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Optimal similarity judgments in intertemporal choice (and beyond)

Fabrizio Adriani and Silvia Sonderegger*

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Abstract

We use a simple cost-benefit analysis to derive optimal similarity judgments – addressing the question: when should we expect a decision maker to distinguish between different time periods or different prizes? Our key premise is that cognitive resources are costly and are to be deployed only where they really matter. We show that this simple insight can explain a number of observed anomalies, such as: (i) time preference reversal, (ii) magnitude effects, (iii) interval length effects. For each of these phenomena, our approach allows to identify the direction of the bias relative to the benchmark case where cognitive resources are costless. Finally, we show that, when applied to choice under risk, the same insights predict anomalies such as the ratio and certainty effects, and rationalize Rabin's risk aversion paradox. This suggests that the theory may provide a parsimonious explanation of behavioral anomalies in different contexts.

JEL Codes: D01, D03, D11, D90.

Keywords: Similarity judgments, Intertemporal Choice, Rational Inattention, Choice Under Risk.

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"*After all, tomorrow is another day.*"

Scarlett O'Hara.

1 Introduction

The notion that people often simplify problems by bunching together distinct objects has a a long history in cognitive psychology. Early formalizations include Luce (1962) and, most notably, Tverski (1977). More recently, Leland (2002) and Rubinstein (2003) point out that observed anomalies in intertemporal choice can be accounted for if people fail to distinguish between different time horizons. The salience literature (such as Kőszegi and Szeidl, 2012, Bordalo et. al., 2012) likewise acknowledges that agents may underweight some aspects of a problem in which their available options look similar, while focusing disproportionately on other aspects.[1] Although these models provide key insights into observed behavioral phenomena, they treat similarity (or salience) as primitives. This effectively shifts the "why" question one step back, from the observed choice to the agent's perception of the choice problem (see, e.g., Stevens, 2016). Here, we directly tackle the explanatory question. We use a simple cost/benefit analysis to derive *optimal* similarity judgments when distinguishing involves cognitive costs. Our approach thus follows the long standing tradition in Economics of using optimality as a tool to reduce the need for *ad-hoc* assumptions.

Similar to Woodford (2012a, 2012b) and Gabaix (2014), we consider a two-stage process where the decision maker's limited perception of the world in the second stage is the outcome of an optimization exercise in the first stage.[2] An Agent (A) faces a choice between two prize/delivery time bundles (x, t) and (x', t') (*Stage 2*). Prior to this and without knowing the actual choice that A will face, a Principal (P) optimally decides which differences A will be able to perceive both in the prize and the time dimensions (*Stage 1*). A chooses between the two bundles based on the dimensions, if any, in which they are perceived as different.

[1]See also the heuristic attributes tradeoff model by Scholten and Read 2010. Another approach (Steiner and Stewart, 2016, Gabaix and Laibson, 2017, Khaw et al., 2019) argues that choice biases may be an optimal response to (exogenous) imperfections in the agent's perception or representation of decision problems.

[2]A two-stage approach of this type is followed in different contexts by Bénabou and Tirole (2002) and Brunnermeier and Parker (2005), among others. Gabaix (2014) uses a two-stage setting to model inattention. Rayo and Becker (2007), Netzer (2009) and Rayo and Robson (2014) consider two-stage setups where a decision maker's preferences are the solution to a constrained optimization problem faced by Nature. Both these interpretations are possible in our setup. A further interpretation is in terms of a dual systems, in the spirit of Kahneman (2003): the principal is the system operating in the unconscious background, while the agent is the conscious mind.

The interests of the two parties are fully aligned: P's objective is to maximize A's underlying payoff from consumption, taking into account the cognitive costs of making distinctions.

The underlying theme of our analysis is simple: people tend to distinguish when there is more to be gained from distinguishing. This basic insight generates a rich set of implications.

First, optimal similarity relations take the form of similarity intervals: given a point in time, all sufficiently close time periods will be perceived as indistinguishable to it (and likewise for prizes, see Figure 1). Intuitively, distinguishing between two prizes or two delivery times is worthwhile only if they are sufficiently different.

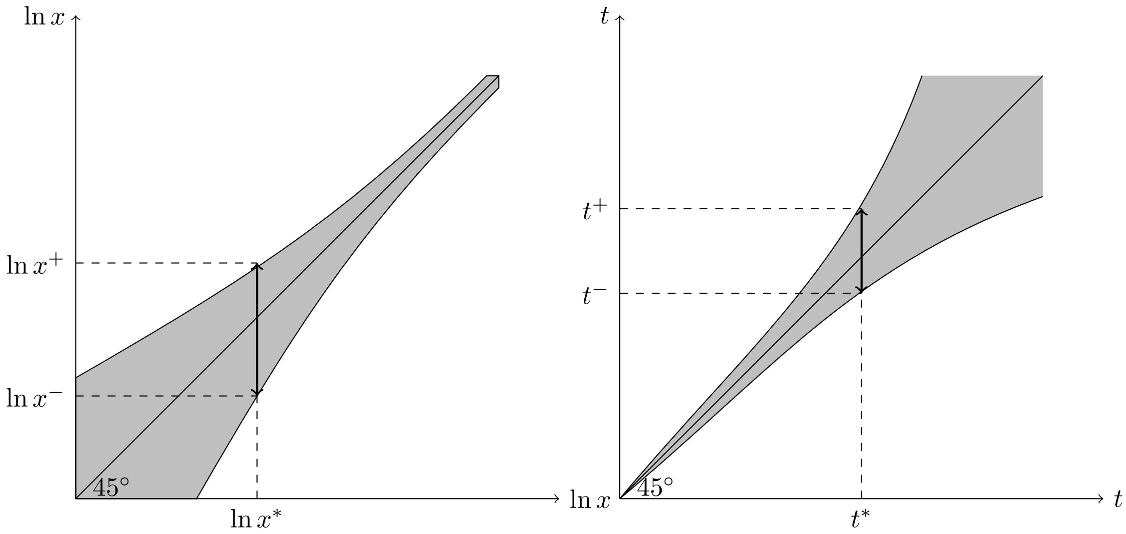

Figure 1: Optimal similarity correspondences for log-prizes, $\ln x$, and times, t.

Second, and less obvious, under standard assumptions on payoffs similarity intervals display the monotonicity properties illustrated in Figure 1. More precisely,

- In the time dimension, the theory predicts *diminishing absolute sensitivity.* As time periods move into the future, similarity intervals expand, i.e., there is a lower propensity to make fine distinctions. That's because the gain from distinguishing two distant periods is realized only in the distant future, and its present value is therefore small. It may thus happen that people distinguish between today and tomorrow, but do not distinguish between an horizon of one year and one year and one day from now.

- In the prize dimension, the theory predicts *augmenting proportional sensitivity.* As prizes increase on a log-scale, similarity intervals shrink, i.e., people make finer distinctions.[3] Intuitively, distinguishing between \$1 and \$2 yields at most a gain of \$1, but distinguishing between \$100 and \$200 could yield a gain of \$100. In other words, fixing the ratio between prizes, distinguishing generates higher returns when stakes are large.

Third, although our main focus is intertemporal choice, the analysis also produces useful insights for choice under risk. Restricting attention to choices between simple lotteries, we show that similarity judgments exhibit *augmenting proportional sensitivity* also in the domain of probabilities. Fixing the ratio between probabilities, the propensity to distinguish is higher when probabilities are larger. Intuitively, there is more to be gained from distinguishing between a probability of 0.45 and one of 0.9 than between 0.05 and 0.1.

These features of optimal similarity relations have precise implications for the type of distortions that may arise compared to the benchmark where the cost of distinguishing is zero. In particular, changing the bundles' features (e.g., by moving both delivery times closer to the present, or by multiplying both prizes by the same constant) may induce the agent's preferences to "switch" when, absent cognitive costs, he would consistently prefer either the smaller/sooner or the larger/later bundle. We show that these preference reversals must follow specific patterns, and that these patterns match a number of empirical stylized facts that are at odds with standard models.[4] These include,

1) *time preference reversal/decreasing impatience*: people become less impatient when facing money/delay tradeoffs that are further in the future,

2) *magnitude effect*: people become less impatient when stakes are larger,

3) *interval length effects*: people discount short horizons differently than long horizons,

4) *ratio effect and certainty effect*: if the winning odds of two gambles are increased proportionately, people become less willing to choose the riskier option.

5) *Rabin's paradox of risk aversion:* people exhibit "unreasonable" risk aversion in gambles with small stakes.

[3]The terminology "diminishing absolute sensitivity" and "increasing proportional sensitivity" is borrowed from Scholten and Read (2010), who impose these properties on their time weighting function and their outcomes value function, respectively. Our model obtains them *endogenously*, as features of optimal similarity judgments.

[4]Time inconsistency and the magnitude effect are two of the classic anomalies in intertemporal choice identified by Loewenstein and Thaler (1989) and Loewenstein and Prelec (1992), Frederick et al. (2002). Read (2001) and Scholten and Read (2006, 2010) discuss evidence on preference reversals within the context of case 3 above (interval length effects). The surveys by Manzini and Mariotti (2009) and Cohen et al. (2019) on intertemporal choice provide additional references for all three phenomena. The ratio effect and its special case, the certainty effect, are classic anomalies in the domain of risk (see e.g., Starmer 2000). For Rabin's paradox, see Rabin (2000).

Of course, if similarity judgments are of the "right type", a model of exogenous similarity could replicate our empirical predictions, but, at the same time, by appropriately modifying similarity judgments, it could also predict the opposite. What we do here goes beyond fitting the model to the data. Optimal similarity judgments predict some patterns of behavior but, crucially, rule out others. This makes the theory easily falsifiable.

Each of anomalies 1)-5) has been modelled separately.[5] What we add to the literature is the observation that a very simple insight – i.e., that people are more likely to distinguish when there is more to be gained from distinguishing – can provide a unifying interpretation for these superficially disjoint phenomena. The notion that behavioral anomalies in risk and intertemporal choice domains may be linked is reminiscent of Halevy (2008), who argues that patience (or lack thereof) reflects the inherent uncertainty of future consumption. People thus exhibit diminishing impatience because they display the common ratio and certainty effects in uncertain environments (see also Saito, 2011). The point we make here is rather different. We show that the same underlying mechanism – optimal similarity judgments – may cause anomalous behavioral patterns in both the contexts of risk and intertemporal choice. Furthermore, optimal similarity judgments are not only consistent with these behavioral anomalies, they actually rule out the opposite patterns, such as increasing impatience in the time domain and certainty aversion in the risk domain.

A further advantage of the approach is that it provides a clear benchmark against which we can assess behavioral anomalies, thus contributing to the debate on welfare analysis under non-standard behavior (see e.g., Bernheim and Rangel, 2009, Masatlioglu et al., 2012). For each behavioral anomaly, we answer the question: what would the agent prefer if he could perfectly distinguish in all dimensions? This may deliver surprising insights, at least at first glance. For instance, compared to the benchmark, we find that time preference reversal reflects *excessive patience* when confronting delivery times that are distant from the present. This is because, unlike today and tomorrow, two sufficiently faraway time periods are perceived as indistinguishable. The result thus suggests a counterargument to the common view that the agent's preferences when comparing distant horizons are a better

[5] Well-known examples in the intertemporal context include hyperbolic discounting (Loewenstein and Prelec 1992, Laibson 1997), Manzini and Mariotti's (2006) (σ, δ) −model, Benhabib et al.'s (2010) fixed cost model. In the Supplementary Appendix, we carefully compare our predictions with those of existing leading models. In the context of risk, important examples include the models of the certainty effect by Gilboa (1988), Jaffray (1988), Schmidt (1998) and Cerreia-Vioglio et al. (2015), as well as the model of small stakes risk aversion by Khaw et al. (2019).

reflection of his true welfare, since they are not influenced by "inefficient" urges for immediate gratification.[6]

The remainder of the paper is organized as follows. We present the model in the next section and illustrate the main result with examples in Section 3. Section 4 provides the main theorem and analyzes its implications. Section 5 focuses on time preference reversal, magnitude effect, and interval length effects. Section 6 presents a number of extensions, including choice under risk. Concluding remarks are in Section 7.

