$$\ell^e\left(b,s,K\right) - \ell^o\left(b,s,K\right) \geq \frac{2(K-s)+1}{s}\frac{s-K}{2(T-1)} - \frac{1}{2s(T-1)} + \frac{(s-K)^2}{s(T-1)} = 0 \ .$$

Combining these two observations with (10) and (11), we obtain

$$d_{t+\tau-1}\left(\Delta_1^*\right) + d_{t+\tau}\left(\Delta_2^*\right) \leq 2b - \frac{1}{T-1} - \ell^o(b,\tau,K) + \ell^e\left(b, s-\tau+1, K'\right) + \frac{2t+\tau+K}{T-1} \ .$$

Hence,

$$\begin{aligned}\Upsilon(b) &\equiv 2\left(b + \frac{t+\tau-1}{T-1}\right) - \left[d_{t+\tau-1}\left(\Delta_1^*, b\right) + d_{t+\tau}\left(\Delta_2^*, b\right)\right] \\ &\geq \frac{\tau-K-1}{T-1} + \ell^o(b,\tau,K) - \ell^e\left(b, s-\tau+1, K'\right) \ .\end{aligned}$$

Tedious calculations (available in the online Appendix) reveal that the right-hand side of the above inequality is nonnegative when (12) is true. This implies that $\Upsilon(b) \geq 0$, contrary to (9). As $t \leq T-s$ was chosen arbitrarily, this proves that the chain cannot break when the agent is offered an element of $\mathcal{D}^*_{s+1,t}(b)$, for every $t \leq T-s$.

Proof of Lemma 4

Let $\Delta = (\delta_t, \ldots, \delta_{t+K}) \in \mathcal{D}^*_{s,t}(b)$ satisfy the chain property. First of all, in order to simplify notation, we normalize t to 0. Thus, $\Delta = \{\delta_0, \ldots, \delta_K\}$. Moreover, in what follows we will indulge in a slight abuse of notation and define T_k and t_k as follows:

$$\begin{aligned}T_k &\equiv \{\tau \in \mathbf{T}_{s,t} : d_\tau\left(\Delta, b\right) = \delta_k\} \ , \\ t_k &\equiv \max\{\tau : \tau \in T_k\} \ .\end{aligned}$$

for each $k = 0, \ldots, K$.

Any delegation set is trivially top loaded if $s \leq 2$; so suppose that $s > 2$.

Suppose that

$$\delta_{k+1} < b + \frac{t_k+1}{T-1} \tag{13}$$

for every $k < K$. If Δ is not top loaded then there is $k < K$ such that $|T_k| > 1$. By (13), this implies that $\delta_k < b + \frac{t_k-1}{T-1}$. As Δ satisfies the chain property, this in turn implies that

$$\delta_{k+1} = 2b + \frac{2t_k}{T-1} - \delta_k > b + \frac{t_k+1}{T-1}:$$

contrary to (13).

What remains to be proved, therefore, is that (13) is true. We do so with a series of claims.

Claim 1: $\delta_{k+1} \leq b + \frac{t_k+1}{T-1}, \forall k < K$.

Let $k < K$ and suppose, contrary to Claim 1, that $\delta_{k+1} > b + \frac{t_k+1}{T-1}$. This implies that there is an integer $q \geq 1$ such that

$$b + \frac{t_k + q}{T-1} < \delta_{k+1} \leq b + \frac{t_k + q + 1}{T-1} \ .$$

Combining this with the chain property:

$$\delta_k = 2\left(b + \frac{t_k}{T-1}\right) - \delta_{k+1} \ ,$$

we obtain

$$b + \frac{t_k - q - 1}{T-1} \leq \delta_k < b + \frac{t_k - q}{T-1} \ .$$

We need to consider two cases separately:

• *Case 1: Either* $k \neq K-1$ *or* $t_{K-1} + q \leq s - 1$. Let $\upsilon > 0$ be such that the agent is indifferent between δ_{k+1} and $\delta_{k+1} - \upsilon$ in state $t_k + q$. Note that, by construction, this implies that

$$\delta_k = b + \frac{t_k - q}{T-1} - \frac{\upsilon}{2} \ , \tag{14}$$

and that the agent is indifferent between δ_k and $\delta_k + \upsilon$ in state $t_k - q$.

Suppose now that the principal deviates from Δ to $\Delta \cup \{\delta_k + \upsilon, \delta_{k+1} - \upsilon\}$. The agent takes the same decisions in states $\{t : t \leq t_k - q \text{ or } t \geq t_k + q + 1\}$ as before the principal's deviation; now takes decision $\delta_k + \upsilon$ in states $\{t : t_k - q + 1 \leq t \leq t_k\}$; and now takes decision $\delta_{k+1} - \upsilon$ in states $\{t : t_k + 1 \leq t \leq t_k + q\}$.

We now decompose the change in the principal's expected loss after his deviation. First, the change in loss in states t_k and $t_k + q$ is given by

$$\left(\delta_k + \upsilon - \frac{t_k}{T-1}\right)^2 - \left(\delta_k - \frac{t_k}{T-1}\right)^2 + \left(\delta_{k+1} - \upsilon - \frac{t_k + q}{T-1}\right)^2 - \left(\delta_{k+1} - \frac{t_k + q}{T-1}\right)^2$$

$$= 2\upsilon\left(\upsilon + \delta_k - \delta_{k+1} + \frac{q}{T-1}\right) = 2\upsilon\left[2\left(\delta_k - b - \frac{t_k}{T-1} + \frac{\upsilon}{2}\right) + \frac{q}{T-1}\right] = -\frac{2\upsilon q}{T-1} < 0 \ ,$$

where the third equality follows from (14).

The change in the principal's loss in the other states is given by

$$\sum_{i=1}^{q-1}\left\{\left(\delta_k + \upsilon - \frac{t_k - i}{T-1}\right)^2 - \left(\delta_k - \frac{t_k - i}{T-1}\right)^2 + \left(\delta_{k+1} - \upsilon - \frac{t_k + i}{T-1}\right)^2 - \left(\delta_{k+1} - \frac{t_k + i}{T-1}\right)^2\right\} \ .$$

For each $i = 1, \ldots, q-1$, the bracketed term is equal to

$$2\upsilon\left(\upsilon + \delta_k - \delta_{k+1} + \frac{2i}{T-1}\right) = 2\upsilon\left[2\left(\delta_k - b - \frac{t_k}{T-1} + \frac{\upsilon}{2}\right) + \frac{2i}{T-1}\right] = -\frac{4\upsilon(q-i)}{T-1} < 0 \ .$$

As a consequence, the deviation from Δ is strictly profitable to the principal: a contradiction to Δ being an ODS.

