CENTRE FOR DECISION RESEARCH & EXPERIMENTAL ECONOMICS

Discussion Paper No. 2021-06

Maria Montero and
Alex Possajennikov

November 2021

An Adaptive Model of Demand Adjustment in Weighted Majority Games

CeDEx Discussion Paper Series
ISSN 1749 - 3293

CENTRE FOR DECISION RESEARCH & EXPERIMENTAL ECONOMICS

The Centre for Decision Research and Experimental Economics was founded in 2000, and is based in the School of Economics at the University of Nottingham.

The focus for the Centre is research into individual and strategic decision-making using a combination of theoretical and experimental methods. On the theory side, members of the Centre investigate individual choice under uncertainty, cooperative and non-cooperative game theory, as well as theories of psychology, bounded rationality and evolutionary game theory. Members of the Centre have applied experimental methods in the fields of public economics, individual choice under risk and uncertainty, strategic interaction, and the performance of auctions, markets and other economic institutions. Much of the Centre's research involves collaborative projects with researchers from other departments in the UK and overseas.

Please visit http://www.nottingham.ac.uk/cedex for more information about the Centre or contact

Suzanne Robey
Centre for Decision Research and Experimental Economics
School of Economics
University of Nottingham
University Park
Nottingham
NG7 2RD
Tel: +44 (0)115 95 14763
suzanne.robey@nottingham.ac.uk

The full list of CeDEx Discussion Papers is available at

http://www.nottingham.ac.uk/cedex/publications/discussion-papers/index.aspx

An Adaptive Model of Demand Adjustment in Weighted Majority Games

Maria Montero
University of Nottingham

Alex Possajennikov*
University of Nottingham

31 October 2021

Abstract

This paper presents a simple adaptive model of demand adjustment in cooperative games, and analyzes this model in weighted majority games. In the model, a randomly chosen player sets his demand to the highest possible value subject to the demands of other coalitions members being satisfied. This basic process converges to the aspiration set. By introducing some perturbations into the process, we show that the set of separating aspirations, i.e. demand vectors in which no player is indispensable in order for other players to achieve their demands, is the one most resistant to mutations. We then apply the process to weighted majority games. We show that in symmetric majority games and in apex games the unique separating aspiration is the unique stochastically stable one.

Keywords: demand adjustment, aspirations, stochastic stability

1 Introduction

Consider a situation in which there are three players, any pair of players can cooperate and generate 30 money units but the addition of the third

*Corresponding author. School of Economics, University of Nottingham, Nottingham NG7 2RD, United Kingdom. e-mail: alex.possajennikov@nottingham.ac.uk.

player to the pair does not bring additional benefits. The situation can be seen as a weighted majority game with three symmetric players dividing a budget: two are enough to form a coalition and agree on a division; the third player's participation does not increase the budget. Each of the three players may formulate a payoff demand, with the understanding that the player is willing to join any coalition that satisfies the demand. A coalition can only form if the demands of its members are satisfied. Clearly, not all demand combinations are equally stable. If the first two players make a demand of 15 and the third player makes a demand of 20, the third player will find that no coalition can satisfy his demand and may reduce it. Similarly, if the first two players make a demand of 15 and the third player makes a demand of 10, the third player may realize that it is possible to increase his demand and still find coalitions that can satisfy it. Demand combinations such that each player is making the highest demand that can still be satisfied are called *aspirations* in the literature on cooperative games.[1]

Even if we restrict ourselves to the set of aspirations, not all demand combinations appear equally stable. For example, suppose the first two players demand 20 each whereas the third player demands 10. There are two feasible coalitions, both of which contain the third player. Because the third player is indispensable, we expect him to be able to increase the demand. There are several solution concepts defined on the space of aspirations, all of which assume that competition for scarce players will drive their price (demand) up. The main ones are the set of partnered aspirations (Albers, 1979; Bennett, 1983) and the set of balanced aspirations, also known as the aspiration core (Cross, 1967).

The research agenda of making connections between cooperative solution concepts and noncooperative games is known as the Nash (1953) program. Our paper contributes to this approach by explicitly modeling the process of adjustment of players' demands in a multilateral Nash demand game,

[1] The terminology comes from Bennett (1983); earlier papers on aspirations include Cross (1967) and Albers (1979).

with the aim of providing foundations for one of the aspiration solution concepts. The game is played repeatedly, but players are myopic and do not take into account the effect of their decisions on future periods. The way in which we model demand adjustment is that, at every period, a player is randomly chosen and "selects from the whole set of feasible coalitions that one which will give him the highest possible return given the demands or payoff expectations of the necessary allies" (Cross, 1967). In doing so, the player's demand is adjusted to the residual value after paying the coalition partners' demands. Bennett et al. (1997) show that processes of this kind converge to the set of aspirations.

To be able to select a subset of the set of aspirations, we introduce small mutations into the process. In particular, we assume that with a small probability, a player experiments with a different demand, which is most likely to be a higher demand than the original one. We look for the set of aspirations that is stochastically stable (see e.g. Young, 1998) under such mutations. If a set of aspirations is stochastically stable, the process spends most of the time in this set as the probability of mutations becomes small. Intuitively, if getting out of the set requires more mutations than reaching the set from outside, the set is stochastically stable. We find that the set of aspirations which is robust to the mutation of *one* player coincides with the set of *separating* aspirations (a subset of partnered aspirations). In a separating aspiration no player is indispensable to another player; each player has several coalitions to satisfy his demand. Thus, separating aspirations are the prime candidates for being stochastically stable.

The literature on demand adjustment in cooperative games (reviewed in Section 6 of Newton, 2018, and discussed in section 4) focuses on games with a nonempty core. In contrast, we study a particular class of games with an empty core, weighted majority games (the example at the beginning of the introduction is an example of such a game). Unlike the core, aspiration solutions concepts are non-empty in these games, thus allowing to make predictions about possible outcomes. Within this class, we show that in symmetric majority games and in apex games there is a unique stochastically stable aspiration, which coincides with the unique separating aspiration. However, in other weighted majority games there may be other sets of aspirations among which the process can move with mutations of one player, never reaching a separating one.

2 The Model

2.1 Aspirations in cooperative games

Let (N, v) be a transferable utility cooperative game, where $N = \{1, 2, ..., n\}$ is the set of players and $v : 2^N \to \mathbb{R}$ with $v(\varnothing) = 0$ is the characteristic function. Any subset S of the player set N is called a *coalition*. We assume that the game is zero-normalized, $v(\{i\}) = 0\ \forall i \in N$. A *demand vector* is an n-tuple $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$. Let $x(S) := \sum_{i \in S} x_i$. The following concepts will be useful:

Definition 1 *A demand vector x is an **aspiration** if it is maximal ($\forall S$ $x(S) \geq v(S)$) and feasible ($\forall i\ \exists S \ni i$ such that $x(S) \leq v(S)$).*

Definition 2 *For given aspiration x the **generating collection** $GC(x) = \{S : x(S) = v(S)\}$ is the set of coalitions that can satisfy the demands of its members.*

With some demand vectors (aspirations), one player, i, may be able to satisfy his demand only if coalitions that satisfy this demand also include one particular another player, j, while player j can satisfy his demand without player i. The following defines aspirations where this cannot happen. Let $\mathcal{C}$ be a collection of coalitions. For each $i \in N$ let $C_i = \{S \in \mathcal{C} : i \in S\}$.

Definition 3 *A collection $\mathcal{C}$ of coalitions is **partnered** if C_i is nonempty for all i and for any i, j in N :*

$$C_i \subseteq C_j \Rightarrow C_j \subseteq C_i.$$

Definition 4 *An aspiration x is **partnered** if $GC(x)$ is partnered.*

There are two ways in which an aspiration can be partnered: either both i and j need each other (in which case $C_i = C_j$), or none of the two players need each other (in which case $C_i \backslash C_j$ and $C_j \backslash C_i$ are both nonempty). The latter of these conditions will be important in our analysis.