2 Model

We consider the following two-stage process. In the first stage, a principal (P) decides which periods (or which prizes) should be perceived by an agent (A) as similar or distinct. In the second stage, A makes a consumption choice using the similarity judgments that were selected in the first stage.

Agent A is a decision maker who, in the present, must select between two two-dimensional bundles $\boldsymbol{y} = (y_1, y_2)$ and $\boldsymbol{y}' = (y_1', y_2')$, with $y_i \in [\underline{y}_i, \overline{y}_i] \subseteq \mathbb{R}^+$, $i = 1, 2$. For most of the paper, we will identify dimension 1 as the *prize* dimension (denoted with x) and dimension 2 as the *delivery time* dimension (denoted with t). In Section 6.3, dimension 2 will be interpreted as the *probability* to win the prize (denoted with π). The bundles featuring in A's problem are randomly selected by Nature, who draws prizes and delivery times from a joint density $\boldsymbol{f}(y_1, y_1', y_2, y_2')$, $\boldsymbol{f} : \mathcal{Y}^2 \to \mathbb{R}^+$, $\mathcal{Y} := \prod_{i=1,2} [\underline{y}_i, \overline{y}_i]$. We assume that,

Assumption 1. *The bundle-generating process satisfies:*

1. (Independence across dimensions) $(y_1, y_1') \perp\!\!\!\perp (y_2, y_2')$*.*

2. (Exchangeability within dimensions) For any dimension $i = 1, 2$*,* (y_i, y_i') *are distributed with joint density* p_i*, where* p_i *is continuous, has full support and is invariant with respect to permutations of its arguments, i.e.* $p_i(y_i, y_i') = p_i(y_i', y_i)$ *for all* $(y_i, y_i') \in [\underline{y}_i, \overline{y}]^2$*.*

This ensures that the two bundles are *a priori* identical – so that labelling a bundle $\boldsymbol{y}$ or $\boldsymbol{y}'$ is inconsequential – and that the bundles' realized values in one dimension provide no information about the bundles' attributes in the other dimension. Section 6.4 discusses what happens when Assumption 1 is relaxed.

[6] See e.g., O'Donoghue and Rabin (2000).

Principal P sets the stage for A's decision-making by determining similarity relations. These specify what A perceives as identical and what he perceives as distinct, and are chosen before the bundles featuring in A's choice are drawn. More precisely, for all possible (y_i, y_i') pairs and all dimensions $i = 1, 2$, P decides whether A distinguishes between y_i and y_i' ($y_i \not\approx y_i'$) or not ($y_i \approx y_i'$). We assume,

Assumption 2. *(Separability of similarity judgments) For all $i = 1, 2$, $j \neq i$, and $(y_i, y_i') \in [\underline{y}_i, \overline{y}_i]^2$, whether $y_i \approx y_i'$ or $y_i \not\approx y_i'$ is independent of (y_j, y_j').*

In words, if A distinguishes between y_i and y_i', he will always perceive these values as distinct, independently of the attributes in the other dimension. Several experimental and neuro-science works provide indirect support for separability between different dimensions, such as the temporal and the magnitude aspects of consumption.[7] In addition, if similarities in one dimension could be made contingent on ex-post realizations in other dimensions, this would necessarily lead to extreme conclusions, such as A never perceiving differences in more than one dimension. This is because P would always ensure that A picks the "right" bundle by letting him distinguish only in one dimension, where the desired bundle is superior to the alternative.

Since P moves before the bundles featuring in A's choice are drawn, when determining A's similarity judgments, P only knows the joint distribution, $\boldsymbol{f}$, of the characteristics of the two bundles. There is no conflict of interests between the principal and the agent; P simply chooses A's similarity judgments to maximize an *underlying payoff* generated by A's choice (defined below), net of cognitive costs.

Cognitive cost Making distinctions entails costly cognitive effort, paid in the present (when A's choice is made). More precisely, distinguishing between any pair (y_i, y_i') carries a fixed cognitive cost $c_i > 0$. The cognitive costs of distinguishing are thus independent of the values of the attributes to be distinguished. This helps to clarify that the effects we highlight arise entirely from expected benefit considerations, rather than from the (untestable) features of

[7]Using functional magnetic resonance imaging, Ballard and Knutson (2009) identify distinct patterns of brain activity associated with each dimension. Activation in the mesolimbic projection regions correlates with increasing the magnitude of future rewards, while activation in lateral cortical regions correlates with increasing delays of future rewards (see also Pine et al., 2009 and Kable and Glimcher, 2007, 2010). Amasino et al. (2018) provide similar evidence using eye-tracking measures. Finally, for choices between lotteries, Arieli at al. (2011) argue that the eye patterns of decision makers suggest that they compare prizes and probabilities one at a time.

the cognitive cost technology.[8]

Underlying payoff The underlying payoff associated with bundle $\boldsymbol{y} = (y_1, y_2)$ is separable in the time and prize dimensions, and is given by $u(\boldsymbol{y}) = \phi_1(y_1)\,\phi_2(y_2)$, with $\phi_i : [\underline{y}_i, \overline{y}_i] \to \mathbb{R}^+$, $i = 1, 2$, continuous and strictly monotone. In what follows, we will first state a number of results that hold generally, for any $\phi_1(.)$ and $\phi_2(.)$ that satisfy these conditions. Further results are then obtained by assuming standard exponential discounting/isoelastic consumption utility (EDIU).

***A*'s preferences** We assume that, when faced with the choice between two bundles, A's preferences are Monotone Along Distinguished Dimensions (MADD), as follows.

1. if A distinguishes between the two bundles in both dimensions, he prefers the bundle yielding the highest underlying payoff.

2. if A distinguishes between the two bundles only in dimension $i = 1, 2$, he prefers the bundle that is better in that dimension, i.e. the one corresponding to $\max\{\phi_i(y_i), \phi_i(y_i')\}$.

3. if A does not distinguish between the two bundles in any dimension, he is indifferent.

MADD preferences are closely related to Rubinstein's (2003) (*) procedure, and are an example of an Intra-Dimensional Comparison (IDC) heuristic, first documented by Tversky (1969). They satisfy the Separability axiom introduced in Tserenjigmid's (2015) model of the IDC heuristic. In Section 6.5, we argue that, in a more general setup where P optimally selects A's preferences *as well as* similarity judgments, MADD preferences are optimal under intuitive restrictions.

Tie breaking rule We assume that, whenever A perceives no difference between two bundles in any dimension (as well as in the knife-edge cases where the two bundles generate the same payoff in the dimensions that A distinguishes), he selects either bundle with probability 1/2. Note however that all of the formal statements in the paper would largely remain unaffected if we instead considered a slightly modified setup, in which failing to distinguish in any dimension leads to randomization *only* if neither of the two bundles is strictly dominated,

[8] Assuming that the cost depends on the values of the attributes to be distinguished in an "intuitive" fashion, e.g., it is higher when these are closer to one another, would only reinforce our results. These would also continue to apply if we adopted a more general cost specification, provided that we rule out pathological cases, such as, e.g., that the cost of distinguishing between time periods or prizes decreases sharply as these get close to one another, or that the cost of distinguishing faraway periods is lower than that of distinguishing periods close to the present.

and A chooses the dominant bundle with probability one otherwise. As will become clear, none of our results require that A selects strictly dominated bundles.

Principal's problem For all pairs of attributes $(y_i, y_i') \in [\underline{y}_i, \overline{y}_i]^2$ and $i = 1, 2$, P chooses whether $y_i \approx y_i'$ or $y_i \not\approx y_i'$ to maximize

$$Q(\boldsymbol{y}, \boldsymbol{y'})u(\boldsymbol{y}) + (1 - Q(\boldsymbol{y}, \boldsymbol{y'}))u(\boldsymbol{y'}) - c_1 I_1(y_1, y_1') - c_2 I_2(y_2, y_2'), \tag{1}$$

where $I_i : [\underline{y}_i, \overline{y}_i]^2 \to \{0, 1\}$ takes value 1 if $y_i \not\approx y_i'$ and 0 otherwise. $Q : \mathcal{Y}^2 \to \{0, 1/2, 1\}$ is a function returning the probability that A chooses $\boldsymbol{y}$ when $\boldsymbol{y'}$ is also available,

$$Q(\boldsymbol{y}, \boldsymbol{y'}) = \begin{cases} 1 & \text{if } \boldsymbol{y} \succ \boldsymbol{y'} \\ \frac{1}{2} & \text{if } \boldsymbol{y} \sim \boldsymbol{y'} \\ 0 & \text{if } \boldsymbol{y} \prec \boldsymbol{y'}, \end{cases} \tag{2}$$

which depends on $I_i(y_i, y_i')$ via MADD preferences:

$$\boldsymbol{y} \succsim \boldsymbol{y'} \Leftrightarrow \prod_{i=1,2} \phi_i\left(y_i\right)^{I_i\left(y_j, y_j'\right)} \geq \prod_{i=1,2} \phi_i\left(y_i'\right)^{I_i\left(y_i, y_i'\right)}. \tag{3}$$

Given Assumption 2 (separability), similarity relations in dimension $i = 1, 2$ are determined independently of dimension $j \neq i$. In practice, this implies that P chooses whether $y_i \approx y_i'$ or not to maximize the expected value of (1), where the expectation is taken with respect to the attributes in dimension j.

Timing To sum up, the decision process in our setup is determined by the following stages.

1. P decides what differences A perceives in the prize and time dimensions;

2. Bundles $(\boldsymbol{y}, \boldsymbol{y'}) \in \mathcal{Y}^2$ in A's problem are drawn;

3. Given the similarity relation in 1., A chooses a bundle and incurs cognitive costs $\sum_{i=1}^{2} c_i I_i(y_i, y_i')$.[9]

[9]Note that we are assuming that the cost of distinguishing in dimension i is the same, independently of whether A also distinguishes in dimension $j \neq i$. This is immaterial. Assuming, for instance, that the cost increases in the number of dimensions that A distinguishes would slightly complicate the analysis without changing any substantial result.

3 Two simple properties

To build intuition, we start with two examples that illustrate the logic behind our main results. Suppose that the underlying payoff is $u(x,t) = xe^{-\delta t}$, $\delta > 0$. To abstract from the prize dimension, we fix the prizes of the two bundles and assume that the bundles featuring in A's problem are of the type $(1,t)$ and $(2,t')$, where delivery times $t \geq 0$ and $t' \geq 0$ are randomly drawn and $1 \not\approx 2$. If ex-post A distinguishes between any t and $t' = t + \ell$, $\ell > 0$, he will choose the bundle that maximizes the underlying payoff, i.e. $(1,t)$ if $1 \geq 2e^{-\delta\ell}$ and $(2,t')$ otherwise. If he doesn't distinguish in the time dimension, he will go for the bundle with the larger prize, $(2,t')$. The benefit from distinguishing between t and $t' = t + \ell$ is thus 0 if $2e^{-\delta\ell} \geq 1$ and

$$e^{-\delta t}(1 - 2e^{-\delta\ell}) \tag{4}$$

otherwise. Keeping ℓ constant, (4) is decreasing in t. This implies that the benefit from distinguishing between two time periods is consistent with *diminishing absolute sensitivity:* fixing the lag separating the two periods, it decreases as these are pushed into the future. Intuitively, the gain from distinguishing between two distant periods will only be realized in the distant future, and therefore has a small present value.

