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Richard Cornes is Professor, School of Economics, University of Nottingham and Roger Hartley is Professor, Department of Economics, Keele University.

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Abstract

Noncooperative games in which each player's payoff function depends on an additively separable function of every player's choice variable may be transformed into an aggregative game, which may be analysed using the concept of 'share functions'. The resulting approach avoids the proliferation of dimensions as the number of players is increased. We show how this approach may be exploited to provide a simple treatment of existence, uniqueness and comparatvie statics in common models that arise in analyses of monopolistic competition, public goods, and rent-seeking contests.

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DISGUISED AGGREGATIVE GAMES

RICHARD CORNES
School of Economics
University of Nottingham
University Park
Nottingham NG7 2RD
England

ROGER HARTLEY
Department of Economics
Keele University
Keele
ST5 5BG
England

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1 Introduction

The noncooperative game that lies at the heart of many economic applications has a great deal of special structure that may be exploited to simplify their analysis. Specifically, player i's payoff function in such games takes the form $u_i(x_i; X)$, where x_i is player i's own chosen quantity, and X is the unweighted sum of every player's quantity, including that of player i. We call such games *aggregative*.

The typical analysis of such games uses the players' best response functions, which define player i's most preferred choice as a function of the choices of all other players. This approach presents the economist with the problem of exploring the properties of a mapping from E^n to E^n where n, the number of players, may be very large. Recently, several authors have exploited the structure of aggregative games to simplify their analysis. Okuguchi (1993) and Cornes, Hartley and Sandler (1999) use the replacement function, $x_i = r_i(X)$. This defines each player's most preferred choice as a function of the aggregate X. Using this approach, a Nash equilibrium may be characterized as a value of X such that $\sum r_i(X) = X$. By avoiding the proliferation of dimensions as the number of players is increased, this approach dramatically simplifies the analysis and permits a better intuitive and geometric feel for the precise way in which the properties of the model depend on the assumptions. The replacement function is a very convenient tool for analyzing games with monotonic best response functions. However, it is less so for games in which best response functions fail to be monotonic and when considering issues such as uniqueness and comparative statics. However, in many such games it turns out that the function $s_i(X) = r_i(X)/X$ is monotonic in X. These include those of open access resources, rent-seeking contests, and Cournot oligopoly. For this reason, Cornes and Hartley (2000) use the function $s_i(X)$, which they label the share function. Use of share functions retains the dimensional attractions of the replacement function and at the same time extends the scope of the analysis to a significantly broader class of applications.

In view of the dramatic simplification achieved by working with replacement and share functions, it is natural to consider whether their scope can be extended beyond games with the reducible structure alluded to above. Our first aim is to persuade the reader that the class of aggregative games is far from exhausting the situations in which the replacement and share functions may profitably be applied. Our second is to identify the class of games for which replacement or share functions may be exploited.

This paper makes two claims. Consider any game in which each player's payoff can be written as $u_i(x_i; t(x))$, where $t(\cdot)$ is an additively separable function of all players' choices, and need not be symmetric. Clearly, any such game may be transformed into an aggregative game, analysis of which can be facilitated by application of the share function. Our first claim is that games in which replacement (or share) functions exist can be transformed into aggregative games, at least if mild regularity conditions are imposed on the function $t(\cdot)$. The unweighted sum is just one member of the family of additively separable aggregator functions $t(x)$: Our second is that the extension of the replacement or share function approach to this larger class of games enables us to exploit the aggregative structure of a number of interesting economic models. Perhaps the simplest and clearest of these is the rentseeking model with general technology studied by Tullock (1980) and others. Another significant model is the symmetric model of monopolistic competition discussed by Spence (1976) and Dixit and Stiglitz (1977), and applied to macroeconomic issues by Blanchard and Kiyotaki (1987). Mankiw and Romer (1991) and Dixon and Rankin (1995) include a number of papers that explore its application to macroeconomic issues, and Sen (1996) and Grossman (1992) provide useful sets of readings dealing with industrial organization and international trade. Our final example generalizes the public good model suggested by Cornes (1993), in which each player cares about her own private good consumption and also her contribution to a public good, the total quantity of which is a CES function of individual contributions.

2 Aggregative Games, Replacement Functions and Share Functions

Consider a simultaneous move game Γ in which I denotes the set of players and X_i the strategy set of player i. If $x_i \in X_i$ for all $i \in I$, we write x for the strategy profile $(x_i)_{i \in I}$ and x_{-i} for the strategy profile $(x_j)_{j \in I \setminus \{i\}}$. Similarly, if $S_i \subseteq X_i$ for all $i \in I$, we write S for $\prod_{j \in I} S_j$;the direct product of the S_i, and S_{-i} for $\prod_{j \in I \setminus \{i\}} S_j$. Each player has a payoff function $\pi : X \to \mathbb{R}$. Throughout we examine only pure strategies.

If f is a function on X, and $S_i \subseteq X_i$ for all $i \in I$, it will prove convenient to use the notation $f(S)$ for the set $\{\theta : \theta = f(s) \text{ for some } s \in S\}$. Note that if any of the S_i is empty, so is $f(S)$.

We begin with a formal definition:

Definition 1 The game G is aggregative if and only if the payoff function of every player may be written as

$$u_i = \pi_i\left(x_i, \sum_{j=1} x_j\right) = \pi_i(x_i, X) \qquad (1)$$

We write individual players' best response functions as

$$x_i^{BR} = b_i(X_{-i});$$

where $X_{-i} \equiv X - x_i$. For any given X we will write $\Psi_i(X)$ for the set of x_i satisfying the conditions $0 \leqslant x_i \leqslant X$ and $x_i = b_i(X - x_i)$:

For a given value of X, this set may be empty or may have many elements. We shall concentrate on situations in which, for all non-negative values of X, $\Psi_i(X)$ contains the single element $r_i(X)$. We refer to $r_i(X)$ as the replacement function of player i. More formally, player i's replacement function is defined as follows:

Definition 2 If, for any value of $X \geqslant 0$, there is a unique value $x_i^{BR} = r_i(X)$ such that x_i^{BR} is a best response to $\left(X - x_i^{BR}\right)$, then the function $r_i(X)$ is the *replacement function of player i.*

The label "replacement function" has an intuitive rationale. Consider a given value of the total, X. Now ask the question: "Given X, is there an amount Y, $0 \leqslant Y \leqslant X$, such that, if the quantity Y were taken away from X, player i's best response to the remaining quantity, $(X - Y)$, would precisely replace Y?" If, for any given quantity, X, there is a unique Y with this property, then it is described by the replacement function.

