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Vincent Anesi and
John Duggan

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Suzanne Robey
Centre for Decision Research and Experimental Economics
School of Economics
University of Nottingham
University Park
Nottingham
NG7 2RD
Tel: +44 (0)115 95 14763
Fax: +44 (0) 115 95 14159
suzanne.robey@nottingham.ac.uk

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Dynamic Bargaining and External Stability with Veto Players

Vincent Anesi* John Duggan†

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Abstract

This note examines the structure of stationary bargaining equilibria in the finite framework of Anesi (2010). The main result establishes a tight connection between the set of equilibrium absorbing points and the von Neumann-Morgestern solutions: assuming that players are patient, that the voting rule is oligarchical, and that there is at least one veto player with positive recognition probability, a set of alternatives corresponds to the absorbing points of an equilibrium if and only if it is a von Neumann-Morgenstern solution. We also apply our analysis of ergodic properties of equilibria to the persistent agenda setter environment of Diermeier and Fong (2012). We show that all equilibria are essentially pure, and we extend their characterization of absorbing sets to allow an arbitrary voting rule and by removing the restriction to pure strategy equilibira.

1 Introduction

Since the seminal work of Baron (1996), bargaining games with an endogenous status quo have become more and more prominent in the literature on dynamic collective decision-making.[1] In these games, each period begins

*School of Economics, University of Nottingham. Email: vincent.anesi@nottingham.ac.uk.

†Department of Political Science and Department of Economics, University of Rochester. Email: dugg@ur.rochester.edu.

[1] To cite a few of the many examples, Kalandrakis (2004), Diermeier and Fong (2011, 2012), Battaglini and Palfrey (2012), Bowen and Zahran (2012), Duggan and Kalandrakis

with a status quo alternative inherited from the previous period, and a player is chosen randomly to propose any feasible alternative, which is then subject to an up or down vote. If the proposal is voted up, then it is implemented in that period and becomes the next period's status quo; if it is voted down, then the ongoing status quo is implemented and remains in place until the next period; this process continues ad infinitum. Anesi (2010) was the first to consider the finite framework, where the set of alternatives is finite and players have strict preferences. His main goal was to provide a noncooperative interpretation for von Neumann-Morgenstern solutions (von Neumann and Morgenstern, 1944), whose rationale in voting context had been questioned by political scientists (e.g., McKelvey et al. 1978). Assuming patient players, he shows that given a von Neumann-Morgenstern solution Y for the voting rule and a sufficiently high discount factor, there is a stationary Markovian equilibrium σ such that the set $A(\sigma)$ of absorbing alternatives under σ is equal to Y. Left open is the opposite logical direction: conditions under which given a stationary Markovian equilibrium σ, the set $A(\sigma)$ of absorbing alternatives is a von Neumann-Morgenstern solution.[2] Concentrating on pure strategy equilibria, Diermeier and Fong (2012) obtain this direction by assuming, in addition to high discount factors, that the same player proposes with probability one in every period.

The main objective of this note is to contribute further to this research program by examining the structure of (mixed-strategy) stationary Markovian equilibria in the finite framework of Anesi (2010). The analysis relies on the characterization of the ergodic properties of equilibria. Namely, we show that when the Nakamura number of the voting rule is high relative to the number of alternatives, all ergodic sets are singleton; in particular, if there is a veto player, then beginning from any given status quo, the equilibrium process transitions with probability one to the set of absorbing alternatives. Moreover, we show that if there is a veto player with positive recognition probability and players are patient, then starting from any given alternative, there is a *unique* absorbing point to which the equilibrium process transitions.

These results allow us to establish a tight connection between the set of equilibrium absorbing points and the von Neumann-Morgestern solutions. Maintaining the assumption that players are patient and there is at least

(2012), Nunnari (2014), Richter (2014), Baron and Bowen (2014), Zápal (2014), Anesi and Seidmann (2015) and Dziuda and Loeper (2015) all use such games.

[2] Anesi (2010) shows by example that, under majority voting, equilibrium absorbing sets may not be von Neumann-Morgenstern solutions.

one veto player with positive recognition probability, we increase the structure of our model in two directions. First, we assume that the voting rule is oligarchical, so that agreement of all veto players is not only necessary, but sufficient for a proposal to pass. Our main result is that under these conditions, a set of alternatives corresponds to the absorbing points of an equilibrium if and only if it is a von Neumann-Morgenstern solution. Second, allowing a general voting rule, we add the assumption that there is a persistent agenda setter, i.e., a fixed player who proposes with probability one in each period. We apply our analysis of ergodic properties of equilibria to show that all equilibria are essentially pure, and we again obtain the equivalence between equilibrium absorbing points and von Neumann-Morgenstern solutions. Thus, we extend Theorem 1 of Diermeier and Fong (2012) by generalizing the quota rules to an arbitrary voting rule and by removing the restriction to pure strategy equilibria.

Noncooperative foundations for von Neumann-Morgenstern solutions in political economy have been investigated in several different institutional settings, including electoral competition (Anesi 2012) and committee voting (Anesi and Seidmann 2014). In particular, Diermeier et al. (2014) consider a discrete version of the divide-the-dollar environment, in which players bargain over allocations of a private good. As in Diermeier and Fong (2012), these authors assume the existence of a veto player and obtain a characterization of pure-strategy equilibria in terms of von Neumann-Morgenstern solutions.

2 Dynamic Bargaining Framework

Consider the following dynamic bargaining model. The set X of alternatives is finite with $|X| \geq 2$, and individuals are numbered $1, \ldots, n$. In each of an infinite number of discrete periods $t = 1, 2, \ldots$, a status quo $q \in X$ is given, and a proposer is drawn from the fixed distribution $(\rho_1, \ldots, \rho_n)$ with $\rho_i > 0$ for each i. The selected individual makes a proposal $x \in X$, and a vote is held. If the group C of individuals who accept belongs to the collection $\mathcal{D}$ of decisive groups, then the outcome for the current period is $z^t = x$; and otherwise, if C is not decisive, then $z^t = q$ is the outcome for the current period. In both cases, the current outcome z^t becomes the status quo in the next period, where the process is repeated. Assume the voting rule $\mathcal{D}$ is nonempty and monotonic. It is *collegial* if there is some individual who belongs to every decisive group and has a veto — that is, if $\bigcap \mathcal{D} \neq \emptyset$ —

and we refer to such an i as a *veto player.* The rule is *oligarchical* if in addition the set of veto players is itself decisive, i.e., $\bigcap \mathcal{D} \in \mathcal{D}$, in which case a coalition is decisive if and only if it contains all veto players.