Focusing now only on the prize dimension, let's fix the delivery times and assume that A has to choose between bundles $(x,0)$ and $(x',1)$, where $x \geq 0$ and $x' \geq 0$ are unknown to P and $0 \not\approx 1$. Let $r \equiv x/x'$. If A does not distinguish between x and rx, he will choose the early bundle $(x,0)$. If he distinguishes, he will select the bundle that maximizes the underlying payoff. The additional benefit accrued from making the distinction is 0 if $e^{-\delta}r \leq 1$ and

$$x(e^{-\delta}r - 1) \tag{5}$$

otherwise. Expression (5) is increasing in x and is thus consistent with *augmenting proportional sensitivity*: fixing the ratio between the prizes, the benefit from the distinction is larger when stakes are higher.

4 Optimal similarity judgments and resulting preferences

In general, P selects what A will distinguish by comparing, for each dimension i and for all possible pairs of realizations y_i and y_i', the expected benefit $b_i\left(y_i, y_i'\right)$ from distinguishing between y_i and y_i' to the cost c_i of making the distinction. This happens in both dimensions simultaneously and without knowing which realized bundles A will face. For instance, when choosing whether to let A distinguish or not between, say, prizes x and x', P does not know which time periods will be associated with these prizes (if they happen to feature in A's choice), and needs to consider all the possibilities. Depending on the realized values of t and t' and on whether A distinguishes between them or not, it is possible that distinguishing between x and x' may improve, have no effect or even worsen the quality of A's choice.[10] Symmetric considerations apply to the decision to let A distinguish between any two time periods. The nature of P's problem implies that uniqueness of the optimal similarity relation is not guaranteed, which complicates the task of providing a full characterization.[11] It is however possible to derive properties that must apply to optimal similarity judgments in a given dimension irrespective of the shape of similarity judgments in the other.

Theorem 1. *Under MADD preferences and Assumptions 1 and 2, for all $i = 1, 2$, the expected benefit $b_i\left(y_i, y_i'\right)$ from distinguishing between any y_i and y_i' takes the form*

$$b_i\left(y_i, y_i'\right) = \max\{\phi_i\left(y_i\right), \phi_i\left(y_i'\right)\} \;\cdot\; B_i\left(\Delta\left(y_i, y_i'\right)\right), \tag{6}$$

where

$$\Delta\left(y_i, y_i'\right) := |\ln \phi_i(y_i) - \ln \phi_i(y_i')|, \tag{7}$$

and $B_i\left(.\right)$ is a continuous, strictly increasing function that satisfies $B_i\left(0\right) = 0$ and that depends on $\left(y_i, y_i'\right)$ only through Δ.

Proof: See Appendix.

The Theorem establishes that the expected benefit from distinguishing between y_i and y_i'

[10] The first two cases are quite straightforward. For an example of the third case, suppose that $x > x'$ and that t and t' are such that $u(x,t) < u(x',t')$ but $t \approx t'$. If $x \not\approx x'$, A selects (x,t). However, if $x \approx x'$, A randomizes, obtaining a larger expected payoff.

[11] An illustrative example with a discrete distribution: let $x \in \{1, 2\}$ and $t \in \{0, 1\}$, $e^{-\delta} = 0.5$, and $c_1 = c_2 = c \in [0.1, 0.19]$. Each bundle is drawn from $\{(1,0), (1,1), (2,0), (2,1)\}$ with uniform probabilities. It is then easy to check that expected payoff net of cognitive costs is maximized by either (i) prize dimension: $1 \not\approx 2$, time dimension: $0 \approx 1$, or (ii) prize dimension: $1 \approx 2$, time dimension: $0 \not\approx 1$.

can always be factored into the product of $\max\{\phi_i(y_i), \phi_i(y_i')\}$ and another function $B_i(.)$ that depends on y_i and y_i' only through Δ. Intuitively, $\max\{\phi_i(y_i), \phi_i(y_i')\}$ reflects the *importance* of dimension i, expressed by the magnitude of the payoff potentially associated with it, while $\Delta(y_i, y_i')$ is a measure of the *distance* in the payoff space between the two bundles in dimension i. The expected benefit is increasing in both distance and importance, and is zero when the bundles are identical in a given dimension. Under the convention that a distinction occurs only if the benefit is strictly higher than the cost, optimal similarity judgments satisfy,

$$y_i \not\approx y_i' \;\Leftrightarrow\; b_i(y_i, y_i') > c_i \tag{8}$$

$$\text{or, equivalently, } y_i \not\approx y_i' \;\Leftrightarrow\; \Delta(y_i, y_i') > B_i^{-1}\left(\frac{c_i}{\max\{\phi_i(y_i), \phi_i(y_i')\}}\right), \tag{9}$$

where $B_i^{-1}(.)$ is strictly increasing and such that $B_i^{-1}(0) = 0$. The optimal similarity relation is reflexive (since $b_i(y_i, y_i) = 0 < c_i$) and symmetric (since $b_i(y_i, y_i') = b_i(y_i', y_i)$). In the Supplementary Appendix, we also prove that it generally admits a numerical representation.[12] A's preferences are given by representation (3) with $I_i(y_i, y_i') = 1$ iff inequality (9) holds. The preference relation $\succ$ is irreflexive and asymmetric, but it is not generally transitive.[13] In this respect, the preferences we obtain are closely related to the *additive difference* model postulated by Tversky (1969). Our microfoundation adds to Tversky's model the feature that the "contribution" of dimension i to A's choice does not only depend on the distance in payoffs, $\Delta(y_i, y_i')$, but also on the dimension's importance, $\max\{\phi_i(y_i), \phi_i(y_i')\}$.

Salience Our theory naturally lends itself to an interpretation in terms of salience. When A perceives y_i and y_i' as identical, dimension i is irrelevant (*not salient*) for A's choice. In contrast, if A distinguishes between y_i and y_i' , then dimension i is accounted for (*salient*) in his decision. We can think of optimal similarity jugments as assigning a weight to each attribute, which is 1 if A distinguishes between the two bundles' realizations in that dimension and 0 otherwise.[14] Clearly enough, *ceteris paribus* a larger distance $\Delta(y_i, y_i')$ makes it more likely that dimension i is salient. This is somewhat reminiscent of Assumption 1 of Kőszegi and Szeidl's (2013) salience model, which posits that the agent focuses more on attributes

[12] We also prove that, within the context of the choice under risk application, the numerical representation of optimal similarity judgments is incompatible with expected utility.

[13] The notion that similarity judgments may be a source of intransitivity has long been understood (see e.g. Armstrong 1951 and Luce 1956).

[14] Probabilistic distinctions are explored in the Supplementary Appendix.

which generate a greater range of utility within the choice set (although their interpretation of what constitutes an attribute differs from ours). Theorem 1 however makes clear that the relationship between payoff distance and salience is mediated by the importance of a dimension, in the sense given above.

We now explore a number of implications of the Theorem.

4.1 Similarity intervals

The first result reflects the idea that the expected benefit from distinguishing necessarily decreases as two attributes become more similar.

Corollary 1. *Given y_i, $b_i(y_i, y_i')$ is U-shaped in y_i', reaching a minimum of zero at $y_i' = y_i$. Optimal similarity judgments thus take the form of similarity intervals.*

Proof: See Appendix.

The logic behind the second statement in Corollary 1 is illustrated in figure 2. For each realization y_i, we can identify an interval around y_i such that A sees all realizations belonging to that interval as indistinguishable from y_i, and all those outside the interval as different from it. *Ceteris paribus*, dimension i is thus more likely to be salient when the (absolute value of the) difference between y_i and y_i' is larger. Note that, from symmetry, if y_i' belongs to y_i's similarity interval, the opposite also holds: y_i belongs to y_i''s similarity interval.

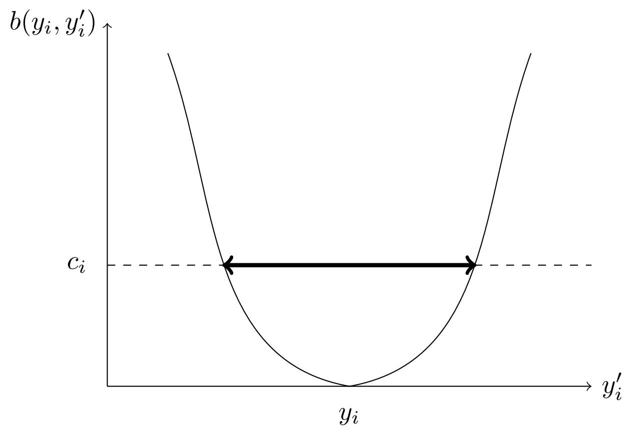

Figure 2: Similarity intervals.

While the Theorem and Corollary 1 are obtained without imposing specific functional forms on the underlying payoff, for concreteness it is instructive to take a stance before proceeding with further characterization. Accordingly, in all the results that follow we focus on,

$$u(x,t) = \frac{x^{1-\rho}}{1-\rho} e^{-\delta t} \tag{10}$$

with $\delta > 0$ and $1 > \rho \geq 0$. This ensures that, in the absence of cognitive costs, our setting reduces to the standard Exponential Discounting Isoelastic Utility (EDIU) model (more general payoffs are discussed in Section 6.1). The EDIU model has the following properties. In the time dimension

$$\Delta(t, t+\ell) = \delta|\ell|, \tag{11}$$

which is independent of t. In the prize dimension

$$\Delta(x, rx) = (1-\rho)\,|\ln r|, \tag{12}$$

which is independent of x. Hence, the expected benefit from distinguishing between t and $t+\ell$ depends on t only through the term $\max\{\phi_2(t), \phi_2(t+\ell)\}$ in (6), and the expected benefit from distinguishing between x and rx depends on x only through $\max\{\phi_1(x), \phi_1(rx)\}$. The two properties identified in Section 3 thus apply more generally, as summarized below.

Corollary 2. *Under EDIU payoff, $b_1(x, rx)$ is strictly increasing in x and $b_2(t, t+\ell)$ is strictly decreasing in t. On a log-scale, the similarity interval of x thus shrinks as x increases (augmenting proportional sensitivity). On a linear scale, t's similarity interval widens as t increases (diminishing absolute sensitivity).*

Proof: See Appendix.

This implies that A's ability to draw distinctions in the prize dimension sharpens as prizes become larger, and, in the time dimension, as delivery times move closer to the present. An optimal similarity correspondence thus has the monotonicity properties illustrated in Figure 1 (see Introduction).[15]

[15] There is an interesting connection between similarity intervals in the prize dimension and the generalized Fechner-Weber Law in psychology (introduced to accommodate the most common deviations from the simple form of the law see, e.g., Norwich 1987), which says that q^+, the threshold for detecting an increment in an initial quantity q is given by: $q^+ = k_0 + k_1 q$, where k_0 and k_1 are positive constants. This implies that $\ln q^+ - \ln q = \ln(k_0/q + k_1)$, which is decreasing in q. Thus, on a log-scale, smaller quantities have larger similarity intervals.