Definition 5 *A collection $\mathcal{C}$ of coalitions is **separating** if $C_i \backslash C_j$ and $C_j \backslash C_i$ are both nonempty for any i, j.*

Definition 6 *An aspiration x is **separating** if $GC(x)$ is separating.*

In a separating aspiration, any pair i, j of players are "separated" in the sense that each of them can find a coalition to satisfy his demand without the other player. Clearly, being separating is a stronger concept for an aspiration than being partnered (indeed, unlike the set of partnered aspirations, the set of separating aspirations can be empty in general games). The term "partnered" comes from Bennett (1983). Payoff vectors that we call "separating" are called "completely separating" in Maschler and Peleg (1966), but are referred to as "minimally partnered" in Reny et al. (2012) (in both these papers the focus is on demand vectors feasible for the grand coalition N of all players, while we consider aspirations, which are not necessarily feasible for N.) We think that "separating" is a better term, emphasizing that any pair of players do not depend on each other, i.e., can be separated.

Another concept that will be useful is the following:

Definition 7 *An aspiration x is **balanced** if x solves the problem*

$$\begin{gathered} min_x \ \sum_{i \in N} x_i \\ s.t. \ x(S) \geq v(S) \ for \ all \ S \subseteq N. \end{gathered}$$

The term "balanced" is from Bennett (1983), although the concept itself is introduced in Cross (1967). It is particularly useful for the weighted majority games that we study.

2.2 The basic demand adjustment process

The process works as follows. Time is discrete: $t = 1, 2, \ldots$. At the beginning of any period t, there is a vector of demands $x^{t-1} = (x_1^{t-1}, \ldots, x_n^{t-1})$; we will drop the superscript when no confusion arises. At $t = 1$, vector x^0 is exogenously given; at $t > 1$, x^{t-1} emerges from the previous period as described below. One of the players is randomly chosen to adjust his demand. We assume that all players have a non-zero probability to be chosen. The chosen player searches for the coalition that leaves him the highest payoff, provided that the demands of all other players in the coalition are satisfied. That is, the player solves

$$\max_{S:i \in S} \{v(S) - x(S \backslash i)\}. \tag{1}$$

Denote the maximum value for the above problem by y_i. Note that $y_i \geq 0$, since player i can always choose $S = \{i\}$, in which case $v(S) - x(S \backslash i) = 0$. Player i proposes one of the coalitions that solve the maximization problem, say coalition Q, and sets his demand to y_i.[2] Hence the new vector of demands is $x^t = (x_1^t, \ldots, x_n^t)$, where $x_i^t = y_i$ and $x_j^t = x_j^{t-1}$ for $j \neq i$. We assume that all coalitions that solve the maximization problem are proposed with a positive probability. The actual payoffs to the players at period t are

$$u_j^t = \begin{cases} x_j^t \text{ for } j \in Q, \\ 0 \text{ for } j \notin Q. \end{cases} \tag{2}$$

i.e. players in Q get their demands, while players outside Q receive nothing. The state of the process at the end of period t is described by the demand vector $x^t = (x_1^t, \ldots, x_n^t)$. We refer to a state as an *aspiration state* if x^t is aspiration.

Player i's behavior can be described as adaptive in that i plays a best response to the other players' past choices. Since the other players are not

[2] In particular, if no coalition involving other players is feasible given their demands, player i forms a singleton coalition and sets $y_i = v(\{i\}) = 0$.

able to change their demands in the current period, we can also view i's decision as rational (though myopic, since the effect of actions on future periods is not taken into account). We interpret coalition Q as a transitory arrangement that exists for period t only; it plays no role in subsequent decisions of the players.

We will denote by Ψ the correspondence that, for a given state x^t at time t, assigns the set of states that can result at time $t+1$ with positive probability according to the process described above, so that $\Psi(x)$ denotes the set of states that can be reached from x in one step.

Let $\mathcal{S}$ be the set of all possible states of the process. Given $\mathcal{A} \subseteq \mathcal{S}$, $\Psi(\mathcal{A}) := \cup_{x \in \mathcal{A}} \Psi(x)$ is the set of states that can be reached in one step from a state in the set $\mathcal{A}$.

Definition 8 *A set of states $\mathcal{A} \subseteq \mathcal{S}$ is **absorbing** if $\Psi(\mathcal{A}) = \mathcal{A}$. An absorbing set $\mathcal{A}$ is **minimal** if no strict subset of $\mathcal{A}$ is absorbing.*

Definition 9 *The **absorbing set solution** is the union of all minimal absorbing sets.*

A set of states is absorbing if, starting from a state in this set, the process cannot get out of the set. The absorbing set solution contains all the long-run outcomes of the process since the process will eventually reach one of the minimal absorbing sets starting from outside the absorbing set solution (if this was not the case, the complement of the absorbing set solution would also be absorbing, therefore it would contain a minimal absorbing set which would have to be included in the absorbing set solution, a contradiction).[3]

We now show that the absorbing set solution for this process coincides with the set of all aspirations. Given a demand vector x, player i's demand is not feasible if $\forall S \ni i \; x(S) > v(S)$. Player i's demand is not maximal if $\exists S \ni i$ such that $x(S) < v(S)$.

[3]We have taken the term *absorbing set solution* from Inarra et al. (2005); this concept also appears in Shenoy (1979) as *dynamic solution*.

Lemma 1 *Let x^{t-1} be the demand vector at $t-1$. Suppose i is randomly selected at time t to adjust his demand. Then:*

(i) If player i's demand x_i^{t-1} is not feasible, $x_i^t < x_i^{t-1}$.

(ii) If player i's demand x_i^{t-1} is not maximal, $x_i^t > x_i^{t-1}$.

Proof. Player i always sets $x_i^t = \max_{S:i\in S}\{v(S) - x^{t-1}(S\backslash i)\}$ (recall that this value is always nonnegative because i can always choose $S = \{i\}$).

(i) Because $\max_{S:i\in S}\{v(S) - x^{t-1}(S\backslash i)\} < x_i^{t-1}$ (given that x_i^{t-1} is not feasible), it follows that $x_i^t < x_i^{t-1}$.

(ii) Because $\max_{S:i\in S}\{v(S) - x^{t-1}(S\backslash i)\} > x_i^{t-1}$ (given that x_i^{t-1} is not maximal), it follows that $x_i^t > x_i^{t-1}$. ■

Our process is therefore a variant of the process of Bennett et al. (1997), since the demand adjustment part of it satisfies their three assumptions:

(i) only one player adjusts at a time;

(ii) a player will increase his demand if some coalition can support the larger demand, given the demands of others;

(iii) a player will decrease his demand if no coalition can support his current demand, given the demands of others.

Note that Bennett et al. (1997) assume that demands adjust in this way, but do not make any explicit assumptions about coalition formation. Since our demand adjustment satisfies their three assumptions, the results of Bennett et al. (1997) that demands converge to the set of aspirations also hold in our model. The underlying myopic rationality of choosing the coalition with the maximum available surplus simplifies proofs considerably.

Proposition 1 *If x^t is an aspiration, $x^{t+1} = x^t$.*

Proof. Consider any state x^t that is an aspiration. Suppose player i is randomly chosen at period $t+1$ to adjust his demand. By feasibility, $\exists S \ni i$ such that $x^t(S) = v(S)$, or equivalently $v(S) - x^t(S\backslash i) = x_i^t$. By maximality, any coalition $Q \ni i$ satisfies $v(Q) - x^t(Q) \leq 0$, which implies $v(Q) - x^t(Q\backslash i) \leq x_i^t$. From these two conditions it follows that $y_i = x_i^t$. Then player i proposes some coalition S that leaves x_i^t to him, thus $x_i^{t+1} = x_i^t$. The demands of the other players do not change, therefore $x^{t+1} = x^t$. ■

It follows that the set of aspirations is absorbing. Indeed, each aspiration vector x is a minimal absorbing set. The following proposition shows that there are no other minimal absorbing sets, hence the absorbing set solution is precisely the set of aspirations.

Proposition 2 *For any initial demand vector x^0 $\exists T$ such that there is a positive probability that $\forall t > T$ x^t is an aspiration.*

Proof. Let H^t denote the set of players whose demands are *not* feasible given x^t, and let L^t denote the set of players whose demands are *not* maximal given x^t.