• *Case 2: $k = K-1$ and $t_{K-1} + q > s-1$.* Define $\tilde{v}$ such that the agent is indifferent between δ_K and $\delta_K - \tilde{v}$ in state $s-1$. Let $\tilde{q} \equiv s - t_{K-1} - 1$. By construction, the agent is indifferent between δ_k and $\delta_k + \tilde{v}$ in state $t_{K-1} - \tilde{q}$. We can then repeat the same argument as in Case 1 — just substitute $\tilde{q}$ and $\tilde{v}$ for q and v, respectively — to obtain the same contradiction.

Claim 2: If $\delta_{k+1} \leq b + \frac{t_k+1}{T-1}, \forall k < K$ then $\delta_{k+1} < b + \frac{t_k+1}{T-1}, \forall k < K$.

Suppose that there is k such that

$$\delta_{k+1} = b + \frac{t_k + 1}{T-1} \ . \tag{15}$$

To obtain a contradiction, we will first prove that (15) implies that

$$\delta_l = b + \frac{t_l - 1}{T-1} \tag{16}$$

for every $0 \leq l < K$. The chain property implies that

$$\delta_k = 2b + \frac{2t_k}{T-1} - \delta_{k+1} = b + \frac{t_k - 1}{T-1} \ .$$

From Claim 1, this in turn implies that $t_k - 1$ is the smallest state in which the agent takes δ_k. Indeed, if she took decision δ_k in state $t < t_k - 1$ then

$$\delta_k = b + \frac{t_k - 1}{T-1} > b + \frac{t}{T-1} \geq b + \frac{t_{k-1} + 1}{T-1} \ ,$$

contrary to Claim 1; and if $t_k = t_{k-1} + 1$ then $\delta_{k-1} = b + \frac{t_{k-1}}{T-1}$, contrary to the chain property.

Consequently, $t_{k-1} = t_k - 2$ and the agent is indifferent between δ_{k-1} and δ_k in state $t_k - 2$. This in turn implies that

$$\delta_{k-1} = 2b + \frac{2t_{k-1}}{T-1} - \delta_k = b + \frac{t_{k-1} - 1}{T-1} \leq b + \frac{t_{k-2} + 1}{T-1} \ ,$$

where the last inequality follows from Claim 1.

We can then proceed recursively to obtain (16) for $0 \leq l \leq k$. Furthermore,

$$\delta_{k+1} = b + \frac{t_k + 1}{T-1} = 2b + \frac{2t_{k+1}}{T-1} - \delta_{k+2}$$

implies that

$$\delta_{k+2} = b + \frac{2t_{k+1} - t_k - 1}{T-1} \leq b + \frac{t_{k+1} + 1}{T-1} \ ,$$

so that $t_{k+1} \leq t_k + 2$. But we must have $t_{k+1} = t_k + 2$ for $k < K-1$: for if not, then $t_{k+1} = t_k + 1$. This would imply that the agent takes her ideal decision, $b + \frac{t_k+1}{T-1} = \delta_{k+1}$ in state t_{k+1}, and could therefore not be indifferent between δ_{k+1} and δ_{k+2}, contrary to the chain property. Thus,

$$\delta_{k+2} = b + \frac{t_{k+1}+1}{T-1} = 2b + \frac{2t_{k+2}}{T-1} - \delta_{k+3} \ .$$

We can then proceed recursively to obtain (16) for $k \leq l \leq K$.

Combined with the chain property, (15) also implies that $|T_k| = 2$ for $1 \leq k \leq K-1$, and $|T_k| \leq 2$ for $k = 0, K$. We then distinguish between two cases:

• *Case 1:* $|T_0| = 1$. In this case, the agent takes $\delta_0 = b - \frac{1}{T-1}$ in state 0; takes decision $\delta_i = b + \frac{2i-1}{T-1}$ in states $2i-1$ and $2i$: for every $0 < i < K$; and takes $\delta_K = b + \frac{2K-1}{T-1}$ at every state in T_K.

Note, also, that $b \geq \frac{1}{T-1}$ — else $\delta_0 < 0$, and the principal could improve on Δ by raising δ_0 — and that K must be odd.

If $|T_K| = 1$ then $s = 2K$ and the expected loss incurred by the principal is

$$\lambda(\Delta, b) \equiv K \left[b^2 + \left(b - \frac{1}{T-1} \right)^2 \right] \ .$$

Consider the delegation set $\Delta^* \equiv \left\{ \delta_0^*, \ldots, \delta_{s-1}^* \right\}$, where

$$\delta_t^* \equiv b + \frac{2t-1}{2(T-1)} : \forall t = 0, \ldots, s-1 \ .$$

It is easy to check that the loss incurred by the principal when he chooses Δ^* is

$$\lambda(\Delta^*, b) \equiv s \left[b - \frac{1}{2(T-1)} \right]^2 = \lambda(\Delta, b) - \frac{s}{4(T-1)^2} < \lambda(\Delta, b) \ ,$$

contrary to the supposition that Δ is an ODS.

If $|T_K| = 2$ then suppose that the principal deviates to $\Delta'_\varepsilon = \{\delta'_0, \ldots, \delta'_K\}$, where $\delta'_i = \delta_i + \varepsilon$ if i is even, $\delta'_i = \delta_i - \varepsilon$ if i is odd, and $\varepsilon > 0$. The change in the principal's loss is then $\varepsilon\left[(2K+1)\varepsilon - 2b\right]$, which is negative for every small enough ε: a contradiction.

• *Case 2:* $|T_0| = 2$. In this case, the agent takes decision $\delta_i = b + \frac{2i}{T-1}$ in states $2i$ and $2i+1$: for every $i = 0, \ldots, K-1$; and she takes decision $\delta_K = b + \frac{2K}{T-1}$ at states in T_K. Moreover, K must be even.

If $|T_K| = 1$ then the change in the principal's loss if he deviates to Δ'_ε (as defined above) is $\varepsilon\left[(2K+1)\varepsilon - 2b\right]$, which is negative if ε is small enough.

If $|T_K| = 2$ then $s = 2(K+1)$ and the principal's loss is

$$\lambda(\Delta, b) = (K+1) \left[b^2 + \left(b - \frac{1}{T-1} \right)^2 \right] .$$

It is easy to check that a principal who offers Δ^* (as defined above) loses

$$\lambda(\Delta^*, b) \equiv s \left[b - \frac{1}{2(T-1)} \right]^2 = \lambda(\Delta, b) - \frac{s}{4(T-1)^2} < \lambda(\Delta, b) \ :$$

a contradiction. This completes the proof of Claim 2.

Combining Claims 1 and 2, we obtain (13).