4.2 Rescaling of prizes and uniform shifts in delivery times

We now highlight some comparative statics properties of optimal similarity judgments. If A optimally distinguishes between two prizes or two delivery times, he will continue to do so if, say, both prizes are doubled, or if both periods shift closer to the present by one year (although, crucially, the opposite does not necessarily hold). Intuitively, these changes increase the importance of the prize or time dimensions, whilst leaving the distance Δ unaffected. The expected benefit from distinguishing is accordingly larger. Formally,

Corollary 3. *Under EDIU payoff:*
In the time dimension, for all $s > 0$*:* $t + s \not\approx t' + s \Rightarrow t \not\approx t'$.
In the prize dimension, for all $\alpha > 1$*:* $x \not\approx x' \Rightarrow \alpha x \not\approx \alpha x'$.

Proof: See Claim 4 in Appendix.

This has direct consequences in terms of A's preferences over bundles.

Corollary 4. *Under EDIU payoff:*
i) For $t > t'$ *and* $s > 0 : (x, t + s) \prec (x', t' + s) \Rightarrow (x, t) \prec (x', t')$.
ii) For $x > x'$ *and* $\alpha > 1 : (x, t) \succ (x', t') \Rightarrow (\alpha x, t) \succ (\alpha x', t')$.

Proof See Claim 5 in Appendix.

To see the logic behind the result, suppose that A strictly prefers \$2 tomorrow to \$1 today. This requires that A perceives \$2 and \$1 as different, or else he would never prefer the bundle with the greater (but delayed) prize. Moreover, either A does not distinguish between today and tomorrow or, if he does, he is happy to delay consumption by one day in order to double the prize. Corollary 3 then implies that A also prefers \$4 tomorrow to \$2 today (since he must also distinguish between these two prizes). Similarly, if A prefers \$1 in *one year* to \$2 in *one year and one day*, then he must also prefer \$1 *today* to \$2 *tomorrow.*

4.3 Distortions compared to the benchmark

Finally, we discuss how A's preferences may differ from the benchmark case where cognitive costs are zero (in which case A always maximizes the underlying payoff). We focus on distortions that alter the direction of A's strict preferences compared to the benchmark. At first glance, these distortions may appear to be negligible, since, ex-ante, the expected loss in payoff is bounded by the cognitive cost, and thus likely to be small. However, while small, these distortions are *systematic*, and may thus have a large cumulative effect. The first observation is that, from (3), A never strictly prefers a dominated bundle. Consider then a choice between a larger/later bundle (x, t) and smaller/sooner bundle (x', t'), where $x > x'$ and $t > t'$. Two cases are in principle possible.

a. (*Early gratification bias*) $u(x, t) > u(x', t')$ but similarity judgments are $x \approx x'$ and $t \not\approx t'$, so that $(x, t) \prec (x', t')$.

b. (*Delayed gratification bias*) $u(x', t') > u(x, t)$ but similarity judgments are $x \not\approx x'$ and $t \approx t'$, so that $(x', t') \prec (x, t)$.

Hence, distortions may take the form of making A more (case a.) or less (case b.) willing to engage in early gratification compared to the benchmark. Corollary 2 implies that case a. is more likely to arise when prizes are small and consumption in the smaller/sooner bundle is immediate or nearly so (as in the case of small impulse purchases). Case b. occurs when prizes are relatively large and the periods involved are faraway (as in the case of careerist workaholics, who commit in advance to too many projects).

5 Empirical "anomalies"

Corollaries 3 and 4 can help us shed light onto a number of empirical "anomalies" that cannot be easily explained by standard models. To this purpose, we consider two different choice frames with the property that, in the standard EDIU model with zero cognitive costs, A would consistently prefer either the sooner/smaller or the larger/later bundle in *both* frames.

5.1 Time preference reversal

Our first result deals with the observed tendency by decision makers to exhibit greater impatience when the money/delay tradeoffs they are faced with are closer to the present – a phenomenon that is commonly (and somewhat imprecisely) referred to as (time) preference reversal.[16] Consider the following two frames. For some $s > 0$, $t > t'$ and $x > x'$,

[16]The reversals we focus on here occur when the same person makes decisions now both for the near future and for the distant future, and the resulting pattern violates standard models (what Ericson and Laibson, 2019 call semi strong preference reversals). As noted e.g., by Manzini and Mariotti (2009), however, although commonly used, the "reversal" terminology may be misleading, in the sense that, even if A switches preferences when delivery times are closer to the present, strictly speaking nothing is really reversed, as the bundles he is facing are different. That's why Halevy (2008) simply uses the term "decreasing impatience."

(a) A must choose between (x, t) and (x', t') [Near term frame.]

(b) A must choose between $(x, t+s)$ and $(x', t'+s)$ [Distant frame.]

Given EDIU, $u(x,t) \lesseqgtr u(x',t') \Leftrightarrow u(x,t+s) \lesseqgtr u(x',t'+s)$ and, hence, if cognitive costs were zero, A would prefer either the smaller/sooner or the larger/later bundle in both frames. With positive cognitive costs, however, a reversal of preferences may occur.

Proposition 1. *(Time preference reversal) Any reversal of strict preferences between the two frames consistent with optimal similarity judgments must take the form:* $(x,t) \prec (x',t')$ *and* $(x,t+s) \succ (x',t'+s)$*. Preference reversal occurs if and only if, given the similarity relation, prizes and delivery times in the two frames satisfy: (i)* $x \not\approx x'$*, (ii)* $t \not\approx t'$*, (iii)* $t+s \approx t'+s$ *(iv)* $u(x,t) < u(x',t')$*.*

Proof: See Claims 6 and 7 in Appendix.

Hence, any reversal between the two frames must necessarily take the form of *decreasing impatience.* A switches from preferring the smaller/sooner bundle in a) to preferring the larger/later bundle in b). Intuitively, while in the near term frame the time dimension is salient ($t \not\approx t'$), in the distant frame it is not ($t+s \approx t'+s$), which causes A's preferences to differ in the two frames. Importantly, the theory predicts that decreasing impatience is the *only* type of preference reversal that arises. The opposite pattern (increasing impatience) would require delivery times being salient when they are faraway ($t+s \not\approx t'+s$) but not when they are close to the present ($t \approx t'$), which would contradict Corollary 3.

Consider now the benchmark case where cognitive costs are zero. What would A consistently prefer in that case? The smaller/sooner or the larger/later bundle? Part (iv) of the Proposition provides the answer.

Remark 1. *Whenever time preference reversal occurs, A would prefer the smaller/sooner bundle in both frames in the absence of cognitive costs.*

In other words, the inability to perfectly distinguish causes A to be *excessively patient* in the distant frame. The issue of how to interpret preference reversal in terms of welfare is clearly important, but it is also far from straightforward.[17] In our model, the reversal arises as the solution to an optimization problem which takes cognitive costs into account, and, in that sense, it is an optimal phenomenon. At the same time, it is also clear that it reflects a

[17]This is highlighted, e.g., by Bernheim (2009), Bernheim and Rangel (2009).

bias compared to the benchmark. This naturally raises questions on the nature of this bias. Does it induce A to postpone consumption in the distant frame, or to bring consumption forward in the near term frame? This is a counterfactual question that cannot be answered by empirical evidence. However, starting from Strotz (1956), the explicit or implicit narrative attached to time preference reversal is that it takes the form of A being "too impatient" in the near term frame. Our analysis provides a counterargument to this logic. When evaluating faraway periods, A perceives them as indistinguishable. His preferences over bundles are therefore based on an incomplete perception of reality, and are thus distorted compared to the case where he perfectly distinguishes in both dimensions. This of course should not be taken to imply that cognitive costs never distort A's choices towards early gratification. This type of distortions can arise, as seen in Section 4.3 (and in the next section), but never in conjunction with time preference reversal.

Finally, note that, as highlighted in part (i) of the Proposition, for time preference reversal to occur the prize dimension must be salient in both frames ($x \not\approx x'$). Applying Corollary 3, this delivers the following prediction,

Corollary 5. *If time preference reversal occurs when prizes are x and x', then it also occurs when prizes are αx and $\alpha x'$, where $\alpha > 1$, but not vice-versa.*

5.2 Magnitude Effect

Consider the following two frames. For some $\alpha > 1$, $t > t'$ and $x > x'$,

(a) A must choose between (x, t) and (x', t') [Small stakes frame.]

(b) A must choose between $(\alpha x, t)$ and $(\alpha x', t')$ [Large stakes frame.]

EDIU implies $u(\alpha x, t) \lesseqgtr u(\alpha x', t') \Leftrightarrow u(x, t) \lesseqgtr u(x', t')$ and, hence, if cognitive costs were absent, A's preferences would be consistent across the two frames. With cognitive costs,

Proposition 2. *(Magnitude effect) Any reversal of strict preferences between the two frames consistent with optimal similarity judgments must take the form: $(x, t) \prec (x', t')$ and $(\alpha x, t) \succ (\alpha x', t')$. Preference reversal occurs if and only if, given the similarity relation, prizes and delivery times in the two frames satisfy: (i) $t \not\approx t'$, (ii) $x \approx x'$, (iii) $\alpha x \not\approx \alpha x'$ and (iv) $u(x, t) > u(x', t')$.*

Proof: See Claims 6 and 7 in Appendix.

Any preference reversal must take the form of a *magnitude effect.* A becomes more patient as stakes increase, switching from preferring the smaller/sooner bundle in (a) to preferring the larger/later in (b). Intuitively, the prize dimension is salient in the large, but not in the small stakes frame ($\alpha x \not\approx \alpha x'$ and $x \approx x'$). Note that the magnitude effect is the *only* preference reversal that may occur between the two frames. The opposite pattern (A becoming *less patient* as stakes increase) would violate Corollary 4, as it would require prizes being salient when they are small but not when they are large ($x \not\approx x'$ and $\alpha x \approx \alpha x'$).

The observation that people become more patient when larger stakes are involved is well documented by lab experiments, but is incompatible with standard models of linear/isoelastic consumption utility.[18] This is problematic, not least because, as many argue, for small amounts (as monetary prizes in experiments tend to be), the utility function ought to be linear. Even leaving this issue aside, Noor (2011) shows that the curvature of utility required to accommodate the evidence may be too extreme to be realistic in *any* environment. Our theory provides an alternative explanation of magnitude effects, based on optimal similarity.

Next, we assess the nature of the distortion compared to the benchmark case of zero cognitive costs. From part (iv) of the Proposition,

Remark 2. *Whenever a magnitude effect occurs, A would prefer the larger/later bundle in both frames in the absence of cognitive costs.*

Intuitively, when prizes are small, A fails to correctly perceive the difference between them, and this induces a bias compared to the benchmark case. The distortion thus takes the form of excessive impatience in the small stakes frame. Note that part (i) of the Proposition suggests that, for a magnitude effect to arise, the time dimension must be salient in both frames ($t \not\approx t'$). Applying Corollary 3, this implies,

Corollary 6. *If a magnitude effect occurs when delivery times are t and t', then it also occurs when delivery times are $t - s$ and $t' - s$, where $s > 0$, but not vice-versa.*

[18] Many studies obtain these magnitude effects by eliciting indifference points. For instance, Benhabib et al. (2010) ask questions of the type: "What amount of money y would make you indifferent between x today and y in τ days?", where x is equal to \$10, \$20, \$30 etc., and τ is equal to 3 days, 2 weeks, 1 month etc., depending on the treatment. In the Supplementary Appendix we show that our setup can rationalize the evidence obtained with this type of data.

5.3 Super/subadditivity (Interval Length Effects)

Our third observation addresses the issue of interval length effects. Consider the following two frames. For some $r > 1$, $\ell > 0$, and $\kappa > 1$,

(a) A must choose between (x, t) and $(rx, t + \ell)$ [Short horizon frame.]

