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Fiscal rules and the maximum sustainable size of the public debt in the Diamond overlapping generations model

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Abstract

We show that the size of the maximum sustainable public debt in the Diamond (1965) OLG model depends on the choice of fiscal instrument. If tax revenue is exogenous, there is a maximum at an interior or bifurcation point, as in Rankin and Roffia's (2003) initial paper but at a generally higher level than for their case of an exogenous debt. Conversely, if income tax rates are exogenous, the technical maximum is at a corner-point of degeneracy, provided the first *Inada* condition holds. However, as this implies notional average taxes of 100%, a *Laffer Curve* becomes important, giving rise to another concept of the maximum debt. More generally, for this closed economy case, the form of the production function in conjunction with the fiscal rule will determine whether the maximum debt is reached at a technical point of bifurcation or at a more behavioural one where tax payers *en masse* deny further payments to bond holders.

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1. Introduction

Large fiscal deficits have been a general concern with a traditional focus on their sustainability in terms of satisfying an intertemporal government budget constraint. For example, see papers by Nielsen (1992), Bohn (1995) and Chalk (2000). With the recent experience of escalating public debt to GDP ratios, attention has switched from deficits to debt and to the question whether excessively large levels may be accommodated? The OECD average for *general government gross financial liabilities* as percentage of GDP rose from 68.7% to 111.9% over the period 2001-2013. For the UK and the US there were steeper climbs from 40.4% and from 54.5% both to 109.1%, but with higher ascents for Greece to 183.7% and for Japan to 228.4%.[3] One might think that debt to GDP ratios cannot continually rise and a day of financial reckoning might eventually come.[4] Although very high current levels of debt may eventually fall to lower steady state values, it is of some interest to investigate the limits required for convergence.[5]

In a timely paper, Rankin and Roffia (2003) investigated the maximal sustainable level of the public debt in the Diamond (1965) overlapping generations model. This model, which was originally geared towards analysing the welfare implications of public debt, but which has proved remarkably versatile ever since both for public finance and for finite-horizon macroeconomics, had, surprisingly, never this particular question asked of it. The answer provided by Rankin and Roffia with respect to a Cobb Douglas production technology is that there exists a maximum level of public debt as a point of bifurcation or of fold-catastrophe, which results from the nonlinearity of the model, that is, from the curvature of the saving function. The capital stock, as the outcome of previous household saving, nonlinearly feeds back on to current saving through determining factor prices, particularly, the wage. If the public debt should ever exceed its maximal sustainable level, the dynamics of capital stock adjustment would imply a step-wise implosion of the economy.

[3] These figures could even be understated on the basis of the claim by Reinhart and Rogoff (2013) of a tendency for some part of public debt to be hidden away from the official accounts.
[4] The historic high for the UK was 260% of GDP in 1822 after a century of war. Ferguson (2008) argues that besides protracted warfare leading to high debt levels, a nation's ability to successfully wage war relies on its ability to issue debt, which in turn depends on the sophistication of and on public confidence in its financial institutions.
[5] The concern here is with feasibility rather than optimality, and is analogous to determining the boiling point of a liquid rather than the preferred temperature for drinking it, assuming it is drinkable.

Other authors have extended this analysis in various directions. Braeuninger (2007) presents a linear AK model with a similar bifurcation property and two steady-states, arising out of a separate, although standard, backward-looking adjustment process for the debt that introduces nonlinear feedback. Farmer and Zotti (2010) extend the Rankin and Roffia model to an open economy, which allows one country to hold part of another's country's debt, where the basic results are carried over.

The general approach, with the exception of Braeuninger (2005), emanating from the seminal paper by Diamond, is to treat the public debt not only as constant but as *exogenous*. That is to say, the size of the public debt is predetermined, so that in a two asset model the capital stock emerges as the residual from saving. This more, hypothetical treatment serves as a useful analytical benchmark but, in practice, probably better suits either a war-time scenario where a patriotic demand for the national debt trumps more ordinary concerns with rates of return or another one of financial repression with inelastic returns. However, a great deal of public debt generally consists of flexibly priced bonds, which suggests that any maximum ought to be determined with respect to an asset market. The joint determination of the capital stock and the maximum value of a flexibly-priced public debt value is the concern of this paper.

This requires that these two are determined by the simultaneous solution of a debt-adjusted investment-saving equality and of no-arbitrage condition between public debt and private capital, which we label, respectively, as the *macroeconomic equilibrium condition* (MEC) and the *asset equilibrium condition* (AEC). In a preliminary section, where the public debt is deemed to be exogenous, the latter plays a more subsidiary role in determining the levels of some other variables after the public debt as a share of wealth has been determined by sole recourse to the former. Subsequently, the two-equation model is not only, naturally, more complicated, but also an additional source of bifurcation maxima. If we assign the stylised, Cobb-Douglas, capital income share value of one-third, thus making its inverse an integer, we may obtain closed-form solutions to the model, albeit polynomial one that may be of an order that is as high as six. Of the technical solutions, there is always an economically meaningful one, whether this bifurcates at an interior point or extends further to one of degeneracy.[6]

[6] A bifurcation is considered from a reversed perspective where the two become one.

Initially the main results of Rankin and Roffia (2003) where the debt is exogenous are replicated but with a consideration of proportional income taxes. A bond-pricing model is then introduced to allow for the public debt to be endogenous, so that either the amount of tax revenue or the income tax rates may be exogenous. For the first case, the qualitative results in Rankin and Roffia are found to carry over but with higher levels of the maximal public debt. The rationale is that making the debt endogenous causes its asset crowding-out effect to become multiplicative rather than additive. For the second case, however, there is a qualitative change in the nature of the results to the extent that the bifurcations are replaced by degeneracies, where economic activity is driven down to zero, also a theoretical possibility that is also discussed by these authors. In our model, this occurs because the exogeneity of tax rates causes revenue too to be endogenous of income, so that the contractionary effect of debt-servicing also becomes multiplicative instead of additive. It also depends on the presence of the first Inada condition, as ensured by the maintained choice of a Cobb-Douglas technology.

As this final case implies untenably high average income tax rates of 100%, it is difficult not to make a case for the operation of a *Laffer Curve*. The peak of this *curve* then defines an alternative concept of the maximum public debt. Bond-holders will then expect to receive what tax-payers *en masse* are prepared to pay rather than what has promised them as inscribed on their bond receipts.[7] When the size of the public debt is practically constrained by the supply of revenue through tax compliance, then the technical bifurcation result is relegated to being a more of a theoretical possibility. By inference, the issue of whether the public debt of a country is held nationally or globally – and, thus, whether its economy is open or closed – becomes irrelevant in the absence of international fiscal flows.

Having obtained the main results for a model that is concerned with asset demands and factor income taxation, it would be remiss to overlook land as another asset to be held that yields another income stream to be taxed. As a non-producible factor, land is interesting both as another source of capital crowding-out and as a buffer, but a limited one, against the further crowding-out effect of public debt. Likewise, consideration is also given to uncertainty in the returns to capital, which enables a model that deals with strictly aggregate asset demands to

[7] In current parlance of receiving "haircuts", with reference to a steady state of an OLG model, each generation must visit the hairdressers.

be converted into one of individual portfolio balance. Both the presence of land and of uncertain returns to capital can potentially raise the level of the maximum debt.

The rest of the paper is set-up as follows. In Section 2, we present the basic model. Section 3 rehearses the exogenous debt case of Rankin and Roffia (2003) but with various possibilities of proportional income taxation on capital and labour. Section 4 augments the model by introducing a bond market, that allows the public debt to be endogenous to set the stage for Sections 5 and 6 where the exogeneity of tax revenue and rates are considered in turn. Section 7 considers the further issues of land and of uncertain returns, while Section 8 summarises the analysis, gives some caveats, and makes some conjectures.

2. The model

2.1 Households

All of the standard assumptions of the Diamond (1965) model apply. Households live for two periods and derive utility from consumption, c_t^Y and c_{t+1}^O, when young and old. The utility function, $U_t = \theta \ln c_t^Y + (1-\theta)c_{t+1}^O$, is logarithmic, where θ is a relative time-preference factor, $0 < \theta < 1$. Households are of measure one and inelastically supply a unit of labour when young, for which they receive a wage, w_t, net of a tax ρT_t, where T_t is the total tax burden and ρ is its share falling on labour. They save by acquiring financial deposits, d_t, and public debt, b_t: $s_t = b_t + d_t$. The net-of-tax return on deposits is $R_{t+1}^K - (1-\rho)T_{t+1}/d_t$, where $1-\rho$ is the remaining hare of tax raised on deposits or capital. We assume that the return on government debt is left untaxed at R_{t+1}^B. The periodic household budget constraints are $c_t^Y = w_t - \rho T_t - s_t$ and $c_{t+1}^O = R_{t+1}^B b_t + R_{t+1}^K d_t - (1-\rho)T_{t+1}$. Utility maximization leads to a level of saving

$$s_t = (1-\theta)(w_t - \rho T_t), \tag{1}$$

The interest rate does do not appear in the saving function, because a logarithmic utility function means that the income and substitution effects exactly cancel out, and because the absence of a future earned household income precludes the possibility of a discounting effect.