Each individual i has a stage utility function $u_i \colon X \to \Re$ such that for all distinct $x, y \in X$, we have $u_i(x) \neq u_i(y)$. Define the dominance relation $\succ$ such that $x \succ y$ if and only if $\{i : u_i(x) > u_i(y)\} \in \mathcal{D}$. A *von Neumann-Morgenstern solution* (or *vNM-solution*) is a set $S \subseteq X$ satisfying both internal stability (for all $x, y \in S$, $\neg(x \succ y)$) and external stability (for all $x \notin S$, there exists $y \in S$ with $y \succ x$). Given a sequence $\mathbf{z} = (z^1, z^2, \ldots)$ of outcomes, the payoff to player i is the normalized discounted utility

$$U_i(\mathbf{z}) \quad = \quad (1-\delta) \sum_{t=1}^{\infty} \delta^{t-1} u_i(z^t),$$

where $\delta \in [0, 1)$ is a common discount factor. We extend payoffs to probability distributions over such sequences via expected utility. The status quo in period 1 is an exogenously given alternative x^0.

The above elements define a dynamic game, and we focus on subgame perfect equilibria in stationary Markov strategies. Specifically, a stationary Markov strategy for player i is a pair of mappings $\sigma_i = (\pi_i, \alpha_i)$ such that $\pi_i \colon X \times X \to [0, 1]$ and $\alpha_i \colon X \times X \to [0, 1]$, where:

- $\pi_i(x, y)$ is the probability that player i proposes y given status quo x,
- $\alpha_i(x, y)$ is the probability that player i accepts alternative y when it is on the floor and the status quo is x.

We term π_i the *proposal strategy* and α_i the *voting strategy* of player i, and we let $\sigma = (\sigma_1, \ldots, \sigma_n)$ denote a profile of stationary Markov strategies. A proposal (resp. voting) strategy is *pure* if for all $x, y \in X$, we have $\pi_i(x, y) \in \{0, 1\}$ (resp. $\alpha_i(x, y) \in \{0, 1\}$). Let $\alpha(x, y)$ be the probability that y passes if proposed given status quo x, i.e.,

$$\alpha(x, y) \quad = \quad \sum_{C \in \mathcal{D}} \left(\prod_{i \in C} \alpha_i(x, y) \right) \left(\prod_{i \notin C} (1 - \alpha_i(x, y)) \right).$$

Given such a profile σ, let $P(\cdot|\sigma)$ denote the stochastic transition process engendered by σ, so that

$$P(x, y|\sigma) \quad = \quad \sum_i \rho_i \pi_i(x, y) \sum_{C \in \mathcal{D}} \left(\prod_{j \in C} \alpha_j(x, y) \right) \left(\prod_{j \notin C} (1 - \alpha_j(x, y)) \right)$$

is the probability that next period's outcome is y given that the outcome in the current period is x. Then $P(x, Y|\sigma) = \sum_{y \in Y} P(x, y|\sigma)$ is the probability that next period's outcome belongs to Y given current outcome x. In general, define $P^1(\cdot|\sigma) = P(\cdot|\sigma)$, and given $t \geq 2$, let $P^t(\cdot|\sigma)$ be the t-step transition defined by

$$P^t(x, y|\sigma) \;\; = \;\; \sum_{z \in X} P^1(x, z|\sigma) P^{t-1}(z, y|\sigma),$$

so that $P^t(x, \cdot|\sigma)$ gives the distribution over outcomes t periods in the future, given outcome x in the current period.

The expected discounted payoff, or *dynamic payoff,* from outcome x in a given period for player i is

$$V_i(x|\sigma) \;\; = \;\; (1 - \delta) u_i(x) + \delta \sum_{t=1}^{\infty} \delta^{t-1} \sum_{z \in X} u_i(z) P^t(x, z|\sigma).$$

Of course, this dynamic payoff is the unique solution to the recursion

$$V_i(x|\sigma) \;\; = \;\; (1 - \delta) u_i(x) + \delta \sum_{y \in X} P(x, y|\sigma) V_i(y|\sigma). \tag{1}$$

A stationary Markov profile σ is a *stationary bargaining equilibrium* if proposals and votes are optimal for all histories; that is, if (i) for all $x \in X$ and all $i \in \{1, \ldots, n\}$, $\pi_i(x, \cdot)$ puts positive probability on solutions to

$$\max_{y \in X} \alpha(x, y) V_i(y|\sigma) + (1 - \alpha(x, y)) V_i(x|\sigma),$$

and (ii) for all $x, y \in X$ and all $j \in \{1, \ldots, n\}$, we have $\alpha_j(x, y) = 1$ if $V_j(y|\sigma) \geq V_j(x|\sigma)$, and we have $\alpha_j = 0$ if $V_j(y|\sigma) < V_j(x|\sigma)$. Note that the optimality condition (ii) on voting strategies incorporates the refinement that players do not cast stage-dominated votes. Moreover, the condition assumes *deferential* voting strategies, so that a player votes for a proposed alternative if indifferent between acceptance and rejection. Because of this assumption, existence of a stationary bargaining equilibrium does not follow from known existence results for Markov perfect equilibria in stochastic games. Nevertheless, existence is not an issue: our first theorem does not require the deferential voting restriction; the others assume that $\mathcal{C}$ is collegial and that the discount factor is high, in which case results by Muto (1984) and Anesi (2010) imply that the game possesses a stationary bargaining equilibrium.[3]

A set $Y \subseteq X$ of alternatives is *invariant* under σ if for all $x \in Y$, we have $P(x, Y|\sigma) = 1$, and it is *ergodic* if it is minimal among invariant sets according to set inclusion. We let $\mathcal{E}(\sigma)$ denote the collection of ergodic sets under σ. An alternative x is *absorbing* if $P(x, x|\sigma) = 1$, or alternatively, $\{x\}$ is ergodic. If there is some t such that $P^t(x, y|\sigma) > 0$, then y is *reachable* from x. Let $A(x|\sigma)$ be the set of absorbing points that are reachable from x.

These concepts can be reformulated in graph-theoretic terms. Define the *graph* of σ, denoted $\Gamma(\sigma)$, as follows: for all $x, y \in X$, we have $x\Gamma(\sigma)y$ if and only if $P(x, y|\sigma) > 0$. Let $\Gamma^1(\sigma) = \Gamma(\sigma)$, and for each $t = 2, 3, \ldots$, define $\Gamma^t(\sigma)$ as follows: for all $x, y \in X$, we have $x\Gamma^t(\sigma)y$ if and only if there exists $z \in X$ such that $x\Gamma(\sigma)z\Gamma(\sigma)^{t-1}y$. The *transitive closure* of Γ, denoted $\Gamma^\infty(\sigma)$, is defined as $\Gamma^\infty(\sigma) = \bigcup_{t=1}^{\infty} \Gamma^t(\sigma)$. Then Y is ergodic if and only if for all $x, y \in Y$, we have $x\Gamma^\infty(\sigma)y$ and $y\Gamma^\infty(\sigma)x$; an alternative x is absorbing if and only if for all $y \in X$, $x\Gamma(\sigma)y$ implies $y = x$; and y is reachable from x if and only if $x\Gamma^\infty(\sigma)y$.