(b) A must choose between (x, t) and $(r^{\kappa}x, t + \kappa\ell)$ [Long horizon frame.]

Note that, absent cognitive costs, A would consistently prefer the smaller/sooner bundle in both frames if $r^{1-\rho}e^{-\delta\ell} < 1$ (and the larger/later bundle under the reverse inequality). Borrowing from the literature, we use the following terminology.

Definition 1. *A's preferences exhibit subadditivity if $(x, t) \succ (rx, t+\ell)$ and $(x, t) \prec (r^{\kappa}x, t+\kappa\ell)$; A's preferences exhibit superadditivity if $(x, t) \prec (rx, t + \ell)$ and $(x, t) \succ (r^{\kappa}x, t + \kappa\ell)$.*

Proposition 3. *(Interval length effects) Both sub and superadditivity are consistent with optimal similarity judgments. Given an optimal similarity relation, subadditivity occurs if and only if prizes and delivery times in the two frames satisfy: (i) $t \not\approx t+\ell$, (ii) $x \approx rx$, (iii) $x \not\approx r^{\kappa}x$ and (iv) $u(x, t) < u(rx, t+\ell)$. Superadditivity occurs if and only if (i) $t \approx t + \ell$, (ii) $t \not\approx t + \kappa\ell$, (iii) $x \not\approx rx$ and (iv) $u(x, t) > u(rx, t + \ell)$.*

Proof: See Claim 8 in Appendix.

Here, preference reversal may occur in both directions. The empirical literature, starting from Read (2001), documents subadditivity. Subsequent work such as Scholten and Read (2006) also reports the opposite tendency, superadditivity.[19] Our results suggest that subadditivity is more likely when the short interval (i.e., ℓ) is "long" (so that $t \not\approx t + \ell$), while superadditivity is more likely when it is "short" (so that $t \approx t + \ell$). This is in line with the findings of Scholten and Read (2006). Finally, for a given r, our results predict that subadditivity should be more likely when stakes are smaller, since this increases the likelihood that $x \approx rx$, while the opposite holds for superadditivity. This corresponds to the *inseparability anomaly* identified empirically by Scholten and Read (2010). Consider now the benchmark case where A faces no cognitive costs of making distinctions.

Remark 3. *In the case of subadditivity, if cognitive costs were absent A would prefer the larger/later bundle in both frames. In the case of superadditivity, if cognitive costs were*

[19]Similar to the case of magnitude effects, many empirical studies obtain interval length effects by eliciting indifference points. In the Supplementary Appendix we show our setup can rationalize the evidence obtained with this type of data.

absent A would prefer the smaller/sooner bundle in both frames.

In other words, in both sub and superadditivity, A's choice is distorted when faced with the short, but not the long horizon frame. Intuitively, in the long horizon frame, the differences in prizes and delivery times between two bundles are larger, and hence A distinguishes between the two bundles in both dimensions. This may help understand the effects of lock-in saving clauses or illiquid savings (as in Laibson 1997).[20] Consider for instance an individual who may consume x now or may save it for later consumption, and let us denote the gross interest rate as $R > 1$. If the individual knows that he won't be able to withdraw until an amount of time t has passed, his choice effectively becomes one between $(x, 0)$ and $\left(R^t x, t\right)$. If t is small, then x and $R^t x$ may be undistinguishable. However, if t is sufficiently large, then $R^t x$ falls outside x's similarity interval. As a result, A perceives a sensible difference between the two quantities, and may thus be less inclined to inefficiently favor immediate consumption over saving. This provides a possible mechanism through which lock-in clauses or illiquidity may help reduce choice distortions due to cognitive costs.[21]

6 Discussion and extensions

6.1 Relaxing EDIU

Our results give precise predictions on how similarity judgments (and, thus, A's preferences over bundles) are affected by larger prize magnitudes or earlier delivery times. The EDIU model is a natural benchmark, as it is standard in economic theory and in the experimental literature. Exponential discounting also has the additional advantage of highlighting how optimal similarity judgments may generate preferences that display decreasing impatience even when this is not directly in-built. To what extent do our predictions rely on the specific functional forms we have selected for ϕ_1 and ϕ_2? Clearly enough, so long as (i) ϕ_1 is such that the expected benefit from distinguishing between x and rx *is everywhere* increasing in x, all our results in the prize domain would remain unchanged. Similarly, so long as (ii)

[20] A careful analysis should take into account that P may be aware that A may have the option to adopt a commitment device, something we currently do not allow. We conjecture that this should make P even less inclined to let A make fine distinctions in the short horizon, thus strengthening the case for introducing lock-in clauses. However, this intuition should be formally verified.

[21] This shares similarities with the theoretical analysis of Bernheim at al. (2015), who identify a potential role of lock-up clauses to address self-control problems. The forces at work in their model are however different from ours.

ϕ_2 is such that the expected benefit from distinguishing between t and $t+\ell$ is *everywhere* decreasing in t, all our results in the time dimension would continue to apply. This holds for exponential discounting but also for alternative specifications such as hyperbolic discounting, since both ensure that $\Delta(t, t+\ell)$ is non-increasing in t. In the prize dimension, (i) holds when ϕ_1 is isoelastic (as in our application), but also when ϕ_1's elasticity is increasing in x since, in both cases, $\Delta(x, rx)$ is non-decreasing in x.

Note however, while sufficient, these conditions are not necessary for our results. Suppose for instance that ϕ_1's elasticity is decreasing, so that an increase in x reduces $\Delta(x, rx)$.[22] In this case, the elasticity effect works as a countervailing force and the overall effect of an increase in x on the benefit from distinguishing between x and rx is ambiguous. If the elasticity of ϕ_1 tends to drop when prizes become very large, we expect that our predictions will continue to hold for small and medium stakes (which are typical in lab and field experiments), but for very large stakes the elasticity effect will dominate. It is thus possible to envisage situations where similarity intervals initially shrink in prize magnitude, and then eventually start expanding as prizes become very large. This might perhaps explain why we are equally bad at perceiving the difference between ¢1 and ¢2 and between \$10 and \$20 billion, while most people have no problem in distinguishing between \$100 and \$200.

6.2 Generalizability to other setups

Our main theorem is proved for two dimensional environments. As shown in the Supplementary Appendix, it easily generalizes to $N > 2$ dimensions with underlying payoff $\prod_{i=1}^{N} \phi_i(y_i)$. In the Appendix, we also show that our comparative statics results can be stated in more general form. The mechanism behind our findings can be described as follows. Suppose that we wish to isolate the effect of changes in a given dimension (prize, time, ..) for similarity judgments and preferences. To this purpose we consider two frames,

(a) A must choose between bundles $\boldsymbol{y}$ and $\boldsymbol{y}'$,

(b) A must choose between $\boldsymbol{z}$ and $\boldsymbol{z}'$.

In all dimensions except the the one we are interested in, $\boldsymbol{y}$ is identical to $\boldsymbol{z}$ and, similarly, $\boldsymbol{y}'$ is identical to $\boldsymbol{z}'$. In the remaining dimension (say, dimension i), the two frames differ

[22] To see this, consider $r > 1$ (the argument for $r < 1$ is analogous). Then, $\Delta(x, rx) = \ln \phi_1(rx) - \ln \phi_1(x)$, and $\frac{\partial \Delta(x,rx)}{\partial x} = \frac{r\phi_1'(rx)}{\phi_1(rx)} - \frac{\phi_1'(x)}{\phi_1(x)}$. This is ≤ 0 if $\frac{xr\phi_1'(rx)}{\phi_1(rx)} \leq \frac{x\phi_1'(x)}{\phi_1(x)}$ i.e. ϕ_1's elasticity is decreasing.

but the distance Δ is kept constant in both frames. Then,

(1) In dimension i, P is more likely to let A distinguish when that dimension is more important in the sense of Theorem 1, i.e. when it potentially generates a larger payoff. For prizes, this happens in the large stake frame, while for delivery times it happens in the near term frame.

(2) Because of (1), in the frame where dimension i is relatively less important, A is more likely to ignore it when comparing bundles. This leads to preference reversals between the two frames.

These observations hold generally (as shown in the proofs provided in the Appendix), but, of course, the nature of the application will determine their implications in terms of observables. Below, we discuss choice under risk.[23]

6.3 Implications for choice under risk

Rubinstein (1988) was the first to point out that similarity may provide the key to understanding "puzzling" phenomena such as the certainty effect or the common ratio effect.[24] The theory, however, falls short of providing an explanation for why similarity judgments should take a particular form. Here, we investigate the implications of optimal similarity judgments for choice under risk. Suppose that (as in Rubinstein 1988) A has to choose between two simple lotteries.

	Probability	Prize		Probability	Prize
$\mathcal{L}_1$:	π	x	$\mathcal{L}_2$:	π'	x'
	$1-\pi$	0		$1-\pi'$	0

[23] Whilst leaving a careful analysis to future research, we believe that another potentially interesting application is choice between *sequences.* Intuitively, P may find it optimal to induce A to distinguish sequences more finely than single outcomes. That's because the benefit from distinguishing between, x and x' is accrued only once, while that of distinguishing between x, x, x, x and x', x', x', x' is accrued four times and is hence larger. It is thus possible that when choosing between \$1000 on Dec 1st OR \$997 on Nov 1st, the agent may perceive \$997 and \$1000 as similar (and thus chooses based on delivery times), while if facing,

April 1	July 1	Oct 1	Dec 1	OR	March 1	June 1	Sept 1	Nov 1
\$1000	\$1000	\$1000	\$1000		\$997	\$997	\$997	\$997

he may perceive a sensible difference in prizes and thus select the option with the larger prizes, as in Rubinstein (2003).

[24] These anomalies refer to the tendency to choose the safer lottery when winning probabilities are large, while reverting to the riskier one when they are small, and are well documented by the experimental literature (see e.g., Starmer 2000, and references therein). Loomes (2010) proposes a descriptive theory where the perception of the ratio between two probabilities is affected by their difference, thus generating behavior that is consistent with the ratio effect.

Each lottery is fully described by two numbers, the prize (x) and the winning probability (π). If, at the ex-ante stage, prizes and winning probabilities are statistically independent, and we let $u(x, \pi) = \pi\phi_1(x)$ (as in standard expected utility), then Theorem 1 applies, with $\Delta(\pi, r\pi) =\mid \ln r \mid$. The benefit from distinguishing between π and $r\pi$ is thus increasing in π. Consider then the following frames. For some $\alpha > 1$, $1 \geq \pi' > \pi > 0$ and $x > x' > 0$,

(a) A must choose between $(x, \alpha\pi)$ and $(x', \alpha\pi')$ [Large odds.]

(b) A must choose between (x, π) and (x', π') [Small odds.]

Proposition 4. *(Common ratio effect) Any reversal of preferences between the two frames must take the following form:* $(x, \alpha\pi) \prec (x', \alpha\pi')$ *and* $(x, \pi) \succ (x', \pi')$.

Proof: See Claim 6 in Appendix.