Proof of Theorem 1

If $b \leq \frac{1}{2(T-1)}$, the result is obvious. Suppose $b > \frac{1}{2(T-1)}$. Part (i) is a direct consequence of $(\mathbf{H_2})$ and Lemmas 2-4. Note, however, that more is needed to establish parts (ii) and (iii) because Lemma 2 only establishes necessity.

We know from Lemma 1 that there is an ODS for every $b > \frac{1}{2(T-1)}$. Let Δ be an ODS. Define $\Delta^*(b, K)$ as the unique solution to the principal's minimization problem in (7): that is, $\Delta^*(b, K)$ is optimal among the delegation sets of cardinality $(K+1)$ satisfying the chain property and top loading.

Suppose, first, that $b \in \left(b^{\min}(K,T), b^{\max}(K,T)\right)$, but that Δ contains $K'+1$ decisions where $K' \neq K$. From Lemma 2, this implies that $b \in \left[b^{\min}(K',T), b^{\max}(K',T)\right]$: a contradiction. Consequently, Δ must be equal to $\Delta^*(b, K)$.

Now suppose that $b = b^{\min}(K,T)$. Applying Lemma 2, an ODS must contain either K or $K+1$ decisions. The computations in the proof of Lemma 2 reveal that, when $b = b^{\min}(K,T)$, $\Delta^*(b,K)$ and $\Delta^*(b,K-1)$ yield the same expected loss to the principal. Therefore, $\mathcal{D}_T^*(b)$ consists of $\Delta^*(b,K-1)$ and $\Delta^*(b,K)$. A parallel argument establishes the result when $b = b^{\max}(K,T)$.

APPENDIX B: PROOFS OF PROPOSITIONS AND COROLLARIES

Proof of Corollary 1

Let $\Delta \in \mathcal{D}_T^*(b)$, and let $|\Delta|-1 = K$. Lemma 2 implies that $b \in \left[b^{\min}(K,T), b^{\max}(K,T)\right]$.

(i) Necessity is easy to establish: If $b \leq \frac{1}{2(T-1)}$ then the first-best decision rule is incentive compatible, so $\Delta \in \mathcal{D}_T^*(b)$ implies that

$$d_t(\Delta, b) = \frac{t}{T-1}\ , \forall t \in \mathbf{T}\ .$$

To establish sufficiency, suppose that $b > \frac{1}{2(T-1)}$. Lemma 2 implies that we have to show that $\ell\left(b,T,K\right) > 0$ if t is even, and $2b - \ell\left(b,T,K\right) > \frac{1}{T-1}$ if t is odd. We must distinguish between four cases:

• Case 1: *t even and K even.* Since ℓ^e is strictly increasing in b $(K < T)$,

$$\ell^e\left(b,T,K\right) > \ell^e\left(b^{\min}(K,T),T,K\right) = \frac{T-K-1}{2(T-1)}$$

for all $b > b^{\min}(K,T)$. If $K < T-1$, we directly obtain the result from the equation equation because $b \geq b^{\min}(K,T)$. If $K = T-1$ then $b^{\min}(K,T) = \frac{1}{2(T-1)}$; so $b > b^{\min}(K,T)$. Thus, $\ell^e\left(b,T,K\right) > \ell^e\left(b^{\min}(K,T),T,K\right) = 0$.

• Case 2: *t odd and K even.* Since $2b - \ell^e(b,T,K)$ is strictly increasing in b,

$$2b - \ell^e\left(b,T,K\right) > 2b - \ell^e\left(b^{\min}(K,T),T,K\right) = \frac{T-K-1}{2(T-1)} + \frac{1}{T-1}$$

for all $b > b^{\min}(K,T)$. The argument is then the same as in Case 1.

• Case 3: *t even and K odd.* Since ℓ^o is strictly increasing in b,

$$\ell^o(b,T,K) > \ell^o\left(b^{\min}(K,T),T,K\right) = \frac{T-K-1}{2T}$$

for all $b > b^{\min}(K,T)$. The argument is then the same as in Case 1.

• Case 4: *t odd and K odd.* Since $2b - \ell^o(b,T,K)$ is strictly increasing in b,

$$2b - \ell^o(b,T,K) > 2b - \ell^o\left(b^{\min}(K,T),T,K\right) = \frac{(T-K-1)(T+1)}{2(T-1)} + \frac{1}{T-1}$$

for all $b > b^{\min}(K,T)$. The argument is then the same as in Case 1.

(ii) Let $\Delta = \{\delta_0, \ldots, \delta_K\}$. For every $t \in \mathbf{T}$, there is k such that $t \in T_{k+1}(\Delta, b)$. Claims 1 and 2 in the proof of Lemma 4 then imply that

$$d_t(\Delta,b) = \delta_{k+1} < b + \frac{t_k(\Delta,b)+1}{T-1} \le b + \frac{t}{T-1} \ .$$

(iii) This follows immediately from the requirement that the agent be indifferent between δ_0 and δ_1 in state 0.

(iv) The condition in the premise implies that the ODS contains T decisions, the largest of which is 1 if $b = 1/2(T-1)$, and must offer a larger decision when the bias is larger, but small enough that the ODS contains T decisions. The agent must be indifferent between the two highest decisions in state $\frac{T-2}{T-1}$ for bias in this range; so $d_{T-2}(\Delta,b) < b + \frac{T-2}{T-1} < 1$.

Now suppose that $b > 1/(T-1)$, so $K < T-1$. The second highest decision in the ODS is then bounded above by the limit of δ_{K-1} as b approaches $\frac{T-K+1}{2(T-1)}$. We can show that the principal expects to lose less if the agent takes decision δ_{K-1} than if she takes any decision above 1, conditional on the state being at least $K/(T-1)$. Consequently, the maximal decision in an ODS must be less than 1 whenever $b > 1/(T-1)$.

Proof of Proposition 1

We distinguish between two cases:

(a) $b > 1/2$. In this case, we have $b > \frac{1}{2} = \lim_{T\to\infty} \frac{T}{2(T-1)}$. Consequently, $b > T/2(T-1)$ for large enough T. Theorem 1 then implies that there is a unique ODS, namely $\{1/2\}$, as in Holmstrom's model.