It is well-known that from any outcome x, the equilibrium Markov chain eventually leads to an ergodic set with probability one. To formalize this claim, let $P_\infty(x, Y|\sigma) = \liminf P^t(x, Y|\sigma)$. Then for all x, we have

$$P_\infty\left(x, \bigcup \mathcal{E}(\sigma)|\sigma\right) = 1,$$

so that with probability one the set $\bigcup \mathcal{E}(\sigma)$ is entered from x and remains in that set. In graph-theoretic terms, for all $x \in X$, there exist an ergodic set Y and $y \in Y$, such that $x\Gamma^\infty(\sigma)y$.

3 Absorbing Alternatives

Our first result establishes that when the set of alternatives is small relative to the Nakamura number of the bargaining game, every ergodic set is a singleton, the lone element being an absorbing alternative. To begin, we define the *Nakamura number* of the voting rule, denoted $\mathcal{N}(\mathcal{D})$, in two cases.

[3]The former shows that there exists a (unique) vNM solution if $\mathcal{D}$ is collegial; the latter that a (pure strategy) stationary bargaining equilibrium exists if there is a vNM solution.

First, in case the rule is non-collegial, let

$$\mathcal{N}(\mathcal{D}) \quad = \quad \min\left\{|\mathcal{G}| \;\mid\; \mathcal{G} \subseteq \mathcal{D} \text{ and } \bigcap \mathcal{G} = \emptyset \;\right\}.$$

In words, $\mathcal{N}(\mathcal{D})$ is the size of the smallest collection of decisive coalitions having empty intersection. Second, in case the rule is collegial, the set $\mathcal{N}(\mathcal{D})$ is equal to the cardinality of the integers. It is known that when the number of players is either three or strictly greater than four, the Nakamura number of majority rule is three. In general, for a quota rule with quota q, the Nakamura number is

$$\mathcal{N}(\mathcal{D}) \quad = \quad \left\lceil \frac{n}{n-q} \right\rceil,$$

so it becomes arbitrarily high when the number of players is large and the quota becomes large relative to n.

Theorem 1: *Assume $|X|(|X|-1) < \mathcal{N}(\mathcal{D})$, and consider any stationary bargaining equilibrium σ. For every $Y \subseteq X$, Y is ergodic if and only if there is an absorbing alternative x such that $Y = \{x\}$.*

Proof: One direction is obvious. For the other, suppose there is an ergodic set Y with $|Y| = k \geq 2$, and enumerate the elements as $y_1, \ldots, y_k$. For each $h = 1, \ldots, k$, let

$$Y_h \quad = \quad \{z \in X \setminus \{y_h\} \mid P(y_h, z|\sigma) > 0\},$$

and enumerate the elements of this set as $z_{h,1}, \ldots, z_{h,\ell_h}$. For each $h = 1, \ldots, k$ and each $\ell = 1, \ldots, \ell_h$, there exists $C_{h,\ell} \in \mathcal{D}$ such that for all $i \in C_{h,\ell}$, we have

$$V_i(z_{h,\ell}|\sigma) \quad \geq \quad V_i(y_h|\sigma).$$

Since $|Y| \leq |X|$ and $|Y_h| \leq |X| - 1$, we have $|Y|(\max_h |Y_h|) < \mathcal{N}(\mathcal{D})$, so by assumption there exists $i \in \bigcap_{h=1}^{k} \bigcap_{\ell=1}^{\ell_h} C_{h,\ell}$. Let y_m maximize player i's dynamic payoff over Y, i.e., $V_i(y_m|\sigma) = \max_{h=1,\ldots,k} V_i(y_h|\sigma)$. Then for all $\ell = 1, \ldots, \ell_m$, we must have

$$V_i(z_{m,\ell}|\sigma) \quad = \quad V_i(y_m|\sigma).$$

Since Y is ergodic, this argument in fact implies that for all $h = 1, \ldots, k$, we have $V_i(y_h|\sigma) = V_i(y_m|\sigma)$, so that the dynamic payoff of player i is constant on Y, and we can denote this by $\overline{V}$. But choosing any y_h and $z_{h,\ell}$, we then have

$$\begin{aligned}(1-\delta)u_i(y_h)+\delta\overline{V} &= V_i(y_h|\sigma)\\ &= V_i(z_{h,\ell}|\sigma)\\ &= (1-\delta)u_i(z_{h,\ell})+\delta\overline{V},\end{aligned}$$

which implies $u_i(y_h)=u_i(z_{h,\ell})$, a contradiction. We conclude that for every ergodic set Y, we have $|Y|=1$, so that there is an absorbing alternative x such that $Y=\{x\}$. Q.E.D.

Note that when the voting rule $\mathcal{D}$ is collegial, the Nakamura number takes an infinite value, so the conditions of Theorem 1 are satisfied. Because the Nakamura number of majority rule is three, the result does not apply unless there are just two alternatives; cycles can arise and ergodic sets with multiple elements can be supported in equilibrium as Example 1 below illustrates. But for a large set of players and quota rules with higher quotas, the result does apply.

Example 1: Let $n=3$, $X=\{x,y,z\}$, and suppose that the players' utilities and discount factor satisfy the following inequalities:

$$\begin{aligned}&(3-\delta)^{-1}\left[(3+\delta)u_1(y)-\delta u_1(x)\right] < u_1(z) < u_1(y) < u_1(x)\ ;\\ &(3-\delta)^{-1}\left[(3+\delta)u_2(x)-\delta u_2(z)\right] < u_2(y) < u_2(x) < u_2(z)\ ;\\ &(3-\delta)^{-1}\left[(3+\delta)u_3(z)-\delta u_3(y)\right] < u_3(x) < u_3(z) < u_3(y)\ .\end{aligned}$$