Preference reversal takes the form of A switching from the safer option in (a), when odds are large, to the riskier option in (b), when odds are small, matching the evidence on the ratio effect. The opposite pattern is incompatible with our theory, as it would contradict a general version of Corollary 4. In the special case where $\pi' = 1$, any preference reversal takes the form of a *certainty effect,* again matching the evidence.[25]

Similar to Bordalo et al. (2012), while prizes are salient in both cases ($x \not\approx x'$), probabilities are salient when odds are large but not when they are small ($\alpha\pi \not\approx \alpha\pi'$ and $\pi \approx \pi'$). What we add to the story is the reason for why this happens: because the expected benefit from distinguishing between two small probabilities is low. Moreover, similar to Corollary 5, if the common ratio effect occurs when prizes are x and x', then it necessarily occurs also when prizes are κx and $\kappa x'$, where $\kappa > 1$, but not vice-versa. *Ceteris paribus*, the common ratio effect is thus more likely when prizes are large.[26] Finally, it is straightforward to prove that, in the absence of cognitive costs, A would prefer the safer gamble in both frames. The bias due to cognitive costs thus takes the form of inducing too much risk taking when odds are small.

It is also possible to investigate the implications of rescaling prizes – the equivalent of what we did in Section 5.4 for intertemporal choice. Suppose that $u(x, \pi) = \pi x^{1-\rho}/(1-\rho)$

[25] In that case, A's problem becomes: (a) choose between (x, π) and $(x', 1)$ [Certain vs uncertain], and (b) choose between $(x, \alpha\pi)$ and (x', α) [Both uncertain]. The observation that people display a disproportionate preference for certainty when choosing between certain and uncertain options dates back at least to Allais (1953), and has been widely confirmed by experimental evidence (see, e.g. Tversky and Kahneman 1986).

[26] The available evidence on the Allais paradox broadly goes in this direction, see e.g. Huck and Mueller (2012), but we are not aware of any specific test of this hypothesis within the broader context of the ratio effect.

(i.e., the underlying payoff corresponds to standard CRRA), and consider the following two frames. For some $\alpha > 1$, $1 \geq \pi' > \pi > 0$ and $x > x' > 0$,

(a) A must choose between (x, π) and (x', π') [Small stakes gambles.]

(b) A must choose between $(\alpha x, \pi)$ and $(\alpha x', \pi')$ [Large stakes gambles.]

Given $u(x, \pi)$, in the absence of cognitive costs A would consistently choose either the riskier or the safer gamble in both frames. With cognitive costs,

Proposition 5. *(Risk aversion in the large and small) Any reversal of preferences between the two frames must take the following form:* $(x, \pi) \prec (x', \pi')$ *and* $(\alpha x, \pi) \succ (\alpha x', \pi')$.

Proof: See Claim 6 in Appendix.

Preference reversal must take the form of A preferring the safer lottery in (a), when stakes are small, while preferring the riskier lottery (with a larger prize) in (b), when stakes are large. Intuitively, while probabilities are salient in both frames ($\pi \not\approx \pi'$), prizes are not salient in (a) but they are in (b) ($x \approx x'$ and $\alpha x \not\approx \alpha x'$). Proposition 5 may shed light on Rabin's (2000) observation that the degree of risk aversion displayed by subjects in small stakes gambles is too high to yield anything but paradoxical implications with large stakes – the so-called "paradox of risk aversion in the large and small".[27] Our explanation is that, when stakes are small (but not when they are large), A perceives them as indistinguishable and chooses between the gambles based on winning probabilities alone. This makes him appear very risk averse. However, it is straightforward to see that, if cognitive costs were absent, A would prefer the riskier gamble in both large and small stakes frames. When stakes are small, the presence of cognitive costs thus makes A take on less risk compared with the benchmark.[28]

6.4 Systematic tradeoffs

What happens when Assumption 1 is relaxed, so that the characteristics of the bundles in one dimension provide information about their attributes in the other? Here, we want to

[27]Rabin's (2000) original argument was based on a thought experiment, but subsequent studies explicitly consider evidence from subjects making choices in small- and large-stakes bets (see e.g., Cox et al. 2013 and Khaw et al. 2019).

[28]This shares similarities to Woodford (2012) and Khaw et al. (2019), where "excessive" risk aversion results from a perceptual bias which is however optimal (in the sense that it maximizes expected payoff subject to cognitive constraints). The underlying mechanisms at work in these models, however, are different from ours.

focus in particular on environments where A *systematically* faces tradeoffs that take the form of trading an egg today against a chicken at a future date.

A potential problem that arises when relaxing the independence assumption is that the expected benefit from distinguishing between time periods may no longer decrease when these are pushed further in the future, since later delivery times may be associated with larger expected prizes, creating a countervailing effect. For the same reason, the expected benefit from distinguishing between prizes may no longer increase when these are scaled up. That said, Assumption 1 is only sufficient for our results. In particular, it is possible to retrieve most of the results in settings where the time lag separating the two delivery times is correlated with the size of the later prize relative to the sooner one. This fully captures the idea that "good things come to those who wait". In our setting, this would imply that t and x are still independent, while allowing for positive correlation between ℓ and r. In the Supplementary Appendix, we establish a weaker version of the main theorem for a class of bundle-generating processes that allow for such tradeoffs.

6.5 Optimality of MADD preferences

So far, we have taken as a given that A's preferences follow the MADD principle: A prefers the bundle that generates the largest underlying payoff along the salient dimensions. In this section, we ask, if P could design A's preferences, under what conditions would he choose MADD preferences? For our purposes, a preference relation can be fully described by the function Q given in (2). Let $\mathcal{Q}$ denote the set of functions $Q : \mathcal{Y}^2 \to \{0, 1/2, 1\}$. Suppose then that P provides A with a preference relation (a function $Q \in \mathcal{Q}$) in order to maximize the net payoff,

$$Q(\boldsymbol{y}, \boldsymbol{y'})[u(\boldsymbol{y}) - u(\boldsymbol{y'})]. \tag{13}$$

We restrict attention to the following class of Q functions.

Assumption 3. *For all pairs $(\boldsymbol{y}, \boldsymbol{y'}) \in \mathcal{Y}^2$, if $(\boldsymbol{z}, \boldsymbol{z}') \in \mathcal{Y}^2$ are identical to $(\boldsymbol{y}, \boldsymbol{y'})$ in all dimensions i such that $y_i \not\approx y'_i$, and $z_i \approx z'_i$ whenever $y_i \approx y'_i$, then $Q(\boldsymbol{y}, \boldsymbol{y'}) = Q(\boldsymbol{z}, \boldsymbol{z'})$.*

If P could make A's preferences depend on things that A does not distinguish, there would be no need to incur the cognitive cost of making distinctions. P could simply "hard wire" the desired choice into A's preferences. By requiring that A's preferences can only depend on what he can distinguish, Assumption 3 makes similarity judgments relevant. The next result establishes that, under this constraint, MADD preferences emerge endogenously as a solution to P's augmented optimization problem.

Lemma 1. *Under Assumptions 1 and 3, Q maximizes (13) if A's preferences are MADD.*

Proof. See Lemma A.1 in the Supplementary Appendix.

The result is stated in terms of "if" rather than "if and only if" simply because optimality has no bite when A fails to distinguish in any dimensions (so that Q need not be equal to 1/2 in those cases). Finally, note that Assumption 1 is needed to rule out peculiar cases of correlation across dimensions where, from P's perspective, the expectation of gains along a non-salient dimension justifies accepting certain losses in the salient one, thus making it optimal to choose the bundle that looks actually worse in the salient dimension.

7 Related literature and concluding remarks

The main contribution of our paper consists in providing a foundation to the literature on similarity (Tverski, 1977, Leland 2002, Rubinstein 1988, 2003), by endogenizing similarity judgments. Optimal similarity judgments arise from an optimization exercise in which the benefits of acquiring more information are weighed against cognitive/attention costs. This approach is clearly aligned with the literature on rational inattention, which has been steadily gaining popularity following the seminal contribution by Sims (2003) – Caplin and Dean (2015) and de Olivera et al. (2016) are recent contributions that specifically address the implications of rational inattention for revealed preferences and choice data. A recent paper by Alaoui and Penta (2015) shows that a cost-benefit representation of cognition may apply to a large class of reasoning processes. Our novelty with respect to this literature is the application of this type of analysis to similarity judgments.

Our paper is also related to the body of work on the evolutionary foundations of preferences.[29] A common theme of that literature is that apparent behavioral anomalies may actually be the solution of an optimization problem in which Nature maximizes individual fitness, subject to some physiological, cognitive or informational constraints. The accounts

[29]Important contributions in this literature include Robson (2001), Samuelson (2004), Rayo and Becker (2007), Herold (2012), Rayo and Robson (2014).

by Dasgupta and Maskin (2006) and Netzer (2009) specifically focus on time preference reversal, but the mechanisms at work in their settings are very different from ours. Moreover, these works are silent about other behavioral anomalies.

Our simple theory generates rich implications for optimal similarity judgments, which shed light on a number of seemingly unrelated stylized facts not easily explainable with standard models. While these have all been independently modelled, to our knowledge no existing single setting has addressed all the phenomena explained by our model. There are a few approaches which, similar to us, have tried to produce a unified explanation for many different behavioral anomalies. One is the dual-self literature, such as, e.g. Fudenberg and Levine (2006, 2011). Our two-stage setting can be thought of in terms of dual-self, albeit with an important difference, namely that the agent's "myopia" is endogenous and depends on what he does/does not distinguish. Other works have emphasized the portability of the rational inattention framework (see e.g. Woodford 2012a, 2012b, Gabaix 2014, 2019), and the model we consider is certainly in this spirit. Although the quest for a parsimonious, unifying account of behavioral anomalies is far from over, our results indicate that optimal similarity judgments may be part of the picture.

8 Appendix

Proof of Theorem 1 In what follows, it will be convenient to transform bundle realizations $(\boldsymbol{y}, \boldsymbol{y}')$ into their log-payoff values. Specifically, let $u(\boldsymbol{v}) = \exp(v_1 + v_2)$, where $v_i = \ln \phi_i(y_i)$ and $\boldsymbol{v} = (v_1, v_2)$. We will use $v_i \not\approx v_i'$ ($v_i \approx v_i'$) as a shorthand, to mean that $y_i \not\approx y_i'$ ($y_i \approx y_i'$). Note that $\Delta(y_i, y_i') = |v_i - v_i'|$ and that MADD preferences imply $\sum_{i=1,2}(v_i - v_i')I_i(v_i, v_i') \geq 0 \Leftrightarrow \boldsymbol{y} \succsim \boldsymbol{y'}$. Note also that the independence/exchangeability assumptions (Assumption 1) on (Y_i, Y_i') trivially extend to random variables (V_i, V_i') in the log-welfare space.

The proof is divided into several claims.

Claim 1. *Given $v > 0$ and $\Delta > 0$, let $\ln \phi_i(y_i) = v$ and $\ln \phi_i(y_i') = v + \Delta$. Then, $b_i(y_i, y_i') = e^{v+\Delta} B_i(\Delta)$ where B_i is continuous and independent of v.*

Given MADD preferences, A's choices are as follows:

1. Suppose $v \not\approx v + \Delta$. Then, for $j = 1, 2 \neq i$, $\boldsymbol{y'} \succ \boldsymbol{y} \Leftrightarrow v_j \approx v_j' \vee v_j - v_j' < \Delta,$; $\boldsymbol{y} \succ \boldsymbol{y'} \Leftrightarrow v_j \not\approx v_j' \wedge v_j - v_j' > \Delta$. Finally, $\boldsymbol{y'} \sim \boldsymbol{y} \Leftrightarrow v_j \not\approx v_j' \wedge v_j - v_j' = \Delta$.

2. Suppose $v \approx v + \Delta$. Then, $\boldsymbol{y'} \succ \boldsymbol{y} \Leftrightarrow v_j \not\approx v'_j \wedge v_j - v'_j < 0$; $\boldsymbol{y} \succ \boldsymbol{y'} \Leftrightarrow v_j \not\approx v'_j \wedge v_j - v'_j > 0$. Finally, $\boldsymbol{y'} \sim \boldsymbol{y} \Leftrightarrow v_j \approx v'_j \vee v_j - v'_j = 0$.

