A Model of Intensional Type Theory in SSet

In the mid 1990s Hofmann and Streicher have shown that groupoids give rise to a model of Martin-Löf Type Theory where $\text{Id}(A, t, s)$ is the discrete groupoid of all morphisms from $t$ to $s$ in $A$. But the syntax of intensional type theory only gives rise to a weak $\infty$-dimensional groupoid structure on every type.

M. Warren has shown that strict $\infty$-dimensional groupoids provide a model of type theory. In our account we start from the observation that Kan complexes within the topos $\text{SSet}$ of simplicial sets provide an appealing notion of weak $\infty$-dimensional groupoid. Moreover, a notion of family of types is provided by the notion of Kan fibration. Since $\text{SSet}$ is a topos and Kan fibrations are closed under $\Sigma$ and $\Pi$ one may interpret the corresponding type-theoretic concepts in a straightforward way.

Hofmann and Streicher have shown how to lift Grothendieck universes to type-theoretic universes in presheaf toposes $\text{Set}^{\text{op}}$ when $\mathcal{C}$ is a category within the Grothendieck universe. Based on some results by V. Voevodsky we show how this can be adapted to build a universe $U$ of Kan complexes which are small in the sense of the Grothendieck universe. Now working in the fibre over $U$ and factoring there the diagonal on the generic type $a : U \vdash \text{El}(a)$ as an anodyne cofibration followed by a Kan fibration gives rise to an interpretation of $a : U, x, y : \text{El}(a) \vdash \text{Id}(\text{El}(a), x, y)$. Performing this construction in the slice over $U$ avoids in a simple way the usual problems with stability under substitution.

Finally, we discuss Voevodsky’s Equivalence Axiom which allows one to reason in a type theory identifying equality with isomorphism.