We further assume that $\mathcal{D}$ is majority rule — i.e., $\mathcal{D}=\{C\subseteq N \mid |C|\geq 2\}$ — and that the three players have the same recognition probability — i.e., $\rho_i = 1/3$ for each $i=1,2,3$. Under these assumptions, there is a (pure strategy) stationary bargaining equilibrium such that: given status quo x, players 2 and 3 propose and accept y, whereas player 1 maintains the status quo; given status quo y, players 1 and 2 propose and accept z, whereas player 3 maintains the status quo; and given status quo z, players 1 and 3 propose and accept x, whereas player 2 maintains the status quo. (We provide the precise details of the equilibrium construction in the supplementary appendix.) The induced Markov chain on X is depicted in Figure 1. Observe that there is no absorbing alternative and that the only ergodic set is the entire set of alternatives X. $\square$

An important implication of Theorem 1 for a stationary bargaining equilibrium σ is that for each alternative x, we have $A(x|\sigma)\neq\emptyset$. This in turn

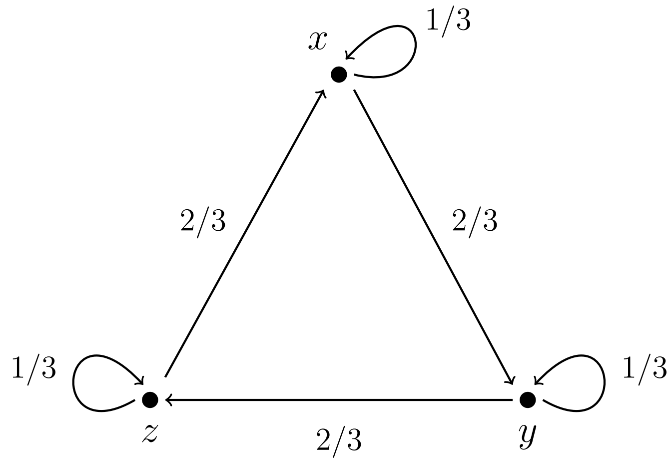

Figure 1: Equilibrium Markov chain with a Condorcet cycle.

implies that beginning from any given alternative, the equilibrium Markov chain eventually transitions to an absorbing alternative with probability one.

4 Bargaining Equilibria and vNM-Solutions

We now increase the structure imposed on the analysis by considering the case of patient players and assuming that there is at least one veto player with positive recognition probability. The following result establishes that for each alternative, there is a unique alternative to which it is absorbed.

Theorem 2: *Assume that there is at least one veto player with positive recognition probability, i.e., there exists $i \in \bigcap \mathcal{D}$ with $\rho_i > 0$. Then there exists $\overline{\delta} \in (0,1)$ such that for all $\delta \in (\overline{\delta}, 1)$, all stationary bargaining equilibria σ, and all $x \in X$, there exists $y \in X$ such that $A(x|\sigma) = \{y\}$.*

Proof: By Theorem 1, we know that for all $x \in X$, we have $A(x|\sigma) \neq \emptyset$. To deduce a contradiction, suppose there are a sequence of discount factors $\{\delta^k\}$ converging to one and corresponding stationary bargaining equilibria $\{\sigma^k\}$ such that for each k, there exist x_k with $|A(x_k|\sigma^k)| \geq 2$. Then $\{\Gamma(\sigma^k)\}$ is the corresponding sequence of equilibrium graphs. Since X is finite, we can go to a subsequence (still indexed by k) on which these alternatives and graphs are constant, and henceforth we write $x = x_k$ for the given alternative and $\Gamma = \Gamma(\sigma^k)$ for the equilibrium graph. Abusing notation slightly, let $A = A(\sigma^k)$ denote the set of absorbing alternatives of σ^k, and let $A(y) = A(y|\sigma^k)$ denote the absorbing alternatives reachable from an alternative y; these sets are constant along the sequence, and we have $|A(x)| \geq 2$. Let w minimize the stage payoff of player i among absorbing alternatives reachable from x, i.e., $u_i(w) = \min_{y \in A(x)} u_i(y)$. Let $\{y_1, y_2, \ldots, y_m\}$ be a path from x to w, so that

$$x = y_1 \Gamma y_2 \cdots \Gamma y_{m-1} \Gamma y_m = w,$$

and let y_ℓ be the highest indexed alternative such that $|A(y_\ell)| \geq 2$, and note that $\ell < m$. For all k, we have

$$V_i(y_{\ell+1}|\sigma^k) \quad \geq \quad V_i(y_\ell|\sigma^k),$$

and since the equilibrium Markov chain eventually transitions from $y_{\ell+1}$ to w with probability one, we have $V_i(y_{\ell+1}|\sigma^k) \to u_i(w)$. With the preceding inequality, we also have $V_i(y_\ell|\sigma^k) \to u_i(w)$. By construction, there exists $z \in A(y_\ell) \setminus \{w\}$, so that $u_i(z) > u_i(w)$. Then there exist $z_1, \ldots, z_h \in X$ such that $y_\ell = z_1 \Gamma z_2 \ldots \Gamma z_{h-1} \Gamma z_h = z$. Thus, there exist $C_1, \ldots, C_h \in \mathcal{D}$ such that for each $k = 1, 2, \ldots$, for each $r = 1, \ldots, h-1$, and for each $j \in C_r$, we have $V_j(z_r|\sigma^k) \leq V_j(z_{r+1}|\sigma^k)$. It follows that if player i proposes z_{r+1} given status quo z_r, the proposal will pass with probability one (deferential voting). Note, moreover, that the probability that i is recognized as proposer is $\rho_i > 0$. Since player i's equilibrium proposal strategy is optimal, it follows that

$$V_i(y_\ell|\sigma) \quad \geq \quad \rho_i^{h-1} u_i(z) + (1 - \rho_i^{h-1}) u_i(w),$$

which is bounded strictly above $u_i(w)$, a contradiction. Q.E.D.

Note that Theorem 2 does not establish that for each x, there is a unique *path* of alternatives leading to the absorbing alternative y. In fact, the following example illustrates the possibility of multiple absorbing paths.

