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Jake Bradley and Junggie Lee

Produced By:

Centre for Finance, Credit and Macroeconomics
School of Economics
Sir Clive Granger Building
University of Nottingham
University Park
Nottingham
NG7 2RD

www.nottingham.ac.uk/cfcm

From Wages to Wealth: How Trade Policy Reallocates Across the Life Cycle

Jake Bradley[*] [†]   Junggie Lee[*]

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Abstract

This paper studies the heterogeneous distributional effects of trade liberalization. We develop a tractable heterogeneous agent general equilibrium model in which individuals differ by income, wealth, age, and employment status, while firms endogenously evolve in productivity following a stochastic process with fixed export costs. In the model, trade openness raises the return to labor and deepens the capital stock, lowering returns on assets. These shifts generate systematic differences in preferences over trade: workers whose income relies primarily on labor gain from openness, while retirees and asset-dependent households may lose. Using microdata from the Brexit referendum in the United Kingdom, we document empirical patterns consistent with the model's predictions: individuals with higher labor income shares were significantly less likely to support leaving the European Union. By linking micro-level heterogeneity to macro-level trade outcomes, the model offers a useful tool for evaluating the political economy and welfare consequences of globalization.

[*]University of Nottingham

[†]IZA

1 Introduction

It is well established that, across a broad class of models, trade generates aggregate welfare gains (see Arkolakis et al. (2012)). However, these gains are *not necessarily distributed evenly* across individuals.[1] In this paper, we develop a tractable heterogeneous agent model to examine who benefits from trade and who may be left worse off. In equilibrium, workers differ along multiple dimensions— including income, asset holdings, age, and employment status— which shape their exposure to and preferences over trade openness.

The model predicts that trade liberalization leads to an increase in the return to labor and a deepening of the capital stock. As capital becomes more abundant, the return on assets falls. This redistribution of returns across factors of production has important implications for individual preferences: those who rely more heavily on labor income stand to gain from trade openness, while individuals whose income derives primarily from capital or transfers may see diminished returns. As a result, we find that the greater the labor income share, the stronger the preference for trade. Using microdata on income and voting behavior from the Brexit referendum in the United Kingdom, we find empirical evidence consistent with these model predictions: individuals with a higher share of labor income were systematically less likely to support leaving the European Union— a policy widely understood as a move toward reduced trade openness.

There is substantial heterogeneity in the model on both the household and firm sides of the economy. Despite this complexity, the model remains analytically tractable, with many components admitting closed-form solutions. This tractability allows for clear insights into the general equilibrium mechanisms through which trade policy affects the returns to labor and capital. In particular, changes in the return on capital are shaped jointly by heterogeneity among workers— who differ in income sources, life cycle positions, and asset holdings— and among firms, whose productivity evolves endogenously.

On the worker side, individuals differ by age, wealth, and employment status. While employed, workers earn labor income and accumulate savings. Upon exiting the labor force, they become inactive and rely on their accumulated assets and publicly funded transfers. Workers solve intertemporal consumption and savings problems subject to life cycle constraints, generating endogenous differences in wealth accumulation and trade preferences. In particular, this structure induces a natural divide: employed workers— whose welfare depends primarily on wages— tend to benefit more from trade openness, while the inactive— whose welfare depends more on asset returns— may prefer more protectionist policies. On

[1] A well-documented case of the uneven impact of trade liberalization is the regional and sectoral exposure to rising Chinese import competition, often referred to as the "*China shock*" (see Autor et al. (2013)).

the firm side, producers begin with a common productivity level that evolves stochastically over time. At any point, firms can pay a labor cost to reorganize production and reset their productivity to the initial level. Firms use capital and labor to produce intermediate goods and may serve foreign markets, subject to fixed and variable trade costs in the spirit of the Melitz (2003) framework. These features endogenize the firm productivity distribution and generate heterogeneity in export participation, investment behavior, and labor demand— all of which feed back into the equilibrium return on capital.

This paper is not the first to depart from the representative agent benchmark in macro-trade models. Several distinct approaches have introduced household heterogeneity to study distributional impacts of trade. Some, such as Fajgelbaum et al. (2011), emphasize heterogeneous preferences over goods, showing that trade affects welfare through differences in consumption baskets. Others, including Cosar et al. (2016) and Fajgelbaum (2020), model wage heterogeneity via search frictions, capturing how trade influences employment and earnings inequality. A third line of work, Caliendo et al. (2019), models sector-specific exposure to trade shocks, where workers in different industries or regions face varying degrees of import competition, as in the China shock studied by Autor et al. (2013). Papers in a similar vein to this are Carroll and Hur (2020) and Waugh (2023). Like this model, both have workers save and consume in an incomplete markets model giving rise to dispersion in household's asset holdings.

In line with Fajgelbaum et al. (2011), Waugh (2023) introduces random taste dispersion across individuals. In the calibrated model, agents with higher price elasticity hold fewer assets on average. Increased trade openness lowers goods prices, disproportionately benefiting these price-sensitive agents, who are relatively asset-poor. The study most closely related to this paper is Carroll and Hur (2020), which, unlike the present work, distinguishes between tradable and non-tradable goods. In their framework, trade liberalization lowers the relative price of tradable goods, benefiting the asset-poor for two reasons: (i) tradable goods comprise a larger share of their consumption basket; and (ii) capital is produced using tradable goods. The latter implies that trade openness increases the supply of capital, reducing its return and thereby adversely affecting agents with large asset holdings.

Our paper differs from Carroll and Hur (2020) and Waugh (2023) in two key respects. First, we introduce a life cycle component in which individuals earn labor income during the early part of life and rely on savings and transfers in later periods. Second, we explicitly model the dynamics of firm productivity, generating an endogenous distribution of productivity that determines the utilization of labor and provides one mechanism through which asset returns decline with trade openness. Qualitatively, and in line with the aforementioned studies, we find that individuals with fewer assets benefit more from trade openness. However, unlike Carroll and Hur (2020) and Waugh (2023), this does not imply that trade is necessarily *pro-poor*. What matters is not income level per se, but its source. In our calibration, a low-asset individual outside the labor force would oppose trade openness, whereas a wealthier employed worker would favor a more liberal trade regime.

In both Carroll and Hur (2020) and Waugh (2023), agents face uninsurable idiosyncratic income risk and accumulate assets for self-insurance within the Bewley-Huggett-Aiyagari framework (Bewley, 1986; Huggett, 1993; Aiyagari, 1994). Our approach introduces household heterogeneity through two distinct mechanisms. First, we adopt a perpetual youth framework (Blanchard, 1985; Yaari, 1965), where agents face a constant mortality risk and smooth consumption over their expected lifetime. As noted by Farhi and Werning (2019), this framework offers greater analytical tractability than standard incomplete markets models, making it well-suited to a range of macroeconomic applications.[2] Second, although agents are *perpetually youthful* in the sense of facing a constant mortality rate, we embed a life cycle component by allowing agents to retire. This introduces buffer-stock saving behavior, individuals accumulate assets during their working life to finance their consumption in retirement. Although the inclusion of a life cycle component adds modest complexity, the model remains analytically tractable.

The firm side of the model adopts the international trade framework of Melitz (2003), incorporating both a fixed cost of exporting and a variable iceberg trade cost. Firm productivity dynamics follow the approach of Luttmer (2007), in which productivity evolves according to a geometric Brownian motion (GBM) process. Two notable papers that combine these elements are Impullitti et al. (2013) and Perla et al. (2021). Our framework differs from both in subtle but important ways. Compared to Impullitti et al. (2013), the main distinction lies in the treatment of fixed export costs. While they model export costs as sunk, we assume they are recurrent. This distinction will be discussed in more detail later in the paper. Briefly, our approach allows for closed-form solutions to the firm's optimization problem and yields a more tractable expression for the endogenous productivity distribution, requiring fewer equilibrium conditions. Relative to Perla et al. (2021), our framework diverges in the modeling of firm entry. In their model, new entrants draw productivity from the existing (incumbent) distribution. Given certain distribution refinements, including a Pareto initial distribution, this implies the productivity distribution retains its Pareto shape in equilibrium. In contrast, we assume that all new firms enter with identical productivity and then evolve according to the GBM process, conditional on their investment decisions. This results in an endogenous productivity distribution that exhibits a *quasi*-Pareto shape

[2]For example, Richter (2015) embed this structure in a New Keynesian model, and Moll et al. (2022) apply it to study the effects of automation on capital allocation.

as an equilibrium object.

The remainder of the paper is organized as follows. Section 2 introduces the general equilibrium model and outlines its theoretical structure. Section 3 describes the calibration strategy and presents the main quantitative results. Section 4 concludes. Additional proofs, empirical details, and supplementary quantitative analysis are collected in the Appendix.

2 Model

The Environment

The model is set in continuous time, assuming a steady state equilibrium throughout. There are N identical countries, each populated by a unit measure of ex-ante identical workers who supply labor and collectively own the domestic firms and capital in the economy. A worker's life cycle follows a Markov process. Workers begin life in employment— earning income from labor services and returns on savings. At a constant Poisson rate η, a worker exits the labor force, in which case, rather than supply labor services, they receive benefits funded by a labor income tax on the employed. Irrespective of one's labor force status workers die at Poisson rate δ.

Firms hire labor and rent capital from workers to produce intermediate goods, which can be traded domestically or exported to one of the $(N-1)$ other countries, subject to trade costs. The structure of trade costs follows Melitz (2003): firms must pay a fixed cost to access each foreign market and face an iceberg-type marginal cost for physical exportation. Firm entry is endogenous, and all entrants are initially identical. However, ex-post productivity diverges due to a stochastic process and firms' endogenous decisions. As productivity evolves, firms face an optimal stopping problem: they may choose to pay a cost to upgrade their productivity if it deteriorates sufficiently. This decision, made in equilibrium, can be interpreted as an investment in reorganizing the production process.

Crucially, the model operates in general equilibrium, with wages and the rental rate of capital determined endogenously. The extent to which an individual worker benefits from trade depends on the equilibrium adjustment of these returns. The rest of this section spells out each agent's environment and decisions carefully, and ends with the equilibrium definition. Proofs and derivations are collected in the appendix.

2.1 Worker Problem

Workers discount the future at a rate ρ, and their life cycle follows the Blanchard-Yaari model of perpetual youth (Blanchard, 1985; Yaari, 1965). Mortality occurs at a constant

Poisson rate δ, which is independent of age. Employed workers exit the labor force at rate η, after which they become permanently inactive — there is no possibility of re-entry into the labor market. The parameter η is calibrated to match observed retirement patterns in the data.

Each country is populated by a unit mass of individuals. When a worker dies, they are immediately replaced by a newborn entrant who begins life employed and with zero wealth. There are no bequest motives; instead, the assets of deceased workers are redistributed proportionally to the living via an annuity mechanism.

Workers earn income from two sources: capital and labor while employed, and capital and a fixed pension benefit w_o when out of the labor force. Preferences are defined over consumption, with flow utility exhibiting constant relative risk aversion (CRRA). The instantaneous utility function is given by

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}, \tag{1}$$

where $\gamma > 1$ denotes the coefficient of relative risk aversion.[3]

Each worker decides how much of their income to consume and how much to save in order to maximize lifetime utility, as defined in equation (2). This optimization is subject to a borrowing constraint (3) and the stochastic income process described in (4). The parameter w_o is exogenous and denotes the pension income received upon retirement. The return on assets is denoted by $\tilde{r}$, while w captures firms' gross wage payments. After labor income is taxed at rate τ, the net return to work is $w(1-\tau)$. Throughout the paper, we assume that net income while employed exceeds the income received when out of the labor force, such that $w(1-\tau) > w_o$. The values of w, $\tilde{r}$, and τ will be determined endogenously in the

[3]The utility function exhibits risk aversion for all positive values of γ. However, the restriction $\gamma > 1$ is imposed to ensure the validity of certain theoretical results presented below.

general equilibrium analysis that follows.[4]

$$\max_{\{c_t\}_{t\geq 0}} \int_0^\infty e^{-(\rho+\delta)t} u(c_t) dt \tag{2}$$

$$\text{subject to, } \frac{da_t}{dt} = \tilde{r}a_t + y_t - c_t \quad \text{and} \quad a_t \geq -\frac{y_t}{\tilde{r}} \quad , \quad c_t \geq 0 \tag{3}$$

$$\text{if employed, } y_t = \begin{cases} w(1-\tau) & \text{with probability } 1-\eta dt \\ w_o & \text{with probability } \eta dt \end{cases} \quad , \quad \text{if inactive, } y_t = w_o \tag{4}$$

The problem outlined in (2)-(4) can be written succinctly as two Hamilton-Jacobi-Bellman (HJB) equations. The present value $\tilde{v}(a)$ and consumption choice $\tilde{c}$ of the inactive are denoted with a tilde to distinguish them from the employed. Derivation of the HJB equations follows standard arguments and is performed in Appendix A.1.

$$(\rho+\delta)v(a) = \max_{0\leq c\leq a+\frac{w(1-\tau)}{\tilde{r}}} \langle u(c) + v_a(a)(\tilde{r}a + w(1-\tau) - c) + \eta(\tilde{v}(a) - v(a))\rangle \tag{5}$$

$$(\rho+\delta)\tilde{v}(a) = \max_{0\leq \tilde{c}\leq a+\frac{w_o}{\tilde{r}}} \langle u(\tilde{c}) + \tilde{v}_a(a)(\tilde{r}a + w_o - \tilde{c})\rangle \tag{6}$$

By imposing the optimality condition $u'(\tilde{c}) = \tilde{v}_a(a)$ on the HJB equation for retired individuals, equation (6) can be reformulated as a first-order differential equation, for which an analytical solution is available. Lemma 1 characterizes the optimal consumption and savings policy of a retired worker, with the derivation provided in Appendix A.2.

Lemma 1 *The optimal consumption and saving behavior of an inactive (retired) agent is given by:*

$$\tilde{c}(a) = \left(\tilde{r} - \frac{\tilde{r}-\rho-\delta}{\gamma}\right)\left(a + \frac{w_o}{\tilde{r}}\right) \tag{7}$$

$$\textit{and} \quad \tilde{s}(a) := \dot{a} = \frac{\tilde{r}-\rho-\delta}{\gamma}\left(a + \frac{w_o}{\tilde{r}}\right). \tag{8}$$

The consumption policy implies that retired individuals consume a constant fraction of their *effective wealth*, defined as the sum of their actual asset holdings and the present value of future pension income. Under the parameter restrictions that $\tilde{r}$, ρ, and δ are all positive, and $\gamma > 1$, consumption remains strictly positive and increases monotonically with asset

[4] The return on assets, denoted by $\tilde{r}$, employs a tilde to distinguish it from the return to capital, r, which will be formally defined in subsequent sections. The return on assets differs from the return to capital due to the presence of an annuity mechanism that reallocates the wealth of deceased individuals among surviving agents. In equilibrium, the return on assets is the sum of the return on capital and the mortality rate, such that $\tilde{r} = r + \delta$.

holdings. The direction of savings behavior is governed by the relationship between the return on assets, $\tilde{r}$, and the effective discount rate, $\rho + \delta$. If $\tilde{r} > \rho + \delta$, retirees accumulate assets over time. Conversely, if $\tilde{r} < \rho+\delta$, retirees dissave, and their asset holdings eventually converge to the borrowing constraint in the long run.

The value function for employed individuals is more complex due to the option value embedded in the possibility of retirement. To the best of our knowledge, a fully analytical characterization is not feasible. Nevertheless, to gain insight into consumption and savings behavior, Lemma 2 provides the Euler equations governing optimal consumption growth for both employed and inactive (retired) individuals. The derivation of Lemma 2 is in Appendix A.3.

Lemma 2 *Optimal consumption growth for employed and inactive (retired) individuals is characterized by the following differential equations:*

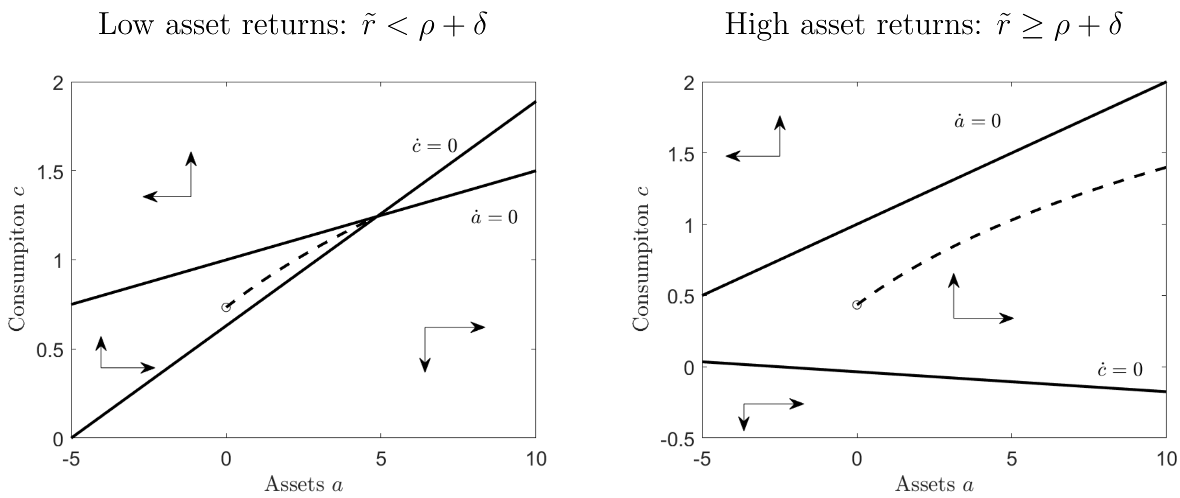
$$\frac{\dot{c}}{c} = \frac{1}{\gamma}\left(\tilde{r} - \rho - \delta - \eta + \eta\frac{c^\gamma}{\tilde{c}^\gamma}\right) \tag{9}$$

$$\frac{\dot{\tilde{c}}}{\tilde{c}} = \frac{1}{\gamma}\left(\tilde{r} - \rho - \delta\right) \tag{10}$$

The final term in the Euler equation for the employed (9) is the constant Poisson retirement rate η multiplied by the ratio of the marginal utility of consumption in employment relative to retirement. Given that income while employed exceeds retirement income, consum

ption during employment is likely to be higher than during retirement ($c > \tilde{c}$). This implies that consumption for employed individuals grows more rapidly, driven by their incentive to accumulate savings in anticipation of retirement.

Figure 1: Phase Diagram



The figure is for illustrative purposes only. The solid lines represent the locus of points for which assets are fixed $\dot{a} = 0$ in equation (3) and consumption fixed $\dot{c} = 0$ in equation (9). The circle is the initial condition and the dashed line is the path of optimal consumption. Model calibration is as follows: $\delta = 0.02$, $\eta = 0.025$, $\rho = 0.05$, $\gamma = 1$, $w(1-\tau) = 1$, $w_o = 0.25$. Two return scenarios are considered: a low return case with $\tilde{r} = 0.05$, and a high return case with $\tilde{r} = 0.1$.

The Euler equation (9) coupled with the wealth accumulation equation (3) allows one to plot the dynamic evolution of consumption and assets of the employed (Figure 1). Depending on the return on assets, the dynamics are different. Employed workers start on zero assets, and both assets and consumption grow. If returns ($\tilde{r}$) are lower than the effective discount rate ($\rho + \delta$), the employed target a level of buffer stock savings to smooth their consumption following retirement. Conversely, if the returns are higher than $\rho + \delta$, assets and consumption grow indefinitely, and in the limit savings and consumption becomes linear in *effective wealth*. The limiting behavior is formalized in Lemma 3, the derivation of which is provided in Appendix A.4.