Deposit saving provides the funds required for the next period's capital stock through an implicit process of financial intermediation, while the holding of public debt is a form of non-productive saving and a source of crowding-out. It takes one period both for financial deposits to materialise into physical investment goods and also for the capital stock to fully depreciate, so that $k_{t+1} = d_t$. With a constant population, capital accumulation is then

$$k_{t+1} = d_t = s_t - b_t \tag{2}$$

2.2 *Production*

Firms produce by using a constant return to scale, Cobb-Douglas and labour-augmenting technology, $y_t = A_t^{1-\alpha} k_t^{\alpha}$, $0 < \alpha < 1$, $A_t = GA_{t-1}$, where total factor productivity, A_t, grows exogenously by the factor $G \geq 1$. Under profit maximization, the gross factor returns for labour and capital are

$$R_t^K = \alpha A_t^{1-\alpha} k_t^{\alpha-1}, \quad w_t = (1-\alpha) A_t^{1-\alpha} k_t^{\alpha} \tag{3}$$

2.3 *Equilbrium*

The three numbered equations above give the macroeconomic equilibrium condition (MEC),

$$k_{t+1} = (1-\theta)\left((1-\alpha) A_t^{1-\alpha} k_t^{\alpha} - \rho \mathrm{T}_t\right) - b_t . \tag{4}$$

Public debt generally has two contractionary effects: the asset crowding-out effect in b_t and the reduction in saving caused by taxing wages, ρT_t, to the service the for debt. Naturally, only the former occurs, if only capital is taxed ($\rho = 0$). Secondly, there is a no-arbitrage between holding untaxed government debt and taxed capital/deposits or the asset market equilibrium condition (AEC)

$$R_t^B = R_t^K - (1-\rho) T_t / k_t . \tag{5}$$

All trending variables within a balanced-growth equilibrium are denoted by *tildes*, so that the above two equations become

$$G\tilde{k}_t = (1-\theta)\left((1-\alpha)\tilde{A}_t^{1-\alpha}\tilde{k}_t^{\alpha} - \rho\tilde{\mathrm{T}}_t\right) - \tilde{b}_t \tag{6}$$

$$R^B = R^K - (1-\rho)\tilde{T}_t \big/ \tilde{k}_t \tag{7}$$

2.3 Parameter values

We benchmark the model by assigning values for the rates of both growth and the steady-state interest rate for when both debt and taxes are absent. The steady-state of the interest factor is solved from equations (3) and (6) as $R^K(0) = \alpha G/(1-\alpha)(1-\theta)$. We fix the length of the period, L, at 35 years in order to compute annual measures. Setting the capital income share at its stylized value of one-third ($\alpha = 1/3$) and the discount parameter at $\theta = 5/6$ ensures that if $G = 2$, so that the annual growth rate approximates 2%, then $R^K(0) = 6$, which implies an annual interest rate of 5.25%. To repeat,

$$\alpha = 1/3, \quad \theta = 5/6, \quad G = 2, \quad L = 35. \tag{8}$$

3. An exogenous public debt

3.1 The government financing requirement

In this first analytical section, we return to Rankin and Roffia (2003) by treating the level of debt as exogenous. The government financing requirement is

$$b_t - b_{t-1} = (R_t^B - 1)b_{t-1} - \mathrm{T}_t \tag{9}$$

Public debt is raised whenever there is a shortfall in the tax revenue, T_t, required for servicing debt interest payments, $(R_t^B - 1)b_{t-1}$. No primary expenditures are assumed for the same reason government debt interest is left untaxed: to secure the highest possible value of the debt. Equation (9) implies that tax revenue in the steady state is

$$\tilde{\mathrm{T}}_t = (R_t^B G^{-1} - 1)\tilde{b}_t, \tag{10}$$

3.2 The distribution of the tax burden

Solving equations (7) and (10) simultaneously through the elimination of R^B gives an expression for the total tax burden, $\tilde{T}_t = (R_t{}^K G^{-1} - 1)\tilde{b}_t \big/ \left(1 + (1-\rho)\tilde{b}_t (G\tilde{k}_t)^{-1}\right)$, of which the proportion ρ falls on labour. Then using the solution for the return on capital in equation (3) with the MEC in equation (6) gives

$$G\tilde{k}_t = (1-\theta)\left((1-\alpha) A_t^{1-\alpha} \tilde{k}_t^{\alpha} - \rho \left(\frac{\alpha A_t^{1-\alpha} \tilde{k}_t^{\alpha} G^{-1} - \tilde{k}_t}{\tilde{k}_t + (1-\rho) G^{-1} \tilde{b}_t} \right) \tilde{b}_t \right) - \tilde{b}_t \tag{11}$$

The objective is to determine the maximum value of $\tilde{b}_t$ or, in the presence of economic growth, one for the indexing factor, μ^B,

$$\tilde{b}_t = \mu^B A_t, \tag{12}$$

The public debt remains exogenous insofar as it is tied to an exogenous although trending variable, A_t, rather to an endogenous one, like capital, output or saving.[8] Solutions may be obtained for three subcases of taxation: where only capital is taxed, $\rho = 0$; where only labour is taxed, $\rho = 1$; and where both factors are taxed at the same rate, so that the relative tax burdens equate with the relative factor shares: $\rho = 1 - \alpha$.

3.3 Solution

It is more meaningful to obtain scale-free solutions for the public debt and for capital as shares of saving,

$$b_t \equiv \lambda s_t \,, \quad G k_t \equiv (1-\lambda) s_t \,. \tag{13}$$

These may be interpreted also as shares of wealth with which saving is synonymous in any two-period model without a bequest motive, in which the old merely disburse the assets they had previously accumulated from a single period of saving. These shares also relate to individual portfolios in an extension of the model where risk-averse agents face uncertain

[8] If alternatively, growth were endogenous with productivity arising from an investment spill-over, $A_t = k_t$, as in the Romer (1980) or Lucas(1988) model, public debt too would necessarily be endogenous in any balanced-growth equilibrium.

asset returns. The assumption $\alpha = 1/3$, which makes α^{-1} an integer, enables equation (11) to be generally solved as a polynomial in λ, the details of which are given in *Appendix One*.

The procedure is then to search numerically for a maximum value μ^B, whether this is at a degeneracy, $0 < \lambda < 1$, or a bifurcation, $\lambda \to 1$. We briefly sketch out an example of the general procedure for the simplest case where only capital is taxed, $\rho = 0$, which limits the polynomial to order three, $\lambda^3 - \lambda^2 + \left((1-\theta)(1-\alpha)\right)^{-3} G {\mu^B}^2 = 0$. Starting with $\mu^B = 0$ gives $\lambda^3 - \lambda^2 = 0$, so there are three solutions, a degenerate one at $\lambda = 1$ and two at $\lambda = 0$. Of the last two, one is economically meaningful in having over a range of values the property $\partial\lambda / \partial\mu^B > 0$, where the share of debt is increasing in the debt parameter.[9] But, *it is over a range of values* because it ends at a point of bifurcation where $\lambda = 2/3$, which is the ratio of share of debt that is associated with the maximum value. At this point the "economic solution" converges with the first degenerate solution, and thereafter these two solutions take on imaginary parts.

Having determined the value of λ at which the bifurcation occurs, additional solutions may then be obtained for the return on capital, for the tax share of GDP and for the debt-GDP ratio. Equations (3), (11) and (12) imply

$$R^K = \alpha\left((G/\mu)(\lambda/(1-\lambda))\right)^{1-\alpha} \tag{14}$$

$$T/y = \left((R^K G^{-1} - 1)/\left(1 + (1-\rho)\,\lambda/(1-\lambda)\right)\right) b/y \tag{15}$$

$$\beta \equiv b/y = \alpha\left(\lambda/(1-\lambda)\right)\left(G/R^K\right) \tag{16}$$

The last of these is identically equal to b/k multiplied by k/y. The first term, according to the definitions in (13), is equivalent to $G\lambda/(1-\lambda)$, while the second term must equal α/R^K due to the constant factor income shares property of the Cobb-Douglas technology.