The replacement function suggests a simple characterization of a Nash noncooperative equilibrium. The value X^* is associated with a Nash equilibrium if and only if $\sum_{j \in I} r_j(X^*) = X^*$. In contrast to the best response function approach, which models a Nash equilibrium of an n-player game as a fixed point of a mapping from E^n to E^n, the replacement function models it as a fixed point associated with a function defined on the real line. Having derived each player's replacement function from her own optimizing problem, we simply add the replacement functions together and look for points where the graph of the resulting aggregate replacement function, $\Phi(X) = \sum_j r_j(X)$, crosses the 45° line.

For reasons alluded to in the introduction - for example, when proving uniqueness - it may be more convenient to work with the share function rather than the replacement function. Player i's share function is defined as follows:

Definition 3 Let player i have a replacement function, $x_i = r_i(X)$. Then, for all $X > 0$, the function $s_i(X) = r_i(X)/X$ is the **share function** of player i.

A Nash equilibrium is an allocation at which $\sum_{j \in I} s_j(X^*) = 1$. The simplicity of the share function arises from the following observations. If (i) each player's share function is continuous and monotonic decreasing wherever the share value is strictly positive, (ii) all share functions approach or equal 0 for large X, and (iii) each player's share function has an least upper bound of unity, then it follows that there exists a unique Nash noncooperative equilibrium. Like the replacement function, the share function allows us to work with functions defined on the real line. It has the attraction of being conveniently monotonic in a wide range of applications, and it has the added attraction of often possessing a natural interpretation - for example, in Tullock rentseeking games, player i's share is simply the probability that i wins the rent.

3 Disguised Aggregative Games: The Main Theorem

This section characterizes those aggregative games for which a replacement function (or correspondence) exists. First, however, we sketch the intuition that lies behind our analysis. If we are to avoid Bellman's curse of dimensionality, we certainly need to be able to determine each player's behaviour with knowledge only of some single aggregate, or 'sufficient statistic', of the game. Thus, we need to be able to write each player's payoff function in the form $\pi_i[x_i; t(x)]$ where x_i is player i's choice variable, $x \equiv (x_1; x_2; ...; x_n)$, and the function $t(x)$ is some aggregate of all players' choice variables which may be thought of as a 'sufficient statistic'. Knowledge of the single value $t(x)$, together with the player's own choice x_i, is sufficient to determine player i's payoff. However, the existence of such an aggregate is not, by itself, sufficient to imply the existence of a replacement function. Not only the payoff function, but also the best response of player i, $\hat{b}_i$, should depend only on the value of the sufficient statistic $t(x)$; and not on the details of the values of its arguments. To see this, consider the first order condition that characterizes a player's best response. If we suppose that $\hat{b}_i > 0$, then it is determined implicitly by the requirement that

$$\frac{\partial \pi_i[\hat{b}_i; t(x)]}{\partial x_i} + \frac{\partial \pi_i[\hat{b}_i; t(x)]}{\partial t}\frac{\partial t(x)}{\partial x_i} = 0:$$

Suppose, for example, that $t(x) = x_1^2 + x_2^3 + x_3^4$. Then the first-order condition for player 1's problem implies that $\frac{\partial \psi_1[\hat{b}_1; t(x)]}{\partial x_1} + \frac{\partial \psi_1[\hat{b}_1; t(x)]}{\partial t} 2x_1 = 0$, which has the form $\omega[\hat{b}_1; t(\cdot)] = 0$. By contrast, if $t(x) = x_1 x_2 + x_3^2$, then the first-order condition implies that $\frac{\partial \psi_1[\hat{b}_1; t(x)]}{\partial x_1} + \frac{\partial \psi_1[\hat{b}_1; t(x)]}{\partial t} x_2 = 0$, which has the form $\omega[\hat{b}_1; x_2; t(\cdot)] = 0$. In this latter case, it is clear that a replacement function cannot be defined, since knowledge not only of the aggregate $t(\cdot)$, but also of x_2, is required in order to determine the value of $\hat{b}_1$. If $\hat{b}_i$ is to depend on the value of $t(\cdot)$, but be independent of the individual x_j's, $j \neq i$, these examples suggest that $t(\cdot)$ be additively separable, so that $\frac{\partial t(x)}{\partial x_i} = \phi_i[x_i; t(x)]$. This conclusion is verified by our central proposition. First, however, our demonstration is significantly simplified if we make one further assumption, the nature of which is clarified in the following definition:

Definition 4 The sufficient statistic $t(x)$ is said to be *regular* if and only if it is twice continuously differentiable and all first partial derivatives are positive for all $x \in X$

Note that Definition 4 really only requires all first partial derivatives to have the same sign. If this were negative, we could satisfy the definition by replacing t with $-t$ and redefining the function $\psi_i(\cdot)$. Armed with this definition, we can now state our central proposition:

Proposition 1 Suppose each X_i is an interval of real numbers. Suppose $t(x)$ is regular and, for $i = 1, \ldots, n$ and all $x \in X$, $\partial t / \partial x_i$ is a positive function of x_i and t alone. Then there exist strictly increasing functions $H : \mathbb{R} \to \mathbb{R}$ and $F_i : X_i \to \mathbb{R}$ for $i \in I$ such that

$$t(x) = H^{-1}\left[\sum_{i=1} F_i(x_i)\right] \quad \text{for all } x \in X. \tag{2}$$

Proof. By hypothesis, there exist functions $\phi_i(x_i; \tau)$ for $i \in I$ such that

$$\frac{\partial t(x)}{\partial x_i} = \phi_i(x_i; t(x)). \tag{3}$$

The proof proceeds by finding the general solution to these equations via a process of successive refinement of the functions $\phi_i(\cdot)$. We start by showing that they are separable. First, observe that, for all x,

$$\begin{aligned} \frac{\partial^2 t(x)}{\partial x_j \partial x_i} &= \frac{\partial \phi_i(x_i; t(x))}{\partial \tau} \frac{\partial t(x)}{\partial x_j} = \frac{\partial \phi_i(x_i; t(x))}{\partial \tau} \phi_j(x_j; t(x)) \\ &= \frac{\partial^2 t(x)}{\partial x_i \partial x_j} = \frac{\partial \phi_j(x_j; t(x))}{\partial \tau} \phi_i(x_i; t(x)) \end{aligned} \tag{4}$$

Dividing both sides by $\phi_i\phi_j$, this can be rewritten

$$\frac{\partial}{\partial \tau}\ln\left[\phi_i(x_i;\tau)\right]=\frac{\partial}{\partial \tau}\ln\left[\phi_j(x_j;\tau)\right]$$

so that

$$\frac{\partial^2}{\partial x_i \partial \tau}\ln\left[\phi_i(x_i;\tau)\right]=0.$$