Lemma 3 *The asymptotic behavior of assets and consumption is given by equations* (11) *and* (12). *These expressions are computed by assuming that a worker remains in employment* *indefinitely.*

$$c \sim \begin{cases} \overline{c} := \frac{C_1}{C_1 - \tilde{r}} \left(w(1-\tau) - w_o \right), & \textit{for } \tilde{r} < \rho + \delta \\ \left(\tilde{r} - \frac{\tilde{r}-\rho-\delta}{\gamma} \right) (a + \phi), & \textit{for } \tilde{r} \geq \rho + \delta, \end{cases} \tag{11}$$

$$a \sim \begin{cases} \overline{a} := \frac{w(1-\tau) - C_1 \frac{w_o}{\tilde{r}}}{C_1 - \tilde{r}}, & \textit{for } \tilde{r} < \rho + \delta \\ \infty, & \textit{for } \tilde{r} \geq \rho + \delta, \end{cases} \tag{12}$$

where

$$C_1 = \left(1 - \frac{\tilde{r} - \rho - \delta}{\eta} \right)^{1/\gamma} \left(\tilde{r} - \frac{\tilde{r} - \rho - \delta}{\gamma} \right),$$

and

$$\phi = \alpha \frac{w_o}{\tilde{r}} + (1 - \alpha) \frac{w(1-\tau)}{\tilde{r}}, \quad \alpha \in (0, 1).$$

Here, C_1 is a composite term that depends on the interest rate $\tilde{r}$, the subjective discount rate ρ, mortality δ, the elasticity parameter η, and the intertemporal elasticity of substitution γ. The parameter ϕ represents the present value of an income stream that lies between employment income and retirement income, constructed as a convex combination of the retirement wage $w_o/\tilde{r}$ and the after-tax employment wage $w(1-\tau)/\tilde{r}$, weighted by a unique α.

Distribution of Assets

The previous section characterized individual workers' optimal decisions. The next objective is to examine how these decisions aggregate across the population. Since workers enter the labor market without assets, Lemmas 1 and 3 jointly determine the relevant domain for asset holdings. In a low-return environment — defined by $\tilde{r} < \rho + \delta$ — employed workers accumulate assets up to an upper bound $\overline{a}$, as established in Lemma 3. Therefore, for the employed, asset holdings lie in the interval $a \in [0, \overline{a})$.[5] Upon retirement, workers begin dissaving according to the consumption rule derived in Lemma 1, extending the domain to $a \in \left(\frac{-w_o}{\tilde{r}}, \overline{a} \right)$ for the inactive. In contrast, when returns are high ($\tilde{r} \geq \rho + \delta$), both employed and retired individuals engage in net positive savings, leading to unbounded asset accumulation. Consequently, the asset domain for all individuals becomes $a \in [0, \infty)$, regardless of labor market status.

The evolution of the asset distribution is governed by the Kolmogorov Forward Equations

[5] Strictly speaking, if the return on assets $\tilde{r}$ is sufficiently low, even employed workers would dissave. This occurs when the intersection of the fixed asset and consumption loci in Figure 1 lies in the region with negative asset holdings and negative consumption. Such behavior would imply a negative net supply of assets, which is not sustainable in equilibrium. Hence, we abstract from this possibility and assume that employed workers always save initially.

(KFE), equations (13) and (14), which track the density dynamics for the employed and inactive populations, respectively. Let $G(a)$ and $\tilde{G}(a)$ denote the cumulative distribution functions (CDFs) of asset holdings for employed and inactive (retired) workers, respectively, scaled such that $\lim_{a\to\infty} G(a) = \overline{L}$ and $\lim_{a\to\infty} \tilde{G}(a) = 1-\overline{L}$, where $\overline{L}$ is the total measure of employed workers and $1-\overline{L}$ is the total measure of inactive workers. In (13), the first term (proportional to δ) captures the inflow of new labor market entrants, all of whom start with zero assets, thus shifting the CDF upward at the origin. The remaining terms account for endogenous savings behavior, mortality-induced exits, and transitions between employment and inactivity, with the final term specifically reflecting the flow from employment into retirement.

The KFE for the retired (14) mirrors that of the employed, with two key distinctions. First, workers do not begin life in retirement, so the δ term is omitted. Second, transitions into retirement occur at rate η and assets are inherited from employment. This only affects $\tilde{G}(a)$ for strictly positive asset levels, since employed workers are never indebted.

$$\frac{\partial G(a)}{\partial t} = \delta - s(a)g(a) - (\delta+\eta)G(a) \tag{13}$$

$$\frac{\partial \tilde{G}(a)}{\partial t} = \begin{cases} -\tilde{s}(a)\tilde{g}(a) - \delta\tilde{G}(a), & \text{for } a<0 \\ -\tilde{s}(a)\tilde{g}(a) - \delta\tilde{G}(a) + \eta G(a), & \text{for } a\geq 0 \end{cases} \tag{14}$$

An individual worker's asset level and labor market status are subject to change over time. However, equilibrium is defined as a state in which the aggregate distribution remains stable. Lemma 4 derives the solution to the differential equations (13) and (14) by imposing the steady state condition $\frac{\partial G(a)}{\partial t} = \frac{\partial \tilde{G}(a)}{\partial t} = 0$.

Lemma 4 *Given a steady state equilibrium* $\frac{\partial G(a)}{\partial t} = \frac{\partial \tilde{G}(a)}{\partial t} = 0$*, the measure of assets of the employed that satisfies the KFE* (13) *is given by,*

$$g(a) = \frac{\delta}{s(a)} e^{-\int_0^a \frac{\delta+\eta}{s(a')}da'} \qquad \text{for } a\in[0,\overline{a}) \quad \text{when } \tilde{r}<\rho+\delta$$
$$\qquad\qquad\qquad\qquad \text{for } a\in[0,\infty) \quad \text{when } \tilde{r}\geq\rho+\delta,$$

and 0 *otherwise. For the inactive, the density is given as*

$$\tilde{g}(a) = \begin{cases} -C\left(\frac{\delta}{\tilde{s}_0}\right)\left(a+\frac{w_0}{\tilde{r}}\right)^{-\frac{\delta}{\tilde{s}_0}-1} & \text{for } a<0 \\ -\frac{\delta}{\tilde{s}_0}\left(a+\frac{w_0}{\tilde{r}}\right)^{-\frac{\delta}{\tilde{s}_0}-1}\left[\frac{\eta}{\tilde{s}_0}\int_0^a G(u)\left(u+\frac{w_0}{\tilde{r}}\right)^{\frac{\delta}{\tilde{s}_0}-1}du + C\right] + \frac{\eta}{\tilde{s}_0}G(a)\left(a+\frac{w_0}{\tilde{r}}\right)^{-1} & \text{for } a\geq 0. \end{cases}$$

The domain of which is $a \in \left(\frac{w_o}{\tilde{r}}, \overline{a}\right)$ *for* $\tilde{r} < \rho + \delta$ *and* $a \in [0, \infty)$ *for* $\tilde{r} \geq \rho + \delta$*. The constant* C *also depends on the value of* $\tilde{r}$ *and is given vy*

$$C = 0 \quad \text{for } \tilde{r} \geq \rho + \delta \text{ and}$$
$$C = \frac{\eta}{\delta + \eta}\left(\overline{a} + \frac{w_o}{\tilde{r}}\right)^{\delta/\tilde{s}_0} - \frac{\eta}{\tilde{s}_0}\int_0^{\overline{a}} G(a')\left(a' + \frac{w_o}{\tilde{r}}\right)^{\frac{\delta}{\tilde{s}_0}-1} da' \quad \text{for } \tilde{r} < \rho + \delta.$$

The parameter $\overline{a}$ *is as defined in Lemma 3 and* $\tilde{s}_0 := \left(\frac{\tilde{r}-\rho-\delta}{\gamma}\right)$*. Integrating the measures over their full support yields the mass of employed and inactive workers such that,*

$$\overline{L} := \int_0^{\infty} g(a)da = \frac{\delta}{\delta + \eta} \quad \text{and} \quad \int_{-\frac{w_o}{\tilde{r}}}^{\infty} \tilde{g}(a)da = \frac{\eta}{\delta + \eta}.$$

Figure 2 visualizes the two measures derived in Lemma 4. The density of employed workers declines monotonically with asset holdings, as individuals either retire or exit the labor force due to mortality. In the low return environment, this decline is steeper because workers reduce savings as they near their target wealth level, approaching the upper bound of the support, $\overline{a}$. In contrast, the asset density of inactive individuals initially increases with assets before eventually declining. In the high return environment, asset holdings remain strictly positive. The upward-sloping portion of the density reflects net inflows exceeding the outflows due to permanent labor force exit. The downward-sloping portion is unbounded and exhibits a Pareto tail. Under the low return calibration, the density has a mode at zero. Across this distribution, inactive individuals dissave, and in the limit, their asset holdings approach the borrowing constraint $w_o/\tilde{r}$.

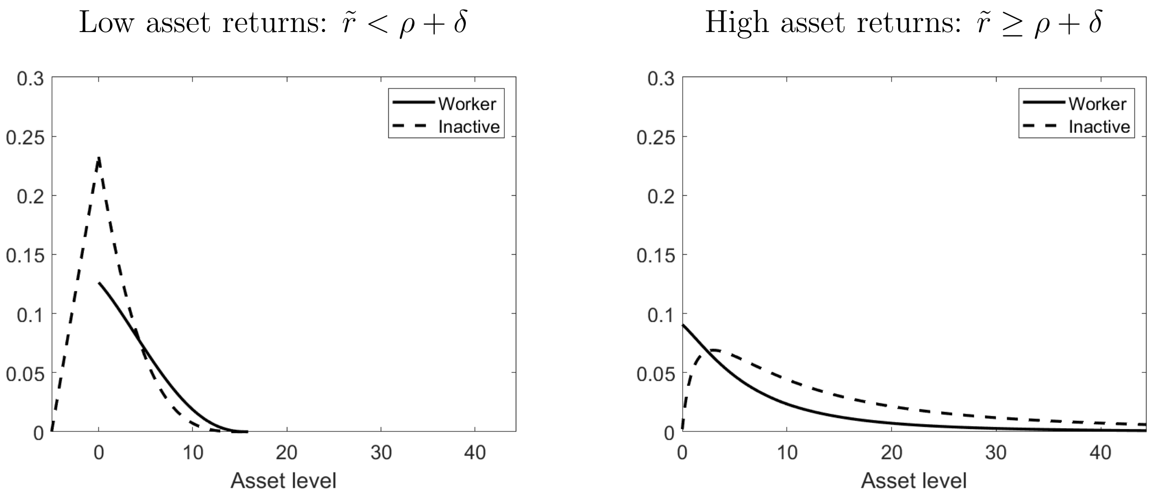
Finally, the computed measure $\overline{L}$ determines the equilibrium labor income tax rate needed to balance the government's budget. In equation (15), the left-hand side represents total tax revenue, while the right-hand side corresponds to total government expenditures, which, in this model, are entirely allocated to funding pensions for retired workers.

$$w\tau\overline{L} = w_o(1 - \overline{L}). \tag{15}$$

2.2 Firm Problem

In each of the N countries there is a perfectly competitive final goods producer that supplies a final consumption good using intermediate inputs, following a constant elasticity of substitution (CES) production function. Intermediate goods are produced by firms with heterogeneous productivity, employing a Cobb-Douglas production function, where both capital

Figure 2: Asset Densities



The figure plots the measures defined by Lemma 4 normalized by the mass of workers — the two plots are $g(a)/\overline{L}$, and $\tilde{g}(a)/(1-\overline{L})$. The model calibration is for illustrative purposes only and is as described in the notes of Figure 1.

and labor are freely adjustable and cost r_k and w per unit, respectively.[6]

Trade follows the framework of Melitz (2003), where firms can export subject to both fixed costs and variable '*iceberg*' trade costs. In order to export one needs to hire ϕ_x workers to facilitate trade, at a cost of $\phi_x w$, where w is the prevailing market wage. The variable cost requires trading $d \geq 1$ units to deliver one unit of the good to the export market. In the Melitz class of models, the distribution of firm productivity plays a crucial role, determining which firms survive, export, or exit, ultimately driving trade-induced resource reallocation. In this model, the productivity distribution, $f(z)$, is endogenously determined. All firms begin with a common fixed productivity level z_0, after which their productivity diverges due to: (i) a stochastic process and (ii) investment decisions aimed at upgrading productivity. The stochastic process for productivity z follows Geometric Brownian motion (GBM) with drift μ and volatility σ. At any time a firm can hire a mass χ workers, at a cost χw, to re-organize the production process and bring the firm to productivity $z = z_0$. Finally, firms die at a Poisson rate δ_f and are replaced by new entrants. The number of intermediate goods producers is determined by a free-entry condition, where the number of workers to setup a firm is $\chi(1+\epsilon)$. Where $\epsilon > 0$, and hence setting up a new firm incurs a greater cost than re-organizing an existing one.

[6]The rental rate of capital, r_k, may differ from the return on capital, r, due to capital depreciating at rate δ_k. Depreciation creates a wedge between the cost of capital faced by the firm and the net return received by the worker, such that $r_k = r + \delta_k$.

The firm's optimization problem yields closed-form analytical solutions for all policy and value functions. Proofs and derivations of the firm problem are collected in Appendix B. We first consider the final goods producer. Next, we consider the static problem facing intermediate firms, which is how much capital and labor to use as inputs, and whether or not to export. Finally, we consider the dynamic problem of the firm, when to invest in the productivity process and the value associated with entry.

Final Goods Producer

The competitive final goods producer utilizes each unique variety v to produce the consumption good using a CES production function, characterized by an elasticity of substitution across varieties, $\omega > 1$. Formally, the firm maximizes its profits as specified in equation (16).

$$\max_{q_{ij}(v)} \left[\sum_{i=1}^{N} \int_{\Omega_{ij}} q_{ij}(v)^{(\omega-1)/\omega} \right]^{\omega/(\omega-1)} - \sum_{i=1}^{N} \int_{\Omega_{ij}} p_{ij}(v) q_{ij}(v) \tag{16}$$

The measure Ω_{ij} defines the endogenous set of varieties consumed in country i and produced in country j. The problem faced by the final goods producer results in the well-known variety demand and price index equations ((17) and (18)). The demand for a particular variety is proportional to total output Y. When computing the price index, one has to distinguish between the price of domestic goods $p_d(z)$, and the price of goods imported from the other $N-1$ countries $p_x(z)$. In equilibrium, an intermediate producer exports if their productivity $z \geq \hat{z}$. In practice, the price index will act as the numeraire of the economy, and be normalized to one.

$$q(z) = \left(\frac{p(z)}{P} \right)^{-\omega} \frac{Y}{P} \tag{17}$$

$$P^{1-\omega} = \Omega \left(\int_{z_l}^{\infty} p_d(z)^{1-\omega} dF(z) + (N-1) \int_{\hat{z}}^{\infty} p_x(z)^{1-\omega} dF(z) \right) \tag{18}$$

Static Problem

Turning to the producers of the individual varieties. The static problem facing these firms is fairly standard. There are a continuum in each country, and they vary by their productivity level z. The production function is assumed to be a constant returns to scale Cobb-Douglas function of the form $q = z\ell^{\beta} k^{1-\beta}$. The variety is supplied monopolostically to the final good producer. The firm maximizes flow profit, and to do so chooses the price of the good, the amount of labor and capital inputs used in production, and whether or not to export the good. Since all of these costs are freely adjustable we can treat this part of the firm's problem as if it were in a static environment.

Since this is fairly standard all derivation is provided in Appendix B.1. Lemma 5 characterizes optimal firm policy in terms of capital and labor inputs and pricing. Lemma 6 presents the associated profit functions.

Lemma 5 *Pricing policy and the volume of labor and capital inputs for domestic production are given by:*

$$\begin{aligned}
p_d(z) &= \frac{\omega}{\omega-1}\frac{C}{z} \\
\ell_d(z) &= \frac{1}{w}\frac{\omega-1}{\omega}\beta Y\left(\frac{\omega}{\omega-1}\frac{C}{P}\right)^{1-\omega} z^{\omega-1} \\
k_d(z) &= \frac{1}{r_k}\frac{\omega-1}{\omega}(1-\beta)Y\left(\frac{\omega}{\omega-1}\frac{C}{P}\right)^{1-\omega} z^{\omega-1}
\end{aligned}$$

If a firm exports, the pricing policy and additional volume of labor and capital inputs per foreign market is given by:

$$p_x(z) = p_d(z)d, \quad \ell_x(z) = \ell_d(z)d^{1-\omega}, \quad k_x(z) = k_d(z)d^{1-\omega}.$$

The constant C is the firm's marginal cost of production and given by,

$$C = w^{\beta}r_k^{1-\beta}\left(\frac{1}{\beta}\right)^{\beta}\left(\frac{1}{1-\beta}\right)^{1-\beta}. \tag{19}$$

Lemma 6 *The flow profit of a firm of productivity z is given by the sum of their domestic, and foreign profits.*

$$\pi(z) = \pi_d(z) + (N-1)\pi_x(z) \tag{20}$$

Where profit in the domestic market is given by,

$$\pi_d(z) = \pi_0 z^{\omega-1} \tag{21}$$

and per country export profit is

$$\pi_x(z) = \max\{\pi_d(z)d^{1-\omega} - \phi_x w, 0\}. \tag{22}$$

The parameter $\pi_0 := \frac{1}{\omega}\frac{Y}{P}\left(\frac{\omega}{\omega-1}\frac{C}{P}\right)^{1-\omega}$.

It immediately follows from Lemma 6 that there exists a threshold productivity $\hat{z}$, that for $z \geq \hat{z}$ firms export, and for $z < \hat{z}$ firms only supply goods domestically. By symmetry across countries, if it is profitable for a firm to export to one destination, it is equally profitable to export to all. The export cutoff productivity level, $\hat{z}$, is defined at the point where per-country export profits are driven to zero, i.e., $\pi_x(\hat{z}) = 0$:

$$\hat{z} = d \left(\frac{\phi_x w}{\pi_0} \right)^{\frac{1}{\omega - 1}}. \tag{23}$$

A key assumption underlying the characterization in (23) is that a firm's export decision can be analyzed statically. This follows from the structure of the fixed cost of exporting, $\phi_x w$, which does not depend on dynamic considerations. Specifically, this cost is incurred only while the firm is actively exporting and contains no sunk component. In standard models of heterogeneous firms, such as Melitz (2003) or Chaney (2008), this assumption is largely inconsequential because firm productivity is static. However, in a dynamic context where productivity evolves over time, the presence or absence of sunk costs materially affects the form of the equilibrium. For instance, in Impullitti et al. (2013), the fixed cost is sunk, giving rise to two productivity thresholds: a higher one for entering export markets and a lower one for exiting. In our setting, we interpret the export cost as arising from the recruitment of global supply chain managers or logistics operators who facilitate the sale of goods to third countries. These individuals are engaged on a pro-rata basis, consistent with the assumption of non-sunk, flexible costs.

Dynamic Problem

Upon entering the market, a firm begins with an exogenously determined initial productivity level, z_0. In the calibration, it is found that $z_0 < \hat{z}$, implying that firms do not export immediately. Firm productivity evolves according to a geometric Brownian motion with drift μ and volatility σ. Firm exit is exogenous, and occurs at a constant Poisson rate δ_f. At any point, a firm can choose to upgrade its productivity to a new level z_0 by incurring a cost. This requires hiring $\chi > 0$ workers at a total cost of χw, where w represents the prevailing market wage.