Example 2: Let the set of alternatives be $X = \{x, y, a, b\}$, let there be four players, each with recognition probability $p = 1/4$, and let the voting rule be such that a coalition is decisive if and only if it contains $\{1, 2\}$ and at least one other player, i.e., $\mathcal{D} = \{C \mid 1, 2 \in C \text{ and } |C| \geq 3\}$. This voting rule makes players 1 and 2 veto players, but it is not oligarchical. Stage payoffs satisfy

$$u_1(x) < u_1(b) < u_1(a) < u_1(y) \ , \ u_2(x) < u_2(a) < u_2(b) < u_2(y) \ ,$$
$$u_3(b) < u_3(y) < u_3(a) < u_3(x) \ , \text{ and } u_4(a) < u_4(y) < u_4(b) < u_4(x) \ .$$

We further assume that

$$3u_3(x) + u_3(a) < 5u_3(y) - u_3(b) \ , \text{ and } 3u_4(x) + u_4(b) < 5u_4(y) - u_4(a) \ .$$

It is worth remarking that the veto players both prefer y to x, but no other player agrees, and thus it is not the case that $y \succ x$. It is readily checked that if the discount factor δ is sufficiently large, then the Markov chain depicted in Figure 2 corresponds to a stationary bargaining equilibrium such that:[4] given status quo x, each player proposes her favorite alternative in $\{y, a, b\}$, and this proposal passes; given status quo y, all players maintain the status quo; given status quo a, player 3 maintains the status quo, whereas all other players obtain the outcome y; with analogous actions at status quo b. In particular, given status quo x, players 3 and 4 are willing to vote for proposal y in order to avoid obtaining their least preferred alternative, which occurs with probability 1/4 if the status quo is maintained — our assumptions guarantee that, for sufficiently large δ, $V_i(x|\sigma) < V_i(y|\sigma)$ for each $i = 3, 4$. And given status quo x, players 1 and 2 are willing to vote for any of a and b in order to avoid remaining at their least preferred alternative for another period. □

The next result strengthens the assumptions of Theorem 2 to provide a tight connection between equilibrium ergodic sets and von Neumann-Morgenstern solutions: if the voting rule is oligarchical, then a set of alternatives is obtained as the absorbing points of a stationary bargaining equilibrium if and only if it is a von Neumann-Morgenstern solution.

Theorem 3: *Assume $\mathcal{D}$ is oligarchical, and that there is at least one veto player with positive recognition probability, i.e., there exists $i \in \bigcap \mathcal{D}$ with $\rho_i > 0$. Then there exists $\overline{\delta} \in (0, 1)$ such that for all $\delta \in (\overline{\delta}, 1)$ and all subsets $Y \subseteq X$, there is a stationary bargaining equilibrium σ with $A(\sigma) = Y$ if and only if Y is a von Neumann-Morgenstern solution.*

Proof: One direction follows from Anesi (2010). For the other, suppose toward a contradiction that there are a sequence $\{\delta^k\}$ of discount factors

[4] Details are provided in the supplementary appendix.

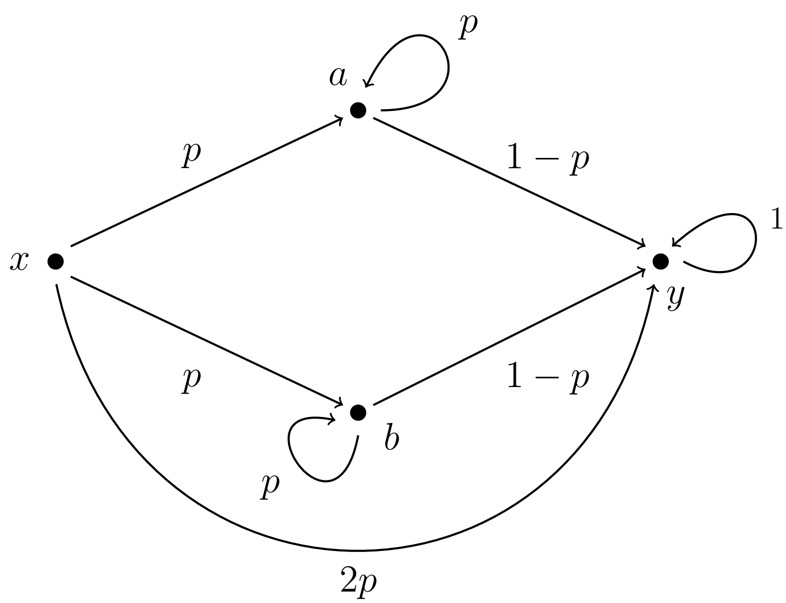

Figure 2: Multiple paths.

converging to one and a sequence $\{\sigma^k\}$ of stationary bargaining equilibria such that for each k, the set $A(\sigma^k)$ of absorbing points of σ^k is not a von Neumann-Morgenstern solution. Clearly, the set $A(\sigma^k)$ is internally stable, for else there exist $x, y \in A(\sigma^k)$ such that $x \succ y$, but then given status quo y, player i could successfully propose x, contradicting the fact that y is an absorbing alternative. It follows that $A(\sigma^k)$ violates external stability, so there exists $x_k \in X \setminus A(\sigma^k)$ such that for all $y \in A(\sigma^k)$, it is not the case that $y \succ x_k$. By Theorem 2, for sufficiently high k, there exists y_k such that $A(x_k|\sigma^k) = \{y_k\}$, so that starting from x_k, the equilibrium Markov process is absorbed into y_k. Going to a subsequence (still indexed by k), we can assume that these alternatives are constant, and henceforth we write $x = x_k$ and $y = y_k$. Let $w_{k,1}$ minimize player i's dynamic payoff over the outcomes distinct from x that occur with positive probability given status quo x, i.e.,

$$V_i(w_{k,1}|\sigma^k) \quad = \quad \min_{z \neq x : P(x,z|\sigma^k)>0} V_i(z|\sigma^k),$$

and recursively, given $w_{k,1},\ldots,w_{k,h}$, let

$$V_i(w_{k,h+1}|\sigma^k) \quad = \max_{z\neq w_{k,h}:P(w_{k,h},z|\sigma^k)>0} V_i(z|\sigma^k).$$

This generates a finite sequence $w_{k,1},\ldots,w_{k,\ell_k}$ with $w_{k,\ell_k} = y$. Since i is a veto player, it follows that $V_i(w_{k,1}|\sigma^k) \geq V_i(x|\sigma^k)$ and that for each $h = 1,\ldots,\ell_k - 1$, we have $V_i(w_{k,h}|\sigma^k) \leq V_i(w_{k,h+1}|\sigma^k)$. In particular, we have $u_i(y) \geq V_i(w_{k,1}|\sigma^k)$. Now suppose in order to deduce a contradiction that for arbitrarily high k, we have $u_i(x) > V_i(w_{k,1}|\sigma^k)$. For such k, we have

$$\begin{aligned}
&V_i(x|\sigma^k)\\
&\quad = \quad (1-\delta)u_i(x) + \delta P(x,x|\sigma^k)V_i(x|\sigma^k) + \delta\sum_{z\neq x} P(x,z|\sigma^k)V_i(z|\sigma^k)\\
&\quad > \quad (1-\delta)V_i(w_{k,1}|\sigma^k) + \delta P(x,x|\sigma^k)V_i(x|\sigma^k) + \delta\left[1 - P(x,x|\sigma^k)\right]V_i(w_{k,1}|\sigma^k),
\end{aligned}$$

which implies

$$V_i(x|\sigma^k) \quad > \quad V_i(w_{k,1}|\sigma^k),$$

a contradiction. We conclude that for sufficiently high k, we have $u_i(y) \geq V_i(w_{k,1}|\sigma^k) \geq u_i(x)$. Thus, we have $u_i(y) \geq u_i(x)$, which further implies $u_i(y) > u_i(x)$. For sufficiently high k, this inequality holds for every veto player, and since the voting rule is oligarchical, we conclude that $y \succ x$. This final contradiction completes the proof. Q.E.D.