[9] The other solution is degenerate and has the response $\partial\lambda/\partial\mu < 0$, implying negative values of the debt where $\mu > 0$.

3.4 *Obtaining annualized debt-GDP ratios*

There is some appeal in looking at some more familiar, annualised measures of the debt-GDP ratio by making an appropriate adjustment to equation (16), which as follows. While α , the capital share , is parametrically constant, λ the debt share is constant by definition of a steady state; and the logic of using $G^{1/L}$ and $R^{K^{1/L}}$ as annualised measures for the growth and the interest factors, respectively, extends to using the two in ratio form in order to construct an equivalent measure for annualised the debt-GDP ratio. [10]

$$\beta^{pa} = (\alpha\lambda/(1-\lambda))\left(G/R^K\right)^{1/L}, \tag{16*}$$

3.5 *The results where the debt is exogenous*

The results, tabulated below, give the bifurcation maxima for the public debt for the three cases of taxation. The values for a base case where both factor incomes are taxed equally are presented in the central column and *emboldened*. The value of λ that is associated with the maximum debt is higher, the smaller the burden of taxation falling on wages, since only this, under the model specification, reduces saving.

Table One: Values at the bifurcation where public debt is exogenous[11]

	only capital income taxed ($\rho = 0$)	both factors taxed ($\rho = 0.66$)	only labour income taxed ($\rho = 1$)
debt-wealth, λ	0.667	**0.459**	0.346
Return on capital, R^K	18.000	**13.655**	11.755
R^K as an annual rate	8.61%	**7.76%**	7.29%
Income tax rate, τ^F	59.2%	**18.8%**	22.0%
debt-GDP , β	7.41%	**4.14%**	3.00%
β as an annual rate	62.6%	**26.8%**	16.8%

[10] Since $\left(G/R^K\right)^{1/35} \approx 1$, $\alpha\lambda/(1-\lambda)$ is a good approximation of β^{pa}.

[11] The respective μ^B values are 0.0100802047, 0.0064721573, 0.0050527877.

If only capital is taxed, the wealth share of public debt, λ, is highest at two-thirds with an imputed annual debt-GDP ratio of 62.6%. If both factors are equally taxed, the annual debt-GDP ratio is lower at 26.8%; and lower still at 16.8%, if capital manages to escape taxation.

3.6 An alternative solution method

Some additional insight can be gleaned through considering an alternative method for determining the value of the maximum debt, although this is only feasible for certain cases. One is these is the present one where debt is the exogenous and where only capital is taxed ($\rho = 0$). The MEC may then be written as the special case of equation (11)

$$\tilde{b}_t = (1-\theta)(1-\alpha)A_t^{1-\alpha}\tilde{k}_t^{\alpha} - G\tilde{k}_t \tag{11A}$$

The procedure, which makes mathematical although not economic sense, is to maximise $\tilde{b}_t$ with respect to $\tilde{k}_t$. There is an interior maximum, because the debt variable is a *non-monotonic* function of $\tilde{k}_t$, a point, while obvious now, is useful for comparative purposes later. It is straightforward to show that the first-order condition for a maximum, $\alpha(1-\theta)(1-\alpha)A_t^{1-\alpha}\tilde{k}_t^{\alpha-1} = 1$, is equivalent to the solution of the tangency conditions for the phase-path, $\partial(k_{t+1}/A_{t+1})/\partial(k_t/A_t) = 1$ and $k_{t+1}/A_{t+1} = k_t/A_t$. Algebraically it can be shown that in this case the maximum debt is associated with a value λ of $1-\alpha$, which is valued at 0.667, according to the assumption in (8), and which appears in the first row and first column of *Table One*. Thus, it can be seen that the value of the maximum $\tilde{b}_t$ depends on the degree of curvature in the wage/saving function, which is inversely related to α. The maximum debt is lowest in the limit if linearity where $\alpha \to 1$.

3.7 An exogenous proportion for government debt?

To complete this *section* and to establish some intuition for the rest of the analysis, we consider a hypothetical case where the amount of debt is endogenous in the sense of being a fixed proportion of saving, λ. Again, returning to the benign example of saving-neutral taxes ($\rho = 0$), the fiscal policy equation, $\tilde{b}_t = \lambda(1-\theta)(1-\alpha)A_t^{1-\alpha}\tilde{k}_t^{\alpha}$, implies a MEC of

$$\tilde{k}_t = \left((1-\lambda)(1-\theta)(1-\alpha)/G\right)^{1/(1-\alpha)} A_t, \tag{11B}$$

In contrast with (11A), there is evidently now a monotonically negative relationship between the chosen debt parameter, λ, and the capital stock, k, $\partial k / \partial \lambda < 0$, so that the maximum must be at degeneracy where $\lambda \to 1$ and $\tilde{k}_t \to 0$. This particular case of an exogenous λ gives some insight into one of the endogenous cases of λ to be considered in *Section 6*.

In this same vein, the possibility of a degeneracy also depends on the first *Inada* condition. To show this, we write the MEC in more general terms as $Gk = (1-\lambda)zf(k)$, where saving is a proportion z of income, which is now expressed as a general function of capital $f(k)$. Dividing through k, gives $G = (1-\lambda)z\, f(k)/k$, which shows that the possibilitity of a degeneracy, defined where $\lambda \to 1$ and $k \to 0$, requires that $f(k)/k \to \infty$ for $G > 0$ to be satisfied. *L'hopital*'s rule then implies that this may be succinctly stated where the first *Inada* condition holds, $f'(k) \to \infty$. Thus, the assumption of a Cobb-Douglas technology is significant for the qualitative nature of the results.

3.8 Discussion: beyond a Cobb-Douglas production technology technology

Returning to the consideration of an exogenous debt, it can be shown that a constant elasticity of substitution (CES) generalisation of the production function may also be problematic for the existence of a steady state with public debt. *Appendix Two* gives some more details, but some basic arguments may be made here. An inelastic degree of factor substitution ($\sigma < 1$) is known to be problematic generally for the existence of an equilibrium in the neoclassical growth model,[12] since the first *Inada* condition then fails to hold. This is even more a problem in the Diamond version of it, because saving then depends on wage income alone, while if $\sigma < 1$, the share of wage income is increasing in the capital stock. Thus, from the starting position of a low capital stock, any increase may never be sufficient to raise the wage and, thereby, saving to the threshold required to sustain the hypothetical rise in capital.[13] Thus, the analysis of a public debt in the Diamond model cannot be separated from the consideration of what is an endemic existence problem, which it may also tend to exacerbate.

[12] This raises the question of its size? Empirical findings, for example, Rowthorne's (1999) earlier summary and the revised approach of Klump, McAdam and Willman (2007) broadly concur that its value may be closer to one half rather than to unity.

[13] This issue is considered by De La Croix and Michel (2002, p.33). Jones and Manuelli (1992) also discuss a related issue that sustained growth in not possible in the OLG model with a convex technology.

Secondly, the alternative case of an degree of elastic factor substitution, ($\sigma > 1$), which does at least ensure the possibility of equilibrium with debt, may allow one with very little of it through the implied limitation of the curvature of the wage/saving function. Raising the value of σ, while maintaining the assumption of constant returns to scale, is roughly equivalent to raising the value of the parameter α with a given Cobb Douglas technology.[14]

Thus, while the assumption of a Cobb-Douglas production technology is often regarded as a good place to start for many macroeconomic analyses, it may also be a good place to end up for this particular one. The fact that the Diamond model is analytically tractable and the evidence that debt-GDP ratios can be very high together suggests that maintaining the Cobb-Douglas assumption may be an expedient proxy for other conditions that are favourable to sustaining a large debt, and that some other dimension of generalisation might be chosen for analysing this model with debt. This paper makes an attempt in this direction by investigating alternative fiscal policy rules, starting by including a process for bond price determination.

4. The bond market

Debt typically comprise bonds that pay fixed coupons, z, and which are sold at a flexible market price, v_t, giving a factor of return,

$$R^B_{t+1} = \left(z + v_{t+1}\right)/v_t \ , \tag{17}$$

that allows for capital gains or losses where $v_{t+1} \neq v_t$. In the present context of a two-period overlapping generations model, the young may save also by buying long-maturity bonds at the price v_t either from the old or as new issues from the government in the anticipation of selling them on in the next period at the price v_{t+1}, when it is their turn to become old.

The government budget financing requirement, again without primary expenditures, is now

[14] This relates to the prospect of endogenous growth in the Solow model where $\sigma > 1$, as shown by Pitchford (1960).

$$(n_{t+1} - n_t)v_{t+1} = zn_t - T_{t+1}, \tag{18}$$

Debt servicing expenditure in period $t+1$ consist of the coupon payment per bond, z, times the number of bonds extant from the previous period, n_t. Any shortfall in tax revenue, T_{t+1}, to finance this is made up by issuing new bonds $n_{t+1} - n_t$ at the current market price, v_{t+1}.