To enter the market, a firm must hire a total workforce of $\chi(1 + \epsilon)$, where $\epsilon > 0$. A proportion χ of these workers is necessary to establish production at the foundational productivity level, while the additional workers are required to set up the broader infrastructure essential for a new firm's operations. Equation (24) shows the evolution of productivity z, where t_0 is time of last upgrade, and $W(\cdot)$ is a standard Wiener process.

$$z(t - t_0) = z_0 + \mu(t - t_0) + \sigma W(t - t_0) \tag{24}$$

There are two equilibrium conditions that arise from firms' dynamic decisions. First, potential entrants must decide whether to enter the market. In equilibrium, entry occurs if the present value of doing so exceeds the entry cost. The measure of firms, Ω, is determined such that the free entry condition holds, ensuring that the marginal gain from entry is zero, as given by equation (25). The value function $V(\cdot)$ remains to be defined and is evaluated at z_0, representing a firm's initial productivity level.

$$V(z_0) = \chi(1+\epsilon)w \tag{25}$$

The second decision is made by incumbent firms, specifically regarding when to hire χ workers and upgrade their technology back to z_0. This decision constitutes an optimal stopping problem, wherein firms must determine the threshold productivity level, z_l, at which they choose to incur the upgrade cost. A similar problem is analyzed in Stokey (2008).[7]

Hamilton-Jacobi-Bellman (HJB) equation:

$$(r+\delta_f)V(z) = \pi(z) + \left(\mu + \frac{\sigma^2}{2}\right) z\frac{\partial V(z)}{\partial z} + \frac{\sigma^2}{2} z^2 \frac{\partial^2 V(z)}{\partial z^2} \tag{26}$$

Value matching conditions:

$$V(z_0) - V(z_l) = \chi w \tag{27}$$

$$\lim_{z\to\hat{z}^-} V(z) = \lim_{z\to\hat{z}^+} V(z) \tag{28}$$

Smooth pasting conditions:

$$\frac{\partial V(z_l)}{\partial z} = 0 \tag{29}$$

$$\lim_{z\to\hat{z}^-} \frac{\partial V(z)}{\partial z} = \lim_{z\to\hat{z}^+} \frac{\partial V(z)}{\partial z} \tag{30}$$

The Hamilton-Jacobi-Bellman equation (26) characterizes the value function over the support $z \in [z_l, \infty)$. The value matching condition (27) ensures that the firm's value increases in a way that renders it indifferent between upgrading and not upgrading. The smooth pasting condition (29) serves as an optimality condition, ensuring a seamless transition at the upgrade threshold. Additionally, two further conditions must be satisfied at the exporting threshold, $\hat{z}$. While the profit function (20) is continuous, its derivative is not defined at the threshold for exporting. To address this, the conditions (28) and (30) ensure that the solution to the HJB equation remains differentiable across the full support of productivity z. As shown by Bradley (2025), omitting these interior conditions would neglect the option value of exporting in the future for firms not currently exporting. The differential equation

[7] See Chapters 6.2 and 6.3, pages 116-121.

(26) admits a closed-form solution that ensures conditions (28), (29), and (30) are satisfied. This solution is formally stated in Lemma 7.

Lemma 7 *The solution to the second-order differential equation* (26) *that satisfies the equilibrium conditions* (28)-(30) *is given by*

$$V(z) = \begin{cases} V_d(z) = \mathcal{B}_1 z^{\omega-1} + \mathcal{B}_2 z^{-\mathrm{v}_1} + \mathcal{B}_3 z^{-\mathrm{v}_2}, & \textit{for } z_l < z < \hat{z} \\ V_x(z) = \mathcal{C}_0 + \mathcal{C}_1 z^{\omega-1} + \mathcal{C}_2 z^{-\mathrm{v}_1}, & \textit{for } z \geq \hat{z} \end{cases} \tag{31}$$

The constants $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$, v_1 and v_2 are provided in Appendix B.2. For a detailed derivation and discussion of the solution, see Bradley (2025).

The solution to the value function includes two endogenous parameters: the lower bound of the productivity support, z_l, and the scale parameter for firm flow profits, π_0, as defined in Lemma 6. The lower bound z_l is determined by the value-matching condition (27), which, in the context of the HJB equation, plays a role analogous to an initial condition in an ordinary differential equation— it pins down the value function at a critical boundary. The composite parameter π_0 is determined by the free entry condition (25).

One interpretation of the value function solution (Lemma 7) is that it comprises both static and dynamic components. The *static component* includes the term with coefficient $\mathcal{B}_1$ for non-exporting firms, and the constant $\mathcal{C}_0$ plus $\mathcal{C}_1$ term for exporting firms. These terms represent the value of the firm assuming productivity remains fixed indefinitely. The *dynamic component* captures fluctuations in productivity over time. Specifically, the coefficients $\mathcal{B}_2$, $\mathcal{B}_3$, and $\mathcal{C}_2$ depend on the export threshold $\hat{z}$, reflecting the fact that non-exporting firms today may become exporters in the future as their productivity evolves— and vice versa, exporters may also cease exporting if productivity declines.

Productivity Distribution

To recapitulate, the evolution of a firm's productivity follows the following process. Firms exit at an exogenous rate δ_f and are replaced by new entrants with an initial productivity level of z_0. Thereafter, their productivity evolves according to an exogenous process of geometric Brownian motion. When productivity reaches an endogenously determined threshold, z_l, firms pay a cost χw and increase their productivity to z_0. This process is formally characterized by the Kolmogorov Forward Equation (32) (KFE).

$$\begin{aligned}
\frac{\partial F(z)}{\partial t} &= -A(t) - \delta_f F(z) - \left(\mu - \frac{\sigma^2}{2}\right) z f(z) + \frac{\sigma^2}{2} z^2 f'(z) && \text{for, } z_l < z < z_0 \\
\frac{\partial F(z)}{\partial t} &= -\delta_f F(z) - \left(\mu - \frac{\sigma^2}{2}\right) z f(z) + \frac{\sigma^2}{2} z^2 f'(z) + E(t) + A(t) && \text{for, } z \geq z_0
\end{aligned} \tag{32}$$

Equation (32) describes the evolution of the cumulative distribution function $F(z)$ of firm productivity over time. Given the economic environment, the CDF satisfies the boundary conditions $F(z_l) = 0$ and $\lim_{z\to\infty} F(z) = 1$.

Examining the terms in the KFE (32) individually: $A(t)$ represents the share of firms upgrading their productivity and adopting the new technology. As these firms upgrade, all productivity levels $z \in (z_l, \hat{z})$ experience a decline in their relative rank within the distribution. Firm exit occurs at a proportional rate δ_f, independent of a firm's position in the distribution. The productivity process itself is influenced by both deterministic and stochastic components. The deterministic movement is captured by the drift term $f(z)$, while stochastic fluctuations are reflected in the diffusion term $f'(z)$. Finally, new entrants $E(t)$ and new adopters $A(t)$ enter the distribution at $z = z_0$. The equilibrium is assumed to be in a steady state, meaning that the distribution remains constant over time. Lemma 8 characterizes the probability density function of firm productivity.

Lemma 8 *Assuming the distribution of productivity is fixed over time,* $\frac{\partial F(z)}{\partial t} = 0$. *The density* $f(z)$ *that satisfies the ODE defined by equation (32) is the double Pareto given by (33).*

$$f(z) = \begin{cases} F_0 \left[\left(\frac{z}{z_l}\right)^{\xi_2} \frac{1}{z} - \left(\frac{z}{z_l}\right)^{\xi_1} \frac{1}{z} \right] & z_l < z < z_0 \\ F_0 \left[\left(\frac{z_0}{z_l}\right)^{\xi_2} - \left(\frac{z_0}{z_l}\right)^{\xi_1} \right] \left(\frac{z}{z_0}\right)^{\xi_1} \frac{1}{z} & z \geq z_0 \end{cases} \tag{33}$$

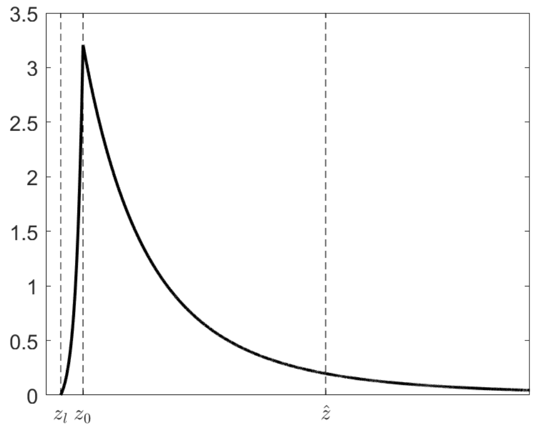
The share of adopters and entrants are given by

$$A = \delta_f \left(\frac{z_l^{\xi_2}}{z_l^{\xi_2} + z_0^{\xi_2}} \right) \quad \textit{and} \quad E = \delta_f.$$

The derivations and definition of the constants F_0, ξ_1 *and* ξ_2 *are provided in Appendix B.3.*

To facilitate visualization of the distribution presented in Lemma 8, Figure 3 plots the corresponding density function. The double Pareto distribution exhibits several key properties: it features a fat right tail; it has zero mass at the minimum; and its density function is non-differentiable at the mode, which corresponds to a firm's initial productivity level, (z_0). Interestingly, the distribution aligns with both the left and right tails of the firm productivity distribution described by Nigai (2017). Nigai argues that the left tail cannot be captured by a Pareto distribution, as it features an increasing density, while the right tail follows a Pareto pattern.

Figure 3: Firm Productivity Distribution



Note: The figure plots the measures defined by Lemma 8. Parameters are calibrated according to the baseline calibration.

At the exporting productivity threshold, however, the density is smooth, and there is no overlap between the distributions of exporting and non-exporting firms. This results from the modeling choice to treat the fixed cost of exporting as non-sunk, allowing firms to transition freely between export statuses. Empirical evidence supports the transient nature of export participation. For instance, Eaton et al. (2008) find that in Colombia, half of all exporting firms had not exported in the previous year, and the majority of these firms cease exporting the following year. Similarly, Besedes and Prusa (2006), using U.S. import data, show that the median duration of exporting a product to the U.S. is between two and four years, with half of all products exported for less than one year.

2.3 Equilibrium

The firm and worker sides of the model have been fully characterized. To complete the model, we connect these components via the product, labor, and asset markets, and formally define the equilibrium. Definitions 1 and 2 specify the partial equilibria for workers and firms, respectively.

Definition 1 Worker-side equilibrium: *Given the return on assets $\tilde{r}$ and wage rate w, the worker-side equilibrium determines optimal consumption and savings, the steady state distribution of assets, and the income tax rate required to balance the government budget. Specifically:*

- *Consumption policies for the employed and the retired, $c(a)$ and $\tilde{c}(a)$, solve the constrained optimization problem described in equations (2)-(4). For retirees, the closed-form solution is provided in Lemma 1, and for employed workers, it follows from the Euler equation in Lemma 2.*

- *The steady state asset distributions for employed and inactive individuals solves the Kolmogorov Forward Equations (13) and (14), with the explicit solutions given in Lemma 4.*

- *The tax rate τ satisfies equation (15), ensuring pension expenditure equals tax revenue, with the solution $\tau = \frac{w_o \eta}{w\delta}$.*

Definition 2 Firm-side equilibrium: *Given total output Y, the rental rate of capital r_k, and the wage w, firm behavior in equilibrium is characterized as follows:*

- *Intermediate firms set domestic and export prices, $p_d(z)$ and $p_x(z)$, as constant markups over marginal cost, as characterized in Lemma 5.*

- *Labor and capital inputs for domestic sales, $\ell_d(z)$ and $k_d(z)$, as well as for each export market, $\ell_x(z)$ and $k_x(z)$, are determined according to Lemma 5.*

- *Firms choose to export if it yields positive flow profits. All firms with productivity $z \geq \hat{z}$ export to all $(N-1)$ foreign markets, where the threshold $\hat{z}$ is defined in equation (23).*

- *A firm's value is given by the expression in Lemma 7. Firms solve an optimal stopping problem defined by equations (26)-(30). Upon reaching productivity level z_l, they may pay a fixed cost χw to upgrade their productivity to z_0. Given the value function, the free entry condition in equation (25) determines the total mass of firms, Ω.*

- *The stationary distribution of firm productivity, $f(z)$, satisfies the Kolmogorov Forward Equation (32). This distribution determines the mass of new entrants, E, and adopters, A, as described in Lemma 8.*

Labor and product market

The equilibrium in the labor market is attained when labor demand is equal to labor supply, as described in equation (34). In this model, labor supply is inelastic and defined by $\overline{L} = \frac{\delta}{\delta+\eta}$. On the demand side, labor is crucial for production targeting both domestic and export markets, as well as for several non-productive tasks. Firms entering the market must hire $\chi(1+\epsilon)$ workers to establish operations, while incumbent firms require χ workers for production reorganization and technological upgrades. Moreover, exporting necessitates the employment of ϕ_x workers for each export market. The per-firm demand for productive and non-productive labor together constitute the left-hand side of equation (34).

$$\underbrace{\int_{z_l}^{\infty} \ell_d(z)dF(z) + (N-1)\int_{\hat{z}}^{\infty} \ell_x(z)dF(z)}_{\text{labor used in production}} + \underbrace{(\delta_f(1+\epsilon)+A)\chi + (N-1)(1-F(\hat{z}))\phi_x}_{\text{non-production labor}} = \frac{\bar{L}}{\Omega} \tag{34}$$

In equilibrium, the balance between labor supply and demand serves to set the equilibrium wage w. The wage rate adjusts to ensure that the quantity of labor supplied matches the quantity of labor demanded.

The product market equilibrium is described by equation (35). Total consumption is equal to total output net of capital expenditure — since capital is assumed to be produced using final goods.

$$\int_{0}^{\infty} c(a)dG(a) + \int_{-\frac{w_o}{\tilde{r}}}^{\infty} \tilde{c}(a)d\tilde{G}(a) = Y - \delta_k\overline{K} \tag{35}$$

To offset depreciation and maintain the capital stock $\overline{K}$ in steady state, expenditure on capital must equal the depreciation rate δ_k multiplied by the steady state stock of capital. The capital stock $\overline{K}$ is the aggregation of the intermediate firm's static problem outlined in Lemma 5 and given by

$$\overline{K} = \int_{z_l}^{\infty} k_d(z)dF(z) + (N-1)\int_{\hat{z}}^{\infty} k_x(z)dF(z).$$

The product market's function is to determine aggregate output Y in equilibrium, rather than prices, because the price level P has been normalized to one (as presented in equation (18)). This normalization allows all results to be interpreted in real rather than nominal terms.

Asset market

The final equilibrium condition to be defined pertains to the asset market. Assets are held by workers, and the aggregate level is given by

$$\overline{S} := \int_0^\infty a dG(a) + \int_{-\frac{w_o}{\tilde{r}}}^\infty a d\tilde{G}(a).$$

The model incorporates four types of assets, all of which are assumed to be risk-free and perfectly liquid. By no-arbitrage, these investments must therefore yield identical returns. An equivalent interpretation is that workers allocate their savings to a mutual fund holding a diversified portfolio across the four asset classes. Similarly, although we assume that workers invest exclusively in domestic assets, permitting costless investment in foreign assets would result in an isomorphic model, given that all N countries are identical, and would therefore leave our results unchanged.

Equilibrium in the asset market requires that the return on total assets equals the total returns generated by firms, the rental yield on the capital stock, and the redistributed wealth of deceased individuals distributed as an annuity.

$$\underbrace{\tilde{r}\overline{S}}_{\text{Return to asset holders}} = \underbrace{(\Pi - E\chi(1+\epsilon)w - A\chi w)\,\Omega}_{\text{Firm profits net of fixed costs}} + \underbrace{r\overline{K}}_{\text{Net return on capital}} + \underbrace{\delta\overline{S}}_{\text{Blanchard annuity}} \tag{36}$$

The four asset classes are:

1. **Equities** — which pay out dividends from firm profits, net of entry and adoption costs. Total distributable profit is $(\Pi - E\chi(1+\epsilon)w - A\chi w)\Omega$, where Π is aggregate firm profit and E and A denote the shares of entrants and adopters, derived in Lemma 8.

2. **Physical capital**— which has an endogenous aggregate supply, $\overline{K}$. The net return on this capital is lower than the rental rate that firm's face r_k, given capital depreciation $\delta_k > 0$. Hence, if workers rent out capital at a cost r_k per unit, their return on the unit is attenuated by the depreciation rate such that $r := r_k - \delta_k$.

3. **Annuities**— financial contracts that reallocate the wealth of deceased individuals among surviving participants, in proportion to their asset holdings.

4. **Debt**— arises when $\tilde{r} < \rho + \delta$, leading some retirees to hold negative asset positions. In such equilibria, the employed effectively lend to the retired.

Since both capital and equities are risk-free and liquid, arbitrage implies a unified return. The asset value of one unit of equity is its dividend divided by its return r. From equation (36), the equilibrium return on assets satisfies:

$$\tilde{r} = r + \delta.$$

The next section analyzes how heterogeneity across workers and firms shapes the equilibrium return $\tilde{r}$.

2.4 The role of heterogeneity

In this paper, we interpret trade openness as changes in the iceberg trade cost parameter, d. A reduction in trade costs increases the profitability of productive firms that export, while reducing the profitability of firms that serve only the domestic market, due to heightened foreign competition. Since the model is symmetric across countries and free of distortions, the aggregate effect of lower trade costs is positive. However, the impact varies across the firm productivity distribution. In essence, changes in trade costs act as a particular form of productivity shock— one that affects firms asymmetrically based on their underlying productivity.

Since all firms recruit from the same inelastically supplied labor pool, a net positive shock— such as a reduction in trade costs d— raises wages. However, the effect on the return to capital from the perspective of workers is far less straightforward. It depends critically on the degree of heterogeneity in labor income, asset holdings, and firm productivity. To understand this relationship more clearly, we begin by imposing simplifying restrictions that eliminate heterogeneity and allow us to characterize how the equilibrium interest rate is determined in different scenarios. These scenarios and their corresponding equilibria are stated in Propositions 1-3, with formal proofs provided in Appendix C. We start with a representative agent version of the model, presented in Proposition 1.

Proposition 1 *Suppose that workers are infinitely lived ($\delta = 0$) and never retire ($\eta = 0$). Then the return on capital, r, is constant and equal to the worker's discount rate, ρ, and is unaffected by changes in trade.*

The assumptions in Proposition 1 correspond to those of the textbook representative agent model. Setting the retirement rate $\eta = 0$ implies that all individuals are actively employed, so labor income is uniformly equal to the equilibrium wage w, and there is no need for taxation since there are no retirees to support. Setting the mortality rate to zero ensures that all workers are infinitely lived, eliminating age based variation. As a result, there is no heterogeneity in asset holdings arising from differences in the time individuals have had to accumulate wealth. Abstracting from off steady state dynamics, this environment supports an equilibrium in which the return on capital is pinned down by the discount rate. The asset market, characterized by equation (36), clears at a point where the supply of savings is perfectly elastic, and all individuals consume their total income each period— that is, they live hand to mouth.

In Proposition 2, we relax the assumption of homogeneous wealth and asset holdings among workers, while maintaining homogeneity in labor income. To further simplify the environment, we also assume that all firms are identical. Recall that in the baseline model, firms are born with initial productivity z_0, which then evolves stochastically with deterministic drift μ and volatility σ. By setting $\mu = 0$ and $\sigma = 0$, firm productivity remains constant at z_0 over time, eliminating firm-level heterogeneity. In this setting, as established in Proposition 2, the steady state return on capital r lies strictly above the worker's discount rate ρ, but remains invariant to changes in trade costs.

Proposition 2 *Suppose that workers never retire ($\eta = 0$), and all firms are identical (i.e., there is no heterogeneity in productivity, so $\mu = 0$ and $\sigma = 0$). Then the return on capital, r, is (locally) independent of trade costs in the sense that where the derivative exists,*

$$\frac{\partial r}{\partial d} = 0,$$

and lies within the interval $r \in (\rho, \rho + \gamma\delta)$, where ρ is the worker's discount rate, γ is the intertemporal elasticity of substitution, and δ is the worker's mortality rate.

Under the restrictions imposed in Proposition 2, the economy settles into one of two regimes: either autarky, where the export threshold satisfies ($\hat{z} > z_0$), or full trade participation, where ($\hat{z} \leq z_0$). In the autarky equilibrium, it is unsurprising that the return on capital is unaffected by the iceberg trade cost d, as no firm engages in trade and therefore no firm incurs the cost. In the trade equilibrium, by contrast, all firms are exposed to d, and a reduction in trade costs increases their demand for capital. Unlike the representative agent environment of Proposition 1, workers in this setting differ in their asset holdings, resulting in an upward-sloping capital supply curve. Nevertheless, because firms are identical, the increase in capital demand is exactly offset by the corresponding increase in capital supply, leaving the equilibrium return on capital unchanged. Proposition 3 relaxes the assumption of firm homogeneity and shows how differences in productivity determine the relative shifts in asset demand and supply— and thus the resulting change in the equilibrium interest rate.