It follows that there are functions $\overline{f_i}(x_i)$ and $\overline{g_i}(\tau)$ for which

$$\ln\left[\phi_i(x_i;\tau)\right]=\overline{f_i}(x_i)+\overline{g_i}(\tau)$$

and therefore

$$\phi_i(x_i;\tau)=\hat{f}_i(x_i)\,g_i(\tau)$$

where

$$\hat{f}_i(x_i)=\exp\left[\overline{f_i}(x_i)\right] \text{ and } g_i(\tau)=\exp\left[\overline{g_i}(\tau)\right]$$

Now observe that $g_i(\cdot)$ can be taken as independent of i since, substituting for ϕ_i in (4), we have

$$\hat{f}_i(x_i)\,g_i'(\tau)\,\hat{f}_j(x_j)\,g_j(\tau)=\hat{f}_j(x_j)\,g_j'(\tau)\,\hat{f}_i(x_i)\,g_i(\tau);$$

which implies that

$$\frac{d}{d\tau}\left\{\ln\left[g_i(\tau)\right]-\ln\left[g_j(\tau)\right]\right\}=0.$$

Hence, $\ln[g_i(\tau)]-\ln[g_j(\tau)]$ is a constant whose value depends on i and j but not on τ. For this to be true, we require that $g_i(\tau)=\alpha_i g_1(\tau)=\alpha_i h(\tau)$, say, for all i. Writing f_i for $\alpha_i\hat{f}_i$, we conclude that ϕ_i has the form

$$\phi_i(x_i;\tau)=f_i(x_i)\,h(\tau) \text{ for all } i.$$

Regularity of t entails positive first derivatives so that $\phi_i(x_i;t(x))$ is positive. Hence $h(t(x))$ cannot vanish. By continuity it must be either positive for all $x\in X$ or negative for all $x\in X$. But the latter case can be transformed into the former be redefining h as $-h$ and each f_i as $-f_i$.

Finally, define

$$H(\tau)=\int_1^{\tau}\frac{1}{h(s)}ds$$

and note that H is a strictly increasing function. Writing $T(x)=H[t(x)]$, we have

$$\frac{\partial T(x)}{\partial x_i}=H'[t(x)]\frac{\partial t(x)}{\partial x_i}=\frac{1}{h(t(x))}\phi_i(x_i;t(x))=f_i(x_i); \tag{5}$$

which implies

$$\frac{\partial^2 T(x)}{\partial x_i \partial x_j} = 0 \text{ for } i \neq j.$$

Therefore T is additively separable. So t can be written as the term in brackets in (2) where, for $i \in I$, (5) gives

$$\frac{dF_i}{dx_i} = f_i > 0;$$

which implies that F_i is a strictly increasing function. ∎

4 Disguised Aggregative Games: Three Examples

In this section we present three models which, when transformed in an appropriate manner, generate aggregative games. In each case we start with a fairly general formulation and confirm the aggregative structure of the transformed game. We then indicate briefly how the imposition of further structure, in the form of specific assumptions concerning preferences or technology, allows us to infer such properties as the existence of a unique equilibrium and to analyze some comparative static properties.

4.1 A Model of Imperfect Competition

Let total demand behaviour for a vector of commodities $(z; q) \equiv (z; q_1; q_2; \ldots; q_n)$ be rationalized by the quasilinear utility function

$$u(z; q) = z + h\left(\sum_k f_k(q_k)\right); \tag{6}$$

where z is the quantity of a numeraire good, q_j is the output of commodity j, $j = 1, \ldots, n$. We assume that the functions $h(\cdot)$ and $f_j(\cdot)$ are everywhere strictly increasing and differentiable. Assume that commodity j is produced by a single firm - which we call firm j - and denote the total cost function of firm j by $c_j(q_j)$. The inverse demand function for commodity j is

$$p_j = \frac{\partial u}{\partial q_j} = h'\left[\sum_k f_k(q_k)\right] f'_j(q_j)$$

and at a Nash-Cournot equilibrium firm j chooses q_j to maximize its profits

$$\pi_j \equiv p_j q_j - c_j(q_j) = h'\left[\sum_k f_k(q_k)\right] f_j'(q_j)\, q_j - c_j(q_j)$$

In order to transform this into an aggregative game, we introduce the variables x_k and X, where $x_k = f_k(q_k)$ and $X = \sum_j x_j$. Substituting into (6), the induced preferences over the x_j variables are

$$\upsilon(z; X) = z + h\left(\sum_{j=1} x_j\right) = z + h(X):$$

The representative consumer regards the transformed variables x_j as perfect substitutes for each other. In terms of the transformed variables, firm j's maximand may be written as

$$\begin{aligned}
\pi_j \quad &\equiv \quad p_j q_j - c_j(q_j) = h'[X]\, f_j'[g_j(x_j)]\, g_j(x_j) - c_j[g_j(x_j)] \\
&= \quad \frac{h'[X]\, f_j'[g_j(x_j)]\, g_j(x_j)}{x_j} x_j - c_j[g_j(x_j)] \\
&= \quad \psi(x_j; X)\, x_j - \chi_j(x_j)
\end{aligned}$$

where $\psi(\cdot)$ is the demand price for the transformed quantity X and $\chi_j(x_j) = c_j[g_j(x_j)]$ is the unit cost of producing x_j. Since the individual firm's profit can be written as a function of the two arguments, x_j and X, we may draw the following inference:

Claim 4.1.1 The imperfect competition model in which demand behaviour is rationalised by (6) may be transformed into an aggregative game.

The trick of transforming variables was first suggested by Spence (1980), who also assumed a utility function in which the argument of $h(\cdot)$ has a CES form:

$$u(z; q) = z + h\left(\sum_k q_k^{\alpha}\right): \tag{7}$$

The inverse demand function for variety j is then

$$p_j = \frac{\partial u(\cdot)}{\partial q_j} = \alpha h'(\cdot)\, q_j^{\alpha - 1}$$

and firm j's revenue is

$$p_j q_j = \alpha h'(X)\, q_j^{\alpha} = \alpha h'(X)\, x_j = \psi(X)\, x_j:$$

Observe that in this case the function $\psi(\cdot)$, which Yarrow (1985) calls the "pseudoinverse demand function" for the industry's output, is independent of the firm's own output. In short, the model has been transformed into a conventional undifferentiated product model where X, the total 'utility output' and $\psi(X)$ is its inverse demand function:

Claim 4.1.2 If the utility function takes the form (7), then the imperfect competition model may be transformed into an aggregative game which is isomorphic to a model of oligopoly with an undifferentiated product.