Equations (17) and (18) together imply $R^B_{t+1} = (T_{t+1} + n_{t+1}v_{t+1})/n_t v_t$. Moreover, as the public debt is now defined as the market valuation of the bond stock, $b_t \equiv n_t v_t$, its return is may be expressed as

$$R^B_{t+1} = (T_{t+1} + b_{t+1})/b_t, \tag{19}$$

which may also be presented in terms of the asset share definitions in (13). The first part, the coupon yield, is given by $T_{t+1}/b_t = ((1-\lambda)/\lambda)T_{t+1}/k_{t+1}$, while the second part, the capital gain , b_{t+1}/b_t, in the steady state, equals k_{t+1}/k_t or G, so that

$$R^B_t = ((1-\lambda)/\lambda)(\tilde{T}_t/\tilde{k}_t) + G \tag{20}$$

The term λ now reflects approximately instantaneous movements in bond prices, v_t, as well as the still remaining, long drawn-out movements in the capital stock, to ensure the AEC or the no arbitrage condition holds at all times, but which in the steady state is

$$R^K_t - (1-\rho)\tilde{T}_t/k_t = ((1-\lambda)/\lambda)(\tilde{T}_t/\tilde{k}_t) + G \quad \text{or}$$

$$R^K_t = (1/\lambda - \rho)(\tilde{T}_t/\tilde{k}_t) + G \tag{21}$$

Thus, the variable λ is tantamount to being an asset price, which is determined from the simultaneous solution of the MEC and the AEC.

5. Exogenous tax revenue

We now turn to the case of exogenous tax revenue. This is equivalent to making debt-servicing exogenous in a static version of the model ($G = 1$) where $b_t = b_{t-1} = \bar{b}$. However, with economic growth arising from exogenous improvements in total factor productivity, A_t, this particular fiscal rule requires that equation (12) is replaced by

$$\mathrm{T}_t = \mu^T A_t \tag{22}$$

in order to ensure a balanced growth path. The MEC and AEC are then given by

$$G\tilde{k}_t = (1-\lambda)(1-\theta)\left((1-\alpha)A_t^{1-\alpha}\tilde{k}_t^{\alpha} - \rho\mu^T A_t\right) \tag{23}$$

$$\alpha A_t^{1-\alpha}\tilde{k}_t^{\alpha-1} - (1-\rho)\mu^T A_t/\tilde{k}_t = \left(\lambda^{-1}-1\right)\mu^T A_t/\tilde{k}_t + G \quad \text{or}$$

$$\alpha A_t^{1-\alpha}\tilde{k}_t^{\alpha-1} = \left(\lambda^{-1}-\rho\right)\mu^T A_t/\tilde{k}_t + G \tag{24}$$

It is key that whereas before the asset crowding out effect of debt in the MEC was *additive* – as represented by the subtraction term $\tilde{b}_t$ in equation (4) – it is now *multiplicative* through the factor $1-\lambda$ in equation (23). It is also of interest that the AEC in equation (24) in being re-presented as $G\tilde{k}_t = \alpha A_t^{1-\alpha}\tilde{k}_t^{\alpha} - (\lambda^{-1}-\rho)\mu A_t$, where $\lambda^{-1}-\rho > 0$, reveals a family resemblance to the MEC in equations (11) and (22); and that in being rewritten as $(\lambda^{-1}-\rho)\mu^T A_t = \alpha A_t^{1-\alpha}\tilde{k}_t^{\alpha} - G\tilde{k}_t$, where the left-hand-side of the expression is non-monotonically related to $\tilde{k}_t$, shows that the AEC is an additional source of a bifurcation. Indeed, it is the *only* source of bifurcation, where only capital is taxed ($\rho = 0$), because the MEC in equation (23) then reduces to a form of equation (11B) giving a monotonic relationship between $\tilde{k}_t$ and λ, which rules-out the possibility of a bifurcation from this particular source. Thus, modelling the asset side in this way adds to rather than subtracts from the possibility of obtaining bifurcation maxima.

These last two equations are solved simultaneously to obtain another polynomial equation in λ, with the some details in *Appendix Three*. Then, equations (3) and (24) and the parameter assumption (8) determine the return on capital as

$$(27/4)(\mu\rho)^2 R^{K^3} - R^{K^2} + 12(1-\lambda)^{-1}R^K - 36(1-\lambda)^{-2} = 0, \tag{25}$$

the solution of which combines with equation (15) to determine taxation as a proportion of GDP,

$$T/y = \mu^T\left(3R^K\right)^{1/2} \tag{26}$$

The results of the present case are presented in *Table Two* alongside the previous results from *Table One* included *italics* and *parentheses* for comparison.

Table Two: Bifurcation values where tax revenue is exogenous and funded through perpetuity bonds *(with the previous values for an exogenous debt)*[15]			
	only capital income taxed: $\rho = 0$	both factors taxed: $\rho = 0.66$	only labour income taxed: $\rho = 1$
debt-wealth, λ	0.716 *(0.666)*	**0.616** *(0.429)*	0.528 *(0.345)*
Return on capital, R^K	21.127 *(18.000)*	**31.250** *(13.655)*	25.424 *(11.755)*
R^K as an annual rate	9.11% *(8.61%)*	**10.33%** *(7.76%)*	9.69% *(7.29%)*
Income tax rate, τ^F	65.6% *(59.2%)*	**23.3%** *(18.8%)*	*29.6% (22.0%)*
debt-GDP , β	7.96% *(7.41%)*	**3.42%** *(4.14%)*	2.93% *(3.00%)*
β as an annual rate	78.6% *(62.6%)*	**49.4%** *(26.8%)*	34.7% *(16.8%)*

Apart from being a possibly better characterisation of debt policy, we see that if revenue is instead made exogenous, the previous qualitative bifurcation results are extended but are at larger values for the public debt both as a proportion of wealth and of annualised GDP.[16] If wage income is taxed, either with or without capital taxation, the annualised debt-GDP ratios roughly double. This is attributed to the fact that the asset crowding-out effect of public debt now enters multiplicatively through $1-\lambda$ rather than additively through the subtraction of $\tilde{b}_t$. The contractionary effect of taxation, however, remains additive, which surely answers why the rise in the value of the maximum debt is more modest where only capital is taxed.

It is also apparent that exogenous revenue leads also to higher rates of income taxation – around a high two-thirds where only capital is taxed, although this is significantly lower where both factors share the burden. A point that may have some relevance here, but which is unavoidable later, is that the amount of revenue raised may depend rather on tax compliance than on policy decisions. Thus, the existence of a *Laffer Curve* will constrain the present solutions at sufficiently high levels of taxation.

[15] The respective values of the indexing parameter μ^T are 0.02741, 0.02406 and 0.02258.

[16] We see that in terms of the period lengths of the OLG model, the debt-GDP ratio may move in an opposite direction to the debt share of wealth across the two policy regimes, because of a differential extent of output crowding-out. This strengthens the case for considering annualised measures of the debt-GDP ratio, which move consistently with the wealth shares, through implicitly taking out the time-compounding effect.

6. Exogenous rates of income taxation

6.1 The basic result

The third main case is where the rates of income tax rates are exogenous. This case conveys more a world of "haircuts" where the holders of public debt are paid what is raised from revenue rather than what they are due according to their bond receipts. Its significance here is that fixing the rates of income taxation makes revenue also to be endogenous of income.

The MEC and AEC for the case of exogenous income tax rates are then given by

$$k_{t+1} = (1-\lambda)(1-\theta)(1-\tau_t^L)(1-\alpha)A_t^{1-\alpha}k_t^{\alpha} \tag{28}$$

$$\frac{(1-\tau^K)\alpha}{(1-\theta)(1-\alpha)(1-\tau^L)(1-\lambda)} = \left(\frac{(1-\alpha)\tau^L + \alpha\tau^K}{(1-\theta)(1-\tau^L)}\right)\frac{1}{\lambda} + 1. \text{ [17]} \tag{29}$$

While the solution for the debt share is now quadratic,

$$\lambda = -\frac{1}{2}\left(\frac{\alpha + (1-\alpha)\tau^L}{(1-\theta)(1-\alpha)(1-\tau^L)} - 1\right) + \sqrt{\frac{1}{4}(..)^2 + \frac{\alpha\tau^K + (1-\alpha)\tau^L}{(1-\theta)(1-\alpha)(1-\tau^L)}}\ , \tag{30}$$

of greater importance is the fact that it is also now monotonically increasing in the rates of income taxation, $\partial\lambda/\partial\tau^L > 0$ and $\partial\lambda/\partial\tau^K > 0$, so that the maxima are to be found at degeneracies, $\tau^L \to 1$ and/or $\tau^K \to 1$ and $k \to 0$, rather than at bifurcations. This arises because the endogeneity of revenue also makes the contractionary effect of debt-servicing multiplicative through the factors $1-\tau^L$ and $1-\tau^K$ rather than additive through the previous subtractions of ρT and $(1-\rho)T$. As discussed earlier in *Subsection 3.7*, the first *Inada* condition is also necessary for this result, so that a feature that may be necessary for obtaining reasonably high maximum values in the earlier bifurcation cases also generates unbounded ones in the present one. *Table Three* shows how the debt-wealth shares vary with the income tax rates.