Proposition 3 *Suppose that workers never retire ($\eta = 0$) and capital does not depreciate ($\delta_k = 0$). Then the effect of a change in iceberg trade costs d on the return to capital r depends on the response of the share of labor allocated to non-production tasks, $\tilde{\ell}$. In particular,*

$$\text{sgn}\left(\frac{\partial r}{\partial d}\right) = -\,\text{sgn}\left(\frac{\partial \tilde{\ell}}{\partial d}\right),$$

$$\textit{where,} \quad \tilde{\ell} = \Omega(\delta_f \chi(1+\epsilon) + A\chi + (1 - F(\hat{z}))(N-1)\phi_x).$$

That is, if an increase in trade costs reduces the labor share in non-production tasks (i.e., $\frac{\partial \tilde{\ell}}{\partial d} < 0$), then the return to capital increases (i.e., $\frac{\partial r}{\partial d} > 0$); conversely, if trade costs raise the non-production labor share, the return to capital falls.

To simplify the derivation, we additionally assume that capital does not depreciate, setting $\delta_k = 0$. The effect of trade on the return to capital depends on how labor is allocated between production and non-production activities, as captured by $\tilde{\ell}$, which denotes the share of labor devoted to tasks not directly involved in production. These non-production tasks include the creation of new firms, the reorganization and upgrading of existing firms' operations, and the coordination of supply chains for export markets.

The return on capital and the non-production labor share $\tilde{\ell}$ are inversely related. When a larger fraction of labor is allocated to non-production activities, the relative value of labor increases compared to capital. As a result, the increase in asset demand is smaller than the increase in asset supply, leading to a decline in the return to capital r. Conversely, when more labor is directed toward production, asset demand rises more strongly relative to supply, thereby increasing the return on capital.

Proposition 3 underscores the importance of endogenizing firm productivity. In the absence of heterogeneity in labor income, the share of workers engaged in non-production tasks is determined by the mass of firms at specific points in the productivity distribution. The canonical heterogeneous firm model introduced by Melitz (2003), and further developed by Chaney (2008), assumes that firm productivity is drawn exogenously from a Pareto distribution. The role of this modeling assumption in shaping trade responses— such as the gains from trade and the number of active firms— has been explored in Arkolakis et al. (2008). These themes are directly relevant for understanding whether the equilibrium interest rate rises or falls in response to a change in trade costs.

The final dimension of heterogeneity we consider is variation in labor income. In the three propositions discussed so far, this channel is shut down by setting the retirement rate $\eta = 0$, ensuring all individuals receive identical labor income throughout life. In this case, the return on assets lies within the interval $r \in (\rho, \rho + \delta\gamma)$, and savings are a constant share of *effective* wealth, given by:

$$s(a) = \frac{r-\rho}{\gamma}\left(a + \frac{w}{r+\delta}\right).$$

In such an environment, a trade shock that affects wages alters the level of savings only through its effect on effective wealth, while the marginal savings rate remains unchanged.

When $\eta > 0$, however, the dynamics change considerably. A trade shock that increases wages not only raises effective wealth but also raises the savings rate— a compounding effect that leads to a larger shift in asset supply and a more pronounced decline in the return to capital r.

This increase in the savings rate can be understood via the Euler equation (9). Retirement consumption $\tilde{c}(a)$ is unaffected by trade policy (as shown in equation (7)), but the increase in effective wealth raises optimal current consumption. According to the Euler equation, this implies faster consumption growth, which requires a higher rate of saving during working life. Intuitively, as the gap between income during employment and retirement widens, individuals smooth consumption over the life cycle by saving a greater fraction of their income while employed.

3 Quantitative Results

3.1 Calibration

We determine model parameters using a combination of normalization, values sourced from the literature, and calibration to empirical data. This calibration strategy is designed to ensure that the model aligns with the empirical motivations of the paper and accurately reflects key features of the UK economy around the time of the Brexit referendum in 2016. Table 1 summarizes the parameter values. The first panel lists parameters normalized for analytical convenience. The second presents values taken directly from the existing literature. The third contains parameters with empirical counterparts that can be calibrated independently of the model solution. Finally, the fourth panel includes internally calibrated parameters, which are estimated by solving the model to match key empirical moments.

The number of countries is fixed at $N = 10$, a relatively arbitrary but inconsequential choice. In the model, export revenues scale proportionally with the number of trading partners, i.e., $(N - 1)$. However, as foreign revenues also decline with the iceberg trade cost d, a higher value of N can be effectively offset by a correspondingly higher d. Thus, N and d jointly determine the scale and profitability of export activity. One must therefore be

Moment	Source	Data	Model	Parameter
Normalization				
Number of countries	—	10		N
Initial productivity	—	1		z_0
Externally calibrated				
Elasticity of substitution:				
Intertemporal	Standard calibration	2		γ
Between goods	Standard calibration	3		ω
Externally calibrated				
Capital depreciation	Penn world table	1%		$\delta_k = 0.01$
Life expectancy	Population survey	81.6 years		$\delta = 0.016$
Average employer age	Business Population Survey	8.3 years		$\delta_f = 0.12$
Internally calibrated				
Return on capital	10-year FTSE 100	5.5%	5.1%	$\rho = 0.047$
Pension replacement rate	Office of National Statistics	22.1%	23.8%	$w_o = 0.041$
Labor share	Penn world table	58.6%	56.9%	$\beta = 0.601$
Labor force participation	Office of National Statistics	60.5%	60.5%	$\eta = 0.010$
Average firm size	Business Population Survey	16.3	16.3	$\epsilon = 6.33$
Export (output) share	National accounts	25%	24%	$d = 3.05$
Export (firm) share	Business Population Survey	9.6%	7.8%	$\phi_x = 1.43$
Firm size distribution:				
1-9 employees	Business Population Survey	81.6%	82.2%	$\chi = 4.34$
10-49 employees		15.4%	14.4%	$\mu = 0.033$
50-249 employees		2.5%	3.2%	$\sigma = 0.024$
250+ employees		0.5%	0.2%	

Table 1: Calibrated Model

held constant while the other is calibrated. The initial productivity level z_0 is normalized to unity, so all subsequent productivity values are interpreted relative to that of a newly entering firm. We adopt a standard calibration of worker preferences, assuming risk aversion with a coefficient of relative risk aversion $\gamma = 2$. The production technology of the final good producer features a constant elasticity of substitution between intermediate goods, with $\omega = 2$.

The depreciation rate of productive capital is taken from UK-specific estimates provided by the Penn World Tables. Maintaining the steady state capital stock requires an annual investment cost equal to one percent of the capital stock's value. Workers and firms exit the economy only for exogenous reasons, occurring at rates δ and δ_f, respectively. Since these are the sole sources of exit in the model, they can be calibrated directly—without solving the full model—by targeting the average ages of workers and firms. Workers are assumed to enter the labor market at age 18, consistent with typical labor market entry in the UK.

Given the worker exit rate δ, the retirement rate η can be pinned down by matching the steady state labor force participation rate, which is given by $\overline{L} = \delta/(\eta + \delta)$.

Some parameters cannot be identified by matching a single empirical moment. While all parameters affect equilibrium moments to some degree, certain ones have especially strong influences on specific outcomes. Clarifying these relationships improves our understanding of the model's identification strategy.

To calibrate the discount rate, we target the 10-year real return on the FTSE 100 index, including dividend payments and deflated using the Consumer Price Index. In the model, the return on all asset classes, including equities, is represented by the risk-free rate r. While the actual return on the FTSE 100 is uncertain and the model abstracts from investment risk, we adopt a return of 5.5% as a reasonable benchmark. The economic intuition is straightforward: a higher discount rate implies that agents value future consumption less, requiring a higher return to defer current consumption. Hence, the discount rate directly influences the equilibrium return on capital.

The pension replacement rate is calculated as the ratio of the basic weekly state pension (£119.30) to median gross weekly earnings (£539), based on data from the April 2016 release of the Annual Survey of Hours and Earnings. In the model, all employed workers earn an identical gross wage w, so the model analogue to the replacement rate is the ratio w_o/w, where w_o denotes the pension benefit. The absolute value of w_o is not directly consequential; what matters is the relative magnitude. Under the chosen calibration, the state pension corresponds to approximately 4% of the productivity of a newly established firm, reflecting its modest size relative to labor income.

The output elasticity of labor in the production function, denoted by β, governs the share of output attributable to labor. In many models, β corresponds directly to the observed labor share and can be externally calibrated. In the present framework, however, β refers specifically to labor used in the production of intermediate goods. Since the model features monopolistic competition and firm profits, and because labor is also employed in activities such as firm entry, technology upgrading, and export participation, β must be interpreted as the *production-specific* labor share after profits are netted out— distinct from the aggregate labor share.

In equilibrium, average firm size is given by the ratio of the mass of firms Ω to the mass of employed workers $\overline{L}$. The equilibrium number of firms is determined by the balance between exogenous firm exits (at rate δ_f) and new firm entry. Lower entry costs encourage greater firm entry, increasing the mass of firms. Entry costs are given by $\chi(1 + \epsilon)w$, where ϵ is a parameter specific to entry. The model also determines the distribution of firm sizes, shaped not only by firm-level labor inputs, $\ell_d(z)$ and $\ell_x(z)$, but also by the productivity distribution $f(z)$, characterized in Lemma 8. The deterministic drift μ and volatility σ of the productivity process directly influence the shape of this distribution. Moreover, the cost of upgrading productivity, χw, influences the lower bound of productivity, z_l, below which firms choose not to operate. This lower bound is essential for accurately capturing the left tail of the firm size distribution.

The final two moments relate to the size of the UK export market: specifically, the share of firms that export and the value of exports as a share of GDP. The share of exporting firms corresponds to the survival function of the firm productivity distribution, evaluated at the export threshold, i.e., $1 - F(\hat{z})$. Since the productivity distribution is already pinned down by other calibrated moments, the threshold $\hat{z}$ — and thus the share of exporting firms — can be targeted by calibrating the fixed cost of exporting, denoted by ϕ_x. Once the number of exporters is determined, the intensity with which they export governs the overall export volume. This intensity depends on both the productivity distribution and the efficiency of exporting, which is influenced by the iceberg trade cost d. A calibrated value of just over three for d implies that three units must be shipped to deliver one unit abroad.

3.2 Results

Factor Prices

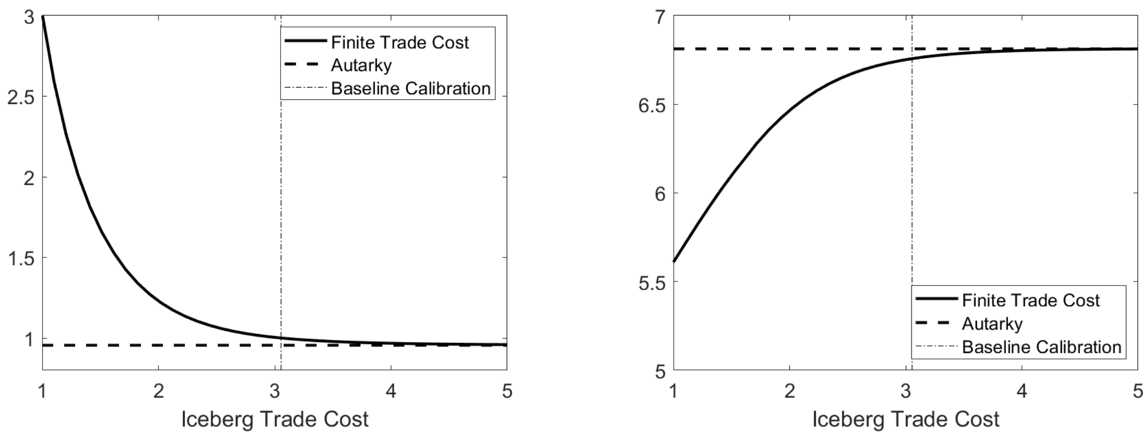
Numerous equilibrium objects are defined through the equilibrium conditions for workers (Definition 1) and firms (Definition 2), as well as through the associated product, labor, and asset markets. From the perspective of an individual worker, however, only two variables are relevant: the net return to labor, given by the after-tax wage $w(1 - \tau)$, and the return on assets, $\tilde{r}$. These are the only endogenous variables that enter the worker's value functions, as shown in equations (5) and (6). To study how these returns respond to changes in trade openness, we vary the *iceberg* trade cost parameter, $d \in [1, \infty)$, and illustrate the resulting effects in Figure 4.

Trade costs range from $d = 1$, where the marginal cost of serving a foreign market equals that of the domestic market, to $d \to \infty$, which corresponds to autarky, as supplying foreign markets becomes prohibitively expensive. As discussed in the context of the role of heterogeneity, a reduction in trade costs can be interpreted as a specific type of positive productivity shock. Given that labor supply is fixed, it is therefore unsurprising that net wages rise as trade costs fall. In contrast, the return on assets declines.

As discussed in Section 2.4, the effect of trade costs on the return to assets depends on the relative shifts in the supply and demand for capital— both of which are shaped by worker and firm heterogeneity. Following an expansion in trade openness, higher wages increase

Figure 4: Worker Returns by Trade Cost

Wages $w(1-\tau)$ by d | Asset returns $\tilde{r} = r + \delta$ by d



Notes: The model is recalibrated for varying values of the *iceberg* trade cost parameter $d \in [1, 5]$. For each value of d, a steady state equilibrium is computed. The figure reports the corresponding equilibrium returns to labor and capital as functions of the trade cost. The vertical dashed line indicates the calibrated value of d, while the horizontal line denotes the return values under autarky, corresponding to the limiting case $d \to \infty$.

the earnings gap between employment and retirement, thereby raising an employed workers incentive to save. This generates a disproportionate increase in the supply of assets, which in turn puts downward pressure on the return to capital as trade costs fall.

On the demand side, the effect of trade liberalization on asset returns is ambiguous and is formally characterized by Proposition 3. A key determinant is the labor allocated to non-productive tasks— namely, firm entry, technology adoption, and export costs. Whether these activities expand or contract depends on the underlying distribution of firm productivity. Different assumptions about this distribution yield different conclusions.[8] Recall that in this framework the distribution of firm productivity is itself an endogenous object and defined by Lemma 8. Figure D.1 in the Appendix illustrates how labor allocated to non-productive tasks evolves with trade costs. As in models with an exogenously fixed Pareto distribution, entry declines with increased trade openness. However, the rise in exporting activity more than offsets this decline, resulting in an overall increase in non-productive labor. This further amplifies the downward pressure on asset returns as trade becomes more open.

Figure 4 displays a smooth evolution of net wages and asset returns as trade costs vary. However, this apparent smoothness masks substantial underlying shifts in the structure of the economy as trade costs decline from the baseline calibrated value. At the baseline

[8] The canonical model often assumes an unbounded Pareto distribution, as in Chaney (2008). Alternative specifications include a log-normal distribution (Head et al., 2014), a bounded Pareto (Melitz and Redding, 2015), and a log-normal-Pareto mixture (Nigai, 2017).

calibration, the return on assets exceeds a worker's effective discount rate, $\tilde{r} > \rho + \delta$. This corresponds to the high-asset environment discussed in Section 2.1, where both employed and retired individuals hold positive assets $a \in [0, \infty)$, and the asset distribution features a fat right tail, as illustrated in the right-hand panel of Figure 2. As trade costs fall, so does the return on assets. When the iceberg cost d drops just below 2— implying that exporting requires producing twice the volume sold— the return on assets falls below the effective discount rate. In this regime, employed workers target a specific buffer stock level of savings, while retired individuals dissave and may eventually become indebted. In equilibrium, asset holdings are thus bounded on $a \in (w_o/\tilde{r}, \overline{a})$, where the supremum $\overline{a}$ is defined in Lemma 3.

Winners and Losers

We have seen in the previous section that, under our calibration, a reduction in trade costs d raises the wage rate w and lowers the return on assets $\tilde{r}$. The relative movements in these factor prices determine the quantitative magnitude of the resulting welfare changes. However, since our calibration implies that $\frac{\partial \tilde{r}}{\partial \tilde{d}} > 0$, Proposition 4 directly establishes the qualitative welfare implications for retirees, regardless of the exact magnitudes. In a high-return environment, which corresponds to the region of the parameter space where our baseline calibration falls, greater trade openness reduces welfare for all individuals outside the labor force. In contrast, when trade costs d are sufficiently low, the economy transitions into a low-return environment, where welfare effects depend on individual asset holdings. In this case, retirees who are in debt or have very low asset holdings benefit from further trade liberalization because the decline in the return on assets $\tilde{r}$ alleviates their borrowing costs. Conversely, retirees with higher asset holdings experience welfare losses as their asset income declines.

Proposition 4 *Suppose that a reduction in trade costs lowers the return on assets, i.e.,*

$$\frac{\partial \tilde{r}}{\partial d} > 0.$$

Then the marginal welfare effect for the inactive (retired) population is given by:

- *High-return regime: if $\tilde{r} \geq \rho + \delta$, then*

$$\frac{\partial \tilde{v}(a)}{\partial d} > 0 \quad \textit{for all } a \in [0, \infty).$$

- *Low-return regime: if* $\tilde{r} < \rho + \delta$*, then*

$$\frac{\partial \tilde{v}(a)}{\partial d} = \begin{cases} < 0 & \text{if } a \in \left(-\frac{w_o}{\tilde{r}}, a^\star\right) \\ \geq 0 & \text{if } a \geq a^\star, \end{cases}$$

where the threshold

$$a^\star = -\frac{w_o}{\tilde{r}} \cdot \frac{\tilde{r} - \rho - \delta}{\gamma \tilde{r}} > 0.$$

Given that the baseline calibration features a rate of return exceeding the effective discount rate ($\tilde{r} > \rho + \delta$) and non-negative asset holdings for all individuals ($a \geq 0$), an increase in the return on assets yields universal gains. Moreover, the magnitude of these gains is increasing in the level of an individual's asset holdings. As in Proposition 4, to assess workers' preferences over trade openness, we examine how their welfare responds to changes in trade costs by computing the derivative of the value functions for both employed (equation 5) and retired individuals (equation 6) with respect to the trade cost d.

$$\frac{\partial v(a)}{\partial d} = \underbrace{\frac{\partial v(a)}{\partial r}\frac{\partial r}{\partial d}}_{+\ \&\ \text{small}} + \underbrace{\frac{\partial v(a)}{\partial w(1-\tau)}\frac{\partial w(1-\tau)}{\partial d}}_{-\ \&\ \text{large}} \qquad \text{(employed)}$$

$$\frac{\partial \tilde{v}(a)}{\partial d} = \underbrace{\frac{\partial \tilde{v}(a)}{\partial r}\frac{\partial r}{\partial d}}_{+\ \&\ \text{small}} \qquad \text{(retired)}$$

The model is solved in steady state. A change in the trade cost d would, in reality, trigger a transition path for both wages and the return on assets, and there is no reason to expect these variables to adjust instantaneously to their new steady state levels. By taking a derivative with respect to d, we are evaluating a marginal change in this primitive. As long as we interpret this exercise as capturing small perturbations around the baseline, it provides a meaningful approximation of workers' preferences over trade openness.

With this caveat in mind, we highlight two key points. First, a qualitative observation: employed individuals derive income from both labor and capital, and thus care about both the after-tax wage $w(1-\tau)$ and the return on capital r. In contrast, retirees rely solely on capital income and are therefore only affected by changes in r. Second, a quantitative observation: because changes in labor income are significantly larger than changes in capital returns (see Figure 4), an increase in trade costs d leads to a much larger welfare loss through lower wages than any offsetting gain through higher returns on capital.

The marginal values are presented as elasticities in consumption-equivalent terms in Figure 5. Across labor market statuses, individuals with greater asset holdings are generally less supportive of trade openness, as lower trade costs reduce the return on capital. For retirees, the marginal value remains strictly positive, indicating a preference for higher trade costs— that is, less trade openness. This reflects the fact that their welfare depends solely on asset returns. In contrast, employed individuals receive income from both labor and capital, and for them, the negative wage effect dominates the positive effect on asset returns. As a result, with the exception of those in the far right tail of the asset distribution, all employed individuals prefer increased trade openness.