The assumption that the voting rule is oligarchical, rather than merely collegial, is needed for the latter result. This is illustrated by the preceding example, in which player 1 is a veto player (with positive recognition probability), yet there are multiple paths from x leading to the unique absorbing point y, and it is not the case that $y \succ x$.

5 Persistent Agenda Setter

It is worthwhile to summarize briefly the steps in the analysis above. First, Theorem 1 assumes that the Nakamura number of the voting rule is high relative to the number of alternatives; then Theorem 2 adds more structure by assuming a veto player. The additional structure of an oligarchic rule is then used in Theorem 3 to obtain a characterization of stationary bargaining equilibria in terms of von Neumann-Morgenstern solutions. Our next theorem adds structure to Theorems 1 and 2 in a different direction: it casts the analysis into the persistent agenda setter model (Diermeier and Fong 2011, 2012), in which some player i is given the sole power to make proposals.

Let $\mathcal{D}^i$ be the voting rule obtained from $\mathcal{D}$ by adding player i to every decisive coalition in $\mathcal{D}$, that is, $\mathcal{D}^i = \{C \cup \{i\} : C \in \mathcal{D}\}$. If player i is the single agenda setter (i.e., $\rho_i = 1$), then any stationary bargaining equilibrium σ with rule $\mathcal{D}$ is also a stationary bargaining equilibrium with collegial rule $\mathcal{D}^i$. To see this, note that if σ is not an equilibrium with rule $\mathcal{D}^i$ then, at some status quo x, i must propose an alternative y that is accepted with rule $\mathcal{D}$ and rejected with rule $\mathcal{D}^i$. This implies that i rejects this proposal and, therefore, that (with rule $\mathcal{D}$) she would have been strictly better off maintaining status quo x rather than proposing y; a contradiction. Hence, Theorems 1 and 2 can be applied directly to any voting rule to yield the following corollary.

Corollary 1: *Assume that there is a persistent agenda setter, i.e., there exists $i \in \{1, \ldots, n\}$ with $\rho_i = 1$. Then there exists $\overline{\delta} \in (0, 1)$ such that for all $\delta \in (\overline{\delta}, 1)$, all stationary bargaining equilibria σ, and all $x \in X$, there exists $y \in X$ such that $A(x|\sigma) = \{y\}$.*

The preceding argument does not directly yield a version of Theorem 3 for the persistent agenda setter model, as the theorem assumes an oligarchic rule. Nevertheless, the structure of a single proposer allows us to obtain a sharper result than Theorem 2, which we will draw on to obtain a characterization of von Neumann-Morgenstern solutions. Our final theorem indeed establishes that this structure, in addition to ensuring the uniqueness of the absorbing alternative y from any status quo x, implies that there is a *unique path* determined in equilibrium from x to y. In particular, the equilibrium graph $\Gamma(\sigma)$ possesses no "branches," so that for every alternative x, there is a unique alternative z such that $x\Gamma(\sigma)z$, precluding equilibrium Markov chains of the sort demonstrated in Example 2. Note the further implication that under the conditions of the theorem, stationary bargaining equilibria are essentially pure, in the sense that for every non-absorbing alternative $x \notin A(\sigma)$, the proposal strategy $\pi_i(x, \cdot)$ puts probability one on the single alternative z such that $x\Gamma(\sigma)z$; if x is an absorbing alternative, then the setter may mix between proposals that are rejected, but mixing in this case is nominal. Thus, we find that Diermeier and Fong's (2012) restriction to pure strategy equilibria is redundant.

Theorem 4: *Assume that there is a persistent agenda setter, i.e., there exists $i \in \{1, \ldots, n\}$ with $\rho_i = 1$. Then there exists $\overline{\delta} \in (0,1)$ such that for all $\delta \in (\overline{\delta}, 1)$, all stationary bargaining equilibria σ, and all $x \in X$, there exist a unique absorbing alternative $y \in X$, a unique natural number $m \geq 1$, and unique alternatives $z_1, \ldots, z_m \in X$ such that*

$$x\Gamma(\sigma)z_1\Gamma(\sigma)\cdots z_{m-1}\Gamma(\sigma)z_m = y.$$

Proof: Consider an arbitrary sequence $\{\delta^k\}$ of discount factors converging to one and a corresponding sequence $\{\sigma^k\}$ of stationary bargaining equilibria. Going to a subsequence (still indexed by k), we can assume that the corresponding graph, Γ is constant. Consider an alternative x. By Corollary 1, there exists a unique alternative y such that $A(x|\sigma^k) = \{y\}$ for all $k \in \mathbb{N}$. It suffices to show that for sufficiently high k, there is a unique path between x and y. We proceed in three steps:

Step 1: The alternative y maximizes player i's utility over the set $R(x)$ of alternatives reachable from x. The proof for this step parallels exactly the proof of Theorem 3 — though i may not be a veto player in this case, being the single agenda setter, she can still (unilaterally) maintain the status quo.

Step 2: Γ is acyclic. Suppose toward a contradiction that there exist $x_1, \ldots, x_m$ such that $x_1\Gamma x_2\Gamma\cdots x_m\Gamma x_1$. This implies that

$$V_i(x_1|\sigma^k) = \cdots = V_i(x_m|\sigma^k).$$

Note that the setter i solves a dynamic programming problem, and σ_i^k is optimal, given σ_{-i}^k. Since x_2 is proposed with positive probability given status quo x_1, it is an optimal choice at x_1, given that future choices are made according to σ_i^k. Then it is optimal to always choose x_2 at x_1, using σ_i^k at all other status quos. Call this strategy $\sigma_i^{k,1}$. We then modify $\sigma_i^{k,1}$ so that at x_2, the setter chooses x_3 with probability one. The resulting strategy, $\sigma_i^{k,2}$, is also optimal. In general, we modify $\sigma_i^{k,j}$ so that at status quo x_j, the setter chooses x_{j+1} with probability one, giving us an optimal strategy at each step. In the end, the strategy $\sigma_i^{k,m}$ is optimal, but following it, the setter just cycles through $x_1, \ldots, x_m$. It follows from Step 1 that the setter's payoff from $\sigma_i^{k,m}$, starting from x_1, is bounded above by

$$\max\{u_i(x_j) \mid j = 1, \ldots, m\} < u_j(y).$$

For sufficiently high k, this payoff is less than the payoff from following σ_i^k, because following that strategy y is eventually reached with probability one. This contradiction completes the step.