Consider a special case in which $u(\cdot) = z + h\left(\sum_j q_j^{\alpha}\right)$, $h(\xi) = 2a\xi - b\xi^2$ and $\alpha = 1/2$. Firm j produces at constant unit cost c_j. Then

$$x_j = q_j^{1/2},\ p_j(\cdot) = h'(X)\,\alpha q_j^{\alpha - 1} \text{ and } \psi_j(\cdot) = a - bX$$

In terms of the transformed variables, firm j's problem becomes

$$\max_{x_j} (a - bX)\,x_j - c_j x_j^2 = \max_{x_j} (a - b[X_{-j} + x_j])\,x_j - c_j x_j^2$$

where $X_{-j} \equiv X - x_j$. This is a simple oligopoly model with linear demand function for a single homogeneous product and a quadratic cost function. Observe that, since x_j is an increasing convex function of q_j, the assumption that q_j is produced at constant costs translates into an increasing cost technology for production of 'utility output'. The first order condition for an interior solution requires that

$$(a - bX) - bx_j - 2c_j x_j = 0;$$

or

$$x_j = \frac{a - bX}{b + 2c_j} \text{ for } X < a/b.$$

Taking account of the nonnegativity constraint, the replacement function is therefore

$$x_j = \max\left\{\frac{a - bX}{b + 2c_j}; 0\right\}.$$

A Nash equilibrium is an allocation at which

$$\sum_{j=1} x_j = \sum_{j=1} \max\left\{\frac{a - bX}{b + 2c_j}; 0\right\} = X.$$

Figure 1 shows the individual and the aggregate replacement functions for a 3-firm example in which $c_1 < c_2 < c_3$. Having solved for the aggregate value Z, we can then readily solve for values of the individual x_j's, and the individual output levels, q_j.

Suppose, instead, that costs are identical across firms, so that $c_j = c$ for all j. Then, for a given value of n, the Nash equilibrium value of X is

$$X = \frac{an}{2c + (n+1)b}$$

This expression allows us to solve for the equilibrium value of X in the competitive limit where n, the number of producers, becomes large: as $n \to \infty$, $X \to a/b$.

4.2 A Generalized Pure Public Good

In an n-player pure public good voluntary contribution game, player i's preferences are represented by an increasing and strictly quasiconcave function $u_i = u_i(y_i; Q)$, where y_i is the quantity consumed of a private good, Q is the aggregate level of a pure public good, and both goods are normal. Q is a strictly increasing concave function of individual contributions: $Q = \Phi(q)$, where $q \equiv (q_1; q_2; \ldots; q_n)$ and $q_i \geq 0$ is i's contribution to the public good. Player i's budget constraint requires that $y_i + q_i \leq m_i$, where m_i is exogenous money income. The individual player's optimizing problem is

$$\underset{y_i; q_i}{\text{Maximize}} \; \{u_i[m_i - q_i; \Phi(q)]\}$$

for which the Kuhn-Tucker firstorder conditions require that

$$\frac{\partial u_i(y_i;Q)/\partial y_i}{\partial u_i(y_i;Q)/\partial Q} \geq \frac{\partial \Phi(q)}{\partial q_i} \tag{8}$$

and

$$\left[\frac{\partial u_i(y_i;Q)/\partial y_i}{\partial u_i(y_i;Q)/\partial Q} - \frac{\partial \Phi(q)}{\partial q_i}\right] q_i = 0. \tag{9}$$

The first and second terms respectively in the square brackets in (9) are, respectively, the marginal rate of substitution and the marginal rate of transformation between y_i and Q. In the canonical pure public good model, $\Phi(q) \equiv \sum_{j=1}^{n} q_i$ and $\partial \Phi(\cdot)/\partial q_i = 1$. The present analysis allows a broader class of aggregator functions. Before introducing our specific functional form, we consider briefly the implications of imposing additive separability on the aggregator function $\Phi(\cdot)$:

- PG.1 Let $\Phi(\cdot)$ be additively separable, and write it as

$$\Phi(\cdot) = G\left[\sum_j f_j(q_j)\right]. \tag{10}$$

Define $z_i = f_i(q_i)$, $i = 1,\dots,n$, and $Z = \sum_{j=1}^{n} z_j$, so that $Q = G(Z)$. We will also write $q_i = f_i^{-1}(z_i) = g_i(z_i) = g_i(\sigma_i Z)$ where $\sigma_i \equiv z_i/Z$ and $Z = G^{-1}(Q) = F(Q)$. Player i's payoff function may be written as

$$\begin{aligned} u_i(y_i; Q) &= u_i\left(m_i - q_i; G\left[\sum_{j=1} f_j(q_j)\right]\right) \\ &= u_i\left(m_i - g_i(z_i); G\left[\sum_{j=1} z_j\right]\right) \\ &= \pi_i(z_i; Z). \end{aligned}$$

This justifies the following claim:

Claim 4.2.1 If the public good aggregator function $\Gamma(\cdot)$ is regular and additively separable, the generalized pure public good model may be transformed into an aggregative game.

Player i's marginal rates of substitution and transformation can be expressed as functions of the transformed variables:

$$\frac{\partial u_i(y_i; Q)/\partial y_i}{\partial u_i(y_i; Q)/\partial Q} = MRS_i(m_i - q_i; Q) \tag{11}$$

$$= MRS_i(m_i - g_i(\sigma_i Z); G(Z)) = \gamma_i(\sigma_i; Z), \text{ say,} \tag{12}$$

and

$$MRT_i = G'\left[\sum_j f_j(q_j)\right] f_i'(q_i) = G'[Z]\, f_i'[g_i(\sigma_i Z)] = \tau_i(\sigma_i; Z) \tag{13}$$

The first order conditions may therefore be rewritten as

$$\gamma_i(\sigma_i; Z) > \tau_i(\sigma_i; Z) \tag{14}$$

$$[\gamma_i(\sigma_i; Z) - \tau_i(\sigma_i; Z)]\, g_i(\sigma_i Z) = 0. \tag{15}$$

We analyze player i's behaviour by examining the first order conditions (14) and (15). Our strategy involves three steps. First, we show that, for any given value of $Z > 0$, the first order conditions are satisfied by only one value of σ_i. This establishes the existence of a share function for player i. Second, we show that the implied share value is negatively related to the value of Z. Finally, we observe that this monotonicity property carries over to the aggregate share function, so that there is only one value Z^* at which $\sum_{j=1}^{n} s_j(Z^*) = 1$. The associated value of Q is the unique Nash equilibrium.