(b) $b \in (0, 1/2]$. As $T/2(T-1)$ is a strictly decreasing function that converges to $1/2$ as $T \to \infty$, we have

$$b \le \frac{1}{2} < \frac{T}{2(T-1)} \ , \forall T \in \mathbb{N} \ . \tag{17}$$

For every $T \in \mathbb{N}$, let

$$\widetilde{K}(T) \equiv \left\{ K \in \mathbb{N} : K \le T-1 \text{ and } \frac{T-K}{2(T-1)} \le b \le \frac{T-K+1}{2(T-1)} \right\} \ .$$

Lemma 2, Theorem 1, and (17) imply that, for every T and every $\Delta_T^*(b) \in \mathcal{D}_T^*(b)$, there exists $K_T \in \widetilde{K}(T)$ such that $\Delta_T^*(b)$ is of the form $\Delta_T^*(b) = \{\delta_0, \ldots, \delta_{K_T}\}$, and

$$\delta_\tau = \begin{cases} \ell\,(b, T, K_T) + \frac{\tau}{T-1} & \text{if } \tau \text{ is even}, \\ 2b - \ell\,(b, T, K_T) + \frac{\tau-1}{T-1} & \text{if } \tau \text{ is odd}, \end{cases}$$

where ℓ is defined before Lemma 2 in Section 2.3.

We want to prove that each $\Delta_T^*(b) \in \mathcal{D}_T^*(b)$ becomes arbitrarily close to $\Delta_\infty^*(b)$ as $T \to \infty$. To do so, we will decompose $\mathcal{D}_T^*(b)$ as follows: $\mathcal{D}_T^*(b) = \mathcal{D}_T^e(b) \cup \mathcal{D}_T^o(b)$, where $\mathcal{D}_T^e(b)$ [resp. $\mathcal{D}_T^o(b)$] is the class of ODSs $\Delta_T^*(b) = \{\delta_0, \ldots, \delta_{K_T}\}$ such that K_T is even [resp. odd].

Suppose, first, that $\Delta_T^*(b) \in \mathcal{D}_T^e(b)$ (i.e., K_T is even). Let

$$E^T \equiv \left\{0, \frac{2}{T-1}, \frac{4}{T-1}, \ldots, \frac{K_T-2}{T-1}, \frac{K_T}{T-1}\right\} ,$$
$$O^T \equiv \left\{\frac{1}{T-1}, \frac{3}{T-1}, \ldots, \frac{K_T-3}{T-1}, \frac{K_T-1}{T-1}\right\} ,$$

and note that we can express $\Delta_T^*(b)$ as

$$\Delta_T^*(b) = \left(\{\ell^e\,(b, T, K_T)\} + E^T\right) \bigcup \left(\left\{2b - \ell^e\,(b, T, K_T) - \frac{1}{T-1}\right\} + O^T\right) . \tag{18}$$

Since $K_T \in \widetilde{K}(T)$, we have

$$\frac{T - K_T}{2(T-1)} \le b \le \frac{T - K_T + 1}{2(T-1)}$$

and therefore

$$T - 2(T-1)b \le K_T \le T - 2(T-1)b + 1 . \tag{19}$$

This in turn implies that

$$\frac{T - 2(T-1)b}{T-1} \le \frac{K_T}{T-1} \le \frac{T - 2(T-1)b + 1}{T-1} .$$

Since both the left-hand and right-hand sides of this inequality converge to $1 - 2b$, we obtain that $K_T/(T-1)$ and $(K_T - 1)/(T-1)$ converge to $1 - 2b$ as $T \to \infty$. This in turn implies that both E^T and O^T converge to the interval $[0, 1 - 2b]$ as $T \to \infty$.

The definition of ℓ^e and (19) imply that

$$\frac{T - 2(T-1)b}{T} b \le \frac{K}{T} b \le \frac{T - 2(T-1)b + 1}{T} b ,$$

where the left-hand and right-hand sides converge to $(1 - 2b)b$ as T becomes arbitrarily large. This proves that $\frac{K_T}{T} b$ also converges to $(1 - 2b)b$ as $T \to \infty$. Using (19) again, we obtain

$$\frac{[2(T-1)b - 1]^2}{2T(T-1)} \le \frac{(T - K_T)^2}{2T(T-1)} \le \frac{[2(T-1)b]^2}{2T(T-1)}$$

where the left-hand and right-hand side both approach $4b^2/2 = 2b^2$. This proves that $\frac{(T-K_T)^2}{2T(T-1)}$ converges to $2b^2$ as $T \to \infty$.

Thus, $\ell^e(b, T, K_T)$, and therefore $2b - \ell^e(b, T, K_T)$, converge to b as $T \to \infty$. By (18), the limit of $\Delta^*_T(b)$ can then be expressed as

$$(\{b\} + [0, 1-2b]) \cup (\{b\} + [0, 1-2b]) \ .$$

We have therefore proved that every $\Delta^*_T(b) \in \mathcal{D}^e_T(b)$ converges to $[b, 1-b]$ — which is equal to $\Delta^*_\infty(b)$ — as $T \to \infty$.

A parallel argument shows that every $\Delta^*_T(b) \in \mathcal{D}^o_T(b)$ converges to $\Delta^*_\infty(b)$ as $T \to \infty$. This proves that $\lim_{T\to\infty} \Delta^*_T(b) = \Delta^*_\infty(b)$ for all $\Delta^*_T(b) \in \mathcal{D}^*_T(b)$.

Proof of Proposition 4

Suppose, first, that $b \geq s/2(T-1)$; so an ODS for $\mathbf{T}_{s,t}$ contains a single decision. By optimality, this implies that $\lambda(s,b) = \lambda^n(s)$, and therefore that $V(s,b) = 0$. As $V \geq 0$, this in turn implies that $V(s+2\upsilon, b) \geq V(s,b)$. This inequality is strict if, in addition,

$$b < \frac{s+2\upsilon}{2(T-1)} = b^{\min}(0, s+2\upsilon) :$$

for Lemma 2 then implies that $\Delta \in \mathcal{D}^*_{s+2\upsilon,t-\upsilon}(b)$ contains at least two elements. Consequently, $\lambda(s+2\upsilon, b) < \lambda^n(s+2\upsilon)$; so

$$V(s+2\upsilon, b) \equiv \lambda^n(s+2\upsilon) - \lambda(s+2\upsilon, b) > 0 = V(s,b) \ .$$

Now suppose that $b < s/2(T-1)$. This implies that there is $K_0 \geq 1$ such that

$$\frac{s-K_0}{2(T-1)} \leq b < \frac{s-K_0+1}{2(T-1)} \ .$$

If the state is distributed according to $p_{s,t}$ then there is a generally unique ODS that contains K_0+1 decisions. Furthermore, if the state is distributed according to $p_{s+2\upsilon,t-\upsilon}$ then there is a generically unique ODS containing $K_{2\upsilon}+1 \equiv K_0+2\upsilon+1$ decisions because

$$b \in \left[b^{\min}(K_0, s), b^{\max}(K_0, s)\right) = \left[b^{\min}(K_{2\upsilon}, s+2\upsilon), b^{\max}(K_{2\upsilon}, s+2\upsilon)\right) \ .$$

To establish the Proposition, we have to show that

$$V(s+2\upsilon) - V(s) \equiv \lambda^n(s+2\upsilon) - \lambda^n(s) + \lambda(s,b) - \lambda(s+2\upsilon, b) > 0 \ . \tag{20}$$

It is easy to see that the change in λ^n is

$$\lambda^n(s+2\upsilon) - \lambda^n(s) = \frac{(s+2\upsilon)^2 - 1 - \left(s^2-1\right)}{12(T-1)^2} = \frac{4\upsilon(s+\upsilon)}{12(T-1)^2} \ ; \tag{21}$$

but we have to distinguish between two cases to determine the change in λ.