Let x^0 be the vector of demands at the beginning of period 1. Player i in H^0 is selected with a positive probability to adjust his demand. Since $\max_{S:i\in S}\{v(S) - x(S\backslash i)\} \geq 0$ (e.g. $S = \{i\}$), the adjusted demand x_i^1 of player i will be feasible. Hence $|H^1| < |H^0|$ and $|L^1| \leq |L^0|$, since player i chooses a maximal coalition. Repeating the argument for players in H^1, H^2, ... with a non-zero probability the process moves to a state with $H^t = \varnothing$.

Suppose now that player $j \in L^t$ is selected. For such a player it holds that $y_i > x_i^t$. Player j increases his demand to claim the maximal surplus available, thus $|L^{t+1}| < |L^t|$. This increase may turn some previously feasible demands unfeasible. However, from the previous paragraph, when such players are selected, the process can reach a period with $H = \varnothing$ without increasing $|L|$. Thus, with a positive probability a situation with $H^r = \varnothing$, $|L^r| < |L^t|$ is reached. Continuing in this fashion, a period T with $H^T = \varnothing$ and $L^T = \varnothing$ is reached, hence x^t for $t > T$ is an aspiration. ■

Thus the process converges to an aspiration with probability 1. The aspiration approach can be criticized because players are assumed to increase their demands whenever they are not maximal, irrespective of the probability of getting those demands. Note that in our model this probability is 1 by construction, since the player who is selected to adjust his demand also selects a feasible coalition to which he belongs.

2.3 The demand adjustment process with mutations

We will assume from now on that the values $v(S)$ are rational numbers. Let m be a common denominator of these numbers, and let $\delta = \frac{1}{lm}$. The number l controls how fine the grid is. We assume that the demands of the players belong to a finite grid $\Gamma_\delta = \{k\delta : k \in \{0, \ldots, K\}\}$ where K is a sufficiently large number (e.g. $K = \frac{V}{\delta}$, where $V = \max_S v(S)$). We consider only demand vectors belonging to the grid. Note that the grid is chosen in such a way that $\forall x \in \Gamma_\delta \times \ldots \times \Gamma_\delta$, if $x(S) < v(S)$, any player $i \in S$ can increase the demand to a point $y_i \in \Gamma_\delta$ so that $x(S) = v(S)$. The state space $\mathcal{S}$ of the demand adjustment process consists of demand vectors x on the grid. With this finite grid, the demand adjustment process is a finite Markov chain. For a sufficiently fine grid the set of aspirations restricted to the grid is non-empty, and contains some partnered aspirations.

Lemma 2 *If $v(S)$ is a rational number for all $S \subseteq N$, there is at least one partnered aspiration with rational coordinates.*

Proof. See Appendix A.1. ∎

The restriction to the finite grid thus retains some aspirations with desirable properties.

Given the state space $\mathcal{S}$, let M be the matrix such that M_{ab} specifies the probability of moving from state a to state b in one step according to the demand adjustment process. Matrix M is the transition matrix of the Markov chain on this state space. A probability distribution on the (finite) state space $\mathcal{S}$ is an $1 \times |\mathcal{S}|$ vector μ, where μ_a is the probability of state a. The vector μ is a stationary distribution of the Markov chain M if $\mu M = \mu$. Note that M may have more than one stationary distribution.

The concept of absorbing sets can be naturally applied to Markov chains.

A set of states $\mathcal{A} \subseteq \mathcal{S}$ is absorbing if for any distribution μ such that the support of μ is in $\mathcal{A}$, it holds that the support of μM is also in $\mathcal{A}$. Because the process cannot permanently stay out of the absorbing set solution, the support of any stationary distribution of the Markov chain must be contained in the support of the absorbing set solution.

From the previous subsection, the absorbing set solution is precisely the set of aspirations (Propositions 1 and 2). Therefore, the support of any stationary distribution consists only of aspiration states. Note that for each particular aspiration x, there is a stationary distribution whose support only includes x. Hence there are many stationary distributions.

We extend the basic process to allow for the possibility of rare occasions in which the players' behavior differs from the one described before. We will refer to such an event as a "mutation". Mutations make the process move between aspirations and may help to select among them. The set of separating aspirations is important because such aspirations will be robust against a mutation by one player, while other aspirations are not.

The basic model assumes that the adjusting player selects the demand that solves the maximization problem (1). We now allow the possibility that this player "mutates". We assume that the player more likely mutates to a higher demand than to a lower demand. That is, with probability $1-\varepsilon$ there is no mutation (and the player adjusts in the usual way), and with probability ε there is a mutation. Conditional on a mutation having occurred, the new demand is within the set $\{x_i^{t-1}, \ldots, V\}$ with probability $1-\varepsilon$ and it is within the set $\{0, \ldots, x_i^{t-1}\}$ with probability ε.[4] Note that if $x_i^{t-1} = V$, then the most likely "mutation" is $x_i^t = x_i^{t-1}$. This model of mutations is similar to *intentional idiosyncratic play* in Naidu et al. (2010): players most likely "experiment" with demands that can give them a higher payoff (if the other players adjust). Note that when $\varepsilon = 0$ the process is the

[4]The conditional probability could be a different value $\nu \neq \varepsilon$, but we can assume that $\nu = O(\varepsilon)$ without changing the results. Setting $\nu = \varepsilon$ allows us to summarize the likelihood of various mutations with one parameter.

same as our basic process. We will consider the case where ε goes to 0.

We denote the transition matrix of the Markov chain of the process with mutation probability ε by M^{ε}. A Markov chain is *irreducible* if there is a positive probability of moving from any state to any other state in a finite number of periods. The introduction of mutations makes the process irreducible, since any vector of demands can arise as a result of n consecutive mutations, one by each player. This implies that the Markov chain M^{ε} has a unique stationary distribution for $\varepsilon > 0$ (see e.g. Young, 1998, pp. 48-49). The states that have a positive probability in the limit of this stationary distribution as ε goes to 0 are much more likely to be visited in the long run. The limit stationary distribution, denoted by $\mu^0 = \lim_{\varepsilon \to 0} \mu^{\varepsilon}$, exists (see Young, 1998, p. 56).

Definition 10 *A state x is* ***stochastically stable*** *if it has a positive probability in the limit stationary distribution as ε goes to 0, that is, $\mu_x^0 > 0$.*

For our model of mutations, the set of separating aspirations is robust to the introduction of one mutation of the most likely type, i.e. from x_i to a higher demand.

Lemma 3 *Consider state x where x is a separating aspiration. Suppose player i mutates, from x_i to a higher demand. Then the adjustment process without mutations will return to state x.*

Proof. Suppose $x_i^t > x_i^{t-1}$. If player i is selected to adjust his demand at $t+1$, because of maximality of the original x^{t-1} he will form a coalition and get x_i^{t-1}, in which case his demand returns to its original value. If another player j is selected to update his demand, he will form a coalition without i and get $x_j^t = x_j^{t-1}$. Since none of the players needed i to achieve their demands, no demands will change until player i is selected to adjust his demand, in which case x_i will return to its original value. ■

On the other hand, if an aspiration is not separating, then a mutation by one player can lead to a different aspiration.

Lemma 4 *Consider state x where x is an aspiration that is not separating. There exists a player i such that, if player i mutates from x_i to $x_i + \delta$, the adjustment process without mutations converges to a different aspiration with a positive probability.*

Proof. Since x is not a separating aspiration, there exist two players, i and j, such that either j needs i to achieve his demand but i does not need j, or i and j both need each other. Now suppose i mutates to $x_i + \delta$. If player j is selected next, he can no longer find a coalition that supports his demand and has to settle for $y_j = x_j - \delta$, supported for example by a coalition Q such that i, $j \in Q$ and Q is in the generating collection of the previous aspiration x. The new state is $(x_1, \ldots, x_i + \delta, \ldots, x_j - \delta, \ldots, x_n)$. This state is not necessarily an aspiration since some of the other players' demands may become unfeasible after an increase in player i's demand. Such players will lower their demands in the next periods with a positive probability, but player i will never lower his demand. Another aspiration will be reached with player i demanding a bit more, and some players, e.g. player j demanding a bit less. ■

That some states are resistant to one (most likely) mutation and other states are not can be helpful in identifying what states can be stochastically stable. If there are sets of states that can be disturbed only with multiple mutations, only such sets can be stochastically stable.