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[17] Growth, G, drops outs as a common factor. The return on capital is $(1-\tau^K)R^K =$ $(1-\tau^K)G/(1-\lambda)(1-\theta)(1-\alpha)(1-\tau^L)$, while the coupon yield on bonds is found by combining $((1-\lambda)/\lambda)((1-\alpha)\tau^L + \alpha\tau^K)y/k$ with $y/k = G/(1-\lambda)(1-\theta)(1-\alpha)(1-\tau^L)$.

Table Three: Public debt as a share of wealth where rates of income taxation are exogenous						
τ		only capital income taxed: $\rho = 0$		both factors taxed: $\rho = 0.667$		only labour income taxed: $\rho = 1$
0		0		0		0
0.1		0.140		0.303		0.208
0.2		0.265		0.476		0.328
0.3		0.378		0.598		0.410
0.4		0.483		0.690		0.472
0.5		0.581		0.765		0.521
0.6		0.673		0.826		0.560
0.7		0.761		0.879		0.593
0.8		0.844		0.925		0.621
0.9		0.924		0.965		0.646
$\rightarrow 1$		$\rightarrow 1$		$\rightarrow 1$		$\rightarrow 0.667$

Considering first labour taxation in isolation, we find that as $\tau^L \rightarrow 1$, $\lambda \rightarrow 2/3$ rather than $\lambda \rightarrow 1$. This is technically because although public debt and output both tend to zero in the limit, the former converges at a faster rate, but economically because of the asset demand effect of leaving capital untaxed. When this is taxed, whether or not accompanied by wage taxation, then, more standardly, $\lambda \rightarrow 1$. However, untenably high average tax rates of 100%, imply an unbounded debt-GDP ratio, $\beta \rightarrow \infty$, through the elimination of economic activity.

6.2 Discussion: Laffer Curves

A*verage* notional tax rates of this magnitude clearly make an inescapable case for the consideration of a *Laffer Curve* or *Curves*.[18] They may arise through disincentive effects on

[18] A *dynamic* Laffer Curve already exists in this model by virtue of the fact that wage taxes reduce the long-run tax base through saving and capital accumulation effects. Equations (2)-(5) and (9) imply that if all incomes are taxed at the same rate, then steady-state tax revenue given by $T = \tau\left((1-\theta)(1-\alpha)(1-\lambda(\tau))(1-\tau)G^{-1}\right)^{\alpha/(1-\alpha)} A$. In the absence of a public debt, $\lambda = 0$, steady-state tax revenue is at a maximum where $\tau = 1-\alpha = 66\%$. In its presence, inspection of the second column of *Table Five* suggests that the relationship is approximated by $\lambda \approx 1-(1-\tau)^2$ where $\tau = 0.5$, so that the peak of the dynamic Laffer Curve would be reached roughly where $\tau = (1-\alpha)/(2-\alpha) = 40\%$ and with a

factor supply, on tax evasion or on both of these.[19 20] It is straightforward within the confines of the present model to incorporate pure evasion effects, for which *Appendix Four* provides a particular example on the broader lines of Allingham and Sandmo (1972).

Although it is beyond the scope of this paper to give space to the wider and complex issues surrounding *Laffer Curves*, other authors have made some attempt to see whether countries might be situated on the "right" or "wrong side" of their *Laffer Curves*, that is, whether a given amount of revenue might be raised with lower taxes and through causing less distortions. Two results stand out and they both pertain to Sweden, one in a seminal paper by Stuart (1981) and another in a more recent paper by Trabandt and Ulrich (2009). Stuart focusses on Swedish labour income taxes to conclude that revenue would have been maximized at a *notional* rate of about 70% instead of the actual notional rate that was closer to 80%. In a more recent analysis that utilises both a more up-to-date technology and data set, Traband and Ulrich find that Sweden and Denmark are both over the "slippery slope" with regard to their rates of capital income taxation but not to their rates of labour income taxation.

For illustrative purposes, we shall merely take a "black box" approach by assuming that these two countries have indeed been near the vicinity of their *Laffer Curve* peaks and that this is reflected by their tendencies to take a high proportion of their GDPs in taxation, amounting to about 50%. In terms of the present model, this is equivalent to imposing a maximum effective tax rate (here considered both on capital and labour) of $\tau = 0.5$, which means truncating the central column of *Table Three* halfway down. The implication is of a maximum debt-wealth ratio of 0.765, which translates to an annualised debt-GDP of 98.8% for the present case of fixed tax rates. *Table Four* summarises all the results where both factor incomes are taxed at the same rate and one which may not exceed 50%, that is a binding factor for the third and final case.

Table Four: Values associated with a maximum values for the three fiscal rules

corresponding debt share of $\lambda = 0.36$. This would be even lower with additional labour supply and tax compliance effects.

[19] In a development context, Sachs (1985) introduced the notion of "debt overhang" where forgiving some amount of international debt may release incentives, leading to an increased payments flow to creditors.

[20] A pure disincentive effect may be defined where taxes are paid but at a reduced level of factor supply, while a pure evasion effect is constituted where supply levels are maintained with households managing to avoid paying tax by moving to a shadow sector or another tax jurisdiction.

with general income taxation			
	exog. Debt	exog. tax revenue	exog. tax rate (a max *effective* rate of 50%
debt-wealth, λ	0.429	0.616	0.765
Return on capital, R^K	13.655	31.250	51.064
R^K as an annual rate	7.76%	10.33%	11.89%
Income tax rate, τ	18.8%	23.3%	50%
debt-GDP , β	4.14%	3.42%	4.25%
β as an annual rate	26.8%	49.4%	98.5%

Making revenue exogenous instead of the debt roughly doubles the value of maximum debt, as represented as a proportion of annualised debt-GDP, while a further move to exogenous tax rates leads to a second, approximate doubling to about 100%.

The relative values of these respective magnitudes under these different fiscal rules should also hold *outside* the steady state, so that the policy maker might resort to the final option in attempting to sustain a burgeoning public debt. Thus, this final option that produces a maximum of maxima may be regarded as the single, operative maximum also *inside* the steady state.

7. Further considerations

7.1 Land as another factor of production and an asset

A focus has been on various forms of income taxation, so it would be remiss to overlook the effect of taxing land rents, not least because this happened to be a major part of UK government public funding in the eighteenth century[21], which was also a time of a rapidly growing national debt. Acquiring land as a store of value, which, although productive, is not producible, is of particular interest to this analysis in providing another means of non-productive saving to complement the holding of public debt. We find that land as an asset

[21] Trevelyan (2002).

both independently crowds out capital and acts, to some extent, as a buffer against further crowding-out flowing the introduction of a public debt.

We modify the production function to include land, m, where $y_t \equiv A_t^{1-\alpha} k_t^{\alpha} m_t^{\beta}$. The rental on land S is solved as $S_t = \beta\, y_t / m_t$ in period t under the maintained assumption of marginal cost pricing. Fixing the area of land at $m = 1$ gives the following asset demands,

$$\begin{aligned}
k_{t+1} &\equiv (1-\gamma-\lambda)(1-\theta)(1-\alpha-\beta)(1-\tau^L) A_t^{1-\alpha} k_t^{\alpha}, \\
b_t &\equiv \lambda_t (1-\theta)(1-\alpha-\beta)(1-\tau^L) A_t^{1-\alpha} k_t^{\alpha}, \\
v_t &\equiv \gamma (1-\theta)(1-\alpha-\beta)(1-\tau^L) A_t^{1-\alpha} k_t^{\alpha},
\end{aligned} \tag{31}$$

where v_t is the price of land of one unit of land.