Suppose, first, that K_0 — and therefore $K_{2\upsilon}$ — is even. For any $\Delta \in \mathcal{D}^*_{s,t}(b)$, we can apply Lemma 2 to obtain

$$\begin{aligned}
s\lambda(s,b) &\equiv \sum_{\tau=t}^{t+s-1}\left(d_\tau(\Delta,b)-\frac{\tau}{T-1}\right)^2 \\
&= \frac{K_0}{2}\ell^e(b,s,K_0)^2+\frac{K_0}{2}\left[2b-\ell^e(b,s,K_0)-\frac{1}{T-1}\right]^2+\sum_{\tau=0}^{s-1-K_0}\left[\ell^e(b,s,K_0)-\frac{\tau}{T-1}\right]^2 \\
&= \frac{K_0}{2}\ell^e(b,s,K_0)^2+\frac{K_0}{2}\left[2b-\ell^e(b,s,K_0)-\frac{1}{T-1}\right]^2+\frac{(2s-2K_0-1)(s-K_0-1)(s-K_0)}{6(T-1)^2} \\
&+(s-K_0)\left[\ell^e(b,s,K_0)-\frac{s-K_0-1}{T-1}\right]\ell^e(b,s,K_0)\ ,
\end{aligned}$$

where $\ell^e(b,s,K_0)$ is defined before Lemma 2. Analogously, for any $\Delta \in \mathcal{D}^*_{s+2\upsilon,t-\upsilon}(b)$:

$$\begin{aligned}
(s+2\upsilon)\lambda(s+2\upsilon,b) &\equiv \sum_{\tau=t-\upsilon}^{t+s+\upsilon-1}\left(d_\tau(\Delta,b)-\frac{\tau}{T-1}\right)^2 \\
&= \frac{K_{2\upsilon}}{2}\ell^e(b,s+2\upsilon,K_{2\upsilon})^2+\frac{K_{2\upsilon}}{2}\left[2b-\ell^e(b,s+2\upsilon,K_{2\upsilon})-\frac{1}{T-1}\right]^2 \\
&+\frac{(2s+4\upsilon-2K_{2\upsilon}-1)(s+2\upsilon-K_{2\upsilon}-1)(s+2\upsilon-K_{2\upsilon})}{6(T-1)^2} \\
&+(s+2\upsilon-K_{2\upsilon})\left[\ell^e(b,s+2\upsilon,K_{2\upsilon})-\frac{s+2\upsilon-K_{2\upsilon}-1}{T-1}\right]\ell^e(b,s+2\upsilon,K_{2\upsilon})\ .
\end{aligned}$$

Substituting out for ℓ^e and rearranging terms, we obtain that

$$\lambda(s)-\lambda(s+2\upsilon) = -\frac{m(b,s,K_0,\upsilon)}{12s^2(T-1)^2(s+2\upsilon)^2}\ , \tag{22}$$

where

$$\begin{aligned}
m(b,s,K_0,\upsilon) \equiv\ & 2(s-K_0)\upsilon\left\{24(s-K_0)(T-1)^2(s+\upsilon)b^2\right. \\
&-12(T-1)\left[2K_0^2(s+\upsilon)-K_0s(3s+2\upsilon)+s\left(s+s^2+2\upsilon\right)\right]b \\
&\left.-2s\left[K_0s(5s+\upsilon)-K_0^2(7s+5\upsilon)+s\left(s\upsilon-s^2-2\right)-4\upsilon\right]\right\}\ .
\end{aligned}$$

Combining (21) and (22), we obtain that (20) is satisfied if and only if

$$M(b,s,K_0,\upsilon) \equiv 4\upsilon(s+\upsilon)s^2(s+2\upsilon)^2 - m(b,s,K_0,\upsilon) > 0\ .$$

To establish the above inequality, we first prove that M is decreasing in b:

$$\begin{aligned}\frac{\partial M\left(b, s, K_0, v\right)}{\partial b} &= -24\left(s-K_0\right)(T-1) v\left\{4\left(s-K_0\right)(T-1)(s+v) b\right. \\ &\quad \left.-2 K_0^2(s+v)+K_0 s(3 s+2 v)-s\left(s+s^2+2 v\right)\right\} \\ &\leq -24\left(s-K_0\right)(T-1) v\left\{2\left(s-K_0\right)^2(s+v)-2 K_0^2(s+v)\right. \\ &\quad \left.+K_0 s(3 s+2 v)-s\left(s+s^2+2 v\right)\right\} \\ &= -24\left(s-K_0\right)(T-1) v s\left(s-K_0-1\right)(s+2 v) \leq 0\ ,\end{aligned}$$

where the first inequality follows from $b \geq \frac{s-K_0}{2(T-1)}$. This implies that

$$\begin{aligned} M\left(b, s, K_0, v\right) &\geq M\left(\frac{s-K_0+1}{2(T-1)}, s, K_0, v\right) \\ &= 4 v\left\{K_0\left[3\left(s-K_0\right)\left(s^2+s+v+2 s^2 v\right)+s\left(K_0^2 s+2 s+2 K_0^2 v+v\right)\right]\right. \\ &\quad \left.+\, s^2\left[3 s^2 v+(2 s+v)\left(4 v^2-1\right)\right]\right\}>0\end{aligned}$$

since $K_0 \leq s-1$ and $v \geq 1$. This inequality establishes the result for the K_0-even case. One can establish the result for the K_0-odd case in like manner (see the online Appendix for a detailed proof), thus completing the proof of sufficiency in the Proposition.

To show necessity, observe that there exist $\Delta_1 \in \mathcal{D}_{s,t}^*$ and $\Delta_2 \in \mathcal{D}_{s+2v,t-v}^*$, both of which contain a single decision, when $b \geq \frac{s+2v}{2(T-1)}$. This implies that $\lambda(s,b) = \lambda^n(s)$ and $\lambda(s+2v,b) = \lambda^n(s+2v)$, which in turn implies that $V(s,b) = V(s+2v,b) = 0$.

Proof of Proposition 5(ii)

Suppose, first, that $\frac{1}{T-1} < b < \frac{1}{2}$, and let $\Delta = \{\delta_0, \ldots, \delta_K\} \in \mathcal{D}_T^*(b)$. Theorem 1 implies that $K > 1$.