Throughout the rest of this example, we make the following assumption:

• PG.2. $\Phi(q)$ takes the form:

$$\Phi(q) = \left[\sum_{j=1} b_j q_j^{\alpha_j}\right]^{\gamma}; \; b_j > 0; \alpha_j \leqslant 1; \gamma \leqslant \frac{1}{\max\{\alpha_1, \dots, \alpha_n\}}. \qquad (16)$$

Observe that the restrictions placed on the parameters guarantee that assumption PG.1. is satisfied. Inspection of (11) reveals that, for any given value of Z - and therefore of Q - an increase in σ_i implies a reduction in y_i and therefore, in view of the normality assumption, an increase in the value of $\zeta_i(\cdot)$. This is reflected in the shape of the graph of $\zeta_i(\cdot)$ in Figure 2. Now consider player i's marginal rate of transformation. From (13), PG.2 and the fact that $z_j = b_j q_j^{\alpha_j}$

$$\begin{aligned} \tau_i(\sigma_i; Z) &= G'[Z] f_i'[g_i(\sigma_i Z)] = \gamma Z^{\gamma - 1} \alpha_j b_j q_j^{\alpha_j - 1} \\ &= \gamma [Z]^{\gamma - 1} \alpha_j b_j^{\frac{1}{\alpha_j}} \sigma_j^{\frac{\alpha_j - 1}{\alpha_j}} Z^{\frac{\alpha_j - 1}{\alpha_j}} = k_j \sigma_j^{\frac{\alpha_j - 1}{\alpha_j}} Z^{\frac{\alpha_j \gamma - 1}{\alpha_j}} \end{aligned} \qquad (17)$$

where $k_j = \gamma \alpha_j b_j^{\frac{1}{\alpha_j}} > 0$. Clearly, since by assumption $\alpha_j \leqslant 1$, $\tau_i(\sigma_i; Z)$ is nonincreasing in σ_i. Again, this is shown in Figure 2[1]. This establishes the following result:

Claim 4.2.2 If PG.2 is satisfied, every player has a well-defined share function $s_i(Z)$.

Now consider how σ_i varies in response to changes in Z. An increase in Z, with σ_i held constant, is associated with an increase in Q and a reduction in $y_i = m_i - g_i(\sigma_i Z)$. On both counts, it increases the value of $\zeta_i(\cdot)$ associated with any given value of σ_i. Thus an increase in Z shifts the graph of $\zeta_i(\cdot)$ upwards. At the same time, since $\alpha_j \gamma - 1 \leqslant 0$, (17) implies that an increase in Z reduces the value of $\tau_i(\cdot)$ associated with any given value of σ_i. It therefore shifts the graph of $\tau_i(\cdot)$ downwards. Figure 2 shows the positions of the graphs of $\zeta_i(\cdot)$ and $\tau_i(\cdot)$ associated with the values Z^0 and $Z^1 > Z^0$. Consequently, if the initial share value is positive, an increase in Z implies a strict reduction in the share value:

Claim 4.2.3 If PG.2 is satisfied, then player i's share function $s_i(Z)$ is everywhere nonincreasing, and is strictly decreasing wherever $s_i(Z) > 0$.

[1] The possibility of $\sigma_i = 0$ arises when the graph of $\zeta_i(\cdot)$ lies everywhere above that of $\tau_i(\cdot)$ given the prevailing value of Z.

This implies that the aggregate share function, $s^{\Sigma}(Z) \equiv \sum_{j=1}^{n} s_j(Z)$, is a monotonic decreasing function wherever $s^{\Sigma}(Z) > 0$.

The implied properties of the aggregate share function lead readily to the following observation:

Claim 4.2.4 *If PG.2 is satisfied, then the generalized public good game possesses a unique Nash equilibrium.*

We finish this example with a brief analysis of the implications of income transfers in the present model. It is well-known that the conventional public good model, in which $\alpha_j = b_j = \gamma = 1$ for all j and consequently $Q = \sum_{j=1}^{n} q_j$, exhibits the neutrality property. A transfer of initial income from one positive contributor to another has no effect on the equilibrium resource allocation. We now show that, more generally, the neutrality property does not hold and we investigate the circumstances under which a transfer from a higher income to a lower income contributor increases the aggregate equilibrium provision of the public good. Our analysis generalizes the result of Cornes (1993), who demonstrates this possibility and the consequent possibility of a Pareto improving redistribution when the aggregator function is a symmetric Cobb-Douglas, obtained by letting $\alpha \to 0$.

At any given Nash equilibrium, a transfer of income from one player to another will increase equilibrium provision, leave it unchanged or reduce it according to whether the sum of the two players' share value rise, remain unchanged or fall at that unchanged provision level. Although we cannot solve explicitly for an individual's share function, we can write down an explicit function for its inverse, which we write as $m_i = \mu_i(\sigma_i; Z)$. The partial derivatives of this function with respect to σ_i provides the information we need concerning the shape of this function. This in turn will allow us to infer the consequences of income redistribution for total provision of the public good.

Let all individuals have the same Cobb-Douglas utility function, $u_i(y_i; Q) = y_i Q$. The public good aggregator function is given by (16) with all $b_i = 1$ and $a_i = a_j$ for all $i \neq j$. At equilibrium, the share value of a strictly positive contributor is determined implicitly by the requirement that $\zeta_i(\cdot) = \tau_i(\cdot)$. Dropping the 'i' subscript, the relationship between an individual's income and that individual's share value at a given equilibrium is:

$$\frac{Z^{\gamma/\alpha}}{m - z^{1/\alpha}} = \gamma\alpha Z^{(\gamma-\alpha)/\alpha} z^{(\alpha-1)/\alpha}$$

or

$$m = \mu(\sigma; Z) = \frac{Z^{1/\alpha}}{\alpha\gamma}\left[\sigma^{(1-\alpha)/\alpha} + \alpha\gamma\sigma^{1/\alpha}\right]. \qquad (18)$$

The partial derivatives of $\mu(\sigma; Z)$ with respect to σ are

$$\frac{\partial \mu}{\partial \sigma} = \frac{Z^{1/\alpha}}{\alpha^2 \gamma}\left\{(1-\alpha)\,\sigma^{(1-2\alpha)/\alpha} + \alpha\gamma\sigma^{(1-\alpha)/\alpha}\right\} > 0 \text{ if } \sigma > 0, \tag{19}$$

$$\frac{\partial^2 \mu}{\partial \sigma^2} = \frac{(1-\alpha)\, Z^{1/\alpha}\sigma^{(1-3\alpha)/\alpha}}{\alpha^3 \gamma}\{1 - 2\alpha + \alpha\gamma\sigma\}. \tag{20}$$

The signs of these expressions depend on the assumed value of the parameter α. It is convenient to consider five cases, according to the value of α. If Z is held constant, and μ is regarded as a function of σ, the following conclusions may be drawn from (19) and (20).