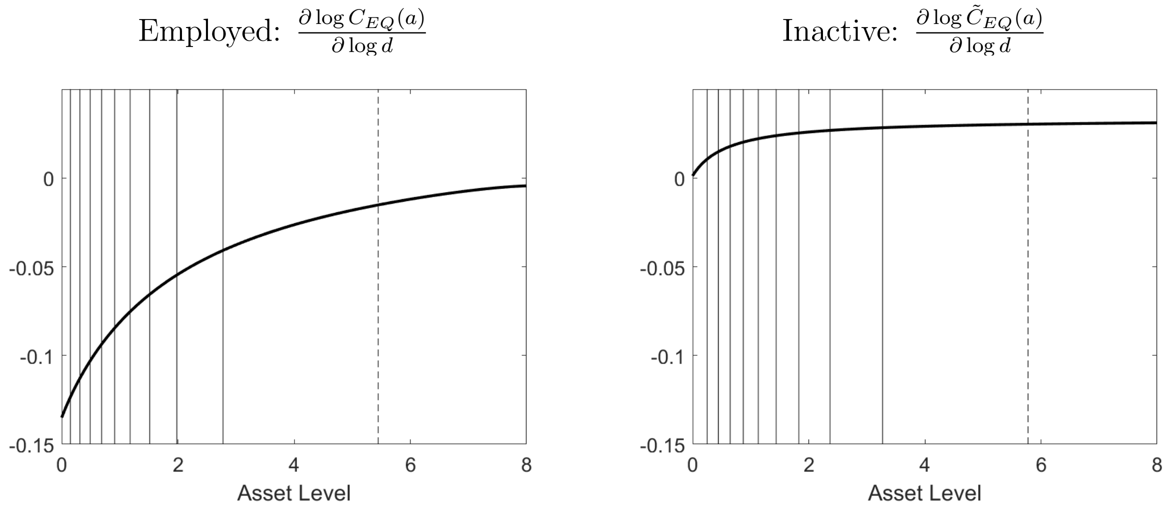


Figure 5: Marginal Returns to Trade Costs: The figure shows the log derivative of the worker value functions (5) and (6), expressed in terms of consumption-equivalent units, with respect to the iceberg cost rate d. Solid vertical lines indicate asset deciles for employed (left panel) and inactive (right panel) individuals, while the dashed line marks the 99$^{\text{th}}$ percentile of the respective asset distribution.

Validation

The heterogeneous agent trade models discussed in the introduction— Waugh (2023) and Carroll and Hur (2020)— both find that trade is pro-poor. Waugh (2023) develops a model with heterogeneous price elasticities and shows that trade benefits the poor because trade liberalization lowers prices, and lower-income households are more price-sensitive in their consumption. The mechanism in Carroll and Hur (2020) is similar in spirit to this paper insofar as trade leads to capital deepening. In Carroll and Hur's setup, capital is purchased with tradable goods, which become relatively cheaper with trade which reduces the return on assets.[9] In contrast, in this model, capital deepening occurs due to increased incentives

[9] Additionally, in Carroll and Hur (2020), tradable goods represent a larger share of the poor's consumption basket. A reduction in their price thus provides a further advantage.

to save for retirement, coupled with a shift of labor toward non-productive tasks. In this framework, it is not straightforward to characterize trade as either pro- or anti-poor. Instead, what determines an individual's support for trade openness is not the level of income per se, but rather the *source* of that income. The larger the share of labor income as a proportion of total income, the stronger the preference for trade openness.

To validate the model's predictions with empirical data, we examine how British workers' preferences regarding EU membership vary according to their source of income. While trade openness was certainly not the only issue at stake in the 2016 referendum, a key consequence of the vote was the United Kingdom's withdrawal from its largest trading partnership.[10] Our analysis uses data from the UK Household Longitudinal Study (UKHLS) spanning 2016 to 2021. The UKHLS is a nationally representative panel dataset that, following the referendum, includes the question: *"Should the United Kingdom remain a member of the European Union or leave the European Union?"* Importantly, the survey also collects detailed information on respondents' income sources. The sample comprises approximately 45,000 individuals aged 16 to 102. For each respondent, we compute their Brexit preference—whether they support leaving the EU—and the share of their total income derived from labor. Full details on the sample construction and computation are provided in Appendix D.3.

Figure 6 plots the relationship between the share of income derived from labor and an individual's willingness to leave the European Union. We observe a monotonic relationship consistent with the model's predictions: support for Brexit— interpreted here as a preference for reduced trade openness— is strongest among those who receive a larger share of their income from non-labor sources, such as asset returns and government transfers.

Aggregate Welfare

The final quantitative result we examine is aggregate welfare, the typical metric in a representative agent trade model. We have compared welfare changes across groups and found that employed individuals experience substantial gains, while retirees incur modest losses when trade costs fall. Here, we abstract from these distributional considerations and focus on the overall effect. The model predicts unambiguous aggregate welfare gains from lower trade costs, with the magnitude of these gains comparable to those predicted by standard trade models without worker heterogeneity. Under our calibration, moving the economy

[10] While the exact nature of the UK's future trade relationship was unclear at the time respondents answered the question, concurrent research during the sample period Dhingra et al. (2017) finds that, even under the most optimistic scenario— i.e., the smallest increase in trade costs—UK imports are projected to fall by 8% and exports by 9% in the long run.

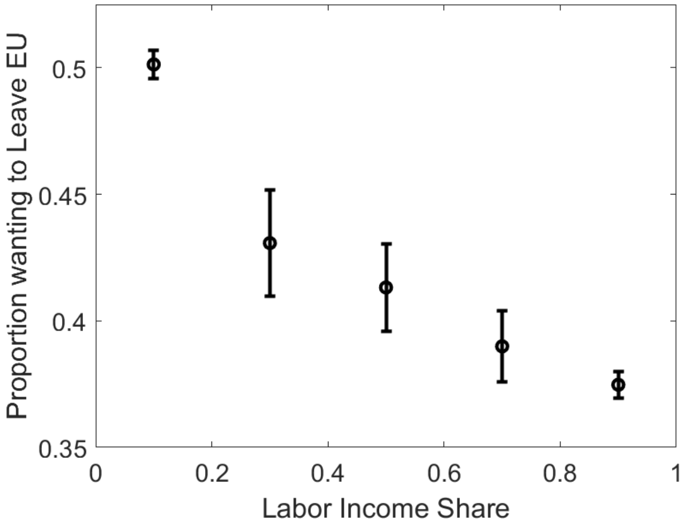


Figure 6: UK Preference for EU Membership by Labor Income Share: Labor income share is computed as the share of total income reported from 2016-2021 earned through paid or self-employment. The five bins represent labor income constituting 0-20%, 20%-40%, 40-60%, 60%-80% and 80%-100% of total income. The points represent the mean, and the whiskers the 95% confidence intervals.

from our baseline calibration to autarky— as $d \to \infty$— would result in an average welfare loss of approximately 3%, measured in consumption-equivalent terms.[11] Figure 7 plots these aggregate gains, computed from steady state equilibria as trade costs d vary, holding all other parameters fixed at their calibrated values shown in Table 1. While, in theory, the efficiency-maximizing policy would be to reduce trade costs multilaterally, in practice such reductions may be difficult to achieve due to geopolitical, logistical, or institutional frictions. Taken together, our results suggest that while trade liberalization delivers aggregate gains, the presence of both winners and losers implies that complementary redistribution policies may be needed to ensure these gains are broadly shared and politically sustainable.

[11] Using the sufficient statistic approach developed in Arkolakis et al. (2012), Arkolakis et al. (2014) compute aggregate welfare gains from trade in models with homogeneous workers and only labor as an input to production across several countries. For the UK, they find a welfare gain of 3.2% from moving from autarky to their baseline calibration.

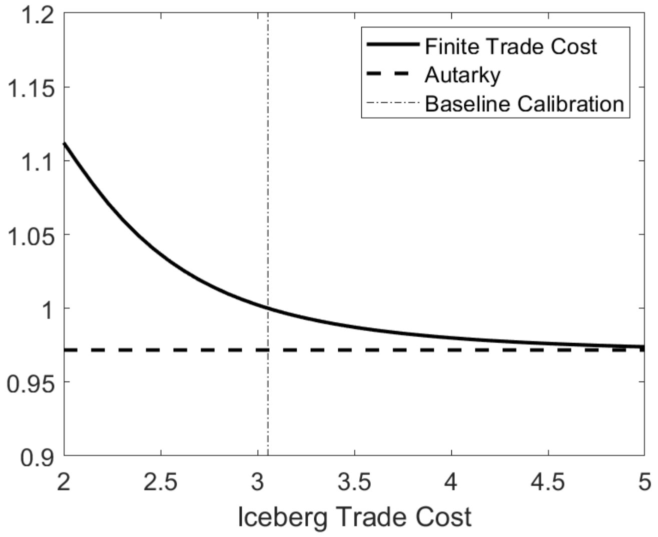


Figure 7: Aggregate Welfare Gains from Trade: The model is recomputed varying the *iceberg* trade cost $d \in [1, 5]$. Aggregate welfare is computed in consumption certainty equivalence. The vertical dashed line indicates the calibrated value of d, while the horizontal line denotes the return values under autarky, corresponding to the limiting case $d \to \infty$.

4 Conclusion

This paper integrates household heterogeneity— arising from incomplete markets and life cycle dynamics— into a dynamic, multi-country trade model. We show that greater trade openness reduces export costs, lowers domestic prices, and raises exporter profits, leading to positive aggregate welfare gains. However, these gains are not distributed uniformly: trade liberalization is not Pareto improving. In our model, asset-rich but economically inactive households prefer higher trade barriers, as they are disproportionately disadvantaged by greater openness. This finding is consistent with empirical evidence from the Brexit referendum, which suggests that British voters acted in accordance with their economic self-interest. Our results underscore the need for policymakers to account for the distributional and intergenerational consequences of trade policy, potentially mitigating adverse effects through targeted transfers.

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A Proofs and Derivations: worker side

A.1 Deriving Hamilton-Jacobi-Bellman equations

The value function of an employed worker is denoted as $v(a)$, while that of an inactive worker is represented by $\tilde{v}(a)$. The objective function (2) can be expressed as a discrete-time Bellman equation with *small* time increments Δ. Ultimately, as $\Delta \to 0$, the model converges to the continuous time environment presented in the model. However, for the present formulation, Δ is assumed to be sufficiently small, ensuring that the discount factor and shock probabilities satisfy:

$$e^{-\rho\Delta} \approx 1 - \rho\Delta \quad \text{and,} \quad (\delta + \eta)\Delta < 1.$$

Under these conditions, the discrete-time Bellman equation for an employed worker takes the form below. The constraint on the level of consumption ensures that consumption is always positive and the borrowing constraint is not violated.

$$v(a_t) = \max_{0 \le c_t \le a_t + \frac{w}{\tilde{r}}} \left\langle \Delta u(c_t) + (1 - \rho\Delta)\Big((1 - (\delta + \eta)\Delta)\, v(a_{t+\Delta}) + \delta\Delta \times 0 + \eta\Delta\tilde{v}(a_{t+\Delta}) + \mathcal{O}(\Delta) \Big) \right\rangle$$

$$\text{where, } a_{t+\Delta} = a_t + \tilde{r}\Delta a_t + \Delta w(1 - \tau) - \Delta c_t.$$

Examining the terms sequentially in the value function, the first term represents the utility flow from consumption and the disutility of work, both realized in the current period. The remaining terms are discounted at rate $(1 - \rho\Delta)$. The four possible outcomes for the worker are as follows. First, no exogenous shock occurs, and the worker remains employed with assets evolving according to the difference equation. Second, with probability $\Delta\delta$, the worker exits the labor market due to death, receiving zero utility in perpetuity. Third, the worker retires and transitions to the inactive state while retaining the same level of assets. Finally, the term $\mathcal{O}(\Delta)$ accounts for misspecification inherent in the discrete-time approximation, particularly the probability of multiple events occurring simultaneously.

Next, we subtract $(1 - \rho\Delta)v(a_t)$ from both sides of the value function, divide by Δ, and manipulate the expression by factoring in the option value of remaining employed using asset changes. This yields:

$$(\rho+\delta)v(a_t) = \max_{0\le c_t\le a_t+\frac{w}{\tilde r}} \left\langle u(c_t) + (1-\rho\Delta)\Bigg((1-(\delta+\eta)\Delta)\, \frac{v(a_{t+\Delta})-v(a_t)}{a_{t+\Delta}-a_t}.\frac{a_{t+\Delta}-a_t}{\Delta} + \eta\left(\tilde v(a_{t+\Delta})-v(a_t)\right) + \frac{\mathcal{O}(\Delta)}{\Delta}\Bigg)\right\rangle$$

Taking the limit as $\Delta \to 0$ leads to the continuous-time Hamilton-Jacobi-Bellman (HJB) equation. The corresponding value function for an inactive worker can be derived analogously in a more straightforward manner:

$$(\rho+\delta)v(a) = \max_{0\le c\le a+\frac{w}{\tilde r}} \langle u(c) + v_a(a)(\tilde r a + w(1-\tau) - c) + \eta(\tilde v(a) - v(a))\rangle \tag{5}$$

$$(\rho+\delta)\tilde v(a) = \max_{0\le \tilde c\le a+\frac{w_o}{\tilde r}} \langle u(\tilde c) + \tilde v_a(a)(\tilde r a + w_o - \tilde c)\rangle \tag{6}$$

A.2 Proof of Lemma 1

We begin with the value function of the retired

$$(\rho+\delta)\tilde v(a) = \max_{0\le \tilde c\le a+\frac{w_o}{\tilde r}} \langle u(\tilde c) + \tilde v_a(a)(\tilde r a + w_o - \tilde c)\rangle\,. \tag{6}$$

The first order condition with respect to consumption implies that $u'(\tilde c) = \tilde v_a(a)$, and given the CRRA utility function defined in (1), the optimality condition is given by:

$$\tilde c = \tilde v_a^{-1/\gamma}. \tag{37}$$

Substituting the optimal consumption condition (37), into the value function (6) results in a differential equation in assets.

$$(\rho+\delta)\tilde v(a) = \frac{\gamma}{1-\gamma}\tilde v_a(a)^{1-\frac{1}{\gamma}} + \tilde v_a(a)(\tilde r a + w_o) \tag{38}$$

The ODE is solved using the method of undetermined coefficients. We hypothesize that the solution takes the form:

$$\begin{aligned} \tilde{v}(a) &= \frac{\mathcal{B}}{1-\gamma}\left(a+\frac{w_o}{\tilde{r}}\right)^{1-\gamma} \\ \implies \tilde{v}_a(a) &= \mathcal{B}\left(a+\frac{w_o}{\tilde{r}}\right)^{-\gamma} \\ \implies \tilde{v}_a(a)^{1-\frac{1}{\gamma}} &= \mathcal{B}^{1-\frac{1}{\gamma}}\left(a+\frac{w_o}{\tilde{r}}\right)^{1-\gamma} \end{aligned}$$

Substituting the three expressions above into the ODE (38) yields:

$$(\rho+\delta)\frac{\mathcal{B}}{1-\gamma}\left(a+\frac{w_o}{\tilde{r}}\right)^{1-\gamma} = \frac{\gamma}{1-\gamma}\mathcal{B}^{1-\frac{1}{\gamma}}\left(a+\frac{w_o}{\tilde{r}}\right)^{1-\gamma} + \mathcal{B}\tilde{r}\left(a+\frac{w_o}{\tilde{r}}\right)^{1-\gamma}.$$

Solving for the undetermined coefficient $\mathcal{B}$

$$\begin{aligned} (\rho+\delta)\frac{\mathcal{B}}{1-\gamma} &= \frac{\gamma}{1-\gamma}\mathcal{B}^{1-\frac{1}{\gamma}} + \mathcal{B}\tilde{r} \\ \frac{\rho+\delta}{1-\gamma} - \tilde{r} &= \frac{\gamma}{1-\gamma}\mathcal{B}^{-\frac{1}{\gamma}} \\ \mathcal{B} &= \left(\tilde{r} - \frac{\tilde{r}-\rho-\delta}{\gamma}\right)^{-\gamma} \end{aligned}$$

Substituting back into our initial guess gives an expression for the value function.

$$\tilde{v}(a) = \frac{1}{1-\gamma}\left(\tilde{r} - \frac{\tilde{r}-\rho-\delta}{\gamma}\right)^{-\gamma}\left(a+\frac{w_o}{\tilde{r}}\right)^{1-\gamma} \tag{39}$$

Using the optimality condition (37) the optimal level of consumption for a retired worker with a assets is given by

$$\tilde{c}(a) = \left(\tilde{r} - \frac{\tilde{r}-\rho-\delta}{\gamma}\right)\left(a+\frac{w_o}{\tilde{r}}\right). \tag{7}$$

Finally, using the wealth accumulation equation defined in (3) one can compute savings as

$$\tilde{s}(a) := \dot{a} = \tilde{r}a + w_o - c = \frac{\tilde{r}-\rho-\delta}{\gamma}\left(a+\frac{w_o}{\tilde{r}}\right). \tag{8}$$

A.3 Proof of Lemma 2

Starting from the value function of the inactive, where $\tilde{c}$ is the optimal level of consumption defined in Lemma 1.

$$(\rho+\delta)\tilde{v}(a)=u(\tilde{c})+\tilde{v}_a(a)(\tilde{r}a+w_o-\tilde{c})$$

Differentiating with respect to the asset level a.

$$(\rho+\delta)\tilde{v}_a(a)=u'(\tilde{c})\frac{\partial\tilde{c}}{\partial a}+\tilde{v}_{aa}(a)\underbrace{(\tilde{r}a+w_o-\tilde{c})}_{\dot{a}}+\tilde{v}_a(a)\tilde{r}-\tilde{v}_a(a)\frac{\partial\tilde{c}}{\partial a}$$

Under optimality $u'(\tilde{c})=\tilde{v}_a(a)$, and hence

$$\tilde{v}_{aa}(a)\dot{a}=-(\tilde{r}-\rho-\delta)\tilde{v}_a(a). \tag{40}$$

Using the optimality condition again, one can write the left hand side of the above in terms of optimal consumption.

$$\tilde{v}_{aa}\dot{a}=\frac{\partial\tilde{v}_a(a)}{\partial t}=\frac{\partial u'(\tilde{c})}{\partial t}=u''(\tilde{c})\dot{\tilde{c}}=-\gamma\tilde{c}^{-\gamma-1}\dot{\tilde{c}}\quad\text{and } v_a(a)=u'(\tilde{c})=\tilde{c}^{-\gamma}$$

Plugging these expressions back into (40) yields the Euler equation for the inactive.

$$\frac{\dot{\tilde{c}}}{\tilde{c}}=\frac{1}{\gamma}\left(\tilde{r}-\rho-\delta\right) \tag{10}$$

Following the same approach for the value function of the employed (5), differentiating with respect to assets.

$$(\rho+\delta+\eta)v_a(a)=u'(c)\frac{\partial c}{\partial a}+v_{aa}(a)\dot{a}+v_a(a)\tilde{r}-v_a(a)\frac{\partial c}{\partial a}+\eta\tilde{v}_a(a)$$

Substituting in the optimality conditions $u'(c)=v_a(a)$ and $u'(\tilde{c})=\tilde{v}_a(a)$ yields:

$$v_{aa}(a)\dot{a}=-u'(c)(\tilde{r}-\rho-\delta-\eta)-\eta u'(\tilde{c})$$

where,

$$v_{aa}\dot{a}=\frac{\partial v_a(a)}{\partial t}=\frac{\partial u'(c)}{\partial t}=u''(c)\dot{c},$$

and hence

$$u''(c)\dot{c} = -u'(c)(\tilde{r} - \rho - \delta - \eta) - \eta u'(\tilde{c}).$$

Finally, given the utility function specified in (1) we can write the Euler function for the employed as

$$\frac{\dot{c}}{c} = \frac{1}{\gamma}\left(\tilde{r} - \rho - \delta - \eta + \eta\frac{c^\gamma}{\tilde{c}^\gamma}\right). \tag{9}$$

A.4 Proof of Lemma 3

Combining the wealth accumulation equation from equation (3) and the Euler equation for employed workers from equation (9), we have:

$$\dot{a} = ra + w(1-\tau) - c$$
$$\dot{c} = \frac{c}{\gamma}\left(\tilde{r} - \rho - \delta - \eta + \eta\frac{c^\gamma}{\tilde{c}^\gamma}\right)$$

We seek a stable saddle-path equilibrium in which both consumption and assets are constant over time, i.e., $\dot{c} = \dot{a} = 0$. This condition holds at the steady state values $(\overline{c}, \overline{a})$, given by:

$$\overline{c} = \tilde{r}\overline{a} + w(1-\tau) \qquad (\dot{a} = 0)$$
$$\overline{c} = C_1\overline{a} + C_1\frac{w_o}{\tilde{r}} \quad \text{, where } C_1 = \left(1 - \frac{\tilde{r} - \rho - \delta}{\eta}\right)^{1/\gamma}\left(\tilde{r} - \frac{\tilde{r} - \rho - \delta}{\gamma}\right). \qquad (\dot{c} = 0)$$

As shown in Figure 1, a stable saddle-path equilibrium requires two conditions. First, the $\dot{a} = 0$ curve must lie above the $\dot{c} = 0$ curve at $a = 0$. This ensures that consumption and asset holdings initially grow. Second, since both relationships are affine, stability requires the $\dot{c} = 0$ locus to be flatter than the $\dot{a} = 0$ locus. These two conditions jointly ensure a unique, stable intersection point.