We assume that only the rental part of the return, S, and not the capital gain is taxed, at the rate τ^M, so that its return is given by $R_{t+1}^M = (1-\tau^M)\, S_{t+1}/v_t + v_{t+1}/v_t$. Using the same reasoning above, its factor of return is $R^M = (1-\tau^M)\big((1-\lambda-\gamma)/\gamma\big)\beta\tilde{y}_t/\tilde{k}_t + G$ in the steady state. The AEC now comprises two equality conditions,

$$(1-\tau^K)\alpha\, \tilde{y}_t/\tilde{k}_t = (1-\tau^M)\big((1-\lambda-\gamma)/\gamma\big)\beta\, \tilde{y}_t/\tilde{k}_t + G = \big((1-\lambda-\gamma)/\lambda\big)\bar{\tau}\tilde{y}_t/\tilde{k}_t + G$$

$$\text{where } \bar{\tau} = \alpha\tau^K + (1-\alpha-\beta)\tau^L + \beta\tau^M \tag{32}$$

From the second of these, we obtain the portfolio shares that satisfy a no-arbitrage between bonds and land, $\gamma = \big(\beta(1-\tau^M)/\bar{\tau}\big)\lambda$, which is a step towards the solution,

$$\begin{aligned}
\lambda = {} & \frac{\alpha\tau^K + (1-\alpha-\beta)\tau^L + \beta\tau^M}{\alpha\tau^L + (1-\alpha-\beta)\tau^L + \beta} \times \\
& \left(-\frac{1}{2}\left(\frac{\alpha+\beta+(1-\alpha-\beta)\tau^L}{(1-\theta)(1-\alpha-\beta)(1-\tau^L)} - 1 \right) + \sqrt{\frac{1}{4}(..)^2 + \frac{\alpha\tau^K + (1-\alpha-\beta)\tau^L + \beta}{(1-\theta)(1-\alpha)(1-\tau^L)}} \right)
\end{aligned} \tag{33}$$

We consider the values $\alpha = 3/10$, $\beta = 2/10$, $G = 2$, which are suggestive of more "historical" parameter values, since land and agriculture have played a greater role in the past.[22] [23] We find that leaving both capital and wages untaxed, $\tau^K = \tau^L = 0$, but taxing only land rents generates a reasonably high value for the debt-wealth ratio of $\lambda = 0.322\tau^M$. As this implies $\bar{\tau} = \beta\tau^M$ and thus $\gamma = (1/\tau^M - 1)\lambda$, we find that the portfolio share of capital is constant at $1-\lambda-\gamma = 1-\lambda/\tau^M = 0.678$. That is to say, in the absence of a public debt, the mere existence of land as a tradable asset crowds-out roughly one third of the capital stock on the basis of these assumed parameter values. Furthermore, the subsequent creation of public debt that is supported through the taxation only of land rents will only crowd out the asset demand for land, while leaving capital stock intact as its pre-debt level.[24]

However, the main interest here is the contribution of rent taxation on the maximum value of the public debt. If we set the effective tax rates on labour and capital incomes at their previously assumed maximum rates of 50%, $\tau^K = \tau^L = 0.5$, we find that the public debt share is given by $\lambda = 0.390 + 0.195\tau^M$. It then follows that switching from a situation where land rents are not taxed at all ($\tau^M = 0$) to one where they are taxed at an inescapable rate of 100% ($\tau^M = 1$) raises the ratio of public debt to wealth from 0.390 to 0.585.

7.2 Uncertainty

7.2.1 Introduction

The asset shares have been determined by an *aggregate* equilibrium condition, where risk-neutral households are naturally indifferent to the composition of their holdings. The model is now extended by considering portfolio diversification by risk-averse agents in response to uncertain and imperfectly correlated asset returns. We find that the introduction of uncertainty within this framework has only a quantitative effect on the results that could alternatively arise from changing the value of any other parameter.

[22] Rhee (1991) presents data to show there has been secular decline in the income share of land from about 9% over 1909-1913 to about 4% over 1981-1984, albeit with a levelling off to contextualise his theoretical point that an asymptotically positive income share is a sufficient condition for dynamic efficiency.

[23] This begs the question of disaggregating the production function.

[24] In a more complex model with a public debt, it may be possible to show that the taxation of land may cause a *crowding-in* of capital. This is naturally the case for a model without a public debt and other non-productive assets. [See Feldstein (1977)]

7.2.2 A revision of the model

We assume that capital is the riskier asset, which is surely plausible at least where the public debt is small.[25] The present framework allows us to incorporate the uncertainty of asset returns in a fairly innocuous way, since the specification of the utility function was chosen to allow closed-form solutions for a limited number of states.

We generalise the model by allowing factor total productivity to be stochastic, so that firms instead maximize expected profit, $E(\Pi_t) = E(A_t^{1-\alpha} L_t^{1-\alpha} K_t^{\alpha}) - w_t L_t - E(R_t^K) k_t$. Workers are assumed to be immune from risk by being paid their *expected* marginal product, $w_t = (1-\alpha)E(A_t^{1-\alpha})k_t^{\alpha}$,[26] while the return on capital is the residual returned after the realisation of the unpredictable TFP shock, $R_t^K = \left(A^{1-\alpha} - (1-\alpha)E(A^{1-\alpha})\right)k_t^{\alpha-1}$ or $R_t^K = \alpha E(A^{1-\alpha})k_t^{\alpha-1}(1+u_t)$ where $u_t \equiv \alpha^{-1}\left(\left(A_t^{1-\alpha} / E\left(A_t^{1-\alpha}\right)\right) - 1\right)$

The term σ^2, $\sigma^2 \equiv E(u_t^2)$, is the variance of the returns to capital as a ratio of their squared mean, and is thus a descaled and multiplicative measure of the (relative) uncertainty of capital. A two-state case permits us to obtain a more tractable solution, while a symmetric distribution where,

$$prob\left(R_{t+1}^K = \bar{R}^K(1+u)\right) = 1/2, \quad prob\left(R_{t+1}^K = \bar{R}^K(1-u)\right) = 1/2 \tag{34}$$

enables uncertainty to be gauged by the single parameter, σ^2.[27]

The set-up with a logarithmic utility function and without future earnings income precludes the possibility of precautionary saving - or dis-saving - in response to interest uncertainty, allowing the saving and portfolio decisions to be separable.[28] [29] We also consider the more straightforward case of exogenous income tax rates in order to limit the complexity of the solution. The MEC remains as given by equation (28) except the expectation, $E(A_t^{1-\alpha})$,

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25 Marketable debt may also be more liquid than capital, and thus performs an important role as a reserve asset for fractional-reserve banking.
26 As the wage determines saving, this assumption also ensures that the model remains deterministic.
27 This will also reflect the effects of all higher-order even moments, ie. kurtosis, etc.
28 See Leland (1968).
29 Another exercise might be to consider the risk-aversion and intertemporal substitution separately through the application of Epstein-Zin (1989) preferences.

trivially replaces the previously certain value, $A_t^{1-\alpha}$. However, for the two-state, symmetric case of uncertainty, with the details given in *Appendix Five*, the AEC is given rather by

$$\lambda_t\left((\overline{R}^K-\overline{R}^B)^2-\overline{R}^K\sigma^2\right)-\overline{R}^K(\overline{R}^K-\overline{R}^B)+\overline{R}^{K^2}\sigma^2=0$$

where $\overline{R}^B=\left(\dfrac{\tau}{\lambda(1-\theta)(1-\alpha)(1-\tau)}+1\right)G$, $\quad \overline{R}^K=\dfrac{\alpha G}{(1-\theta)(1-\alpha)(1-\lambda)}$, (35)

$\overline{R}^B$ and $\overline{R}^K$ being the expected values of the factors of return factors on public debt and capital. This two-state case of uncertainty generates a quartic solution in place of the single-state quadratic solution arising under certainty.

The results are reported in *Table Five* where the rows give values for a common rate of income taxation, τ, and columns varying degrees of capital income uncertainty, where σ^2 is always a mean-preserving spread. The latter ranges from the certainty case of $\sigma^2=0$, where the first column merely repeats the middle column of *Table Three*, to one of maximum uncertainty where $\sigma^2=1$, where there is a 50% chance of a losing the entire proceeds of any investment in capital, that is, principal and interest combined.