Definition 11 *We call a set of states $\mathcal{B}$ **locally stable** if (i) all states in $\mathcal{B}$ are in an absorbing set; (ii) for any $\mathcal{S} \subseteq \mathcal{B}$, after a mutation of one player to a higher demand the basic process converges to a state in $\mathcal{B}$; (iii) there is no proper subset of $\mathcal{B}$ that has this property.*

This definition is based on the definition in Nöldeke and Samuelson (1993). It implies that there is a sequence of mutations, one at a time, that allows to move between any two states in $\mathcal{B}$ (otherwise a subset of $\mathcal{B}$ would be locally stable). It is also related to the "one-deviation" property of Newton and Sawa (2015), although they define this property for more general mutation structures.

Proposition 3 *If state x is stochastically stable, then $x \in \mathcal{B}$, where $\mathcal{B}$ is in a locally stable set of states.*

The proposition is a restatement of Proposition 1 in Nöldeke and Samuelson (1993) and their proof applies. Intuitively, the "cost" (in terms of the number of most likely mutations) of moving away from a locally set $\mathcal{B}$ is more than 1. From states not in a locally stable set, the cost of moving away is 1. If the probability of mutations goes to zero, the process spends almost all the time in those states that are part of a locally stable set.

Lemma 3 shows that each separating aspiration is in a locally stable set, but there may be other (non-singleton and consisting of aspirations that are not separating) locally stable sets. Below we analyze the stochastic stability of aspirations in a class of weighted majority games. We show that in an important subclass of these games separating aspirations are indeed the only ones that are locally, and thus stochastically, stable. However, in other weighted majority games there exist non-singleton locally stable sets, thus aspirations that are not separating can still be stochastically stable.

3 Demand adjustment in weighted majority games

3.1 Weighted majority games

A simple voting game is a transferable utility game (N, v) such that $v(S) = 0$ or 1 for all $S \subseteq N$. We will assume that $v(S) = 1$ implies $v(T) = 1$ for all $T \supseteq S$ (monotonicity). A coalition S is called *winning* if $v(S) = 1$, and *losing* if $v(S) = 0$. The set of winning coalitions is denoted by W. A *minimal winning coalition* S is a coalition that is just large enough to win, that is, S is winning but no $T \subsetneq S$ is winning. The set of minimal winning coalitions is denoted by W^m.

We only consider simple voting games that are *proper*, that is, if $S, T \in W$ then $S \cap T \neq \varnothing$. If a simple game is proper it is not possible for two disjoint coalitions to be winning. A stronger condition is the following:

Definition 12 *A simple voting game is* ***constant-sum*** *if* $v(S) + v(N \backslash S) = 1$.

In a constant-sum game, the partition of the set of players into two sets always results in one winning coalition and one losing coalition.

A *veto player* is a player who is in all winning coalitions. A *null player* is a player such that $v(S) = v(S \cup \{i\})$ for any S; such a player does not belong to any coalition in W^m. We assume henceforth that there are no null players, that is, each player belongs to at least one coalition in W^m.

A simple voting game is *weighted* if it is possible to assign a number of votes (weight) $w_i \geq 0$ to each player and to set a threshold q such that S is winning if and only if $\sum_{i \in S} w_i \geq q$. The combination $[q; w_1, ..., w_n]$ is a *representation* of the voting game. There are many representations $[q; w_1, ..., w_n]$ that are equivalent in that they produce the same set of winning coalitions.

Definition 13 *A representation* $[q; w_1, ..., w_n]$ *is called* ***homogeneous*** *if all minimal winning coalitions have the same total weight* q.

Definition 14 *A game that admits a homogeneous representation is a* ***homogeneous game****.*

For example, $[3; 2, 1, 1, 1]$ is a homogeneous game because each minimal winning coalition has exactly 3 votes. In contrast, $[5; 2, 2, 2, 1, 1, 1]$ is not a homogeneous game. Coalition $\{1, 2, 3\}$ is minimal winning but has 6 votes, while other minimal winning coalitions (such as $\{1, 2, 4\}$) have 5 votes. Moreover, it is not possible to find an alternative representation of this game that would be homogeneous.

Two players, i and j, are of the same *type* if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subset N$, $i, j \notin S$. If $w_i = w_j$, i and j are of the same type, though the converse is not necessarily true. It will sometimes be useful to refer to coalition types by listing the player types that form the coalition, as in $[3; 2, 1, 1, 1]$ having two types of minimal winning coalition, [21] and [111].

Weighted majority games have an empty core unless there are veto players. Constant-sum games have no veto players, except for the trivial case in which there is one veto player who is also a dictator, that is, $\{i\} \in W$.

3.2 Aspirations in weighted majority games

We focus on constant-sum homogeneous games. For games in this class, there is an aspiration vector with desirable properties.

Remark 1 *Let (N, v) be a constant-sum homogeneous game and $[q; w_1, ..., w_n]$ a homogeneous representation of this game. The aspiration vector $\left(\frac{w_i}{q}\right)_{i \in N}$ is balanced, separating and has rational coordinates.*

For constant-sum homogeneous games, Peleg (1968, theorem 3.5) shows that the nucleolus (Schmeidler, 1969) is the only homogeneous representation that has $\sum_{i \in N} w_i = 1$ (hence the homogeneous representation is unique up to a multiplicative constant in this class of games). Given that the nucleolus is a representation, the vector $\left(\frac{w_i}{q}\right)_{i \in N}$, where w is the nucleolus and q is $\sum_{i \in S} w_i$ for any minimal winning coalition S, is an aspiration vector and the generating collection for this aspiration vector is W^m. Peleg (1968) also shows that the nucleolus is proportional to a representation with integer weights, hence $\left(\frac{w_i}{q}\right)_{i \in N}$ has rational coordinates. This aspiration is separating, since for any i and j there is a feasible coalition that contains i but not j. To see this, consider $S \in W^m$ such that $S \ni i$. If $j \notin S$, the result follows. Suppose $j \in S$. Since the game is constant-sum, $N \backslash S$ is losing and $\{i\} \cup N \backslash S$ is winning. Furthermore, since the game is homogeneous, there exists a coalition $T \subseteq \{i\} \cup N \backslash S$ such that $i \in T$ and $w(T) = q$; this coalition is feasible for i and does not involve j. That this aspiration vector is also balanced follows from Kohlberg (1971). It is the only balanced aspiration vector (see Morelli and Montero, 2003, Remark 10).

That the balanced aspiration has rational coordinates allows us to select the grid size δ in such a way that the grid contains the balanced aspiration. Peleg (1968) shows that a constant-sum homogeneous game has a unique integer representation $[q; w_1, \ldots, w_n]$ with $\min_{i \in N} w_i = 1$. If $\delta = \frac{1}{lq}$, then the balanced aspiration is on the grid.

For constant-sum homogeneous games, we have established that there is a unique balanced aspiration vector, which is also a separating aspiration vector and has rational coordinates. There may be many other separating aspirations as the example below illustrates.

Example 1 *(Aspirations that are separating but not balanced) Consider the game* $[4; 2, 2, 1, 1, 1]$. *All demand vectors of the form* $x = (a, a, \frac{1-a}{2}, \frac{1-a}{2}, \frac{1-a}{2})$, *where* $\frac{1}{2} \leq a \leq 1$ *are separating aspirations for this game. If* $a > \frac{1}{2}$, *the only coalitions in* $GC(x)$ *are of the form* $[211]$. *No player depends on any other; in particular, players with* 2 *votes do not depend on any particular player with* 1 *vote to obtain their demands. Aspirations with* $a > 0.5$ *are separating but not balanced, since the aspiration* $(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ *has a smaller total sum.*

The example also shows that separating aspirations can result in a very unequal distribution between types, as in the case of $x = (1, 1, 0, 0, 0)$.

If we relax the assumption that the game is constant-sum and homogeneous, it is possible for an aspiration vector to be balanced but not partnered (and therefore not separating; see Appendix A.2).

3.3 Symmetric majority games

The simplest class of games to which we can apply our adjustment process is the following. Consider the symmetric majority game with n players and $w_i = 1$ for all players:

$$[q; 1, \ldots, 1].$$

If $q = n$, then the game is a *unanimity* game (all players are needed to form a winning coalition; all players are veto players). In this game, there are no separating aspirations and all demand vectors with $x_1 + \ldots + x_n = 1$ are in the core. Therefore we consider $\frac{n}{2} < q < n$. The three-player simple majority example in the introduction is the symmetric majority game with $n = 3$ and $q = 2$.