The first condition holds if:

$$w(1-\tau) > \frac{C_1}{\tilde{r}}w_o$$

The second condition holds if:

$$C_1 > \tilde{r}$$

Since we restrict attention to equilibria in which employment income exceeds retirement income— i.e., $w(1-\tau) > w_o$— a sufficient condition for both requirements is:

$$C_1 > \tilde{r}$$

This in turn is satisfied when:

$$\tilde{r} < \rho + \delta$$

Hence, when asset returns are sufficiently low ($\tilde{r} < \rho+\delta$), the stable steady state is determined by the intersection of the $\dot{a} = 0$ and $\dot{c} = 0$ curves. The corresponding asymptotic values are:

$$c \sim \frac{C_1}{C_1 - \tilde{r}} \left(w(1-\tau) - w_o \right) \tag{41}$$

$$a \sim \frac{w(1-\tau) - C_1 \frac{w_o}{\tilde{r}}}{C_1 - \tilde{r}} \tag{42}$$

In contrast, when the return on assets is high ($\tilde{r} \geq \rho + \delta$), both consumption and asset accumulation grow without bound. To characterize this behavior, we analyze the asymptotic form of the value function. The employed worker's Bellman equation is given by:

$$(\rho + \delta) v(a) = u(c) + v_a(a) \left(\tilde{r}a + w(1-\tau) - c \right) + \eta \left(\tilde{v}(a) - v(a) \right)$$

Under the optimality condition, $c = v_a(a)^{-1/\gamma}$, we can write:

$$u(c) - v_a(a)c = \frac{\gamma}{1-\gamma} v_a(a)^{1-\frac{1}{\gamma}}$$

Substituting this into the Bellman equation and incorporating the expression for retirement value $\tilde{v}(a)$ from Equation (39), we obtain the differential equation:

$$\begin{aligned}(\rho + \delta + \eta) v(a) = &\frac{\gamma}{1-\gamma} v_a(a)^{1-\frac{1}{\gamma}} + v_a(a) \left(\tilde{r}a + w(1-\tau) \right) \\ &+ \frac{\eta}{1-\gamma} \left(\tilde{r} - \frac{\tilde{r} - \rho - \delta}{\gamma} \right)^{-\gamma} \left(a + \frac{w_o}{\tilde{r}} \right)^{1-\gamma}\end{aligned} \tag{43}$$

We conjecture a solution of the form:

$$v(a) \sim \frac{A}{1-\gamma} (a + \phi)^{1-\gamma} \quad \text{as } a \to \infty$$

Given $\gamma > 1$, if $A > 0$ then the value function is negative over the domain $a \in \left(-\frac{w(1-\tau)}{\tilde{r}}, \infty \right)$. Moreover, as $a \to \infty$, we have $v(a) \to 0$, and the differential equation (43) holds in the limit. Since the equation is satisfied asymptotically, our aim is to choose parameter values for A

and ϕ that minimize approximation error for large a. We begin by matching the highest order 'leading' terms to select A.

Computing each term asymptotically, the left-hand side of (43) is:

$$(\rho+\delta+\eta)v(a) \sim \frac{A(\rho+\delta+\eta)}{1-\gamma}(a+\phi)^{1-\gamma}$$

The right-hand side, term by term, satisfies

$$\text{(i)} \quad \frac{\gamma}{1-\gamma}v_a^{1-1/\gamma} = \frac{\gamma}{1-\gamma}A^{1-1/\gamma}(a+\phi)^{1-\gamma}$$

$$\text{(ii)} \quad v_a(a)(\tilde{r}a+w(1-\tau)) = A\tilde{r}\left(a+\frac{w(1-\tau)}{\tilde{r}}\right)(a+\phi)^{-\gamma}$$

$$\text{(iii)} \quad \frac{\eta}{1-\gamma}\cdot\left(\tilde{r}-\frac{\tilde{r}-\rho-\delta}{\gamma}\right)^{-\gamma}\left(a+\frac{w_o}{\tilde{r}}\right)^{1-\gamma}$$

Matching the leading-order terms gives

$$\frac{A(\rho+\delta+\eta)}{1-\gamma} = \frac{\gamma}{1-\gamma}A^{1-1/\gamma} + A\tilde{r} + \frac{\eta}{1-\gamma}\left(\tilde{r}-\frac{\tilde{r}-\rho-\delta}{\gamma}\right)^{-\gamma}.$$

Multiplying both sides by $(1-\gamma)/A$ and rearranging yields

$$\rho+\delta+\eta-(1-\gamma)\tilde{r} = \gamma A^{-1/\gamma} + \eta\frac{1}{A}\left(\tilde{r}-\frac{\tilde{r}-\rho-\delta}{\gamma}\right)^{-\gamma}. \tag{44}$$

Inspection of (44) suggests the candidate solution

$$A = \left(\tilde{r}-\frac{\tilde{r}-\rho-\delta}{\gamma}\right)^{-\gamma}, \tag{45}$$

since this choice simplifies the final term. Substituting (45) into (44) verifies the equality. Hence, our candidate solution is

$$v(a) \sim \frac{1}{1-\gamma}\left(\tilde{r}-\frac{\tilde{r}-\rho-\delta}{\gamma}\right)^{-\gamma}(a+\phi)^{1-\gamma} \quad \text{as } a\to\infty. \tag{46}$$

To determine ϕ, recall that (46) only holds in the limit $a \to \infty$. For finite a, we define the approximation error as $\epsilon(\phi)$. The loss function is constructed by subtracting the right-hand side of (43) from the left-hand side after substituting the asymptotic form (46). Since

$\epsilon(\phi) = 0$ in the limit, the ODE is solved exactly as $a \to \infty$. For finite a,

$$\begin{aligned}\epsilon(\phi) &= A\,(a+\phi)^{1-\gamma}\left[-\frac{\rho+\delta+\eta}{1-\gamma}+\frac{\gamma}{1-\gamma}A^{-1/\gamma}+\frac{\tilde{r}a+w(1-\tau)}{a+\phi}+\frac{\eta}{1-\gamma}\left(\frac{a+\frac{w_o}{\tilde{r}}}{a+\phi}\right)^{1-\gamma}\right]\\ &= A\,(a+\phi)^{1-\gamma}\left[-\frac{\eta}{1-\gamma}-\tilde{r}+\frac{\tilde{r}a+w(1-\tau)}{a+\phi}+\frac{\eta}{1-\gamma}\left(\frac{a+\frac{w_o}{\tilde{r}}}{a+\phi}\right)^{1-\gamma}\right].\end{aligned}$$

Differentiating with respect to ϕ gives

$$\epsilon'(\phi) = -A\,(a+\phi)^{-\gamma}\left[\eta+(1-\gamma)\tilde{r}+\gamma\frac{\tilde{r}a+w(1-\tau)}{a+\phi}\right] < 0,$$

for $\phi \in \left(\frac{w_o}{\tilde{r}}, \frac{w(1-\tau)}{\tilde{r}}\right)$.

Next, we establish that $\epsilon(\phi)$ changes sign within this interval:

$$\epsilon\left(\tfrac{w_o}{\tilde{r}}\right) = A\,\left(a+\tfrac{w_o}{\tilde{r}}\right)^{1-\gamma}\left[-\tilde{r}+\tilde{r}\left(\frac{a+\frac{w(1-\tau)}{\tilde{r}}}{a+\frac{w_o}{\tilde{r}}}\right)\right] > 0,$$

$$\epsilon\left(\tfrac{w(1-\tau)}{\tilde{r}}\right) = A\,\left(a+\tfrac{w(1-\tau)}{\tilde{r}}\right)^{1-\gamma}\left(\frac{1}{1-\gamma}\right)\left[-\eta+\eta\left(\frac{a+\frac{w_o}{\tilde{r}}}{a+\frac{w(1-\tau)}{\tilde{r}}}\right)^{1-\gamma}\right] < 0, \quad \gamma > 1.$$

Hence, to minimize the approximation error, we set

$$\phi \;=\; \alpha\,\frac{w_o}{\tilde{r}} \;+\; (1-\alpha)\,\frac{w(1-\tau)}{\tilde{r}}, \quad \alpha \in (0,1).$$

Finally, with A and ϕ specified, we can characterize the asymptotic behavior of consumption. Using the optimality condition $c(a) = v_a(a)^{-1/\gamma}$, we obtain

$$c(a) \;\sim\; \left(\tilde{r}-\frac{\tilde{r}-\rho-\delta}{\gamma}\right)\big(a+\phi\big), \quad \text{as } a \to \infty.$$

A.5 Proof of Lemma 4

The employed

We can compute the total mass of employed workers $\overline{L}$ as equalizing the flow in and out of employment.

$$\overline{L} = \frac{\delta}{\delta+\eta} \tag{47}$$

Differentiating (13) with respect to a and rearranging yields:

$$s'(a)g(a) + s(a)g'(a) = -(\delta+\eta)g(a)$$
$$\frac{g'(a)}{g(a)} = -\frac{\delta+\eta+s'(a)}{s(a)}.$$

Integrating and rearranging gives us the density of assets held by the employed.

$$\log(g(a)) = -\int_0^a \frac{\delta+\eta}{s(a')}da' - \log(s(a)) + \mathcal{C}_0$$
$$g(a) = \frac{\mathcal{C}_1}{s(a)} e^{-\int_0^a \frac{\delta+\eta}{s(a')}da'}$$

We know from the KFE that $g(0) = \frac{\delta}{s(0)}$, hence $\mathcal{C}_1 = \delta$, and

$$g(a) = \frac{\delta}{s(a)} e^{-\int_0^a \frac{\delta+\eta}{s(a')}da'} \quad \text{and,}$$
$$G(a) = \mathcal{C}_2 - \frac{\delta}{\delta+\eta} e^{-\int_0^a \frac{\delta+\eta}{s(a')}da'}.$$

From Lemma 3, if $\tilde{r} \geq \rho+\delta$, the terminal condition to pin down $\mathcal{C}_2$ is $G(\infty) = \overline{L} = \frac{\delta}{\delta+\eta}$. Instead if $\tilde{r} < \rho+\delta$, the terminal condition to pin down $\mathcal{C}_2$ is $G(\overline{a}) = \overline{L} = \frac{\delta}{\delta+\eta}$. Consider the high return environment,

$$\mathcal{C}_2 = \frac{\delta}{\delta+\eta}$$

so,

$$G(a) = \frac{\delta}{\delta+\eta}\left(1 - e^{-(\delta+\eta)\int_0^a \frac{1}{s(a')}da'}\right).$$

Consider the low return environment,

$$\mathcal{C}_2 = \frac{\delta}{\delta+\eta}\left(1 + e^{-\int_0^{\overline{a}} \frac{\delta+\eta}{s(a')}da'}\right)$$

so,

$$G(a) = \frac{\delta}{\delta+\eta}\left(1 + e^{-(\delta+\eta)\int_0^{\overline{a}} \frac{1}{s(a')}da'} - e^{-(\delta+\eta)\int_0^a \frac{1}{s(a')}da'}\right).$$

The retired

We know from Lemma 1 that the savings rate of the retired is linear in *effective wealth*.

$$\tilde{s}(a) = \underbrace{\left(\frac{\tilde{r}-\rho-\delta}{\gamma}\right)}_{\tilde{s}_0}\left(a+\frac{w_0}{\tilde{r}}\right)$$

Although the retired are only in debt in a low return environment, we consider the general steady state KFE that exhibits a discontinuity at $a = 0$, such that

$$\begin{aligned} 0 &= -\tilde{s}(a)\tilde{g}(a) - \delta\tilde{G}(a), && a < 0 \\ 0 &= -\tilde{s}(a)\tilde{g}(a) - \delta\tilde{G}(a) + \eta G(a), && a \geq 0 \end{aligned}$$

We begin by looking at the KFE for $a < 0$, which can be re-written as

$$\tilde{s}_0\left(a+\frac{w_0}{\tilde{r}}\right)\tilde{G}'(a) + \delta\tilde{G}(a) = 0$$

This is a linear ODE with integrating factor given by

$$IF(a) = \left(a+\frac{w_0}{\tilde{r}}\right)^{\frac{\delta}{\tilde{s}_0}}.$$

Thus, the solution is given by,

$$\tilde{G}(a) = \frac{C_-}{\left(a+\frac{w_0}{\tilde{r}}\right)^{\frac{\delta}{\tilde{s}_0}}} \quad \text{for} \quad a < 0,$$

where C_- is a yet to be determined coefficient of integration. For the case of $a \geq 0$, the steady state KFE can be expressed as

$$\tilde{s}_0\left(a+\frac{w_0}{\tilde{r}}\right)\tilde{G}'(a) + \delta\tilde{G}(a) = \eta G(a).$$

Implementing the same integrating factor as used previously for $a < 0$ we obtain,

$$\frac{d}{da}\left[IF(a)\cdot\tilde{G}(a)\right] = \frac{\eta}{\tilde{s}_0}G(a)\left(a+\frac{w_0}{\tilde{r}}\right)^{\frac{\delta}{\tilde{s}_0}-1}$$

Integrate both sides,

$$IF(a)\cdot\tilde{G}(a) = \frac{\eta}{\tilde{s}_0}\int_0^a G(u)\left(u+\frac{w_0}{\tilde{r}}\right)^{\frac{\delta}{\tilde{s}_0}-1} du + C_+,$$

then,

$$\tilde{G}(a) = \left(a + \frac{w_0}{\tilde{r}}\right)^{-\frac{\delta}{\tilde{s}_0}} \left[\frac{\eta}{\tilde{s}_0} \int_0^a G(u) \left(u + \frac{w_0}{\tilde{r}}\right)^{\frac{\delta}{\tilde{s}_0}-1} du + C_+\right].$$

To recapitulate, the function $\tilde{G}(a)$ and its associated density $\tilde{g}(a)$ are given by:

$$\tilde{G}(a) = \begin{cases} \frac{C_-}{\left(a+\frac{w_0}{\tilde{r}}\right)^{\frac{\delta}{\tilde{s}_0}}} & \text{for } a < 0 \\ \left(a + \frac{w_0}{\tilde{r}}\right)^{-\frac{\delta}{\tilde{s}_0}} \left[\frac{\eta}{\tilde{s}_0} \int_0^a G(u) \left(u + \frac{w_0}{\tilde{r}}\right)^{\frac{\delta}{\tilde{s}_0}-1} du + C_+\right] & \text{for } a \geq 0. \end{cases}$$

$$\tilde{g}(a) = \begin{cases} -C_- \left(\frac{\delta}{\tilde{s}_0}\right) \left(a + \frac{w_0}{\tilde{r}}\right)^{-\frac{\delta}{\tilde{s}_0}-1} & \text{for } a < 0 \\ -\frac{\delta}{\tilde{s}_0} \left(a + \frac{w_0}{\tilde{r}}\right)^{-\frac{\delta}{\tilde{s}_0}-1} \left[\frac{\eta}{\tilde{s}_0} \int_0^a G(u) \left(u + \frac{w_0}{\tilde{r}}\right)^{\frac{\delta}{\tilde{s}_0}-1} du + C_+\right] + \frac{\eta}{\tilde{s}_0} G(a) \left(a + \frac{w_0}{\tilde{r}}\right)^{-1} & \text{for } a \geq 0. \end{cases}$$

Imposing continuity in $\tilde{G}(a)$ at $a = 0$ implies that $C_- = C_+$. Notice, this also implies continuity in $\tilde{g}(a)$ at $a = 0$.

The value of the constants of integration depend on whether the economy is in the low or high return environment, whether $\tilde{r} \lesseqgtr \rho + \delta$.

In the low return environment: we know that $\tilde{G}(\overline{a}) = \frac{\eta}{\delta+\eta}$ and $G(\overline{a}) = \frac{\delta}{\delta+\eta}$, hence

$$C_+ = C_- = \frac{\eta}{\delta + \eta} \left(\overline{a} + \frac{w_o}{\tilde{r}}\right)^{\delta/\tilde{s}_0} - \frac{\eta}{\tilde{s}_0} \int_0^{\overline{a}} G(a') \left(a' + \frac{w_o}{\tilde{r}}\right)^{\frac{\delta}{\tilde{s}_0}-1} da'$$

In the high return environment:

The $\lim_{a\to\infty} \tilde{G}(a) = \frac{\eta}{\delta+\eta}$ for any C_+. To pin down C_+, $\tilde{G}(0) = 0 \implies C_+ = 0$, and hence $C_- = 0$ also

$$\tilde{g}(a) = \begin{cases} -C \left(\frac{\delta}{\tilde{s}_0}\right) \left(a + \frac{w_0}{\tilde{r}}\right)^{-\frac{\delta}{\tilde{s}_0}-1} & \text{for } a < 0 \\ -\frac{\delta}{\tilde{s}_0} \left(a + \frac{w_0}{\tilde{r}}\right)^{-\frac{\delta}{\tilde{s}_0}-1} \left[\frac{\eta}{\tilde{s}_0} \int_0^a G(u) \left(u + \frac{w_0}{\tilde{r}}\right)^{\frac{\delta}{\tilde{s}_0}-1} du + C\right] + \frac{\eta}{\tilde{s}_0} G(a) \left(a + \frac{w_0}{\tilde{r}}\right)^{-1} & \text{for } a \geq 0. \end{cases}$$

$$C = 0 \quad \text{for } \tilde{r} \geq \rho + \delta \text{ and}$$

$$C = \frac{\eta}{\delta + \eta} \left(\overline{a} + \frac{w_o}{\tilde{r}}\right)^{\delta/\tilde{s}_0} - \frac{\eta}{\tilde{s}_0} \int_0^{\overline{a}} G(a') \left(a' + \frac{w_o}{\tilde{r}}\right)^{\frac{\delta}{\tilde{s}_0}-1} da' \quad \text{for } \tilde{r} < \rho + \delta$$

B Proofs and Derivations: firm side

B.1 Firm Static Problem

A firm with productivity level z has an output function $y(z)$. When serving the domestic market, output is governed by a Cobb-Douglas production function:

$$q_d(z) = y_d(z) = z\ell_d(z)^\beta k_d(z)^{1-\beta}$$

When producing for a foreign market, output must reflect *iceberg* trade costs. Specifically, to deliver $q_x(z)$ units abroad, the firm must produce $q_x(z)d$ units domestically. The corresponding output relationship is:

$$q_x(z) = y_x(z)d^{-1} = z\ell_x(z)^\beta k_x(z)^{1-\beta} d^{-1}$$

The total output required to satisfy both domestic and foreign demand across $(N-1)$ export destinations is:

$$q_d(z) + q_x(z)d(N-1) = z\left(\ell_d(z) + (N-1)\ell_x(z)\right)^\beta \left(k_d(z) + (N-1)k_x(z)\right)^{1-\beta}$$

Given total production, the firm chooses the cost-minimizing combination of labor and capital by solving:

$$\min_{\ell_d(z),\ell_x(z),k_d(z),k_x(z)} w\left(\ell_d(z) + (N-1)\ell_x(z)\right) + r_k\left(k_d(z) + (N-1)k_x(z)\right) \tag{48}$$

This cost minimization problem can be reformulated using the following Lagrangian:

$$\begin{aligned}\mathcal{L} &= w\left(\ell_d(z) + (N-1)\ell_x(z)\right) + r_k\left(k_d(z) + (N-1)k_x(z)\right) \\ &\quad + \lambda\left[y_d(z) + (N-1)y_x(z) - z\left(\ell_d(z) + (N-1)\ell_x(z)\right)^\beta \left(k_d(z) + (N-1)k_x(z)\right)^{1-\beta}\right]\end{aligned}$$

The first-order conditions imply the optimal ratio of labor to capital input satisfies:

$$\frac{w\ell_d(z)}{r_k k_d(z)} = \frac{w\ell_x(z)}{r_k k_x(z)} = \frac{\beta}{1-\beta}$$

Substituting this condition into the production function yields the total cost for domestic and export production as:

$$w\ell_d(z) + r_k k_d(z) = \frac{q_d(z)}{z} \cdot w^\beta r_k^{1-\beta} \left(\frac{1}{\beta}\right)^\beta \left(\frac{1}{1-\beta}\right)^{1-\beta} = \frac{C}{z} q_d(z)$$
$$w\ell_x(z) + r_k k_x(z) = \frac{q_x(z)d}{z} \cdot w^\beta r_k^{1-\beta} \left(\frac{1}{\beta}\right)^\beta \left(\frac{1}{1-\beta}\right)^{1-\beta} = \frac{C}{z} q_x(z)d$$

where C is the constant unit cost of producing an effective unit of output.