Let E denote the event $\{K-2, K-1\}$, and consider an amateur who can distinguish between every state except for those in E. Define δ_E as the decision which leaves this amateur indifferent with δ_K in event E: viz.

$$\delta_E \equiv 2b + \frac{2T-3}{T-1} - \delta_K\ .$$

For fixed T, we need to distinguish between cases in which K is odd and K is even.

• If K is even (so $K-1$ is odd) then, by Theorem 1,

$$\begin{aligned}\delta_{K-2} &= \frac{K}{T} b+\frac{K^2+T^2-5 T}{2 T(T-1)}\ ,\ , \\ \delta_{K-1} &= \frac{2 T-K}{T} b-\frac{K^2-4 K T+T^2+3 T}{2 T(T-1)} \\ \text{and } \delta_K &= \frac{K}{T} b+\frac{K^2+T^2-T}{2 T(T-1)} ;\end{aligned}$$

so

$$\delta_E = \frac{2T-K}{T}b + \frac{4KT-5T-K^2-T^2}{2T(T-1)} > \delta_{K-2} \ ,$$

where the inequality implies that the agent prefers δ_{K-3} over δ_E in state $K-3$.

If the principal appoints an expert and offers Δ then, in event E, he loses

$$\left(\delta_{K-2} - \frac{K-2}{T-1}\right)^2 + \left(\delta_{K-1} - \frac{K-1}{T-1}\right)^2$$
$$= \left[\frac{K}{T}b + \frac{(T-K)^2-T}{2T(T-1)}\right]^2 + \left[\frac{2T-K}{T}b - \frac{(T-K)^2+T}{2T(T-1)}\right]^2 \ .$$

Now suppose that the principal offers an amateur a delegation set identical to Δ, except that δ_{K-2} and δ_{K-1} are replaced by δ_E. By construction, the amateur takes the same decision as the expert except in event E, where she takes δ_E. The principal's loss in event E is then

$$\left(\delta_E - \frac{K-2}{T-1}\right)^2 + \left(\delta_E - \frac{K-1}{T-1}\right)^2 = \left[\frac{2T-K}{T}b - \frac{(T-K)^2+T}{2T(T-1)}\right]^2 + \left[\frac{2T-K}{T}b - \frac{(T-K)^2+3T}{2T(T-1)}\right]^2 \ .$$

The principal prefers to appoint this amateur and offer her δ_E if and only if

$$\left[\frac{2T-K}{T}b - \frac{(T-K)^2+3T}{2T(T-1)}\right]^2 < \left[\frac{K}{T}b + \frac{(T-K)^2-T}{2T(T-1)}\right]^2 \ . \tag{23}$$

Now, Theorem 1 implies that $b < \frac{T-K+1}{2(T-1)} < \frac{(T-K)^2+T}{2(T-1)(T-K)}$; so

$$\frac{2T-K}{T}b - \frac{(T-K)^2+3T}{2T(T-1)} < \frac{K}{T}b + \frac{(T-K)^2-T}{2T(T-1)} \ . \tag{24}$$

Furthermore,

$$\delta_{K-2} - \frac{K-2}{T-1} = \frac{K}{T}b + \frac{(T-K)^2-T}{2T(T-1)} > 0$$

because $\delta_{K-2} = d_{K-2}(\Delta, b) > \frac{K-2}{T-1}$ (by Corollary 1).

There are two cases to consider. If the left-hand side of (23) is positive then (24) implies that the principal prefers to appoint this amateur and offer her δ_E; and if the left-hand side of (23) is negative then

$$\frac{(T-K)^2+3T}{2T(T-1)} - \frac{2T-K}{T}b < \frac{K}{T}b + \frac{(T-K)^2-T}{2T(T-1)}$$

if and only if $b > 1/(T-1)$. We therefore conclude that the principal prefers to appoint this amateur and offer her δ_E if $b > 1/(T-1)$.

- If K is odd (so $K-1$ is even) then

$$\begin{aligned}\delta_{K-2} &= \frac{K+1}{T}b + \frac{K^2+T^2-5T-1}{2T(T-1)} \ , \\ \delta_{K-1} &= \frac{2T-K-1}{T}b - \frac{K^2+T^2-4KT+3T-1}{2T(T-1)} \ , \\ \delta_K &= \frac{K+1}{T}b + \frac{K^2+T^2-T-1}{2T(T-1)}, \\ \delta_E &= \frac{2T-K-1}{T}b + \frac{4KT-5T-K^2-T^2+1}{2T(T-1)} < \delta_{K-2} \ ,\end{aligned}$$

where the last inequality exploits $b < \frac{T-K+1}{2(T-1)}$, and implies that the agent must take a decision $\delta \geq \delta_{K-2}$ in event E if she continues to take δ_t in states $t < K-2$.

If $\delta = \delta_{K-2}$ then the amateur outperforms the expert if and only if the principal prefers the agent to take δ_{K-2} rather than δ_{K-1} in state $K-1$: that is, when $b > 1/(T-1)$. Now $\frac{1}{T-1} < \frac{T-K}{2(T-1)} = b^{\min}(K,T)$ if and only if $K < T-1$: so the amateur outperforms the expert for every $b \in \left(\frac{1}{T-1}, \frac{1}{2}\right)$.

Now consider cases in which $b \in \left(\frac{1}{2}, \frac{T}{2(T-1)}\right)$: so $K=1$ (by Theorem 1). The expert would take $\frac{2(T-1)}{T}b - \frac{1}{2}$ in state 0, and $\frac{2}{T}b + \frac{1}{2}$ otherwise; so the principal expects to lose

$$\lambda(\Delta, b) = \frac{1}{T}\left\{\left[\frac{2(T-1)}{T}b - \frac{1}{2}\right]^2 + \left[\frac{2}{T}b + \frac{T-3}{2(T-1)}\right]^2 + \sum_{t=2}^{T-1}\left[\frac{t}{T-1} - \frac{2}{T}b - \frac{1}{2}\right]^2\right\} \ .$$

Consider an amateur who can't distinguish between states 0 and 1. A principal who offers this amateur the delegation set

$$\left\{\frac{2(T-2)}{T}b - \frac{T-3}{2(T-1)}, \frac{4}{T}b + \frac{1}{2}\right\}$$

expects to lose

$$\lambda_E \equiv \frac{1}{T}\left\{\left[\frac{2(T-2)}{T}b - \frac{1}{2}\right]^2 + \left[\frac{2(T-2)}{T}b - \frac{T-3}{2(T-1)}\right]^2 + \sum_{t=2}^{T-1}\left[\frac{t}{T-1} - \frac{4}{T}b - \frac{1}{2}\right]^2\right\} \ .$$

Subtracting terms yields

$$\lambda_E - \lambda(\Delta, b) = \frac{4(T-3)}{T^2}b\left[b - \frac{T}{2(T-1)}\right] \ .$$

Consequently, this amateur strictly outperforms the expert if and only if $T > 3$.