The balanced aspiration is $\left(\frac{1}{q}, \ldots, \frac{1}{q}\right)$, which is also separating. Other aspirations include, for example $(0, \ldots, 0, 1, \ldots, 1)$, with $q - 1$ players demanding 0. There aspirations are clearly non-partnered, with players with demand 1 depending on players with demand 0.

Proposition 4 *The unique stochastically stable state for a symmetric majority game with* $\frac{n}{2} < q < n$ *is the balanced aspiration* $\left(\frac{1}{q}, \ldots, \frac{1}{q}\right)$.

Proof. Consider an aspiration $x = (x_1, \ldots, x_n)$ with $x_m = \min_{i=1,\ldots,n} x_i < x_M = \max_{i=1,\ldots,n} x_i$. Since $\frac{1}{q}$ is on the grid, $x_m \leq \frac{1}{q} - \delta$ (otherwise there are players whose demands are not feasible) and $x_M \geq \frac{1}{q} + \delta$ (otherwise there are coalitions that are not maximal).

In any coalition in $GC(x)$, players with demand x_m are included, and any excluded players demand x_M. Let $x_i = x_m$ and $x_j = x_M$. Suppose player i mutates to $x_m + \delta$. If player j is selected to adjust, he sets demand to $x_M - \delta$. Other players with demand x_M may need to adjust downwards by δ, but in a new aspiration y, $y_m \geq x_m$ and if $y_m = x_m$, then the number of players with demand x_m is smaller in y than in x. Continuing the mutations in this fashion, aspiration with $x_m = \frac{1}{q}$ is reached. Then $x_M = \frac{1}{q}$, and the balanced aspiration is reached.

Since the balanced aspiration is separating, it constitutes a locally stable set. The previous argument shows that there are no other locally stable sets. By Proposition 3, the balanced aspiration is stochastically stable. ■

3.4 Apex games

Apex games are weighted majority games with one major player (the apex player) and $n-1 \geq 2$ minor players (or base players). They can be described as

$$[n-1; n-2, 1, \dots, 1],$$

with the apex player having $n-2$ votes, each of the $n-1$ minor players having 1 vote, and $n-1$ (out of total $2n-3$) votes are needed to win. In terms of the characteristic function, an apex game is given by $v(S) = 1$ if $1 \in S$ and $|S| > 1$, or if $S = \{2, \dots, n\}$, and $v(S) = 0$ otherwise. Player 1 needs only one minor player to form a winning coalition, whereas the only way to win in the absence of the apex player is if all minor players form a coalition. Apex games have received a lot of attention in the literature since von Neumann and Morgenstern (1944), from both theoretical and experimental perspectives.[5]

The set of aspirations in apex games can be divided into several subsets. If $x_1 < \frac{n-2}{n-1}$, then in an aspiration every $x_i > \frac{1}{n-1}$, and $x(\{2, \dots, n\}) > 1$. This implies that $x_i = 1 - x_1 \ \forall i = 2, \dots, n$, with $GC(x) = \{\{1, i\}_{i=2,\dots,n}\}$ if $x_1 > 0$ and $GC(x) = \{\{1\}, \{1, i\}_{i=2,\dots,n}\}$ if $x_1 = 0$. If $x_1 > \frac{n-2}{n-1}$, in an aspiration $\min_{\{2,\dots,n\}} x_i = 1 - x_1 < \frac{1}{n-1}$, $\max_{\{2,\dots,n\}} x_i > \frac{1}{n-1}$, and $\sum_{i=2}^{n} x_i = 1$. The generating collection of such aspirations consists of the coalition of minor players $\{2, \dots, n\}$, and one or more coalitions $\{1, i\}$. If $x_1 = 1$, also some singleton coalitions are feasible. Finally there is aspiration $x = \left(\frac{n-2}{n-1}, \frac{1}{n-1}, \dots, \frac{1}{n-1}\right)$ with $GC(x) = \{\{2, \dots, n\}, \{1, i\}_{i=2,\dots,n}\} = W^m$. This aspiration is the unique balanced aspiration and it is separating.

For our demand adjustment process with mutations, the following proposition holds:

Proposition 5 *The unique stochastically stable state for apex games is the balanced aspiration* $\left(\frac{n-2}{n-1}, \frac{1}{n-1}, \dots, \frac{1}{n-1}\right)$.

[5] See Davis and Maschler (1965), Aumann and Myerson (1988), Bennett and van Damme (1991), Montero (2002), Fréchette et al. (2005) for theoretical developments and Selten and Schuster (1968), Rapoport et al. (1979), Funk et al. (1980), Rapoport (1990), Fréchette et al. (2005) for experimental studies.

Proof. Consider an aspiration x with $x_1 > \frac{n-2}{n-1}$. If there is only one coalition $\{1, i\}$ in $GC(x)$, player 1 needs player i. If player i mutates to $x_i + \delta$ and player 1 is then selected to adjust his demand, player 1 is forced to reduce his demand. Other players may need to lower their demands as well, but in the new aspiration y it holds that $y_1 < x_1$.

If there is more than one coalition $\{1, i\}$ in $GC(x)$, player 1 does not depend on any player, but there is a player k with $x_k = \max_{\{2,\ldots,n\}} x_i > \frac{1}{n-1}$ that does depend on player i. Suppose player i mutates to $x_i + \delta$ and player k is selected to adjust. Player k will propose coalition $\{2, \ldots, n\}$ with probability 1[6], so that player i receives $x_i + \delta$ and player k receives $x_k - \delta$. No other player needs to adjust, but coalition $\{1, i\}$ is not feasible for the new aspiration vector. Repeating the reasoning if necessary, a chain of mutations, happening one at a time, leads to an aspiration x in which only one coalition $\{1, i\}$ is in $GC(x)$.

Repeating the steps of the last two paragraphs, from any aspiration x with $x_1 > \frac{n-2}{n-1}$ there is a chain of mutations, happening one at a time, and possibly adjustment of demands according to the basic process, that lead to the aspiration $\left(\frac{n-2}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right)$.

Consider now aspiration x with $x_1 < \frac{n-2}{n-1}$. Since $\{2, \ldots, n\}$ is not feasible, any minor player i needs player 1. Suppose player 1 mutates to $x_1 + \delta$ and player $j \neq 1$ is selected to adjust. Player j proposes coalition $\{1, j\}$, giving a payoff $x_1 + \delta$ to player 1 and lowering his own demand to $x_j - \delta$. Furthermore, all other minor players also lower their demands when selected because the coalitions with player 1 became unfeasible. When a new aspiration y is reached, it holds that $y_1 > x_1$. Repeating the step if necessary, there is a chain of mutations (happening one at a time) and subsequent adjustment according to the basic process leading to the partnered aspiration $\left(\frac{n-2}{n-1}, \frac{1}{n-1}, \ldots, \frac{1}{n-1}\right)$.

The balanced aspiration itself cannot be upset by one mutation according

[6] since $x_1 \geq \frac{n-2}{n-1} + \delta$ and $x_k \geq \frac{1}{n-1} + \delta$, it cannot be optimal for k to propose $\{1, k\}$.

to Lemma 3, thus it is locally stable. The previous reasoning shows that there are no other locally stable sets. According to Proposition 3, this implies the result. ■

3.5 Stochastic stability in other weighted majority games

Allowing intentional "mutations" works to select the unique separating aspiration in the classes of symmetric majority games and apex games. The players that demand too little can start demanding a bit more, and, since other players depend on them to satisfy their demand, the competition for scarce players drives the demands to the separating aspiration.

However, we will see below that the process does not always lead to this strong result. While for some games only separating aspirations are stochastically stable (Example 1), we show that for other games there exist locally stable sets that do not contain separating aspirations (Examples 2 and 3). Thus the strong result for apex games from the previous subsection does not easily generalize to other weighted majority games.