The firm's flow profits from domestic and foreign markets, net of variable costs, are given by:

$$P\pi_d(z) = p_d(z)q_d(z) - \frac{C}{z}q_d(z)$$
$$P\pi_x(z) = p_x(z)q_x(z) - \frac{C}{z}q_x(z)d$$

The firm sets prices to maximize total profits:

$$\max_{p_d(z),\, p_x(z),\, \hat{z}} \pi_d(z) + \mathbb{1}_{z \geq \hat{z}}(N-1)\left[\pi_x(z) - \phi_x w\right] \tag{49}$$

where $\phi_x w$ is the fixed cost of exporting, and $\hat{z}$ is the productivity threshold above which firms find it profitable to export.

The optimal prices are:

$$p_d(z) = \frac{C}{z} \cdot \frac{\omega}{\omega - 1} \tag{50}$$
$$p_x(z) = p_d(z) \cdot d \tag{51}$$

Using the final demand system (equation (17)), the firm's flow profits can be written as:

$$\pi_d(z) = \frac{1}{\omega} \cdot \frac{Y}{P} \left(\frac{\omega}{\omega - 1} \cdot \frac{C}{P}\right)^{1-\omega} z^{\omega - 1} \tag{52}$$

$$\pi_x(z) = \pi_d(z) \cdot d^{1-\omega} \tag{53}$$

B.2 Firm Value Function

To solve the Hamilton-Jacobi-Bellman (HJB) equation (26), we employ the method of undetermined coefficients. Initially, we hypothesize that the solution assumes the following form:

$$V_d(z) = \mathcal{B}_1 z^{\omega-1} + \mathcal{B}_2 z^{-\text{v}_1} + \mathcal{B}_3 z^{-\text{v}_2} \qquad (z_l < z < \hat{z})$$
$$V_x(z) = \mathcal{C}_0 + \mathcal{C}_1 z^{\omega-1} + \mathcal{C}_2 z^{-\text{v}_1} \qquad (z \geq \hat{z})$$

Subsequently, we determine the value function by substituting our guess into the HJB equation. This involves matching the coefficients and verifying that all smooth pasting and value matching conditions (27)-(30) are satisfied. The resulting solutions for the coefficients are:

$$\text{v}_1 = \frac{\mu + \sqrt{\mu^2 + 2\sigma^2(r+\delta_f)}}{\sigma^2}$$
$$\text{v}_2 = \frac{\mu - \sqrt{\mu^2 + 2\sigma^2(r+\delta_f)}}{\sigma^2}$$
$$\mathcal{C}_0 = -(N-1)\frac{w}{(r+\delta_f)}\frac{\phi_x}{P}$$
$$\mathcal{B}_1 = \pi_0 \left((r+\delta_f) - (\omega-1)\left(\mu + (\omega-1)\frac{\sigma^2}{2}\right)\right)^{-1}$$
$$\mathcal{C}_1 = \pi_0 \left(1 + (N-1)d^{1-\omega}\right)\left((r+\delta_f) - (\omega-1)\left(\mu + (\omega-1)\frac{\sigma^2}{2}\right)\right)^{-1}$$
$$\mathcal{B}_2 = \frac{\text{v}_2}{\text{v}_2 - \text{v}_1}\mathcal{C}_0 \hat{z}^{\text{v}_2} z_l^{\text{v}_1-\text{v}_2} + \frac{\text{v}_2(\text{v}_1+\omega-1)}{\text{v}_1(\text{v}_2-\text{v}_1)}(\mathcal{C}_1 - \mathcal{B}_1)\hat{z}^{\omega+\text{v}_2-1} z_l^{\text{v}_1-\text{v}_2} + (\omega-1)\left(\frac{1}{\text{v}_1}\right)\mathcal{B}_1 z_l^{\text{v}_1+\omega-1}$$
$$\mathcal{C}_2 = \frac{1}{\text{v}_1}(\omega-1)\left((\mathcal{C}_1 - \mathcal{B}_1)\hat{z}^{\omega+\text{v}_1-1} + \mathcal{B}_1 \hat{z}^{\text{v}_1-\text{v}_2} z_l^{\text{v}_2+\omega-1}\right) + \mathcal{B}_2\left(1 - \hat{z}^{\text{v}_1-\text{v}_2} z_l^{\text{v}_2-\text{v}_1}\right)$$
$$\mathcal{B}_3 = \frac{\text{v}_1}{\text{v}_1 - \text{v}_2}\mathcal{C}_0 \hat{z}^{\text{v}_2} + \frac{(\text{v}_1+\omega-1)}{(\text{v}_1-\text{v}_2)}(\mathcal{C}_1 - \mathcal{B}_1)\hat{z}^{\omega+\text{v}_2-1}$$

For a comprehensive discussion of a similar problem, see Bradley (2025).

B.3 Proof of Lemma 8

To restate the problem, we aim to determine the steady state distribution of firm productivity, denoted by the cumulative distribution function $F(z)$, which satisfies the Kolmogorov Forward Equation (32) in its stationary form:

$$\frac{\partial F(z)}{\partial t} = 0$$

The distribution $F(z)$ and associated density $f(z)$ must be valid and satisfy the following boundary conditions:

$$f(z_l) = 0 \tag{54}$$

$$F(z_l) = 0 \tag{55}$$

$$f(z_0^-) = f(z_0^+) \tag{56}$$

$$F(z_0^-) = F(z_0^+) \tag{57}$$

$$\lim_{z \to \infty} F(z) = 1 \tag{58}$$

We conjecture that the solution takes the piecewise power-law form specified and apply the method of undetermined coefficients— following the approach outlined in Appendix B.2:

$$F(z) = \begin{cases} G_1 z^{\tilde{\xi}_1} + G_2 z^{\tilde{\xi}_2} + G_0 & \text{for} \quad z_l < z < z_0 \\ G_3 z^{\xi_1} + G_4 & \text{for} \quad z \geq z_0 \end{cases} \tag{59}$$

Differentiating (59) yields the corresponding candidate probability density function:

$$f(z) = \begin{cases} G_1 \tilde{\xi}_1 z^{\tilde{\xi}_1 - 1} + G_2 \tilde{\xi}_2 z^{\tilde{\xi}_2 - 1} & \text{for} \quad z_l < z < z_0 \\ \xi_1 G_3 z^{\xi_1 - 1} & \text{for} \quad z \geq z_0 \end{cases} \tag{60}$$

To satisfy the upper bound condition (58), we require that $\xi_1 < 0$ and $\underline{G_4 = 1}$. Next, imposing boundary condition (54) gives:

$$G_1 = -G_2 \frac{\tilde{\xi}_2}{\tilde{\xi}_1} z_l^{\tilde{\xi}_2 - \tilde{\xi}_1}. \tag{61}$$

Applying condition (55), we obtain:

$$G_0 = -G_1 z_l^{\tilde{\xi}_1} - G_2 z_l^{\tilde{\xi}_2} \tag{62}$$

Continuity of the density at $z = z_0$ from (56) implies:

$$G_3 = G_1 \frac{\tilde{\xi}_1}{\xi_1} z_0^{\tilde{\xi}_1 - \xi_1} + G_2 \frac{\tilde{\xi}_2}{\xi_1} z_0^{\tilde{\xi}_2 - \xi_1} \tag{63}$$

Imposing continuity of the CDF at $z = z_0$ using (57) yields:

$$G_1 z_0^{\tilde{\xi_1}} + G_2 z_0^{\tilde{\xi_2}} - G_1 z_l^{\tilde{\xi_1}} - G_2 z_l^{\tilde{\xi_2}} = [G_1 \frac{\tilde{\xi_1}}{\xi_1} z_0^{\tilde{\xi_1}-\xi_1} + G_2 \frac{\tilde{\xi_2}}{\xi_1} z_0^{\tilde{\xi_2}-\xi_1}] z_0^{\xi_1} + 1 \tag{64}$$

We hence have four equations (61)-(64) and four unknowns $G_0 - G_3$, which collectively ensure that our candidate solution satisfy the boundary conditions outlined. Solving this simultaneous system yields:

$$\begin{aligned}
G_0 &= F_0 \left(\frac{1}{\tilde{\xi_1}} - \frac{1}{\tilde{\xi_2}} \right) \\
G_1 &= -F_0 \frac{1}{\tilde{\xi_1}} \frac{1}{z_l^{\tilde{\xi_1}}} \\
G_2 &= F_0 \frac{1}{\tilde{\xi_2}} \frac{1}{z_l^{\tilde{\xi_2}}} \\
G_3 &= F_0 \frac{1}{\xi_1} \frac{1}{z_0^{\xi_1}} \left[\left(\frac{z_0}{z_l} \right)^{\tilde{\xi_2}} - \left(\frac{z_0}{z_l} \right)^{\tilde{\xi_1}} \right] \\
\text{where } F_0 &= \left[\left(\frac{z_0}{z_l} \right)^{\tilde{\xi_1}} \left(\frac{1}{\xi_1} - \frac{1}{\tilde{\xi_1}} \right) - \left(\frac{z_0}{z_l} \right)^{\tilde{\xi_2}} \left(\frac{1}{\xi_1} - \frac{1}{\tilde{\xi_2}} \right) - \left(\frac{1}{\tilde{\xi_1}} - \frac{1}{\tilde{\xi_2}} \right) \right]^{-1}
\end{aligned}$$

We can rewrite CDFs as the guessed forms.

$$F(z) = \begin{cases} F_0 \left[\frac{1}{\tilde{\xi_2}} \left(\frac{z}{z_l} \right)^{\tilde{\xi_2}} - \frac{1}{\tilde{\xi_1}} \left(\frac{z}{z_l} \right)^{\tilde{\xi_1}} \right] + F_0 \frac{1}{\xi_1} \frac{1}{z_0^{\xi_1}} \left[\left(\frac{z_0}{z_l} \right)^{\tilde{\xi_2}} - \left(\frac{z_0}{z_l} \right)^{\tilde{\xi_1}} \right] & z_l \leq z < z_0 \\ F_0 \left[\left(\frac{z_0}{z_l} \right)^{\tilde{\xi_2}} - \left(\frac{z_0}{z_l} \right)^{\tilde{\xi_1}} \right] \frac{1}{\xi_1} \left(\frac{z}{z_0} \right)^{\xi_1} + 1 & z \geq z_0 \end{cases} \tag{65}$$

Again, we start from the original differential equations (KFEs). At the steady state, the cumulative distribution $F(z)$ does not change over time and we can drop the time subscript t.

$$\frac{\partial F(z)}{\partial t} = 0 = -A - \delta_f F(z) - \left(\mu - \frac{\sigma^2}{2} \right) z f(z) + \frac{\sigma^2}{2} z^2 f'(z) \qquad \text{for, } z_l \leq z < z_0 \tag{66}$$

$$\frac{\partial F(z)}{\partial t} = 0 = -\delta_f F(z) - \left(\mu - \frac{\sigma^2}{2} \right) z f(z) + \frac{\sigma^2}{2} z^2 f'(z) + E \qquad \text{for, } z \geq z_0 \tag{67}$$

To begin with the case where $z \in [z_l, z_0)$, if we replace $F(z)$, $f(z)$, and $f'(z)$ with corresponding expressions that are derived from the guessed form (66), we have to solve the following general solution for each case:

$$0 = \left(F_0 \left(\frac{z}{z_l}\right)^{\tilde{\xi}_2} \right) \left(-\delta_f \frac{1}{\tilde{\xi}_2} - (\mu - \frac{\sigma^2}{2}) + (\tilde{\xi}_2 - 1)\frac{\sigma^2}{2} \right)$$
$$- \left(F_0 \left(\frac{z}{z_l}\right)^{\tilde{\xi}_1} \right) \left(-\delta_f \frac{1}{\tilde{\xi}_1} - (\mu - \frac{\sigma^2}{2}) + (\tilde{\xi}_1 - 1)\frac{\sigma^2}{2} \right)$$

Therefore, $\tilde{\xi}_1$ and $\tilde{\xi}_2$ are roots of the following quadratic equation.

$$\frac{\sigma^2}{2}\tilde{\xi}^2 - \mu\tilde{\xi} - \delta_f = 0$$

$$\tilde{\xi}_1 = \frac{\mu - \sqrt{\mu^2 + 2\sigma^2\delta_f}}{\sigma^2}, \quad \tilde{\xi}_2 = \frac{\mu + \sqrt{\mu^2 + 2\sigma^2\delta_f}}{\sigma^2}$$

Furthermore, we can derive the particular solution of (66) if we use the boundary condition $F(z_0) = 0,\ f(z_0) = 0$.

$$A = \frac{\sigma^2}{2} z^2 f'(z_l) = \frac{\sigma^2}{2} F_0(\tilde{\xi}_2 - \tilde{\xi}_1)$$

By the same logic, we apply the guessed expression of $F(z)$ to the equation (67) to get the general solution for the case of $z \geq z_0$.

$$0 = \left(F_0 \left[\left(\frac{z_0}{z_l}\right)^{\tilde{\xi}_2} - \left(\frac{z_0}{z_l}\right)^{\tilde{\xi}_1} \right] \left(\frac{z}{z_0}\right)^{\xi_1} \right) \left(-\delta \frac{1}{\xi_1} - (\mu - \frac{\sigma^2}{2}) + (\xi_1 - 1)\frac{\sigma^2}{2} \right)$$

The exponent ξ_1 is also the solution of the following quadratic function, but only one solution can be valid since $\xi_1 < 0$.

$$\frac{\sigma^2}{2}\xi_1^2 - \mu\xi_1 - \delta_f = 0$$

$$\xi_1 = \tilde{\xi}_1 = \frac{\mu - \sqrt{\mu^2 + 2\sigma^2\delta_f}}{\sigma^2} < 0$$

Again, we can derive the particular solution of (67) if we use the boundary condition $F(\infty) = 1$, $f(\infty) = 0$, and $f'(\infty) = 0$.

$$E = \delta_f \tag{68}$$

C Proofs and Derivations: model equilibrium

C.1 Proof of Propositions 1, 2 and 3

We begin by assuming that workers remain employed throughout their lives ($\eta = 0$), which is consistent with the restrictions imposed in Propositions 1, 2 and 3. Under this restriction, the value function (5) for the employed reduces to:

$$(\rho + \delta)v(a) = \max_{0 \leq c \leq a + \frac{w(1-\tau)}{\tilde{r}}} \langle u(c) + v_a(a)(\tilde{r}a + w(1-\tau) - c) \rangle \tag{69}$$

Using the optimality condition $u'(c) = v_a(a)$ and following the same steps as in Appendix A.2, optimal consumption and saving is given by:

$$c(a) = \left(\tilde{r} - \frac{\tilde{r} - \rho - \delta}{\gamma}\right)\left(a + \frac{w}{\tilde{r}}\right) \tag{70}$$

$$s(a) = \frac{\tilde{r} - \rho - \delta}{\gamma}\left(a + \frac{w}{\tilde{r}}\right). \tag{71}$$

Note, that as $\eta = 0$, there is full employment and hence the tax rate to balance the budget is $\tau = \frac{w_o \eta}{w \delta} = 0$. Hence, the tax rate does not appear in optimal consumption or savings behavior. Turning to the KFE (13) when $\eta = 0$, under the steady state, the equilibrium allocation of assets satisifies:

$$s(a)g(a) = \delta\left(1 - G(a)\right).$$

Integrating over the full support of a and substituting in the savings rate (71) yields:

$$\frac{\tilde{r} - \rho - \delta}{\gamma} \int \left(a + \frac{w}{\tilde{r}}\right) g(a)da = \delta \int \left(1 - G(a)\right) da. \tag{72}$$

Looking at the left hand side of (72).

$$\frac{\tilde{r}-\rho-\delta}{\gamma}\int\left(a+\frac{w}{\tilde{r}}\right)g(a)da = \frac{\tilde{r}-\rho-\delta}{\gamma}\underbrace{\int ag(a)da}_{\overline{S}}+\frac{\tilde{r}-\rho-\delta}{\gamma}.\frac{w}{\tilde{r}}\underbrace{\int g(a)da}_{\overline{L}=1}$$

$$= \frac{\tilde{r}-\rho-\delta}{\gamma}\left(\overline{S}+\frac{w}{\tilde{r}}\right) \tag{73}$$

Turning to the right hand side, integration by parts yields:

$$\delta\int(1-G(a))\,da = \delta\left[a(1-G(a))\right]_0^\infty+\delta\underbrace{\int ag(a)da}_{\overline{S}} \tag{74}$$

Under the restriction $\eta = 0$, the cdf is difined for $\tilde{r} > \rho+\delta$ and given by,

$$G(a) = 1-\left(a+\frac{w}{\tilde{r}}\right)^{-\frac{\delta(\tilde{r}-\rho-\delta)}{\gamma}}$$

and hence,

$$a(1-G(a)) = \frac{a}{\left(a+\frac{w}{\tilde{r}}\right)^{\frac{\delta(\tilde{r}-\rho-\delta)}{\gamma}}}$$

$$\left[a(1-G(a))\right]_0^\infty = \lim_{a\to\infty}\frac{a}{\left(a+\frac{w}{\tilde{r}}\right)^{\frac{\delta(\tilde{r}-\rho-\delta)}{\gamma}}}-\frac{0}{\left(\frac{w}{\tilde{r}}\right)^{\frac{\delta(\tilde{r}-\rho-\delta)}{\gamma}}}$$

$$= 0 \qquad \text{(by L'Hôpital's rule)}$$

hence, $\delta\int(1-G(a))\,da = \delta\overline{S}$

Putting together the simplified expressions for the lhs (73) and rhs (74) gives the aggregate asset supply when $\eta = 0$.

$$\overline{S} = \frac{\tilde{r}-\rho-\delta}{\delta(\gamma+1)+\rho-\tilde{r}}\frac{w}{\tilde{r}}$$

In terms of the return absent the annuity, since $\tilde{r} = r+\delta$ is given by

$$\overline{S} = \frac{r-\rho}{\delta\gamma+\rho-r}\frac{w}{r+\delta} \tag{75}$$

For an equilibrium to exist aggregate savings must be positive and finite. This is the case when,

$$r \in (\rho, \rho+\gamma\delta). \tag{76}$$

On the asset demand side, the demand $\overline{D}$, the right hand side of (36) is given by

$$\overline{D} = \underbrace{\overline{K}}_{\text{Capital stock}} + \underbrace{\frac{1}{r}\left(\overline{\Pi} - w\tilde{\ell}\right)}_{\text{Asset value of firms}}.$$

The capital stock is as defined in the main text. The parameter $\overline{\Pi}$ represents aggregate profits net of production costs. The parameter $\tilde{\ell}$ is the share of the labor force who perform non-productive tasks. These include the entry and upgrading costs as well as the fixed costs associated with exporting. The share (since total labor supply is equal to one) is given by,

$$\tilde{\ell} = \Omega(\delta_f \chi(1+\epsilon) + A\chi + (1 - F(\hat{z}))(N-1)\phi_x). \tag{77}$$

Given the Cobb-Douglas production function and constant markups, aggregation of the component parts of asset demands is as follows:

$$\overline{K} = \frac{1}{r+\delta_k}\frac{1-\beta}{\beta}w(1-\tilde{\ell})$$
$$\overline{\Pi} - w\tilde{\ell} = \frac{w}{\beta(\omega-1)}(1-\tilde{\ell}) - w\tilde{\ell} = \frac{w}{\beta(\omega-1)} - \frac{1+\beta(\omega-1)}{\beta(\omega-1)}w\tilde{\ell}$$

So, total asset demand is given by

$$D = \left[\frac{1}{r+\delta_k}\frac{1-\beta}{\beta} + \frac{1}{r}\frac{1}{\beta(\omega-1)}\right]w - \left[\frac{1}{r+\delta_k}\frac{1-\beta}{\beta} + \frac{1}{r}\frac{1+\beta(\omega-1)}{\beta(\omega-1)}\right]w\tilde{\ell}.$$

Since asset demand and supply are both scaled by the wage rate w. Equating the two implies that the equilibrium interest rate r is a function of primitive parameters and the share of labor devoted to non-productive tasks only.

$$\frac{r-\rho}{\delta\gamma+\rho-r}\frac{1}{r+\delta} = \left[\frac{1}{r+\delta_k}\frac{1-\beta}{\beta} + \frac{1}{r}\frac{1}{\beta(\omega-1)}\right] - \left[\frac{1}{r+\delta_k}\frac{1-\beta}{\beta} + \frac{1}{r}\frac{1+\beta(\omega-1)}{\beta(\omega-1)}\right]\tilde{\ell} \tag{78}$$

So far all derivations have been for the general restriction of $\eta = 0$. We now derive individually Propositions 1, 2 and 3 under the specific restrictions.