APPENDIX C: EXAMPLES

C.1 A three-state example

As mentioned in the main text, the following conclusions rely on the supposition that states are equiprobable:

(i) A sufficiently unbiased agent takes a decision above 1 but *only when the ODS contains T decisions, and then in the highest state* (Corollary 1iv)*;* and

(ii) UP holds in the finite-state model (Proposition 4).

To see this, consider a simple example.

Example Suppose that $\mathbf{T} = \{0, 1, 2\}$, and that states 0 and 2 are each realized with the same probability p. As we show in the online Appendix, any ODS $\Delta(b, p)$ must satisfy

$$\Delta(b,p) = \begin{cases} \left\{0, \frac{1}{2}, 1\right\} & \text{if } b < \frac{1}{4} , \\ \left\{\left(2b - \frac{1}{2}\right)(1-2p), 4bp + \frac{1}{2} - p, 2b(1-2p) + \frac{1}{2} + p\right\} & \text{if } \frac{1}{4} < b < \frac{1}{2} , \\ \left\{2b(1-p) - \frac{1}{2}, 2bp + \frac{1}{2}\right\} & \text{if } \frac{1}{2} < b < \frac{1}{2(1-p)} , \\ \left\{\frac{1}{2}\right\} & \text{if } b > \frac{1}{2(1-p)} . \end{cases}$$

This example shows that statements (i) and (ii) may not be true when $p \neq 1/3$:

(i) If p is close to $1/2$ and $1/2 < b < 1$ then the ODS consists of two decisions, the larger of which is taken in states 1 and 2, and exceeds 1.

(ii) The variance of the state is $2p$. The characterization of $\Delta(b, p)$ above implies that

$$\begin{aligned} \lambda(b) &= p\left[2b(1-p) - \frac{1}{2}\right]^2 + 4b^2p^2(1-2p) + p\left(2bp - \frac{1}{2}\right)^2 \\ &= -4b^2p^2 + 4b^2p - 2bp + \frac{1}{2}p. \end{aligned}$$

Consequently, the value of delegation is $2bp(1 - 2b + 2bp)$, which decreases in p if p is small enough: that is, UP fails for every $b \in \left(\frac{1}{2}, \frac{1}{2(1-p)}\right)$.

C.2 The four-state case

There are six amateurs who can't distinguish between connected states. One of these amateurs can't distinguish between any states, and can therefore not outperform the expert. More generally, an amateur who can't distinguish between a connected set of states which includes 1 can also not outperform the expert because she must take the same decision in every state in that event; and the principal can induce an expert to take a common decision in that event.

Write $\Delta_E(b)$ for the delegation set that the principal offers to an amateur who cannot distinguish between states in some collection of events E, and $\lambda_E(b)$ for the loss that he then makes.

We start with amateurs who can't distinguish between states in a single event (so E is unique). We ignore cases in which $b < 1/6$ because the principal can then achieve first best with an expert.

If $E = \{0, 1\}$ then

	$\Delta_E(b)$	$\lambda_E(b)$	$\Lambda_E(b)$
$\frac{1}{6} < b < \frac{1}{3}$	$b + \frac{1}{6}, b + \frac{1}{2}, b + \frac{5}{6}$	$\frac{1}{2}b^2 - \frac{1}{6}b + \frac{1}{36}$	$\frac{1}{36}$
$\frac{1}{3} < b < \frac{2}{3}$	$b - \frac{1}{6}, b + \frac{1}{2}$	$b^2 - \frac{2}{3}b + \frac{5}{36}$	$\frac{5}{36}$
$b > \frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

If $E = \{1, 2\}$ then

	$\Delta_E(b)$	$\lambda_E(b)$	$\Lambda_E(b)$
$\frac{1}{6} < b < \frac{1}{4}$	$0, \frac{1}{2}, 1$	$\frac{1}{72}$	$b^2 + \frac{1}{72}$
$\frac{1}{4} < b < \frac{1}{2}$	$b - \frac{1}{4}, b + \frac{1}{4}, b + \frac{3}{4}$	$b^2 - \frac{1}{2}b + \frac{11}{144}$	$\frac{11}{144}$
$\frac{1}{2} < b < \frac{2}{3}$	$\frac{3}{2}b - \frac{1}{2}, \frac{1}{2}b + \frac{1}{2}$	$\frac{3}{4}b^2 - \frac{1}{2}b + \frac{5}{36}$	$\frac{1}{4}b^2 - \frac{1}{4}b + \frac{5}{36}$
$b > \frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

If $E = \{2, 3\}$ then

	$\Delta_E(b)$	$\lambda_E(b)$	$\Lambda_E(b)$
$\frac{1}{6} < b < \frac{1}{2}$	$\frac{1}{2}b, \frac{3}{2}b, \frac{1}{2}b + \frac{2}{3}$	$\frac{3}{4}b^2 - \frac{1}{3}b + \frac{1}{18}$	$\frac{1}{4}b^2 + \frac{1}{18}$
$\frac{1}{2} < b < \frac{2}{3}$	$\frac{3}{2}b - \frac{1}{2}, \frac{1}{2}b + \frac{1}{2}$	$\frac{3}{4}b^2 - \frac{1}{2}b + \frac{5}{36}$	$\frac{1}{4}b^2 + \frac{5}{36}$
$b > \frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

If $E = \{0, 1, 2\}$ then

	$\Delta_E(b)$	$\lambda_E(b)$	$\Lambda_E(b)$
$\frac{1}{6} < b < \frac{1}{3}$	$\frac{1}{3}, 1$	$\frac{1}{18}$	$b^2 + \frac{1}{36}$
$\frac{1}{3} < b < \frac{2}{3}$	$\frac{1}{2}b + \frac{1}{6}, \frac{3}{2}b + \frac{1}{2}$	$\frac{3}{4}b^2 - \frac{1}{2}b + \frac{5}{36}$	$\frac{1}{4}b^2 + \frac{5}{36}$
$b > \frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

There are also five amateurs who cannot distinguish between disconnected events: $E = \{0, 2\}, \{0, 3\}, \{1, 3\}, \{0, 1, 3\}$ and $\{0, 2, 3\}$.