Example 1 (continued) *Consider the game* $[4; 2, 2, 1, 1, 1]$. *Recall that in this game, separating aspirations are of the form* $(a, a, \frac{1-a}{2}, \frac{1-a}{2}, \frac{1-a}{2})$ *for* $\frac{1}{2} \le a \le 1$ *(this set includes the unique balanced aspiration* $(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.*)*

Consider aspiration x with $x_1 + x_2 = 1$ and $x_1 < x_2$ (the case $x_1 > x_2$ can be analyzed analogously). In x, player 2 depends on player 1: if there is a coalition $S \in GC(x), 2 \in S, 1 \notin S$, then coalition $S \backslash \{2\} \cup \{1\}$ is not maximal. Since $x_1 + x_2 = 1$ and $\frac{1}{2}$ is on the grid, then $x_1 \le \frac{1}{2} - \delta$ and $x_2 \ge \frac{1}{2} + \delta$. If player 1 mutates to $x_1 + \delta$ and player 2 adjusts to $x_2 - \delta$, then a new aspiration y is reached with $y_1 > x_1$ and $y_1 + y_2 = 1$. Continuing if necessary, an aspiration with $x_1 = x_2 = \frac{1}{2}$ can be reached by a sequence of mutations, one player (player 1) at a time.

Consider now aspiration x with $x_1 = x_2 = \frac{1}{2}$ and $x_i = x_m = \min_{k=3,4,5} x_k < x_j = x_M = \max_{k=3,4,5} x_k$. If $x_m \ge \frac{1}{4}$, then player j does not have a feasible coalition. If more than one player demands x_m, then there is a coalition S which is non-maximal. If $x_M \leq \frac{1}{4}$, then there also exists a coalition S which is non-maximal. Therefore $x_m \leq \frac{1}{4} - \delta$ and $x_M \geq \frac{1}{4} + \delta$. Since $x_i < x_j$, player j depends on player i. Suppose player i mutates and player j adjusts. Then a new aspiration y is reached, with $y_m > x_m$ and $y_M < x_M$. Continuing if necessary, the balanced aspiration $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$ is reached.

Consider now an aspiration x with $x_1 + x_2 > 1$. Suppose again $x_i = x_m = \min_{k=3,4,5} x_k < x_j = x_M = \max_{k=3,4,5} x_k$. Since the only feasible coalitions are [211], a maximal such coalition has to include player i, therefore players 1 and 2 depend on player i. Suppose player i mutates to $x_m + \delta$. One of players 1 or 2 adjusts. If then $x_1 + x_2 = 1$, then we are in one of the cases in the previous paragraphs. Continuing if necessary, either an aspiration with $x_1 + x_2 = 1$ is reached, or an aspiration with $x_3 = x_4 = x_5$. If the former, the process continues as described in the previous paragraphs. If the latter, then $x_1 = x_2$ (otherwise a player is not feasible, or a coalition not maximal). A separating aspiration $\left(a, a, \frac{1-a}{2}, \frac{1-a}{2}, \frac{1-a}{2}\right)$ with $\frac{1}{2} \leq a \leq 1$ has been reached. Therefore from any aspiration, a sequence of mutations, one player at a time, can reach the set of separating aspirations $\left(a, a, \frac{1-a}{2}, \frac{1-a}{2}, \frac{1-a}{2}\right)$ with $\frac{1}{2} \leq a \leq 1$. This set is the unique locally stable set. Therefore, stochastically stable states are within this set of separating aspirations.

The previous example shows that there are games other than apex games in which only the separating aspirations are stochastically stable in the demand adjustment process with mutations (even if the set is larger than the unique balanced aspiration), because locally stable sets contain only separating aspirations. However, in other games there are locally stable sets that contain other aspirations (including non-partnered ones).

Example 2 *Consider the game* $[7; 5, 2, 2, 1, 1, 1, 1]$. *Consider aspiration* $x = (0.8, a, 0.7 - a, 0.1, 0.1, 0.1, 0.1)$ *with* $0.2 < a < 0.5$. *In* x, *no player depends on any other player, except players 2 and 3, who depend on each other. Thus it is partnered but not separating.*

Suppose that the process is at x. If a player other than player 2 or 3 mutates upwards, then no other player would need to adjust; the process will return to x. Suppose player 2 mutates upwards to y_2 (mutations by player 3 can be analyzed analogously). The only other player who would need to adjust is player 3. If $y_2 < 0.5$, player 3 adjusts to $y_3 = 0.7 - y_2$ and in the new aspiration players 2 and 3 still depend on each other and there are no other dependencies among the players. If $y_2 \geq 0.5$, then player 3 adjusts to $y_3 = 0.2$ (with coalition $\{1,3\}$). If $y_2 = 0.5$, then y is an aspiration. If $y_2 > 0.5$, player 2's demand is unfeasible and he has to lower the demand to 0.5. In either case aspiration $y = (0.8, 0.5, 0.2, 0.1, 0.1, 0.1, 0.1)$ is reached. This aspiration is not partnered, since player 2 depends on player 3 but not vice versa. If player 3 now mutates upwards, then player 2 would need to adjust, but the adjustment would lead either to aspiration $z = (0.8, 0.2, 0.5, 0.1, 0.1, 0.1, 0.1)$ or to an aspiration like x.

Therefore, the set of aspirations $(0.8, a, 0.7 - a, 0.1, 0.1, 0.1, 0.1)$ with $0.2 \leq a \leq 0.5$ is a locally stable set. The set contains non-partnered aspirations ($a = 0.2$ or $a = 0.5$). Aspirations in the set can be reached one from another by a series of mutations, one at a time, but no aspiration outside of the set (including the unique balanced aspiration $\left(\frac{5}{7}, \frac{2}{7}, \frac{2}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}\right)$) can be reached from it by one mutation.

Example 3 *Consider the game* $[8; 2, 2, 2, 2, 2, 2, 1, 1, 1]$*, with nine players, players 1-6 have two votes each and players 7-9 have one vote each. Minimal winning coalitions in this game can be either four players with two votes* ([2222]) *or three players with two votes and two players with one vote* ([22211])*.*

In this game, the unique balanced aspiration is $\left(\frac{2}{8}, \ldots, \frac{2}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}\right)$. Consider aspiration $x = \left(\frac{2}{8}, \frac{2}{8} + \delta, \ldots, \frac{2}{8} + \delta, \frac{1}{8} - \delta, \frac{1}{8} - \delta, \frac{1}{8} - \delta\right)$, in which only one player with two votes demands $\frac{2}{8}$, while other such players demand δ more. It is non-partnered, with all players depending on player 1. Mutations of players other than player 1 will result in the process going back to x. Suppose player 1 mutates upwards. If any of players 2-6 adjust, the adjustment is to $\frac{2}{8}$. Player 1 then adjusts to $\frac{2}{8}+\delta$, leading to an aspiration that is a permutation of x (within types of players). If player 7 adjusts, the adjustment is to $\frac{1}{8}-2\delta$. Player 1 then adjusts to $\frac{2}{8}+\delta$ and the new aspiration is $y = \left(\frac{2}{8}+\delta, \dots, \frac{2}{8}+\delta, \frac{1}{8}-2\delta, \frac{1}{8}-\delta, \frac{1}{8}-\delta\right)$ (if players 8 or 9 adjusts, the new aspiration is a permutation of y). In y, all players depend on player 7. If player 7 mutates upwards, then either players 8 or 9 adjust to $\frac{1}{8}-2\delta$, leading to an aspiration that is a permutation of y, or any of the players 1-6 adjust to $\frac{2}{8}$, leading to an aspiration that is a permutation of x. The process thus can move between aspirations like x and y with one mutation but cannot reach any other aspiration with one mutation. The set of aspirations that are permutations of x and y is locally stable, even though none of these aspirations is partnered.

Note that the reasoning in the previous paragraph does not depend (much) on the size of δ: if, for example, $\delta' = \delta/2$, the same reasoning applies. There is also nothing special about it being only δ away from the balanced aspiration. Consider aspiration $x' = (\frac{2}{8}+a\delta, \frac{2}{8}+(a+1)\delta, \dots, \frac{2}{8}+(a+1)\delta, \frac{1}{8}-(\frac{3}{2}a+1)\delta, \dots, \frac{1}{8}-(\frac{3}{2}a+1)\delta)$, with integer a divisible by 2 and $0 \leq a \leq \frac{1}{12\delta}-\frac{4}{3}$ (the example in the previous paragraph is obtained by setting $a=0$). In x', all players depend on player 1. Similarly to the discussion in the previous paragraph, mutations of one player can move between permutations of x' and $y' = (\frac{2}{8}+(a+1)\delta, \dots, \frac{2}{8}+(a+1)\delta, \frac{1}{8}-(\frac{3}{2}a+2)\delta, \frac{1}{8}-(\frac{3}{2}a+1)\delta, \frac{1}{8}-(\frac{3}{2}a+1)\delta)$. The set of aspirations which are permutations of x' and y' is again locally stable.