Table Five: Public debt shares for varying degrees of capital interest uncertainty under exogenous income taxes

	$\sigma^2=0$	$\sigma^2=0.33$	$\sigma^2=0.66$	$\sigma^2=1$
$\tau=0$	**0**	**0**	**0**	**0**
$\tau=0.1$	**0.303**	**0.372**	**0.470**	**0.606**
$\tau=0.2$	**0.476**	**0.528**	**0.590**	**0.663**
$\tau=0.3$	**0.598**	**0.633**	**0.673**	**0.716**
$\tau=0.4$	**0.690**	**0.713**	**0.739**	**0.765**
$\tau=0.5$	**0.765**	**0.780**	**0.794**	**0.810**
$\tau=0.6$	*0.826*	*0.835*	*0.844*	*0.852*
$\tau=0.7$	*0.879*	*0.884*	*0.888*	*0.892*
$\tau=0.8$	*0.925*	*0.927*	*0.928*	*0.930*
$\tau=0.9$	*0.965*	*0.965*	*0.965*	*0.966*
$\tau\rightarrow 1$	$\rightarrow 1$	$\rightarrow 1$	$\rightarrow 1$	$\rightarrow 1$

Evidently, the effect of relatively uncertain returns to capital, as might be expected, raises the portfolio demand for public debt, but the relative effect is strongest where taxes are at their lowest while remaining positive. For example, if the exogenous income tax rate is 10%, the switch from certainty to the extreme uncertainty leads to doubling of the portfolio share of public debt from 0.303 to 0.606, but where the effective tax rate is at its assumed *Laffer* maximum of 50% there is a more meagre rise from 0.765 to 0.810. Thus, including asset return uncertainty in a way that is favourable to a portfolio demand for public debt only has a marginal effect.[30]

8. Concluding comments

The contribution of this paper has been to determine the maximum value of the public debt within the Diamond OLG model under various possibilities for the fiscal policy instrument. If tax revenue is exogenous, the qualitative results in Rankin and Roffia (2003) of bifurcation maxima are carried over but at higher values. Making rates of income tax instead exogenous, however, causes a qualitative change with degeneracies replacing the bifurcations – at least for the Cobb Douglas under consideration. As this case implies notional but untenable average tax rates of 100%, some concept of a *Laffer Curve* becomes inescapable as a practical concern, unless political economy considerations come into play even before.[31]

Rankin and Roffia (2003) also point out that their results would analogously arise with the operation of a social security systrem. This present analysis causes us to concur if social security payments are also exogenous, but not necessarily where they are *earnings-related* in the "Bismarckian" sense nor where alternatively, "Beveridgean" benefits are either linked to an aggregate wage measure or else financed through proportional income taxes.[32] Indeed, we suggest that the strongest analogue is to be found in the implementation of large-scale

[30] The effect of uncertainty would be more powerful in the presence of asset transactions costs, causing a possible breakdown in portfolio balance.
[31] Reinhart and Roffia (2011) conclude that default is more prevalent where the debt is held internationally, presumably because only domestic residents get to vote.
[32] Beveridgean benefits tend to be indexed to an aggregate measure of wages in large welfare states like Denmark and the Netherlands.

infrastructure projects, which by the fact of their indivisibility are inherently exogenous of income, but generally determined by unrelated technical factors and utility needs.

We conclude with some caveats why the figures obtained for the maximum steady debt are almost certainly too high. First, in search of a high maximum value for the public debt, we have precluded any primary government expenditure. In practise, the requirement that governments provide public goods, services and transfer payments will inevitable leave left revenue left over for servicing a public debt. Secondly, to reiterate a point already made, much hangs on the specification of the model, which presently favours high values for the bifurcation maxima and the possibility of degeneracies.[33]

Thirdly, there are certain reasons why current levels of public debt may never be able to return to some historical values. If the size of the debt is constrained by the availability of tax revenues, the analysis then turns on one governing the elasticities of factor supply, the opportunities for tax evasion and the social norms for compliance. Going further back in time, a significant amount of debt, at least in England, was serviced by taxes on the most inelastic of factors, land,[34] the income share of which has experienced some considerable secular decline.[35] In addition and by broad consensus, there have been notable increases in the extent to which the factors of production are internationally mobile to the extent that the the term *globalization* is generally used to describe the world So the fact that the UK public debt in 1822 was 262% of GDP may be a misleading, if not dangerous, precedent to follow.

If the economy is open, as considered by Farmer and Zotti (2010), a single country indeed may have the scope to run a larger public debt, since the saving base may be global rather than national. This, however, cannot be the right perspective in the event that the size of a nation's public debt is constrained by its own tax base. Rather, global openness may restrict the scope for running up large national public debts, because international factor mobility, tax competition and a possible "race to the bottom" – at least for capital income – will reduce the scope for raising taxes, leading to lower sustainable debts. Thus, in order for a single country,

[33] Technologies that generate multiple equilibria and development traps surely suggest that the emergence of internal financial markets may require a certain threshold of economic development. The obverse is that the excessive crowding out of capital by public debt could return economies to primitive financial states without the base living standards to sustain these secondary markets.
[34] Trevelyan (2002).
[35] Rhee (1991).

that is vulnerable to international factor flows, to run up a persistently large public debt, it may need either to command commensurate and equally persistent, global fiscal flows or else to retreat to some extent from financial openness.

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Appendix One: Solution for the exogenous debt case

Equation (11) is expanded to

$$\left(Gk + b - (1-\theta)(1-\alpha)A^{1-\alpha}k^{\alpha}\right)\left(Gk + (1-\rho)b\right) + (1-\theta)\rho\left(\alpha A^{1-\alpha}k^{\alpha} - Gk\right)b = 0 \quad \text{(A1)}$$

Dividing through twice by Gk twice and using $b/Gk = \lambda/(1-\lambda)$ gives

$$\begin{aligned}&\left(\left(1+\frac{\lambda}{1-\lambda}\right)H - (1-\theta)(1-\alpha)\left(\frac{\lambda}{1-\lambda}\right)^{1-\alpha}\right)\left(1+(1-\rho)\frac{\lambda}{1-\lambda}\right) \\ &+ (1-\theta)\rho\left(\alpha\left(\frac{\lambda}{1-\lambda}\right)^{1-\alpha} - H\right)\frac{\lambda}{1-\lambda} = 0\end{aligned} \quad \text{where } H \equiv \mu^{1-\alpha}G^{\alpha} \quad \text{(A2)}$$

Defining $Q \equiv \left(\lambda/(1-\lambda)\right)^{\alpha}$ gives

$$\begin{aligned}&\left(\left(1+G^{1/\alpha}\right)H - (1-\theta)(1-\alpha)G^{1/\alpha-1}\right)\left(1+(1-\rho)G^{1/\alpha}\right) \\ &+ (1-\theta)\rho\left(\alpha G^{1/\alpha-1} - H\right)G^{1/\alpha} = 0\end{aligned} \quad \text{(A3)}$$

If $\alpha = 1/3$

$$\left((1+G^{3})H - (1-\theta)(1-\alpha)G^{2}\right)\left(1+(1-\rho)G^{3}\right) + (1-\theta)\rho\left(\alpha G^{2} - H\right)G^{3} = 0 \quad \text{(A4)}$$

Appendix Two: Outside the Cobb-Douglas case

We generalise the production function but for the relatively benign case where only capital is taxed. The MEC for an exogenous debt may be written as $b = (1-\theta)w(k) - Gk$, where $w(k)$ is a more general expression of the wage function. Any interior maximum for the debt occurs where $(1-\theta)\,\partial w(k)/\partial k - G = 0$. Defining ε as the elasticity of wage – or, here, of saving – with respect to the wage, $\varepsilon \equiv \left(\partial w(k)/\partial k\right)\left(k/w(k)\right)$, means that any interior maximum for the level of the debt, $b = (\varepsilon^{-1} - 1)Gk$ – or as the wealth share, $\lambda = 1 - \varepsilon$ – requires that $\varepsilon < 1$. This, is, of course, satisfied for the Cobb-Douglas case where $\varepsilon = \alpha < 1$.

However, this result does not generally apply to a constant elasticity of subtitution (CES) production function. With this elasticity, σ, $0 \le \sigma < \infty$, where the assumption of labour-augmenting technology is maintained, output is given by

$$y=\left(\alpha k^{(\sigma-1)/\sigma}+(1-\alpha)(Al)^{(\sigma-1)/\sigma}\right)^{\sigma/(\sigma-1)}, \tag{B1}$$

If, as before, labour is paid its marginal production, the wage function is

$$w=(1-\alpha)A^{(\sigma-1)/\sigma}\left(\alpha k^{(\sigma-1)/\sigma}+(1-\alpha)A^{(\sigma-1)/\sigma}\right)^{1/(\sigma-1)}, \tag{B2}$$

Which has a capital elasticity of

$$\varepsilon=\frac{\alpha}{\left(\alpha+(1-\alpha)(A/k)^{(\sigma-1)/\sigma}\right)\sigma}. \tag{B3}$$

The condition for an interior maximum, $\varepsilon<1$, requires $(1-\sigma)\alpha<\sigma(1-\alpha)(A/k)^{(\sigma-1)/\sigma}$. Sufficient for this a *non-inelastic* degree of factor substitution, $\sigma\geq 1$, which is consistent with the first Inada condition holding. Even so, any propensity for a model to hold debt will either diminish or vanish altogether, once we allow taxes to affect saving.