If $E = \{0, 2\}$ then the principal would like the agent to take the same decision in E and in state 1, so he offers the same ODS (and makes the same loss) as with an amateur who cannot distinguish between 0, 1 and 2 - for which, see above. Analogously, if $E = \{1, 3\}$ then the principal would like the agent to take the same decision in E and in state 2, so he offers the same delegation set (and makes the same loss) as with an amateur who cannot distinguish between 1, 2 and 3 - who cannot outperform the expert.

If $E = \{0,3\}$ then

	$\Delta_E(b)$	$\lambda_E(b)$	$\Lambda_E(b)$
$\frac{1}{6} < b < \frac{2}{9}$	$\frac{1}{2}b + \frac{1}{2}, \frac{3}{2}b + \frac{1}{6}$	$\frac{3}{4}b^2 - \frac{1}{6}b + \frac{5}{36}$	$\frac{1}{4}b^2 - \frac{1}{3}b + \frac{1}{4}$
$b > \frac{2}{9}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

If $\frac{1}{6} < b < \frac{2}{9}$ then the agent takes $\frac{1}{2}b + \frac{1}{2}$ in state 1, and $\frac{3}{2}b + \frac{1}{6}$ otherwise. In this case, the principal also loses $\lambda_E(b)$ if he offers $\{b + \frac{7}{18}, \frac{3}{2}b + \frac{1}{2}\}$.

Now suppose that $\frac{1}{6} < b < \frac{2}{9}$. If $E = \{0,1,3\}$ then the principal offers $\{b + \frac{7}{18}, \frac{3}{2}b + \frac{1}{2}\}$ and again loses $\lambda_E(b)$; and if $E = \{0,2,3\}$ then the principal offers $\{\frac{3}{2}b + \frac{1}{6}, \frac{1}{2}b + \frac{1}{2}\}$ and again loses $\lambda_E(b)$. If $b > \frac{2}{9}$ then the principal offers the single decision $\frac{1}{2}$ to an amateur who can only distinguish state 1 and to an amateur who can only distinguish state 2.

There are three amateurs who can't distinguish between states in more than one event:

If $E = \{\{0,1\},\{2,3\}\}$ then

	$\Delta_E(b)$	$\lambda_E(b)$	$\Lambda_E(b)$
$\frac{1}{6} < b < \frac{1}{3}$	$\frac{1}{6}, \frac{5}{6}$	$\frac{1}{36}$	$b^2 + \frac{1}{36}$
$\frac{1}{3} < b < \frac{2}{3}$	$b - \frac{1}{6}, b + \frac{1}{2}$	$b^2 - \frac{2}{3}b + \frac{5}{36}$	$\frac{5}{36}$
$b > \frac{2}{3}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

If $E = \{\{0,2\},\{1,3\}\}$ then

	$\Delta_E(b)$	$\lambda_E(b)$	$\Lambda_E(b)$
$\frac{1}{6} < b < \frac{1}{3}$	$b + \frac{1}{6}, b + \frac{1}{2}$	$b^2 - \frac{1}{3}b + \frac{5}{36}$	$\frac{5}{36}$
$b > \frac{1}{3}$	$\frac{1}{2}$	$\frac{5}{36}$	$b^2 + \frac{5}{36}$

Finally, the principal would offer the single decision $\frac{1}{2}$ to an amateur who cannot distinguish between states 0 and 3 or between the other two states.

We now use these calculations to characterize the best amateurs for the principal ($E^P(b)$) and the agent ($E^A(b)$) and the principal's ensuing loss:

	$E^P(b)$	$\lambda_{E^P}(b)$
$\frac{1}{6} < b < \frac{1}{4}$	$\{1,2\}$	$\frac{1}{72}$
$\frac{1}{4} < b < \frac{3}{8}$	$\{1,2\}$	$b^2 - \frac{1}{2}b + \frac{11}{144}$
$\frac{3}{8} < b < \frac{2}{3}$	$\{0,1\}$ and $\{\{0,1\},\{2,3\}\}$	$b^2 - \frac{2}{3}b + \frac{5}{36}$

This table demonstrates that the principal cannot improve on appointing an amateur who can't distinguish between two connected states. However, the best amateur is not uniquely defined when $b \in (\frac{3}{8}, \frac{2}{3})$: for an amateur who can't distinguish between 0 and 1 would take the same decision in states 2 and 3 if offered her loss-minimizing delegation set.

We can also use the calculations above to characterize the best amateurs for the agent ($E^A(b)$) and the agent's ensuing loss for $\frac{1}{6} < b < \frac{2}{3}$:[32]

	$E^A(b)$	$\Lambda_{E^A}(b)$
$\frac{1}{6} < b < \frac{1}{3}$	$\{0,1\}$	$\frac{1}{36}$
$\frac{1}{3} < b < \frac{1}{2}$	$\{1,2\}$	$\frac{11}{144}$
$\frac{1}{2} < b < \frac{2}{3}$	$\{1,2\}$	$\frac{1}{4}b^2 - \frac{1}{4}b + \frac{5}{36}$

The ODS and the principal's and the agent's losses from appointing the expert are

	$\Delta(b)$	$\lambda(b)$	$\Lambda(b)$
$\frac{1}{6} < b < \frac{1}{3}$	$b - \frac{1}{6}, b + \frac{1}{6}$ $b + \frac{1}{2}, b + \frac{5}{6}$	$(b - \frac{1}{6})^2$	$\frac{1}{36}$
$\frac{1}{3} < b < \frac{1}{2}$	$\frac{1}{2}b, \frac{3}{2}b, \frac{1}{2}b + \frac{2}{3}$	$\frac{3}{4}b^2 - \frac{1}{3}b + \frac{1}{18}$	$\frac{1}{4}b^2 + \frac{1}{18}$
$\frac{1}{2} < b < \frac{2}{3}$	$\frac{3}{2}b - \frac{1}{2}, \frac{1}{2}b + \frac{1}{2}$	$\frac{3}{4}b^2 - \frac{1}{2}b + \frac{5}{36}$	$\frac{1}{4}b^2 + \frac{5}{36}$

Comparing $\lambda(b)$ and $\lambda_{E^P}(b)$, we see that an amateur outperforms the expert if and only if $\frac{7}{24} < b < \frac{2}{3}$. If $\frac{1}{2} < b < \frac{2}{3}$ then the expert takes the same decision in all positive states. Contrary to our construction in Proposition 5, the best amateur can't distinguish between states $K-1$ and K.

Comparing $\Lambda(b)$ and $\Lambda_{E^A}(b)$, we see that the agent is indifferent between being an expert and some amateur ($\{0,1\}$) if $b < 1/3$, and prefers to be an amateur ($\{1,2\}$) if $1/3 < b < 2/3$.

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[32]It is easy to confirm that the agent prefers to be the expert if $b < 1/6$, and is indifferent if $b > 2/3$..

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