These last two examples show that it is not necessarily the case that only separating aspirations are contained in a locally stable set. The analysis of stochastic stability in these games then requires going beyond locally stable sets, looking also at mutations that are not most likely ones. We leave this analysis for future research.

4 Related Literature

The starting point of our model is Cross (1967) who presents a first attempt to formalize the competition for players whose "price" (as represented by their demand) is too low. This competition can be thought as driving prices up for players indispensable for others to satisfy their demand. This concept underlies the approach used in Maschler and Peleg (1966) for payoff vectors feasible for the grand coalition and in Albers (1979) and Bennett (1983) for more general payoff vectors.

Treating players' behavior as setting a demand has an obvious connection with Nash (1953) demand game that models two-player bargaining. Young (1993) is the first who uses stochastic stability in this game, in a process of best responding (to finite samples of past observations), finding that the payoff division related to the Nash bargaining solution is the unique stochastically stable one. Since then, other papers analyzed dynamic processes in cooperative games, as reviewed in Section 6 of the survey paper Newton (2018). The most relevant of these papers are also discussed below.

Several papers in this literature study demand adjustment processes in cooperative games (from Green, 1974 and Feldman, 1974 to more recent Agastya, 1997, 1999; Arnold and Schwalbe, 2002; Newton, 2012; Rozen, 2013; Sawa, 2019; Nax, 2019). While these papers differ on the details of the process (such as whether adjustment by coalitions is allowed, whether players only set a demand or also specify coalition partners, or which (if any) mutations are more likely), they all focus on games with a non-empty core. Payoff allocations in the core are obtained by the adjustment process, with the possibility of mutations allowing in some cases further selection within the core, as in Agastya (1999), Newton (2012) and Sawa (2019). Other papers, such as Klaus et al. (2010), Newton and Sawa (2015), Nax and Pradelski (2015, 2016), and Klaus and Newton (2016) obtain similar results for assignment games and matching problems with non-empty core.

The one paper that has a result for games with an empty core is Nax

(2019), where in such games his process cycles through all coalitions structures, including the one with all players being singletons. (In Nax's paper, players' individual demands are called their "aspirations", not the whole demand vectors, as in Bennett, 1983, and in our paper.) The mechanism relies on joint deviations by coalitions: in the process, with a positive probability any (potentially profitable) coalition can be selected to adjust their demands jointly but demands are made individually and may be incompatible, in which case the players split into singletons. In contrast, in the (basic) process used in this paper (which is based on Bennett et al., 1997), only one player adjusts at a time, and a coalition always forms. In weighted majority games, minimal winning coalitions form in our stochastically stable states.

Our model is closely related in spirit to the above mentioned models. We allow only one player to adjust and the players also selects a coalition to form. We further allow for mutations, which are more likely to be demand increases (in the spirit of "intentional mistakes" in Naidu et al., 2010). For some weighted majority games (which all have an empty core), our model allows quite a sharp prediction, selecting among aspirations those that are separating.

5 Conclusion

The paper presented a simple best-reply adaptive model of demand adjustment in cooperative games. The basic process without mutations converges to the set of aspirations; introducing certain mutations allowed us to select a plausible subset of the set of aspirations.

Our model of mutations allowed players to experiment with higher demands. This model identifies the set of separating aspirations, in which no player is indispensable in order for other players to achieve their demands, as the set that is most resistant to change.

For two particular classes of weighted majority games, namely symmetric games and apex games, we show with such infrequent mutations the set of separating aspirations is stochastically stable. In this way we provide sharp predictions for these important classes of games with an empty core.

A Appendix

A.1 Proof of Lemma 2

Lemma 2 *If $v(S)$ is a rational number for all $S \subseteq N$, there is at least one partnered aspiration with rational coordinates.*

Proof. Recall that the set of balanced aspirations is defined as the solution to the following linear programming problem

$$\begin{gathered} \min_x \ \textstyle\sum_{i \in N} x_i \\ \text{s.t. } x(S) \geq v(S) \text{ for all } S \subseteq N. \end{gathered}$$

This problem can be solved by the simplex method to obtain a balanced aspiration with rational coordinates. If this balanced aspiration is also partnered, we have found a partnered aspiration with rational coordinates. If not, we can use the method of Bennett (1983, theorem 6.5) to find a partnered aspiration. This procedure uses the dual linear programming problem

$$\begin{gathered} \max_\lambda \ \textstyle\sum_{S \subset N} v(S)\lambda_S \\ \text{s.t.} \textstyle\sum_{S \ni i} \lambda_S \leq 1 \\ \lambda_S \geq 0 \text{ for all } S \subset N. \end{gathered}$$

Let x be the balanced aspiration we found by solving the primal. By complementary slackness, any coalition that has $\lambda_S > 0$ in the corresponding solution of the dual has $x(S) = v(S)$, that is, it belongs to $GC(x)$. Also, any player with $x_i > 0$ in the balanced aspiration under consideration has $\sum_{S \ni i} \lambda_S = 1$. Other players may in principle have $\sum_{S \ni i} \lambda_S < 1$, but these players must be getting $x_i = 0$, so that $\{i\}$ is in the generating collection of x. We can then take $\lambda_{\{i\}}$ to be as large as needed so that $\sum_{S \ni i} \lambda_S = 1$ holds for all players, while still keeping the property that only coalitions in

$GC(x)$ can have positive values for λ_S. Some coalitions may be in $GC(x)$ and have $\lambda_S = 0$, and, as Bennett (1983) shows and we discuss below, this is the reason why balanced aspirations are not always partnered.

Denote by $\mathcal{C}(x)$ the collection of coalitions with $\lambda_S > 0$. A crucial step of Bennett's argument is that, if these were the only coalitions in the generating collection, the aspiration x would be partnered since the partnership condition holds for $\mathcal{C}(x)$, that is, for all i and j,

$$\exists S' \in \mathcal{C}(x), i \in S', j \notin S' \Longrightarrow \exists S'' \in \mathcal{C}(x), j \in S'', i \notin S''.$$

This is because, if all $S \in G(x)$ that contain i also contain j, but not the reverse, we would have $\sum_{S \ni i} \lambda_S < \sum_{S \ni j} \lambda_S$, hence it would not be possible for both sums to equal 1.

Hence, if x is not partnered, this must be because of a coalition S such that $S \in GC(x)$, $S \notin \mathcal{C}(x)$, $j \in S$, $i \notin S$. We now modify x slightly so that S stops being in $GC(x)$ without any other coalition being added to $GC(x)$. Let y be such that $y_k = x_k$ for all $k \neq i$, j; $y_i = x_i - \delta$, $y_j = x_j + \delta$. If δ is sufficiently small, none of the coalitions involving i that were previously unfeasible will become feasible; also, if δ is chosen to be a rational number, the new vector y will still have rational coordinates.

The vector y is an aspiration and, since all coalitions involving j but not i have become unfeasible, i and j now satisfy the partnership condition. There may be other players that were unpartnered in x and are still unpartnered, and there may even be some previously partnered players that have become unpartnered (this would be the case if player k can form a coalition without player l under both x and y, but all coalitions player l could form without k under x have become unfeasible because they all involved j and excluded i). However, since the partnership condition holds for $\mathcal{C}(x)$, the coalition in $GC(y)$ containing k but not l must have a weight of 0, and the same process can be applied to make that coalition unfeasible so that k and l become partners.

The process can be repeated until a partnered aspiration is reached. Coalitions in $\mathcal{C}(x)$ are not affected by the process, hence the demand vectors remain partnered aspirations when restricted to $\mathcal{C}(x)$. Every time an adjustment is made some coalitions leave $GC(x)$ and no coalitions are added to $GC(x)$. Since $GC(x)$ is a finite set the process eventually terminates, and the resulting aspiration is partnered (and incidentally still balanced, since the total sum of the demands is not altered). ■

A.2 An aspiration vector that is balanced but not partnered

In a constant-sum homogeneous game there is a unique balanced aspiration which is also partnered, see Remark 1. The following example shows that, for games outside this class, it is possible for an aspiration to be balanced but not partnered.