An inelastic value of one-half, $\sigma=1/2$, on which Rowthorne (1999), in an earlier summary, and Klump, McAdam and Willman (2007), in a revised approach, broadly concur, is problematic for the OLG model. The MEC without either debt or taxation is given by $Gk=(1-\theta)(1-\alpha)A^{-1}\left(\alpha k^{-1}+(1-\alpha)A^{-1}\right)^{-2}$, for which the existence condition is $1-\theta>4G\alpha$, This clearly requires a very small value for α, which here determines but does not here define the capital income share.[36] The condition for an equilibrium with debt and taxes would be even more stringent. For example, with just proportional wage taxes, where, $Gk=(1-\tau^L)(1-\theta)(1-\alpha)A^{-1}\left(\alpha k^{-1}+(1-\alpha)A^{-1}\right)^{-2}$, α must be even lower to satisfy $(1-\tau^L)(1-\theta)>4G\alpha$.

Appendix Three: Solution for exogenous tax revenue

Equations (23) and (24) are combined through the elimination of $\tilde{k}_t^{\alpha}$ to obtain

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[36] Analagous to the $\sigma<1$ case of CES is the another one where the production requires a minimum level of capital, $y=A^{1-\alpha}(k-k^{\min})^{\alpha}$, $k^{\min}>0$. If an equilibrium without debt does exist, a bifurcation *with* debt would occur well before $k=k^{\min}$.

$$\tilde{k}_t \equiv \frac{(1-\lambda)(1-\theta)}{G}\left(\frac{(1-\alpha)\lambda^{-1}-\rho}{\alpha-(1-\lambda)(1-\theta)(1-\alpha)}\right)\mu A \tag{C1}$$

This is then substituted back into (23), which is manipulated and then after having the common factor, $\left(A(1-\lambda)(1-\theta)\right)^{1/\alpha}\mu$ removed is

$$\begin{aligned}&\left(1-\rho\lambda-\rho(1-\theta)\lambda(1-\lambda)\right)^{1/\alpha}G\mu^{1/\alpha-1}\\&=(1-\lambda)(1-\theta)\left((1-\alpha)\lambda^{1/\alpha-1}-\rho\lambda^{1/\alpha}\right)\left(\alpha-(1-\alpha)(1-\theta)(1-\lambda)\right)^{1/\alpha-1}\end{aligned} \tag{C2}$$

If $\alpha = 1/3$

$$\begin{aligned}&\left(1-\rho\lambda-\rho(1-\theta)\lambda(1-\lambda)\right)^{3}G\mu^{2}\\&=(1-\lambda)(1-\theta)\left((1-\alpha)\lambda^{2}-\rho\lambda^{3}\right)\left(\alpha-(1-\alpha)(1-\theta)(1-\lambda)\right)^{2}\end{aligned} \tag{C3}$$

Appendix Four: An auxiliary model of tax evasion

The first analysis of tax evasion was provided by Allingham and Sandmo (1972) from the perspective of expected utility, which is applied here. The indirect utility function of a type-one household that pays (notional) labour taxes at the notional rate, $\hat{\tau}^L$, is given by $U^1 = \ln(1-\hat{\tau}^L)+\tilde{U}$, where $\tilde{U}$ comprises everything that affects utility other than the labour tax rate. We assume that type-two evade paying labour income taxes by joining a secondary, informal sector, where they receive the same wage but incur the deadweight cost, c, to obtain the alternative utility, $U^2 = \tilde{U} - c$. The individual would choose the primary, formal sector and pay full labour taxes if $U^1 \geq U^2$ or if $c \geq -\ln(1-\hat{\tau}^L)$. If the exponent of costs, $\exp(c)$, is uniformly distributed across individuals on the support, $(\gamma^{\min},\gamma^{\max})$, then the proportion of type-one households or young households who pay taxes is given by $\mu = \left(\gamma^{\max}-(1-\hat{\tau}^L)^{-1}\right)\left(\gamma^{\max}-\gamma^{\min}\right)^{-1}$ subject to $0<\mu<1$. Setting $\gamma^{\min}=1$ (ie. the lowest cost is zero) gives $\mu = 1-\varpi\,\hat{\tau}^L/(1-\hat{\tau}^L)$, where $\varpi \equiv \left(\gamma^{\max}-1\right)^{-1}$. The effective tax rate on labour income is then

$$\tau^L = \mu\hat{\tau}^L = \hat{\tau}^L - \varpi\,\hat{\tau}^{L\,2}\big/(1-\hat{\tau}^L). \tag{D1}$$

If capital income is taxed at the same notional rate, $\hat{\tau}^K = \hat{\tau}^L = \hat{\tau}$, and this cannot be evaded because it is deducted at source, $\tau^K = \hat{\tau}^K$. Given the constant factor shares property of the Cobb-Douglas production function, the average effective tax rate is

$$\tau = (1-\alpha)\left(\hat{\tau} - \varpi\,\hat{\tau}^2\big/(1-\hat{\tau})\right) + \alpha\hat{\tau} = \hat{\tau} - (1-\alpha)\omega\hat{\tau}^2\big/(1-\hat{\tau}).$$

Total tax revenue is at a maximum where

$$\hat{\tau}^{\max} = 1 - \sqrt{(1-\alpha)\varpi\big/(1+(1-\alpha)\varpi)}\ . \tag{D2}$$

Appendix Five: Solution where the return on capital is uncertain with a common exogenous income tax rate

The household's expected utility, where public debt remains a safe asset with the return factor, $\overline{R}^B$, is

$$U = \theta\ln\left((1-\tau)w_t - s_t\right)$$
$$+(1-\theta)\left(\frac{1}{2}\ln\left((1-\lambda_t)\overline{R}^K_{t+1}(1+\sigma) + \lambda_t\overline{R}^B_{t+1}\right) + \frac{1}{2}\ln\left((1-\lambda_t)\overline{R}^K_{t+1}(1-\sigma) + \lambda\overline{R}^B_{t+1}\right) + \ln s_t\right)$$

Saving is unaffected by uncertainty, because there is no future income to be discounted and the specification of a logarithmic function, where the intertemporal elasticity of substitution and the rate of relative risk-aversion are collapsed into a single parameter valued at unity. [See Leland (1968)]. The first- and second-order conditions are

$$\frac{\partial U_t}{\partial\lambda_t} = -\frac{\beta}{2}\left(\frac{\left(\overline{R}^K_{t+1}(1+\sigma) - \overline{R}^B_{t+1}\right)}{(1-\lambda_t)\overline{R}^K_{t+1}(1+\sigma) + \lambda_t\overline{R}^B_{t+1}} + \frac{\left(\overline{R}^K_{t+1}(1-\sigma) - \overline{R}^B_{t+1}\right)}{(1-\lambda_t)\overline{R}^K_{t+1}(1-\sigma) + \lambda_t\overline{R}^B_{t+1}}\right) = 0$$

$$\frac{\partial^2 U_t}{\partial\lambda_t^{\,2}} = -\frac{\beta}{2}\left(\frac{\left(\overline{R}^K_{t+1}(1+\sigma) - \overline{R}^B_{t+1}\right)^2}{\left((1-\lambda_t)\overline{R}^K_{t+1}(1+\sigma) + \lambda_t\overline{R}^B_{t+1}\right)^2} + \frac{\left(\overline{R}^K_{t+1}(1-\sigma) - \overline{R}^B_{t+1}\right)^2}{\left((1-\lambda_t)\overline{R}^K_{t+1}(1-\sigma) + \lambda\overline{R}^B_{t+1}\right)^2}\right) < 0$$

The former gives

$$\lambda_t\left((\overline{R}^K_{t+1} - \overline{R}^B_{t+1})^2 - \overline{R}^K_{t+1}\sigma^2\right) - \overline{R}^K_{t+1}(\overline{R}^K_{t+1} - \overline{R}^B_{t+1}) + \overline{R}^{K\,2}_{t+1}\sigma^2 = 0 \tag{E1}$$

The expected returns to public debt and capital in the steady-state under the exogenous taxation of labour alone are given by

$$\overline{R}^B = \left(\frac{\tau}{\lambda(1-\theta)(1-\alpha)(1-\tau)} + 1\right)G, \qquad \overline{R}^K = \frac{\alpha G}{(1-\theta)(1-\alpha)(1-\lambda)} \tag{E2}$$