Consider the benefit from distinguishing between v and $v + \Delta$. Comparing 1) and 2), if A distinguishes in dimension j, A would choose $\boldsymbol{y'}$ anyway if $v_j - v'_j < 0$ and would choose $\boldsymbol{y}$ anyway if $v_j - v'_j > \Delta$. Hence, the benefit from $v \not\approx v + \Delta$ is always zero in these cases. Ignoring zero measure cases, distinguishing in dimension i thus affects A's choice either if a) $v_j \approx v'_j$ or b) $v_j \not\approx v'_j \wedge v_j - v'_j \in (0, \Delta)$. In a) distinguishing in dimension i leads A to pick bundle $\boldsymbol{y'}$ with probability one rather than 1/2. In b) $v_i \not\approx v'_i$ leads A to pick $\boldsymbol{y'}$ with probability one instead of $\boldsymbol{y}$. Given Assumption 2, P chooses whether $v \not\approx v + \Delta$ or not considering all possible realizations (v_j, v'_j) in dimension j. Hence, he takes expectations with respect to random variables (V_j, V'_j). Let $\mathcal{E}^{\varnothing}$ denote the event $\{V_j \approx V'_j\}$. For $z \in \mathbb{R}$, let $\mathcal{E}^-_z$ denote the event $\{V_j \not\approx V'_j\} \cap \{V_j - V'_j < z\}$ and $\mathcal{E}^+_z$ denote the event $\{V_j \not\approx V'_j\} \cap \{V_j - V'_j > z\}$. [We can ignore zero probability events like $\{V_j - V'_j = 0\}$ or $\{V_j - V'_j = \Delta\}$.] The expected benefit from distinguishing between v and $v + \Delta$ is thus

$$\mathbb{P}(\mathcal{E}^{\varnothing})\frac{1}{2}E\left[e^{V'_i+V'_j} - e^{V_i+V_j}\middle|\mathcal{E}^{\varnothing}\right] + \mathbb{P}(\mathcal{E}^+_0 \cap \mathcal{E}^-_\Delta)E\left[e^{V'_i+V'_j} - e^{V_i+V_j}\middle|\mathcal{E}^+_0 \cap \mathcal{E}^-_\Delta\right] =$$

$$e^{v+\Delta}\left(\underbrace{\mathbb{P}(\mathcal{E}^{\varnothing})\frac{1}{2}E\left[e^{V'_j} - e^{-\Delta+V_j}\middle|\mathcal{E}^{\varnothing}\right] + \mathbb{P}(\mathcal{E}^+_0 \cap \mathcal{E}^-_\Delta)E\left[e^{V'_j} - e^{-\Delta+V_j}\middle|\mathcal{E}^+_0 \cap \mathcal{E}^-_\Delta\right]}_{:=B_i(\Delta)}\right), \tag{14}$$

where the expectation is taken with respect to (V_j, V'_j). From Assumption 1, all probabilities and expectations are independent of the realized values of (V_i, V'_i), so that B_i is constant with respect to v. Continuity of B_i in Δ (except possibly at $\Delta = 0$) follows from continuity of the distributions of bundle characteristics.

Claim 2. *For all $\Delta > 0$, $B_i(\Delta)$ is increasing.*

We now establish that, given $\Delta' > \Delta'' > 0$, $B_i(\Delta') > B_i(\Delta'')$. From (14),

$$\begin{aligned} B_i(\Delta') - B_i(\Delta'') = \mathbb{P}(\mathcal{E}^{\varnothing})\frac{1}{2}E\left[e^{-\Delta''+V_j} - e^{-\Delta'+V_j}\middle|\mathcal{E}^{\varnothing}\right] &+ \\ \mathbb{P}(\mathcal{E}^+_0 \cap \mathcal{E}^-_{\Delta'})\left(E\left[e^{V'_j}\middle|\mathcal{E}^+_0 \cap \mathcal{E}^-_{\Delta'}\right] - e^{-\Delta'}E\left[e^{V_j}\middle|\mathcal{E}^+_0 \cap \mathcal{E}^-_{\Delta'}\right]\right) &- \\ \mathbb{P}(\mathcal{E}^+_0 \cap \mathcal{E}^-_{\Delta''})\left(E\left[e^{V'_j}\middle|\mathcal{E}^+_0 \cap \mathcal{E}^-_{\Delta''}\right] - e^{-\Delta''}E\left[e^{V_j}\middle|\mathcal{E}^+_0 \cap \mathcal{E}^-_{\Delta''}\right]\right) & \end{aligned} \tag{15}$$

Given $\Delta' > \Delta''$, the term on the first line of (15) is strictly positive whenever $\mathbb{P}(\mathcal{E}^{\varnothing}) > 0$. We thus just need to evaluate the difference between the second and third lines. Given $e^{-\Delta''} > e^{-\Delta'}$, and using $\mathcal{E}^-_{\Delta''} \subset \mathcal{E}^-_{\Delta'}$, the difference is strictly larger than

$$\mathbb{P}(\mathcal{E}^+_0 \cap \mathcal{E}^-_{\Delta'} \cap \mathcal{E}^+_{\Delta''}) E\left[e^{V'_j} - e^{-\Delta' + V_j} \middle| \mathcal{E}^-_{\Delta'} \cap \mathcal{E}^+_0 \cap \mathcal{E}^+_{\Delta''} \right]. \tag{16}$$

where, since Δ' and Δ'' are both positive, $\mathcal{E}^+_0 \cap \mathcal{E}^-_{\Delta'} \cap \mathcal{E}^+_{\Delta''} = \mathcal{E}^-_{\Delta'} \cap \mathcal{E}^+_{\Delta''}$. Then, the expectation in (16) becomes

$$E\left[\exp\left(V'_j\right)\left(1 - \exp\left(V_j - V'_j - \Delta'\right)\right) | \mathcal{E}^-_{\Delta'} \cap \mathcal{E}^+_{\Delta''}\right] > 0 \tag{17}$$

where the inequality comes from the expectation being conditional on $\{V_j - V'_j < \Delta'\}$ through $\mathcal{E}^-_{\Delta'}$. Finally, note that, since from Assumption 1 the joint density of (V_j, V'_j) has full support, $\mathbb{P}(\mathcal{E}^-_{\Delta'} \cap \mathcal{E}^+_{\Delta''}) > 0$ whenever $\mathbb{P}(\mathcal{E}^{\varnothing}) = 0$, so that (15) must be positive.

Claim 3. *B_i is equal to zero for $\Delta = 0$.*

For $\Delta = 0$, the first term in (14) is zero given exchangeability (the second part of Assumption 1). The second term is zero since $\mathbb{P}(\mathcal{E}^-_0 \cap \mathcal{E}^+_0) = 0$. Note also that, for any sequence $\{\Delta_n\}_{n \in \mathbb{N}}$, $\Delta_n > 0$, such that $\Delta_n \to 0$, $B_i(\Delta_n) \to 0$, which implies no discontinuity at $\Delta = 0$.

Finally, note that, since the two bundles are ex-ante identical, one can always set $v = \min\{v_i, v'_i\}$ and $\Delta = \max\{v_i, v'_i\} - \min\{v_i, v'_i\}$ to obtain

$$b_i(y_i, y'_i) = \exp\left(\max\{v_i, v'_i\}\right) B_i\left(\Delta\right) = \max\{\phi_i(y_i), \phi_i(y'_i)\} B_i\left(\Delta\right), \tag{18}$$

where all properties of B_i follow from the claims above. □

Proof of Corollary 1 Assume $y'_i > y_i$, where y_i is constant. If ϕ_i is increasing, then, from Theorem 1

$$b_i(y_i, y'_i) = \phi_i(y'_i) B_i(\ln \phi_i(y'_i) - \ln \phi_i(y_i)), \tag{19}$$

which is increasing in y'_i. If ϕ_i is decreasing, then

$$b_i(y_i, y'_i) = \phi_i(y_i) B_i(\ln \phi_i(y_i) - \ln \phi_i(y'_i)), \tag{20}$$

which is again increasing in y_i'. Suppose now that $y_i' < y_i$. Then, expressions (19) and (20) apply to the cases where ϕ_i is decreasing and increasing, respectively. This in turn implies that $b_i(y_i, y_i')$ is decreasing for all $y_i' < y_i$. Finally, $b_i(y_i, y_i) = 0$ follows from $B_i(0) = 0$. □

Proof of Corollary 2 The first statement directly follows from Theorem 1 once we note that, in the time (prize) dimension, $\Delta = \delta|\ell|$ ($\Delta = (1-\rho)|\ln r|$) is constant with respect to t (x). As for the second statement, let $[t^-, t^+]$ be t's similarity interval. From Theorem 1, t^+ and t^- solve

$$e^{-\delta t} B_2(\delta[t^+ - t]) = c_2, \; e^{-\delta t^-} B_2(\delta[t - t^-]) = c_2, \tag{21}$$

which, having defined $\ell^+ := t^+ - t$ and $\ell^- := t - t^-$, implies

$$B_2(\delta \ell^+) = c_2 e^{\delta t}, \; e^{\delta \ell^-} B_2(\delta \ell^-) = c_2 e^{\delta t}, \tag{22}$$

Since each LHS is an increasing function and each RHS is increasing in t, both ℓ^+ and ℓ^- must be increasing in t. Hence $t^+ - t^- = \ell^+ + \ell^-$ must be increasing in t. Using a log-scale for the prize domain, let $[\ln x^-, \ln x^+]$ denote the similarity interval for $\ln x$. Similar to the time dimension, we have

$$r^+ B_1((1-\rho)\ln r^+) = \frac{c_1}{x}, \; B_1((1-\rho)\ln r^-) = \frac{c_1}{x}, \tag{23}$$

where both $r^+ \equiv x^+/x$ and $r^- \equiv x/x^-$ are decreasing in x, so that $\ln x^+ - \ln x^- = \ln r^+ + \ln r^-$ is decreasing in x. □

Proof of Corollaries 3, 4, and Propositions 1-5

We prove these results by establishing a number of general implications of the main Theorem. Fix two pairs of bundles $(\boldsymbol{y}, \boldsymbol{y'}) \in \mathcal{Y}^2$ and $(\boldsymbol{z}, \boldsymbol{z'}) \in \mathcal{Y}^2$, with $y_i > z_i$, $y_i' > z_i'$, $y_i \neq y_i'$ and $\Delta(y_i, y_i') = \Delta(z_i, z_i')$. Note that this general formulation encompasses the following as special cases

1. Exponential discounting and, for $s > 0$, $\boldsymbol{z} = (x, t)$, $\boldsymbol{z'} = (x', t')$, $\boldsymbol{y} = (x, t+s)$, $\boldsymbol{y'} = (x', t'+s)$ (so that $\Delta(t, t') = \Delta(t+s, t'+s) = \delta|t - t'|$).

2. Isoelastic utility and, for $\alpha > 1$, $\boldsymbol{z} = (x, t)$, $\boldsymbol{z'} = (x', t')$, $\boldsymbol{y} = (\alpha x, t)$, $\boldsymbol{y'} = (\alpha x', t')$ (so that $\Delta(x, x') = \Delta(\alpha x, \alpha x') = (1-\rho)|\ln x - \ln x'|$).