Example 4 *(An aspiration vector that is balanced but not partnered) Consider the game* $[42; 11, 11, 9, 7, 7, 7, 7, 5, 5, 1]$*, which appears in Freixas and Molinero (2009). The aspiration vector* $x = \left(\frac{w_i}{q}\right)_{i \in N}$ *is balanced but not partnered.*

Note that the above game is neither constant-sum nor homogeneous. It is not constant-sum because the majority is 42 out of a total of 58 votes, so for example coalition $\{1, 2\}$ and its complement are both losing. It is not homogeneous because there are minimal winning coalitions such as coalitions of type [11 11 9 7 5] which have more than 42 votes. Note that the only coalitions in $GC(x)$ are the ones that have exactly 42 votes. It can be shown that the aspiration vector x is balanced, but it is not partnered because the player with 9 votes needs the player with 1 vote, but the player with 1 vote can form a coalition of type [11 11 7 7 5 1] without the player with 9 votes.

References

[1] Agastya, M. (1997). Adaptive Play in Multiplayer Bargaining Situations. *Review of Economic Studies* 64, 411-426.

[2] Agastya, M. (1999). Perturbed Adaptive Dynamics in Coalition Form Games. *Journal of Economic Theory* 89, 207-233.

[3] Albers, W. (1979). Core- and Kernel- Variants Based on Imputations and Demand Profiles, in *Game Theory and Related Topics*, ed. by O. Moeschlin and D. Palaschke. North-Holland.

[4] Arnold, T., Schwalbe, U. (2002) Dynamic Coalition Formation and the Core. *Journal of Economic Behavior and Organization* 49, 363-380.

[5] Aumann, R. J., Myerson, R. (1988). Endogenous formation of links between players and of coalitions: an application of the Shapley value, in *The Shapley value, Essays in Honor of Lloyd Shapley* (edited by A. Roth). Cambridge: Cambridge University Press.

[6] Bennett E. (1983) The Aspiration Approach to Predicting Coalition Formation and Payoff Distribution in Sidepayment Games. *International Journal of Game Theory* 12, 1-28.

[7] Bennett, E., van Damme, E. (1991). Demand Commitment Bargaining: The Case of Apex Games, in *Game Equilibrium Models*, Vol III, *Strategic Bargaining* (edited by R. Selten.) 118-140. Berlin: Springer-Verlag.

[8] Bennett E., Maschler M., Zame W. (1997) A Demand Adjustment Process. *International Journal of Game Theory* 26, 423-438.

[9] Cross J. (1967) Some Theoretic Characteristics of Economic and Political Coalitions. *Journal of Conflict Resolution* 11, 184-195.

[10] Davis, M., Maschler, M. (1965). The Kernel of a Cooperative Game. *Naval Research Logistics Quarterly* **12**, 223-259.

[11] Feldman, A.M. (1974) Recontracting Stability. *Econometrica* 42, 35-44.

[12] Fréchette, G., Kagel, J., Morelli, M. (2005) Behavioral Identification in Coalitional Bargaining: An Experimental Analysis of Demand Bargaining and Alternating Offers. *Econometrica* 73, 1893-1937.

[13] Freixas, J., Molinero, X. (2009) On the Existence of a Minimum Integer Representation for Weighted Voting Systems. *Annals of Operations Research* 166, 243-260.

[14] Funk, S. G., Rapoport, A., Kahan, J. P. (1980). Quota vs. Positional Power in 4-person Apex Games. *Journal of Experimental Social Psychology*, 16, 77-93.

[15] Green, J.R. (1974) The Stability of Edgeworth's Recontracting Process. *Econometrica* 42, 21-34.

[16] Inarra, E., Kuipers, J., Olaizola, N. (2005) Absorbing and Generalized Stable Sets. *Social Choice and Welfare* 24, 433-437.

[17] Klaus, B., Klijn, F., Walzl, M. (2010) Stochastic Stability for Roommate Markets. *Journal of Economic Theory* 145, 2218-2240.

[18] Klaus, B., Newton, J. (2016) Stochastic Stability in Assignment Problems. *Journal of Mathematical Economics* 62, 62-74.

[19] Kohlberg, E. (1971) On the Nucleolus of a Characteristic Function Game. *SIAM Journal of Applied Mathematics* 20, 62-66.

[20] Maschler M., Peleg B. (1966) A Characterization, Existence Proof and Dimension Bounds for the Kernel of a Game. *Pacific Journal of Mathematics* 18, 289-328.

[21] Montero, M. (2002) Noncooperative Bargaining in Apex Games and the Kernel. *Games and Economic Behavior* 41, 309-321.

[22] Morelli, M., Montero, M. (2003) The Demand Bargaining Set: General Characterization and Application to Majority Games. *Games and Economic Behavior* 42, 137-155.

[23] Naidu, S., Hwang, S.H., Bowles, S. (2010) Evolutionary Bargaining with Intentional Idiosyncratic Play. *Economics Letters* 109, 31-33.

[24] Nash, J. (1953) Two-Person Cooperative Games. *Econometrica* 21, 128-140.

[25] Nax, H.H. (2019) Uncoupled Aspiration Adaptation Dynamics into the Core. *German Economic Review* 20, 243-256.

[26] Nax, H.H., Pradelski, B.S.R. (2015) Evolutionary Dynamics and Equitable Core Selection in Assignment Games. *International Journal of Game Theory* 44, 903-932.

[27] Nax, H.H., Pradelski, B.S.R. (2016) Core Stability and Core Selection in a Decentralized Labor Matching Market. *Games* 7, 10

[28] Neumann, J. von, Morgenstern, O. (1944). *Theory of Games and Economic Behavior*. Princeton University Press.

[29] Newton, J. (2012) Recontracting and Stochastic Stability in Cooperative Games. *Journal of Economic Theory* 147, 364-381.

[30] Newton, J. (2018) Evolutionary Game Theory: A Renaissance. *Games* 9, 31.

[31] Newton, J., Sawa, R. (2015) A One-Shot Deviation Principle for Stability in Matching Problems. *Journal of Economic Theory* 157, 1-27.

[32] Nöldeke G., Samuelson L. (1993) An Evolutionary Analysis of Backward and Forward Induction. *Games and Economic Behavior* 5, 425-454.

[33] Peleg, B. (1968) On Weights of Constant-Sum Majority Games. *SIAM Journal of Applied Mathematics* 16, 527-532.

[34] Rapoport, A. (1990) Test of the Kernel and Two Bargaining Set Models in Four- and Five-Person Games. Experimental Studies of Interactive Decisions. Theory and Decision Library Volume 5, 1990, pp 183-209

[35] Rapoport, A., Kahan, J., Funk, S., Horowitz, A. (1979). *Coalition Formation by Sophisticated Players.* Springer-Verlag.

[36] Reny, P.J., Winter, E., Wooders, M. (2012) The Partnered Core of a Game with Side Payments. *Social Choice and Welfare*, 521-536.

[37] Rozen, K. (2013) Conflict Leads to Cooperation in Demand Bargaining. *Journal of Economic Behavior and Organization* 87, 35-42.

[38] Sawa, R. (2019) Stochastic Stability under Logit Choice in Coalitional Bargaining Problems. *Games and Economic Behavior* 113, 633-650.

[39] Schmeidler, D. (1969) The Nucleolus of a Characteristic Function Game. *SIAM Journal on Applied Mathematics* 17, 1163-1170.

[40] Selten, R., Schuster, K.G. (1968). Psychological Variables and Coalition-Forming Behavior, in *Risk and Uncertainty* (edited by K. Borch and J. Mossin) 221-241. London: Macmillan.

[41] Shenoy, P.P. (1979) On Coalition Formation: A Game-Theoretic Approach. *International Journal of Game Theory* 8, 133-164.

[42] Young, H.P. (1993) An Evolutionary Model of Bargaining. *Journal of Economic Theory* 59, 145-168.

[43] Young, H.P. (1998) *Individual Strategy and Social Structure: An Evolutionary Theory of Institutions.* Princeton University Press.