CASE I: $\alpha < 0$: $\mu(\sigma; Z)$ is (i) strictly decreasing and (ii) strictly concave.

A player's share is a strictly decreasing and strictly convex function of income. The graph of $\mu(\sigma : Z)$ is shown in Panel (a) of Figure 3. If an equalizing transfer is made from a higher to a lower income individual, the donor's share value will rise, but by less than the fall in that of the recipient. Thus the equilibrium level of Z will fall. Recall that if $\alpha < 0$, Z is a decreasing function of Q. Consequently, we can conclude

Claim 4.2.5 If $\alpha < 0$, then

1. an equalising redistribution from richer to poorer leads to a greater quantity of the public good, and
2. equalising incomes amongst a subset of players increases the quantity of public good.

CASE II: "$\alpha = 0$": We use inverted commas because the function $\Psi(.)$ is not well-defined for $\alpha = 0$. However, by letting $\alpha \to 0$ from above, we generate the Cobb-Douglas form. This is the case analysed by Cornes (1993), who shows that an equalizing transfer from a higher to a lower income individual will lead to an increase in the equilibrium level of Q. Because this requires a different treatment, and has already been analysed, we do not explicitly treat this case here.

CASE III: $0 < \alpha \leqslant 1/2$: $\mu(\sigma : Z)$ is (i) strictly increasing and (ii) strictly convex.

In this, and in the remaining cases, Z is increasing in Q. Since the player's share is a strictly increasing and strictly concave function of income, a transfer from a high to a lower income individual will reduce the former's share value, but will increase that of the latter by more. This is clearly seen in Panel (b) of the figure. Therefore at the initial equilibrium value of Z the aggregate share will increase. Thus the equilibrium value of Z, and therefore of Q, increases. Again we can state

Claim 4.2.6 If $0 < \alpha \leqslant 1/2$, then

1. an equalising redistribution from richer to poorer leads to a greater quantity of the public good.
2. Equalising incomes amongst a subset of players increases the quantity of public good.

CASE IV: $1/2 < \alpha < 1$: $\mu(\sigma : Z)$ is (i) strictly increasing, (ii) strictly concave for $0 \leqslant \sigma < (2\alpha - 1)/\alpha\gamma$ and (iii) strictly convex for $(2\alpha - 1)/\alpha\gamma < \sigma \leqslant 1$. Furthermore, $\mu(\sigma; Z)$ has (iv) slope unbounded above as σ approaches 0, and (v) a point of inflection at $\sigma = (2\alpha - 1)/\alpha\gamma$.

Panel (c) shows the graph of $\mu(\sigma : Z)$. Since there is a critical value of income below which the share function is concave, and above which it is convex, a transfer from a higher to a lower income individual has an ambiguous effect on the equilibrium level of Q. The following claim can be made:

Claim 4.2.7 If $1/2 < \alpha < 1$, then

1. an equalising redistribution from richer to poorer when the richest person involved has income no greater than $m^*(Z)$, where Z is the equilibrium level of Z, leads to a smaller quantity of the public good, and
2. equalising incomes amongst a subset of players when the richest person involved has income no greater than $m^*(Z)$, where Z is the equilibrium level of Z, leads to a smaller quantity of the public good.

Note that if there is a sufficient number of players - say a finite set of types with many of each type or a large number of independent selections from a distribution over income - then all shares will be less than $(2\alpha - 1)/\alpha\gamma$, which means that all incomes are less than $m^*(Z)$. Consequently, we would expect that, in the presence of many players, equalizing transfers will reduce the equilibrium quantity of Q. The ambiguity of this result may seem anomalous, in view of the lack of ambiguity in cases I-III and the neutrality proposition associated with case V. However, further consideration of case V explains the apparent anomaly.

CASE V: $\mu(\sigma : Z)$ is (i) nondecreasing and (ii) piecewise linear.

This is the standard additive public good model. Transfers between positive contributors do not affect the equilibrium level of public good provision. However, for any given level of Z, there is a level of income $\underline{m}$ such that, for all $m < \underline{m}$, an individual will not contribute. Hence the piecewise linear nature of the relationship between σ and m. A transfer between a high income contributor and a low income noncontributor will reduce the share value of the former. However, the share value of the latter will generally either remain unchanged or increase by less than the fall in the share value of the donor. Therefore, the sum of the two players' share value falls.

4.3 Rent-seeking with a General Technology

Our last example is a Tullock rent-seeking contest. In such contests, players' replacement functions are necessarily non-monotonic. By contrast, share functions retain monotonicity. For this reason, and because each player's share value has a natural interpretation as that player's probability of winning the rent, it is convenient to use the share function approach. Suppose that each of n risk neutral contestants applies effort in order to enhance the probability of winning an exogenous rent, R. The effort level applied by player i is denoted by e_i. Then the probability that player i wins the rent is given by

$$p_i = \frac{f_i(e_i)}{f_i(e_i) + \sum_{j \neq i} f_j(e_j)}$$

Player i's payoff is the expected value of wealth. The payoff function is

$$\begin{aligned} u_i(e) &= \left[\frac{f_i(e_i)}{f_i(e_i) + \sum_{j \neq i} f_j(e_j)}\right](I_i + R - e_i) \\ &\quad + \left[1 - \frac{f_i(e_i)}{f_i(e_i) + \sum_{j \neq i} f_j(e_j)}\right](I_i - e_i) \qquad (21) \\ &= \left[\frac{f_i(e_i)}{f_i(e_i) + \sum_{j \neq i} f_j(e_j)}\right] R + (I_i - e_i); \end{aligned}$$

where I_i is i's exogenous initial wealth and $f_i(e_i)$ is player i's effective input as a rentseeker. For simplicity, we assume that $f_i(0) = 0$, and that $f_i'(e_i) > 0$ and $f_i''(e_i) \leq 0$ for all $e_i \geq 0$. The function $f_i(e_i)$ has been called "player i's production function for rent."

To convert this game into one that is aggregative, define $x_i = f_i(e_i)$ for all i. Since $f_i(\cdot)$ is everywhere strictly increasing, we can define its inverse: $e_i = f_i^{-1}(x_i) = g_i(x_i)$. The function $g_i(\cdot)$, which is everywhere increasing and convex, may be interpreted as the total cost of producing the intermediate input x_i that, in turn, influences player i's probability of success. The properties of $f_i(\cdot)$ imply that its inverse has the following properties: $g_i(0) = 0$, and $g_i'(x_i) > 0$ and $g_i''(x_i) > 0$ for all $x_i > 0$. Stated in terms of the variable x_i, player i's objective function is

$$\nu(x_i; X_{-i}) = \left[\frac{x_i}{x_i + \sum_{j \neq i} x_j} \right] R + (I_i - g_i(x_i)); \qquad (22)$$

where $X_{-i} \equiv \sum_{j \neq i} x_j$. Player i's payoff may alternatively be expressed as a function of x_i and $X \equiv x_i + \sum_{j \neq i} x_j$:

$$\pi(x_i; X) = \left[\frac{x_i}{X} \right] R + [I_i - g_i(x_i)]$$

Thus we have:

Claim 4.3.1 The Tullock rent-seeking contest with nonincreasing returns may be transformed into an aggregative game.