Step 3. There is a unique path between x and y in Γ. It follows from the previous step that if $z \in R(x)$, then $R(z) \subsetneqq R(x)$. Say z *branches* if there are distinct alternatives s, t such that $z\Gamma s$ and $z\Gamma t$. Suppose toward a contradiction that some alternative $z \in R(x)$ branches, and choose z so that $R(z)$ is minimal among

$$\{R(w) \mid w \in R(x) \text{ and } w \text{ branches}\}.$$

Then following w are at least two deterministic paths that lead to y. But as $\delta^k \to 1$, it is not possible to maintain the setter's indifference over these paths for arbitrarily high k. Q.E.D.

We close this note by recording an implication of Theorem 4 for the connections between von Neumann-Morgenstern solutions and equilibrium absorbing sets in the persistent agenda setter model. Note that this corollary imposes no restriction on the voting rule $\mathcal{D}$. This result thus extends Theorem 1 in Diermeier and Fong (2012) by generalizing their quota rules to an arbitrary voting rule and by removing the restriction to pure strategy equilibria.

Corollary 2: *Assume that there is a persistent agenda setter, i.e., there exists $i \in \{1, \ldots, n\}$ with $\rho_i = 1$. Then there exists $\overline{\delta} \in (0, 1)$ such that for all $\delta \in (\overline{\delta}, 1)$ and all subsets $Y \subseteq X$, there is a stationary bargaining equilibrium σ with $A(\sigma) = Y$ if and only if Y is a von Neumann-Morgenstern solution for $\mathcal{D}^i$.*

Proof: Sufficiency follows from Diermeier and Fong (2012). For necessity, suppose toward a contradiction that there are a sequence $\{\delta^k\}$ of discount factors converging to one and a sequence $\{\sigma^k\}$ of stationary bargaining equilibria such that for each k, the set $A(\sigma^k)$ of absorbing points of σ^k is not a von Neumann-Morgenstern solution. By the same logic as in the proof of Theorem 3, the set $A(\sigma^k)$ must be internally stable. It follows that $A(\sigma^k)$ violates external stability, so there exists $x^k \in X \setminus A(\sigma^k)$ such that for all $y \in A(\sigma^k)$, it is not the case that $y \succ x^k$. By Corollary 1 and Theorem 4, for sufficiently high k, there exist: an alternative y^k such that $A(x^k|\sigma^k) = \{y^k\}$; a unique path $\{x^k, z_1^k, \ldots, z_{m_k}^k\}$ from x^k to y^k; and a coalition $C^k \in \mathcal{D}^i$ such that $V_j\left(x^k|\sigma^k\right) \leq V_j\left(z_1^k|\sigma^k\right)$ for all $j \in C^k$. Going to a subsequence (still indexed by k), we can assume that this coalition and these alternatives are constant, and henceforth we write $C = C^k$, $x = x^k$, and $z_1 = z_1^k$. Hence, for each k, we have

$$\begin{aligned} 0 &\leq V_j(z_1|\sigma^k) - V_j(x|\sigma^k) \\ &= V_j(z_1|\sigma^k) - (1-\delta^k)u_j(x) - \delta^k V_j(z_1|\sigma^k) \\ &= (1-\delta^k)(V_j(z_1|\sigma^k) - u_j(x)) \end{aligned}$$

for all $j \in C$. This implies $V_j(z_1|\sigma^k) \geq u_j(x)$ for all $j \in C$. Taking limits and using $V_j(z_1|\sigma^k) \to u_j(y)$, we then have $u_j(y) \geq u_j(x)$ and, thus, $u_j(y) > u_j(x)$ for all $j \in C$. This contradicts our supposition that there is no absorbing alternative y such that $y \succ x$. Q.E.D.

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Supplementary Appendix: Details of the Equilibria in Examples 1 and 2

Example 1. Let the strategy profile σ be defined by:

$$\pi_1(x,x) = \pi_1(y,z) = \pi_1(z,x) = \pi_2(x,y) = \pi_2(y,z) = \pi_2(z,z)$$
$$= \pi_3(x,y) = \pi_3(y,y) = \pi_3(z,x) = 1;$$

$$\alpha_2(x,y) = \alpha_3(x,y) = 1-\alpha_1(x,y) = 1;\ \alpha_1(x,z) = \alpha_3(x,z) = 1-\alpha_2(x,z) = 0;$$

$$\alpha_2(y,x) = \alpha_3(y,x) = 1-\alpha_1(y,x) = 0;\ \alpha_1(y,z) = \alpha_2(y,z) = 1-\alpha_3(y,z) = 1;$$

$$\alpha_1(z,x) = \alpha_3(z,x) = 1-\alpha_2(z,x) = 1;\ \alpha_1(z,y) = \alpha_2(z,y) = 1-\alpha_3(z,y) = 0.$$

Hence,

$$\alpha(x,x) = \alpha(x,y) = 1 - \alpha(x,z) = 1\ , \tag{2}$$
$$\alpha(y,y) = \alpha(y,z) = 1 - \alpha(y,x) = 1\ , \tag{3}$$
$$\alpha(z,z) = \alpha(z,x) = 1 - \alpha(z,y) = 1\ . \tag{4}$$

Simple calculations then yield

$$V_i(x|\sigma) = \frac{(3-\delta)^2 u_i(x) + 2(3-\delta)\delta u_i(y) + 4\delta^2 u_i(z)}{3\,(3+\delta^2)}\ ,$$
$$V_i(y|\sigma) = \frac{4\delta^2 u_i(x) + (3-\delta)^2 u_i(y) + 2(3-\delta)\delta u_i(z)}{3\,(3+\delta^2)}\ ,$$
$$V_i(z|\sigma) = \frac{2(3-\delta)\delta u_i(x) + 4\delta^2 u_i(y) + (3-\delta)^2 u_i(z)}{3\,(3+\delta^2)}\ ;$$

so that

$$V_i(x|\sigma) \geq V_i(y|\sigma) \text{ iff } (3+\delta)u_i(x) \geq (3-\delta)u_i(y) + 2\delta u_i(z)\ ,$$
$$V_i(x|\sigma) \geq V_i(z|\sigma) \text{ iff } (3-\delta)u_i(x) + 2\delta u_i(y) \geq (3+\delta)u_i(z)\ , \text{ and}$$
$$V_i(y|\sigma) \geq V_i(z|\sigma) \text{ iff } (3+\delta)u_i(y) \geq 2\delta u_i(x) + (3-\delta)u_i(z)\ ,$$

for each $i = 1,2,3$. Thus, under our assumptions on stage utilities, dynamic payoffs satisfy $V_1(y|\sigma) < V_1(z|\sigma) < V_1(x|\sigma)$, $V_2(x|\sigma) < V_2(y|\sigma) < V_2(z|\sigma)$, and $V_3(z|\sigma) < V_3(x|\sigma) < V_3(y|\sigma)$. Combined with equations (2)-(4), these inequalities imply that σ satisfies the conditions for a stationary bargaining equilibrium.