3. Expected utility and, for $\alpha > 1$, $\boldsymbol{z} = (x, \pi)$, $\boldsymbol{z'} = (x', \pi')$, $\boldsymbol{y} = (x, \alpha\pi)$, $\boldsymbol{y'} = (x', \alpha\pi')$ (so that $\Delta(\pi, \pi') = \Delta(\alpha\pi, \alpha\pi') = |\ln \pi - \ln \pi'|$).

The following result establishes Corollary 3.

Claim 4. *Whenever ϕ_i is increasing, $z_i \not\approx z'_i \Rightarrow y_i \not\approx y'_i$. Symmetrically, when ϕ_i is decreasing, $z_i \not\approx z'_i \Leftarrow y_i \not\approx y'_i$.*

Proof. This follows since, given $\Delta(y_i, y'_i) = \Delta(z_i, z'_i)$, $b_i(y_i, y'_i) > (<) b_i(z_i, z'_i)$ when ϕ_i is an increasing (decreasing) function. □

WLOG, assume now $y_i > y'_i$. The next result establishes Corollary 4.

Claim 5. *If $(\boldsymbol{y}, \boldsymbol{y}')$ and $(\boldsymbol{z}, \boldsymbol{z}')$ are identical in dimension $j \neq i$, then $\boldsymbol{z} \succ \boldsymbol{z'} \Rightarrow \boldsymbol{y} \succ \boldsymbol{y'}$ if ϕ_i is increasing and $\boldsymbol{z} \succ \boldsymbol{z'} \Leftarrow \boldsymbol{y} \succ \boldsymbol{y'}$ if ϕ_i is decreasing.*

Proof. Consider ϕ_i increasing. $\boldsymbol{z} \succ \boldsymbol{z'}$ requires

$$\phi_i(z_i)^{I_i(z_i,z'_i)}\phi_j(z_j)^{I_j(z_j,z'_j)} > \phi_i(z'_i)^{I_i(z_i,z'_i)}\phi_j(z'_j)^{I_j(z_j,z'_j)}. \tag{24}$$

This implies

$$\left(\frac{\phi_i(y_i)}{\phi_i(y'_i)}\right)^{I_i(y_i,y'_i)} \geq \left(\frac{\phi_i(z_i)}{\phi_i(z'_i)}\right)^{I_i(z_i,z'_i)} > \left(\frac{\phi_j(z'_j)}{\phi_j(z_j)}\right)^{I_j(z_j,z'_j)} = \left(\frac{\phi_j(y'_j)}{\phi_j(y_j)}\right)^{I_j(y_j,y'_j)}, \tag{25}$$

where the first inequality comes from $\Delta(y_i, y'_i) = \Delta(z_i, z'_i)$, $y_i > y'_i$, and, given $z_i \not\approx z'_i \Rightarrow y_i \not\approx y'_i$, $I_i(y_i, y'_i) \geq I_i(z_i, z'_i)$. The middle inequality follows from (24). The last equality comes from $y_j = z_j$ and $y'_j = z'_j$. In turn, (25) implies $\boldsymbol{y} \succ \boldsymbol{y'}$. The argument for ϕ_i decreasing is symmetric. □

The next result establishes Propositions 4-5 and the first part of Propositions 1-2.

Claim 6. *If $(\boldsymbol{y}, \boldsymbol{y}')$ and $(\boldsymbol{z}, \boldsymbol{z}')$ are identical in dimension $j \neq i$, any reversal of strict preferences must take the form of $\boldsymbol{y} \succ \boldsymbol{y'}$ and $\boldsymbol{z} \prec \boldsymbol{z'}$ if ϕ_i is increasing and $\boldsymbol{y} \prec \boldsymbol{y'}$ and $\boldsymbol{z} \succ \boldsymbol{z'}$ if ϕ_i is decreasing.*

Proof. Consider again the case where ϕ_i is increasing. Then, if $y_i > y'_i$, we know from Claim 5 that $\boldsymbol{z} \succ \boldsymbol{z'} \Rightarrow \boldsymbol{y} \succ \boldsymbol{y'}$, so that no reversal can occur when $\boldsymbol{z} \succ \boldsymbol{z'}$. A symmetric argument applies to the case where ϕ_i is decreasing. □

The following result establishes the second part of Propositions 1 and 2.

Claim 7. *If $(\boldsymbol{y}, \boldsymbol{y}')$ and $(\boldsymbol{z}, \boldsymbol{z}')$ are identical in dimension $j \neq i$, necessary and sufficient conditions for a reversal of strict preferences are: i) $y_i \not\approx y_i'$, 2) $z_i \approx z_i'$, 3) $z_j \not\approx z_j'$, 4) $\phi_j(z_j) < (>) \phi_j(z_j')$ when ϕ_i is increasing (decreasing).*

Proof. Consider again ϕ_i increasing. $\boldsymbol{z} \prec \boldsymbol{z'}$ and $\boldsymbol{y} \succ \boldsymbol{y'}$ require

$$\begin{aligned} \phi_i(z_i)^{I_i(z_i,z_i')}\phi_j(z_j)^{I_j(z_j,z_j')} &< \phi_i(z_i')^{I_i(z_i,z_i')}\phi_j(z_j')^{I_j(z_j,z_j')} \\ \phi_i(y_i)^{I_i(y_i,y_i')}\phi_j(y_j)^{I_j(y_j,y_j')} &> \phi_i(y_i')^{I_i(y_i,y_i')}\phi_j(y_j')^{I_j(y_j,y_j')} \end{aligned} \tag{26}$$

or, given $z_j = y_j$ and $z_j' = y_j'$,

$$\left(\frac{\phi_i(y_i)}{\phi_i(y_i')}\right)^{I_i(y_i,y_i')} > \left(\frac{\phi_j(y_j')}{\phi_j(y_j)}\right)^{I_j(y_j,y_j')} = \left(\frac{\phi_j(z_j')}{\phi_j(z_j)}\right)^{I_j(z_j,z_j')} > \left(\frac{\phi_i(z_i)}{\phi_i(z_i')}\right)^{I_i(z_i,z_i')}, \tag{27}$$

Note that, since $\Delta(y_i, y_i') = \Delta(z_i, z_i')$, $\phi_i(y_i)/\phi_i(y_i') = \phi_i(z_i)/\phi_i(z_i')$, so that, given $y_i > y_i'$, the above can only hold if $I_i(y_i, y_i') = 1$ and $I_i(z_i, z_i') = 0$ (so that $y_i \not\approx y_i'$ and $z_i \approx z_i'$). Moreover, given $I_i(z_i, z_i') = 0$, the last inequality cannot hold strict if $I_j(z_j, z_j') = 0$ or $\phi_j(z_j) > \phi_j(z_j')$. This implies that $z_j \not\approx z_j'$ and $\phi_j(z_j) < \phi_j(z_j')$ are also necessary. It is then straightforward to check that these conditions are also sufficient. Finally, a symmetric argument applies to the case where ϕ_i is decreasing. □

In order to establish Proposition 3, we can use a similar approach. Let $(\boldsymbol{z}, \boldsymbol{y}, \boldsymbol{y'}) \in \mathcal{Y}^3$ denote three bundles, with the properties that: (a) for all $i = 1, 2$ $y_i > y_i' > z_i$, (b) for $\kappa > 1$, $\Delta(z_i, y_i) = \kappa\Delta(z_i, y_i')$. Let ϕ_2 be decreasing and ϕ_1 increasing. Note that Proposition 3 is a special case since in the short horizon frame, $\Delta(t, t + \ell) = \delta\ell$ and $\Delta(x, rx) = (1 - \rho)\ln r$, while in the long horizon frame $\Delta(t, t + \kappa\ell) = \kappa\delta\ell$ and $\Delta(x, r^\kappa x) = \kappa(1 - \rho)\ln r$.

Claim 8. *Subadditivity ($\boldsymbol{z} \prec \boldsymbol{y} \wedge \boldsymbol{z} \succ \boldsymbol{y'}$) occurs if and only if: i) A always distinguishes in dimension 2, ii) in dimension 1, $z_1 \not\approx y_1$ but $z_1 \approx y_1'$, iii) $\phi_1(z_1)\phi_2(z_2) < \phi_1(y_1')\phi_2(y_2')$. Superaddivity ($\boldsymbol{z} \prec \boldsymbol{y'} \wedge \boldsymbol{z} \succ \boldsymbol{y}$) occurs if and only if: i) A always distinguishes in dimension 1, ii) in dimension 2, $z_2 \not\approx y_2$ but $z_2 \approx y_2'$, iii) $\phi_1(z_1)\phi_2(z_2) > \phi_1(y_1')\phi_2(y_2')$.*

Proof. $\boldsymbol{z} \prec \boldsymbol{y}$ and $\boldsymbol{z} \succ \boldsymbol{y'}$ (Subadditivity) requires

$$\left(\frac{\phi_2(z_2)}{\phi_2(y_2)}\right)^{I_2(z_2,y_2)} < \left(\frac{\phi_1(y_1)}{\phi_1(z_1)}\right)^{I_1(z_1,y_1)} \tag{28}$$

and

$$\left(\frac{\phi_1(y_1')}{\phi_1(z_1)}\right)^{I_1(z_1,y_1')} < \left(\frac{\phi_2(z_2)}{\phi_2(y_2')}\right)^{I_2(z_2,y_2')} \Leftrightarrow \left(\frac{\phi_2(z_2)}{\phi_2(y_2)}\right)^{\kappa I_2(z_2,y_2')} > \left(\frac{\phi_1(y_1)}{\phi_1(z_1)}\right)^{\kappa I_1(z_1,y_1')}, \tag{29}$$

where the second inequality follows from the first and $\Delta(z_i, y_i) = \kappa\Delta(z_i, y_i')$ for all $i = 1, 2$. Note that, given $y_i > z_i$ and the the fact that ϕ_2 is decreasing and ϕ_1 increasing, $\phi_2(z_2)/\phi_2(y_2) > 1$ and $\phi_1(y_1)/\phi_1(z_1) > 1$. Note also that, from Theorem 1, given ϕ_2 decreasing and $B_2(\kappa\Delta) > B_2(\Delta)$, $b_2(z_2, y_2) > b_2(z_2, y_2')$, so that $I_2(z_2, y_2) = 1$ is implied by $I_2(z_2, y_2') = 1$. Suppose first that $\phi_1(z_1)\phi_2(z_2) < \phi_1(y_1')\phi_2(y_2')$ (which implies $\phi_1(z_1)\phi_2(z_2) < \phi_1(y_1)\phi_2(y_2)$). Clearly, for both inequalities (28) and (29) to hold simultaneously, it is necessary and sufficient that $I_2(z_2, y_2) = I_1(z_1, y_1) = I_2(z_2, y_2') = 1$, and $I_1(z_1, y_1') = 0$. To establish that $\phi_2(z_2)\phi_1(z_1) < \phi_2(y_2')\phi_1(y_1')$ is also necessary, suppose that $\phi_2(z_2)\phi_1(z_1) \geq \phi_2(y_2')\phi_1(y_1')$. Then, for both (28) and (29) to hold simultaneously, it must be that $I_2(z_2, y_2) = 0$ and $I_2(z_2, y_2') = 1$. As pointed out above, however, this would violate the main Theorem. Hence, subadditivity cannot arise in this case.

Symmetrically, superadditivity ($\boldsymbol{z} \prec \boldsymbol{y'}$ and $\boldsymbol{z} \succ \boldsymbol{y}$) requires that both inequalities (28) and (29) are reversed. In this case, one can invoke Theorem 1 to establish that $I_1(z_1, y_1') = 1$ implies $I_1(z_1, y_1) = 1$. The rest of the argument is the mirror image of the one given above. $\square$

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