Standard reasoning shows that, if player i's payoff reaches a maximum at a positive value of x_i, then the first order condition requires the value of the implied share, $\sigma_i = x_i / X$, to satisfy the condition

$$(1 - \sigma_i) = \frac{g_i'(\sigma_i X) X}{R}. \qquad (23)$$

Moreover, the convexity of $g_i(\cdot)$ ensures that this condition is not only necessary, but also sufficient, for $\nu(x_i; X_{-i})$ to be maximized by a positive value of x_i. Furthermore, since the left hand side of (23) is decreasing in σ_i and the right hand side is increasing in σ_i, it can be satisfied at most by a single value of σ_i. In summary, we can draw the following inference:

Claim 4.3.2 A share function for player i exists. The share value takes the value 0 if and only if $g_i'(0) > 0$ and $X > R[g_i'(0)]^{-1}$. Otherwise, it is determined by the following condition:

$$(1 - \sigma_i) R = g_i'(\sigma_i X) X$$

Furthermore, the convexity of $g_i(\cdot)$ implies that an increase in X must be accompanied by a fall in σ_i in order to maintain (23):

Claim 4.3.3 In the Tullock rent-seeking game with nonincreasing returns, σ_i is nonincreasing in X, and is strictly decreasing in X wherever $\sigma_i > 0$.

Inspection of (23) reveals a further significant property of the share function. As $X \to 0$, so to does the right hand side of (23), implying that $\lim_{X \to 0} \sigma_i = 0$. It may help to summarize the salient properties of the share function:

Claim 4.3.4 Player i's share function has the following properties:

(i) $s_i(X) \to 1$ as $X \to 0$.

(ii) $s_i(X)$ is continuous

(iii) For all $X < R[g_i'(0)]^{-1}$, $s_i(X)$ is strictly positive and strictly decreasing

(iv) For all $X \geq R[g_i'(0)]^{-1}$, $s_i(X) = 0$

(v) If $g_i'(0) = 0$, $s_i(X) \to 0$ as $X \to \infty$.

In the light of these properties, it is worth observing that if the production function for rent is homogeneous of degree less than zero - an assumption suggested by Tullock and adopted by several other authors - then $\lim_{x_i \to 0} g_i'(\cdot) = 0$ and, as (4.3.4) shows, the share function approaches zero asymptotically as X increases. There is therefore no finite value of X at which player i drops out of the contest. By contrast, if $g_i'(0)$ is strictly positive, then there exists a finite value of X, $\overline{X}_i$, such that $\sigma_i = 0$ for all $X \geq \overline{X}_i$.

Since a Nash equilibrium may be characterized as a value of X, together with implied share values, at which $\sum_{j=1}^{n} s_j(X) = 1$, we can immediately the following result:

Claim 4.3.5 In the rentseeking contest with n risk neutral players, each with a technology characterized by nonincreasing returns to scale, there exists a unique Nash equilibrium.

Finally, we state a number of comparative static properties of the model that may be readily inferred from (4.3.4). First, we consider the effect of an exogenous change in the aggregate value X on player i's payoff. Using (4.3.4),it may be shown that

Claim 4.3.6 Let player i in a rentseeking game be risk neutral and have access to a technology exhibiting nonincreasing returns to scale. Then, if $X^1 > X^0$,

$\pi_i[s_i(X^0); X^0] \geq \pi_i[s_i(X^1); X^1]$

An immediate corollary is the following:

Claim 4.3.7 If additional players enter a rent-seeking game and at least one new player actively participates, then

(i) None of the original players switches from nonparticipation to active participation,

(ii) The probability that an original active participant wins the rent falls,

(iii) All original active participants are made worse off.

Demonstrations of these last two propositions may be found in Cornes and Hartley (2000), which also treats more general rent-seeking contests in which players not only use general convex technologies but also are strictly risk averse. They show that the existence of continuous monotonic share functions carries over to this more general environment and are thereby able to demonstrate the existence of a unique Nash equilibrium, even though players may differ with respect to their attitudes towards risk.

5 Conclusion

The payoff functions of players in a noncooperative n-person game may be written $\pi_i(\mathbf{q})$, where $\mathbf{q}$ is the vector of individual choices. In the absence of further restrictive assumptions, a Nash equilibrium of such a game is modelled as the fixed point of the mapping provided by the players' best response functions. This has the disadvantage of requiring analysis of a mapping in potentially high dimension - as high as the number of players in the game. However, if each player's payoff function can be written in the form $\pi_i\left(q_i; \sum_{j=1}^{n} f_j(q_j)\right)$, an alternative approach is possible. This involves two useful tricks. First, we define a new set of variables, $x_j = f_j(q_j)$ for all j. Then, instead of using best response functions, we introduce and exploit 'share functions'. By these means, we avoid the need to work in a space as large as the number of players in the game. Instead, the relationships we use are all defined on the real line. The resulting analysis can readily handle games with many players, and is not significantly complicated by allowing for the possibility that each player may have idiosyncratic preferences or costs. We have drawn attention to a number of applications in which the addition of heterogeneous players does not give rise to Bellman's 'curse of dimensionality'. Applications include not only symmetric models of monopolistic competition, but also generalized public good models and rent-seeking models of the type suggested by Tullock. Indeed, we have elsewhere used this approach to analyze rent-seeking contests involving many risk-averse players with differing attitudes to risk. In view of the relative ease with which our approach can accommodate a large number of potentially heterogeneous players, we believe that it offers an attractive way of analyzing 'limiting' behaviour of models as the number of players is allowed to increase.