Example 2. The strategy profile σ, described in the example, is defined by

- Policy strategies given status quo x: $\pi_1(x,y) = \pi_2(x,y) = \pi_3(x,a) = \pi_4(x,b) = 1$;
- Policy strategies given status quo y: $\pi_i(y,y) = 1$ for each $i = 1,2,3,4$;
- Policy strategies given status quo a: $\pi_1(a,y) = \pi_2(a,y) = \pi_3(a,a) = \pi_4(a,y) = 1$;
- Policy strategies given status quo b: $\pi_1(b,y) = \pi_2(b,y) = \pi_3(b,y) = \pi_4(b,b) = 1$;
- Voting strategies given status quo x: $\alpha_i(x,y) = 1$ for each $i = 1,2,3,4$,

  $$\alpha_1(x,a) = \alpha_2(x,a) = \alpha_3(x,a) = 1 - \alpha_4(x,a) = 1 \ ,$$
  $$\alpha_1(x,b) = \alpha_2(x,b) = \alpha_4(x,b) = 1 - \alpha_3(x,b) = 1 \ ,$$

  so that
  $$\alpha(x,x) = \alpha(x,y) = \alpha(x,a) = \alpha(x,b) = 1 \ ; \tag{5}$$

- Voting strategies given status quo y: $\alpha_i(y,x) = 0$ for each $i = 1,2,3,4$,

  $$\alpha_1(y,a) = \alpha_2(y,a) = \alpha_4(y,a) = 1 - \alpha_3(y,a) = 0 \ ,$$
  $$\alpha_1(y,b) = \alpha_2(y,b) = \alpha_3(y,b) = 1 - \alpha_4(y,b) = 0 \ ,$$

  so that
  $$\alpha(y,x) = 1 - \alpha(y,y) = \alpha(y,a) = \alpha(y,b) = 0 \ ; \tag{6}$$

- Voting strategies given status quo a:

  $$\alpha_1(a,x) = \alpha_2(a,x) = \alpha_3(a,x) = 1 - \alpha_4(a,x) = 0 \ ,$$
  $$\alpha_1(a,y) = \alpha_2(a,y) = \alpha_4(a,y) = 1 - \alpha_3(a,y) = 1 \ ,$$
  $$\alpha_1(a,b) = 1 - \alpha_2(a,b) = \alpha_4(a,b) = 1 - \alpha_3(a,b) = 0 \ ,$$

  so that

  $$1 - \alpha(a,x) = \alpha(a,y) = \alpha(a,a) = 1 - \alpha(a,b) = 1 \ ; \tag{7}$$

- Voting strategies given status quo b:

$$\alpha_1(b,x) = \alpha_2(b,x) = \alpha_4(b,x) = 1 - \alpha_3(b,x) = 0 \ ,$$
$$\alpha_1(b,y) = \alpha_2(b,y) = \alpha_3(b,y) = 1 - \alpha_4(b,y) = 1 \ ,$$
$$\alpha_1(b,a) = 1 - \alpha_2(b,a) = \alpha_3(b,a) = 1 - \alpha_4(b,a) = 1 \ ,$$
$$1 - \alpha(b,x) = \alpha(b,y) = 1 - \alpha(b,a) = \alpha(b,b) = 1 \ . \tag{8}$$

It is readily checked that the corresponding dynamic payoffs are:

$$V_i(y|\sigma) = u_i(y) \ ,$$
$$V_i(a|\sigma) = \frac{(1-\delta)u_i(a) + \delta(1-p)u_i(y)}{1-\delta p} \ ,$$
$$V_i(b|\sigma) = \frac{(1-\delta)u_i(b) + \delta(1-p)u_i(y)}{1-\delta p} \ ,$$
$$V_i(x|\sigma) = (1-\delta)u_i(x) + \delta p\left[2V_i(y \mid \sigma) + V_i(a \mid \sigma) + V_i(b \mid \sigma)\right]$$

for each $i = 1,2,3,4$. Furthermore, for any two distinct alternatives z and z' in $\{a,b,c\}$, we have

$$\frac{V_i(z|\sigma) - V_i(x|\sigma)}{1-\delta} = u_i(z) - u_i(x) + \frac{\delta p}{1-\delta p}\big[u_i(y) - u_i(z')\big]$$

for every $i = 1,2,3,4$. From our assumptions on u_1 and u_2, it follows that

$$V_1(x|\sigma) < V_1(b|\sigma) < V_1(a|\sigma) < V_1(y|\sigma) \tag{9}$$

and

$$V_2(x|\sigma) < V_2(a|\sigma) < V_2(b|\sigma) < V_2(y|\sigma) \ . \tag{10}$$

In addition,

$$\frac{V_i(y|\sigma) - V_i(x|\sigma)}{1-\delta} = u_i(y) - u_i(x) + \frac{\delta p}{1-\delta p}\big[2u_i(y) - u_i(a) - u_i(b)\big]$$

for each $i = 1,2,3,4$. Combined with these equalities, our assumptions on u_3 and u_4 imply that there exist $\bar{\delta}_3, \bar{\delta}_4 \in (0,1)$ such that, for each $i = 3,4$, $V_i(y|\sigma) > V_i(x|\sigma)$ for all $\delta \in \big(\bar{\delta}_i, 1\big)$; so that

$$V_3(b|\sigma) < V_3(x|\sigma) < V_3(y|\sigma) < V_3(a|\sigma) \tag{11}$$

whenever $\delta > \bar{\delta}_3$, and

$$V_4(a|\sigma) < V_4(x|\sigma) < V_4(y|\sigma) < V_4(a|\sigma) \tag{12}$$

whenever $\delta > \bar{\delta}_4$.

Suppose $\delta > \max\left\{\bar{\delta}_3, \bar{\delta}_4\right\}$. Inequalities (9)-(12) imply that the voting strategies defined above satisfy condition (ii) in the definition of a stationary bargaining equilibrium. Coupled with (5)-(8), these inequalities also imply that the proposal strategies defined above satisfy condition (i) in that definition. This proves that σ is a stationary bargaining equilibrium.