Proposition 1: $\delta = 0$, $\eta = 0$

For the asset market to be in equilibrium, asset supply $\overline{S} := \int a g(a) da$ must equal asset demand. For that, we require a positive and finite value of $\overline{S}$. To see why the only possible equilibrium value of the interest rate r is when it equals the discount rate ρ. Take a worker's savings function, equation (71), under case 1's restrictions:

$$s(a) = \frac{r-\rho}{\gamma}\left(a + \frac{w}{r}\right)$$

If $r < \rho$, savings are negative and all workers assets in the limit hit the borrowing constraint. Hence, the equilibrium aggregate savings rate is negative. If $r > \rho$, savings are always positve, and since workers live indefinitely all workers asset level will be infinite, and hence so will aggregate savings. The only possible equilibrium is the knife edge case when $r = \rho$. Here, savings are equal to zero and all workers will consume their entire income. Hence, the savings rate is indeterminate and a steady state can exist for any positive and finite $\overline{S}$.

Proposition 2: $\mu = 0$, $\sigma = 0$ and $\eta = 0$

Since equation (78) involves only two endogenous variables— the return on capital, r, and the share of labor allocated to non-production tasks, $\tilde{\ell}$— and does not include any trade cost parameters, it suffices to demonstrate that $\tilde{\ell}$ is determined solely by model primitives unrelated to trade. Establishing this result will imply that r is likewise invariant to trade costs. The remainder of this proof derives the expression for $\tilde{\ell}$ under the parameter restrictions stated in Proposition 2.

Under the restriction $\mu = \sigma = 0$, firm productivity is fixed upon entry, making the productivity distribution degenerate with all firms having the same productivity level $z = z_0$. Consequently, the export behavior of firms is determined entirely by whether this common productivity level lies above or below the export cutoff $\hat{z}$, defined in equation (23). Specifically, if $z_0 < \hat{z}$, no firms export; if $z_0 \geq \hat{z}$, all firms export. To formalize this, let $\lambda \in \{0, 1\}$ denote the trade regime, where $\lambda = 0$ indicates autarky (no exports) and $\lambda = 1$ indicates full export participation.

Substituting this classification into the general expression for the share of non-productive labor from equation (77), we obtain:

$$\tilde{\ell} = \Omega(\delta_f \chi(1+\epsilon) + \lambda(N-1)\phi_x). \tag{79}$$

In addition to the simplification of $F(z)$ and the introduction of the parameter λ, the expression can be further reduced by noting that the mass of adopters is zero ($A = 0$) in this framework. Since all firms share the same productivity level z_0, and the cost of upgrading is strictly positive, no firm has an incentive to adopt a new productivity level identical to what it already possesses.

The expression for $\tilde{\ell}$ includes the total mass of firms, Ω. From equation (18), and recalling that the price index P is the numeraire (i.e., $P = 1$), the mass of firms is given by:

$$\Omega = \left(\int_{z_l}^{\infty} p_d(z)^{1-\omega} dF(z) + (N-1) \int_{\hat{z}}^{\infty} p_x(z)^{1-\omega} dF(z) \right)^{-1}.$$

Applying the degenerate distribution assumption and the binary trade regime captured by λ, the expression for the mass of firms simplifies to:

$$\Omega = \left(p_d(z_0)^{1-\omega} + \lambda(N-1) p_x(z_0)^{1-\omega} \right)^{-1}.$$

Substituting the pricing policies derived in Lemma 5 into the expression, the total measure of firms Ω can be written as:

$$\Omega = \left(\left(\frac{\omega}{\omega - 1} \right)^{1-\omega} C^{1-\omega} z_0^{\omega-1} \left(1 + \lambda(N-1) d^{1-\omega} \right) \right)^{-1}.$$

Here, C is the marginal cost of production. We can rearrange the above expression and express the marginal cost as

$$C = \Omega^{\frac{1}{\omega-1}} \left(\frac{\omega}{\omega - 1} \right)^{-1} z_0 \left(1 + \lambda(N-1) d^{1-\omega} \right)^{\frac{1}{\omega-1}}. \tag{80}$$

Under the imposed parameter restrictions and using the trade regime indicator λ, firm profits as described in Lemma 6 can be written as:

$$\pi(z_0) = \pi_0 z_0^{\omega-1} + \lambda(N-1) \pi_0 d^{1-\omega} z_0^{\omega-1} - \lambda(N-1) \phi_x w,$$

where π_0 is the constant defined in Lemma 6 and depends on the marginal cost of production C. Substituting the expression for C from equation (80) into π_0, firm profits simplify to:

$$\pi(z_0) = \frac{1}{\omega} Y \Omega^{-1} - \lambda(N-1) \phi_x w. \tag{81}$$

Profit (81) in turn depends on total output,

$$Y = \underbrace{\left(\Omega z_0^{\omega-1} (1 + (N-1) d^{1-\omega}) \right)^{\frac{1}{\omega-1}}}_{\text{effective productivity}} \left(\underbrace{1 - \tilde{\ell}}_{\text{production labor}} \right)^{\beta} \overline{K}^{1-\beta}.$$

Since all firms are identical, aggregation is straightforward. Substituting the optimal labor and capital allocations from Lemma 5 yields:

$$Y = \left(\Omega z_0^{\omega-1}(1+(N-1)d^{1-\omega})\right)^{\frac{1}{\omega-1}} \left(1-\tilde{\ell}\right) \beta^{-1} w C^{-1}.$$

Substituting in the expression for marginal cost from equation (80), we obtain a simple expression for output:

$$Y = \frac{\omega}{\beta(\omega-1)}(1-\tilde{\ell})w$$

Plugging this back into equation (81), firm profit becomes:

$$\pi(z_0) = \frac{1}{\beta(\omega-1)}(1-\tilde{\ell})w\Omega^{-1} - \lambda(N-1)\phi_x w.$$

Since firm productivity is fixed at z_0, the present value of a firm is given by:

$$V(z_0) = \frac{\pi(z_0)}{r+\delta_f} \quad ,$$

where r is the return on capital and δ_f is the exogenous firm exit rate. Imposing the free entry condition from equation (25) gives:

$$\frac{1}{r+\delta_f}\left(\frac{1}{\beta(\omega-1)}(1-\tilde{\ell})\Omega^{-1} - \lambda(N-1)\phi_x\right) = \chi(1+\epsilon). \tag{82}$$

We are left with two equations— (79) and (82)— and two endogenous variables: the share of non-productive labor $\tilde{\ell}$ and the measure of firms Ω. Solving these simultaneously yields:

$$\begin{aligned} \Omega &= \frac{1}{\delta_f\chi(1+\epsilon)+\lambda(N-1)\phi_x+\beta(\omega-1)[(r+\delta_f)\chi(1+\epsilon)+\lambda(N-1)\phi_x]} \\ \tilde{\ell} &= \frac{\delta_f\chi(1+\epsilon)+\lambda(N-1)\phi_x}{\delta_f\chi(1+\epsilon)+\lambda(N-1)\phi_x+\beta(\omega-1)[(r+\delta_f)\chi(1+\epsilon)+\lambda(N-1)\phi_x]} \end{aligned}$$

Hence, $\tilde{\ell}$ is locally independent of the trade cost d. That is, if the regime indicator λ remains unchanged, the share of non-productive labor does not respond to changes in d. Substituting this into the identity in equation (78), the return on capital r becomes a function of model primitives and the equilibrium regime λ, implying:

$$\frac{\partial r}{\partial d} = \frac{\partial \tilde{\ell}}{\partial d} = 0.$$

Proposition 3: $\delta_k = 0$ **and** $\eta = 0$

Assuming no capital depreciation equation (78) simplifies to

$$\frac{r-\rho}{\delta\gamma+\rho-r}\frac{r}{r+\delta} = \left[\frac{1-\beta}{\beta} + \frac{1}{\beta(\omega-1)}\right] - \frac{\omega}{\beta(\omega-1)}\tilde{\ell}.$$

This equilibrium condition holds for any iceberg trade cost $d \geq 1$, and hence the derivative with respect to d must also hold. Differentiating by d yields:

$$\frac{\delta\left(\left(\gamma-1\right)r^2+\left(2\rho+2\gamma\delta\right)r-\rho^2-\gamma\delta\rho\right)}{\left(r+\delta\right)^2\left(r-\rho-\gamma\delta\right)^2}\frac{\partial r}{\partial d} = -\frac{\omega}{\beta(\omega-1)}\frac{\partial\tilde{\ell}}{\partial d}$$

The coefficient on the derivative of the return to capital on trade cost $\frac{\partial r}{\partial d}$ is positive if,

$$\left(\gamma-1\right)r^2+\left(2\rho+2\gamma\delta\right)r > \rho^2+\gamma\delta\rho.$$

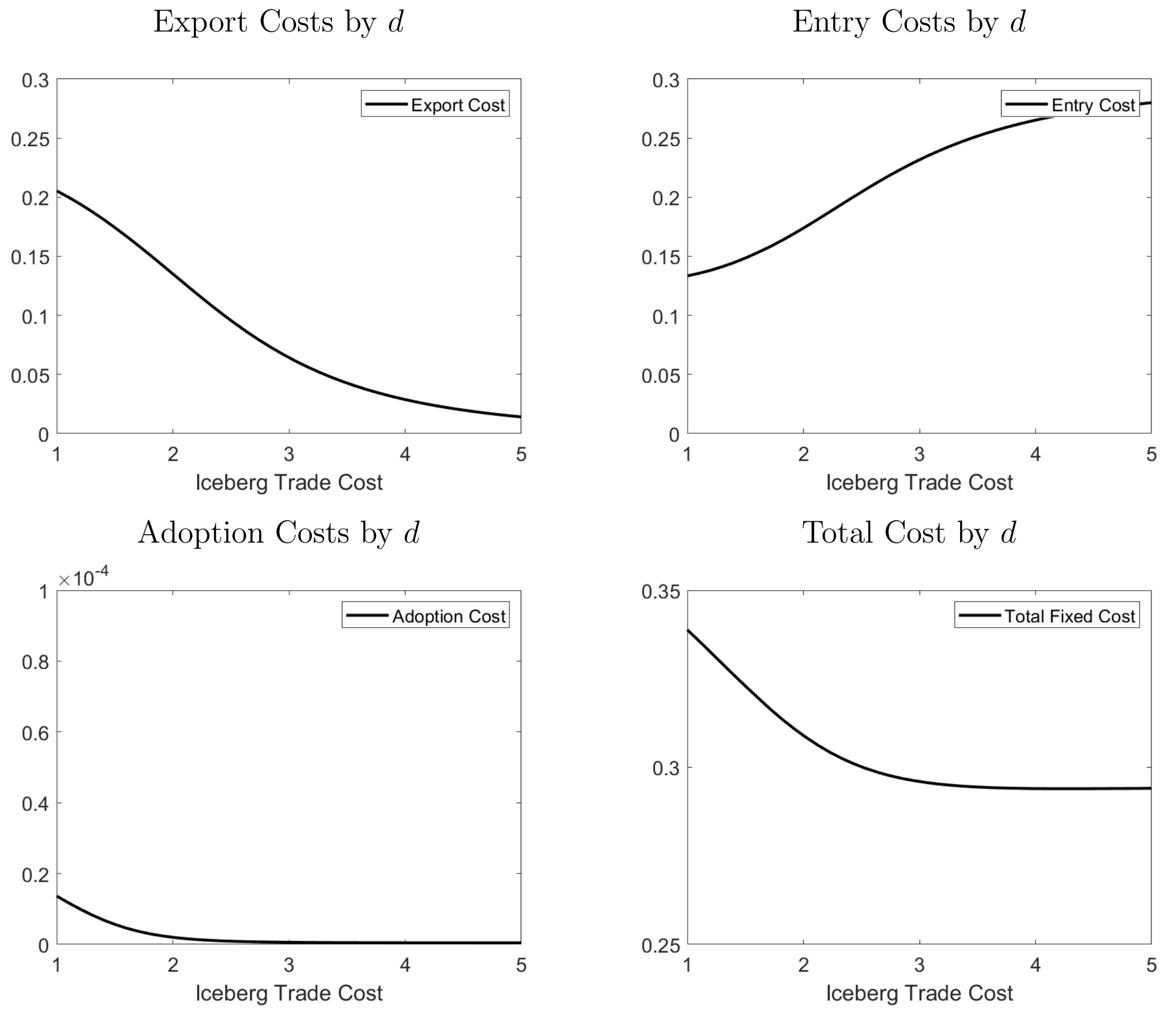
Since $\gamma > 1$, this will always hold for $r \geq \rho/2$. However, we know from (76) that there is $r > \rho$. Hence, the coefficient of coefficient on $\frac{\partial r}{\partial d}$ must be positive. Given $\omega > 1$ and $\beta \in (0,1)$, the coefficient on the share of labor not used in production on trade cost $\frac{\partial\tilde{\ell}}{\partial d}$ is always negative. Hence,

$$\text{sgn}\left(\frac{\partial r}{\partial d}\right) = -\text{sgn}\left(\frac{\partial\tilde{\ell}}{\partial d}\right)$$

D Quantitative Appendix

D.1 Non-productive Labor Tasks

Figure D.1: Non-productive Labor Tasks



Notes: The model is recalibrated across a range of values for the *iceberg* trade cost parameter, $d \in [1, 5]$. For each value of d, a steady state equilibrium is computed. The figure displays the resulting equilibrium shares of labor allocated to three activities by incumbent firms: upgrading productivity (adoption costs), firm entry (entry costs), and exporting (export costs). Since the total population is normalized to one, these values can be interpreted as the fraction of the population engaged in each activity.

D.2 Proof of Proposition 4

Starting from the value function of the retired, given by equation (39) in this appendix:

$$\tilde{v}(a) = \frac{1}{1-\gamma}\left(\tilde{r} - \frac{\tilde{r}-\rho-\delta}{\gamma}\right)^{-\gamma}\left[a + \frac{w_o}{\tilde{r}}\right]^{1-\gamma},$$

differentiating with respect to $\tilde{r}$ yields

$$\frac{\partial \tilde{v}(a)}{\partial \tilde{r}} = \underbrace{\left(\tilde{r} - \frac{\tilde{r}-\rho-\delta}{\gamma}\right)^{-\gamma}}_{>0}\underbrace{\left[a + \frac{w_o}{\tilde{r}}\right]^{-\gamma}}_{>0}\left[\underbrace{\left(\tilde{r} - \frac{\tilde{r}-\rho-\delta}{\gamma}\right)^{-1}\left(a + \frac{w_o}{\tilde{r}}\right) - \frac{w_o}{\tilde{r}^2}}_{\text{sign depends on } a}\right].$$

The first two terms are strictly positive since $\tilde{r} > 0$ and $a > -w_o/\tilde{r}$, which always holds in equilibrium. Thus, the sign of $\frac{\partial \tilde{v}(a)}{\partial \tilde{r}}$ depends only on the bracketed term, which is positive when

$$a > a^\star = -\frac{w_o}{\tilde{r}} \cdot \frac{\tilde{r}-\rho-\delta}{\gamma\tilde{r}}.$$

In the high-return regime, $\tilde{r} > \rho+\delta$ and $a \in [0,\infty)$. Since $\tilde{r} > \rho+\delta$, $a^\star < 0$. Since $a \geq 0$ it follows that $\frac{\partial \tilde{v}(a)}{\partial \tilde{r}} > 0$ for all retirees.

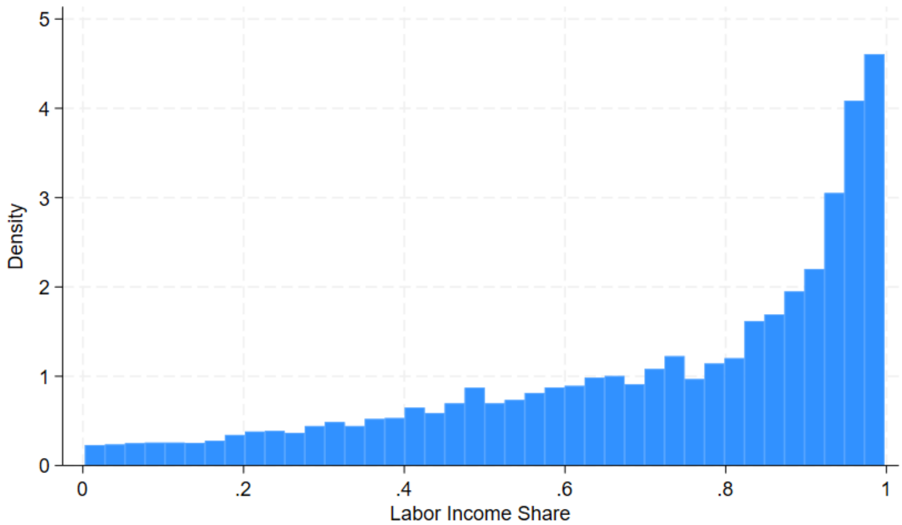
In the low-return regime, $\tilde{r} < \rho+\delta$ and $a \in \left(-\frac{w_o}{\tilde{r}}, \overline{a}\right]$. In this scenario $a^\star > 0$ and thus for $a \in \left(-\frac{w_o}{\tilde{r}}, a^\star\right)$, $\frac{\partial \tilde{v}(a)}{\partial \tilde{r}} < 0$.

D.3 Data on Income and Brexit

Data are from the UK Household Longitudinal Study (UKHLS) for the years 2016 to 2021. To focus on a stable political environment— post-referendum but pre-COVID-19 policy disruptions— the sample is restricted to responses provided before the onset of pandemic-related policies and after the Brexit referendum. Each respondent was asked annually: "*Should the United Kingdom remain a member of the European Union or leave the European Union?*" To minimize variation in the policy context across individuals, we use each person's response closest to June 2019, the midpoint of the sample period.

Income is disaggregated into labor and non-labor components. Labor income includes all earnings from paid employment or self-employment, potentially across multiple jobs. Non-labor income encompasses pension income, social transfers, investment returns, private benefits, and other sources. Since non-labor income can be lumpy and irregular, total income is aggregated over up to five years per individual. Despite aggregation, a substantial share of individuals report either no labor income or only labor income. Of the 45,558 individuals in the sample, 32.4% report zero labor income, while 14.5% report that all their income comes from labor. Figure D.3 shows the distribution of the labor income share, excluding these two extreme cases.

Figure D.3: Labor Income Share Distribution



Notes: Histogram of the labor income share, restricted to $(0, 1)$. Each bin is of equal width. The bar area reflects the proportion of the sample falling within each interval.