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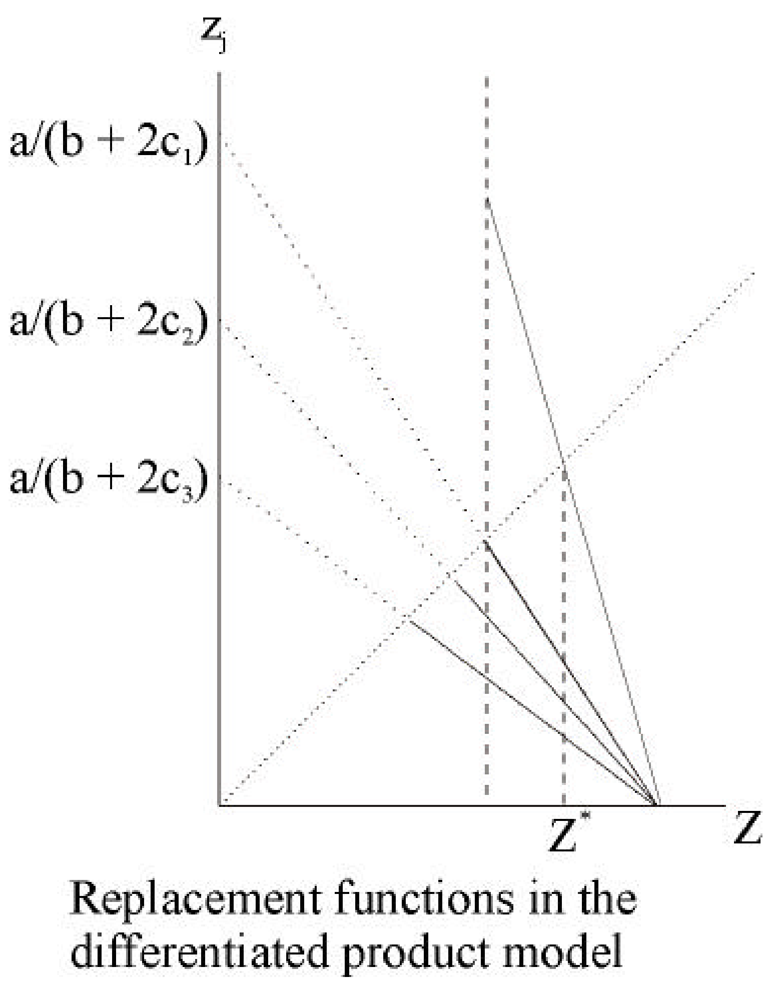
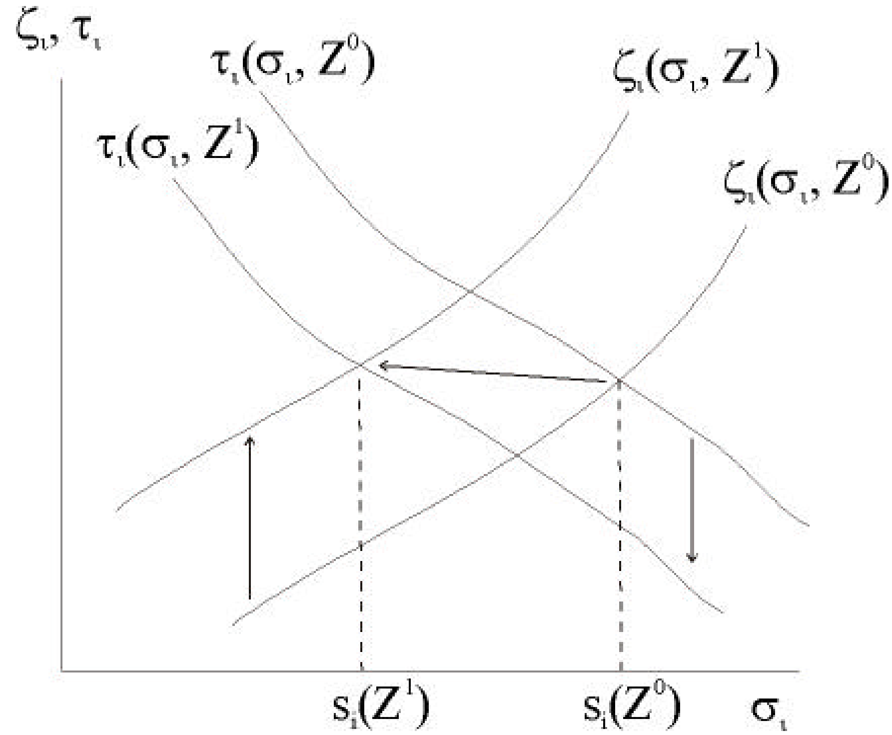

z_j
a/(b + 2c1)
a/(b + 2c2)
a/(b + 2c3)
Z*
Z
Replacement functions in the differentiated product model

Figure 1:



The Public Good Problem

Figure 2:

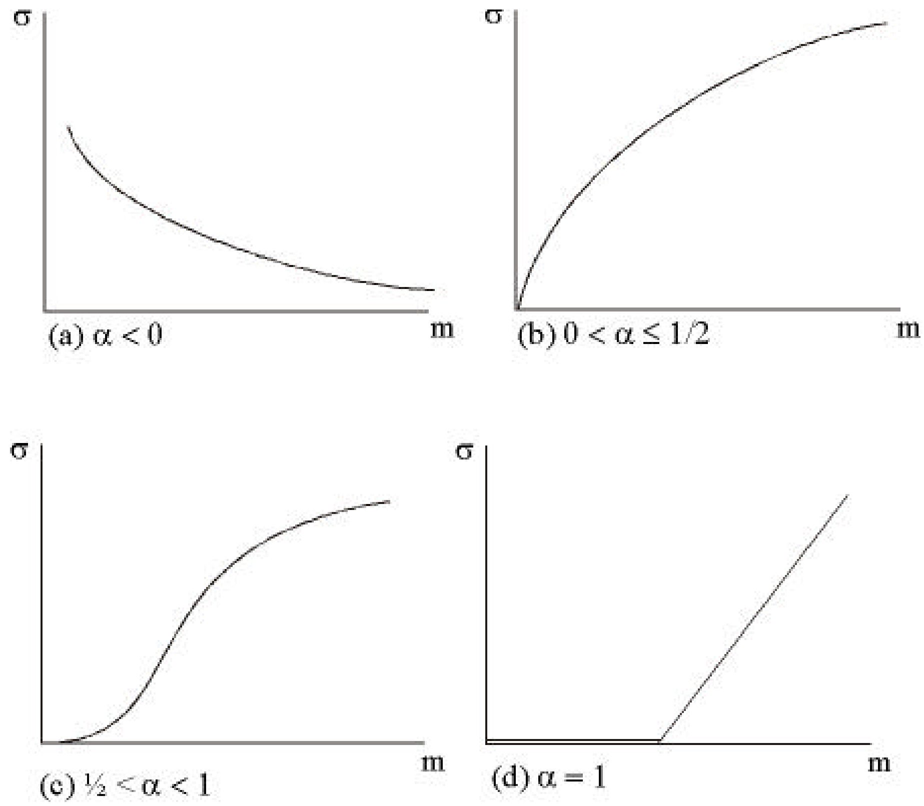


Figure 3: