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Abstract

This paper considers the extent to which expenditure by contestants in imperfectly discriminating rent-seeking contests dissipates all or only part of the rent. In particular, we investigate strategic effects, technological effects and asymmetry under an assumption of diminishing returns to scale. Although asymmetry can reduce dissipation when there are few contestants, we show that this effect disappears in the Nash equilibria of large contests. Similarly, strategic effects are diminished if the cost of entry, which restricts the number of contestants, is fully taken into account. When individual entry costs fall to zero, the reduction in dissipation arising from technological factors is entirely eliminated in the limit. More generally, the dissipation-reducing properties of all three effects operating simultaneously disappear as individual entry fees fall to zero provided the aggregate cost of entry is added to the expenditure of entrants. These conclusions are robust to details of the entry process which can be sequential, in which case the ordering is irrelevant to the limiting results, or simultaneous. Our principal theoretical tool is the share function which expresses the probability of a player winning the contest as a function of aggregate expenditure. However, this methodology has independent interest as it can be applied in many other contexts (not formally analyzed here).

JEL classifications: C72, D72
Key words: entry costs, noncooperative game theory, rent dissipation, rent seeking

1 Introduction

Rent-seeking contests allow participants to expend resources in an attempt to increase their probability of winning the rent, or to increase their share if the rent is divisible. If entry to such contests is free, it is often contended that the aggregate expenditure will be equal to the value of the rent. (See, for example, Becker[3], Krueger[8] and Posner[13].) If this expenditure has no economic value, this leads to the conclusion that the value of the rent measures the waste of resources. However, Tullock[19] argues that this conclusion can "explain too much" and offers examples where there is evidence that aggregate expenditure appears to fall well short of the rent.

Tullock[19] proposed a simple and insightful model of contests in which both under-dissipation and, somewhat controversially, over-dissipation could occur. (In this paper, we concentrate on the former case.) Analysis of this model and its variants have led to the identification of a number of factors leading to incomplete dissipation of the rent. Firstly, strategic effects can reduce expenditure. For example, aggregate expenditure with two identical contestants is half that when there are many similar contestants. (See, for example, Perez-Castrillo and Verdier[12] for a detailed treatment of symmetric Tullock contests.) A second factor reducing rent-seeking is the form of the contest success function (which determines each contestant's probability of winning the rent as a function of the expenditure of each of the contestants). When this takes the generalized logistic form in which the ratio of winning probabilities is equal to the ratio of two functions of the outlays of the players, the proportion of the rent dissipated in large contests depends on the nature of these functions. It is convenient to refer to these as production functions. If all contestants have the same production function which has constant elasticity not exceeding one, the proportion of the rent dissipated is equal to the value of this elasticity. A third factor leading to incomplete dissipation is asymmetry. For example, when there are two types of contestant with different linear production functions, aggregate expenditure can be an arbitrarily small proportion of the rent even if the total number of contestants is large. For indivisible rents, risk aversion can also reduce expenditure (see, for example, Skaperdas and Gan [16] and Cornes and Hartley[5]) but we will assume risk neutrality throughout the present paper.

Our intention here is to explore obstacles to these explanations for partial dissipation. Consider asymmetry. In contests with several distinct types of contestant having bounded marginal products, dissipation will be incomplete when there are few representatives of each type. However, once the numbers of each type become large, the whole rent will be dissipated. The reason is that only one type will make a positive outlay in the large-game limit, effectively rendering the contest a symmetric one in which dissipation is complete. More generally, although technological effects can result in incomplete dissipation even when there are many players, any further reduction due to asymmetry vanishes as the number of contestants of each type becomes large. The reason is similar: one type will dominate the aggregate expenditure on rent-seeking rendering the contest approximately symmetric.

Strategic factors also reduce dissipation when there are few contestants, though this prompts the question of what restricts entry to the contest. When there are barriers in the form of a non-refundable entry fee either paid directly or incurred as an opportunity cost, it is appropriate to count such fees in total expenditure. When a small number of entrants arises as a consequence of a large cost of entry, the latter may dissipate a significant additional portion of the rent, weakening the reduction in dissipation due to strategic effects. Even a small entry fee may be significant if it is paid by many entrants; the aggregate cost of entry can have a finite limit as individual fees approach zero. We investigate this possibility by studying subgame perfect equilibria of a sequential game in which entry decisions are made (sequentially) in the preliminary stages and the Tullock contest is played at the final stage by those players who chose to enter. We show that, typically, when entry costs are small, almost all of the rent is dissipated in equilibrium. In a symmetric contest with bounded marginal product, nearly all this dissipation is in the form of outlays in the final contest. However, when marginal products are unbounded, aggregate entry costs as well as aggregate outlays in the final contest has a finite limit as the entry cost falls to zero. The reduction in dissipation in large games due to technology vanishes once entry costs are factored in. These conclusions persist even when there are several distinct types of player: for small enough entry fees, only one type will choose to enter in equilibrium and we can apply the results for symmetrical contests.

The analysis employs a novel methodology involving extensive use of the share function which expresses the probability of winning the contest as a function of aggregate input (transformed aggregate outlay). We use share functions for expositional convenience in the case of symmetric contests, but they are an essential tool in our study of asymmetric problems and we present new several results about share functions throughout the paper. These results have independent interest, as share functions have wide applicability beyond the context of contest theory although we do not pursue such applications here.

In the next section, we discuss the basic model of contests used throughout the paper as well as introducing and proving several properties of share functions. In Section 3, we investigate rent dissipation in large symmetric contests and generalize well-known results for production functions with constant elasticity to arbitrary concave production functions. In Section 4, we turn to a consideration of asymmetric contests and apply the share-function approach to prove that, when marginal products are bounded, only the most efficient types eventually participate leading to full dissipation in the limit and, when marginal products are unbounded, aggregate expenditure by efficient types swamps that of other types leading again leading to the same level of dissipation as large symmetric contests. In Section 5, we present a model including positive entry costs for symmetric contests and show that subgame perfect equilibria of this model entail (nearly) complete dissipation. Section 6 combines the analyses of Sections 4 and 5 to deduce that dissipation is complete even when there are multiple types possibly with unbounded marginal product. Section 7 concludes and Section 8 contains the proofs of our results.

2 Contests with a finite set of contestants

In this section we review, via a novel analysis, generalized Tullock rent-seeking contests in which the set of contestants is finite, of size $n\,(\geq 2)$, and exogenously determined. Each contestant makes an outlay of x_i in an attempt to secure an indivisible rent of value R. The probability that contestant i wins depends on the complete vector of outlays $\mathbf{x}$ according to a contest success function, $p_i\left(\mathbf{x}\right)$, which satisfies $\sum_{j=1}^{n} p_j\left(\mathbf{x}\right) = 1$ and $p_i\left(\mathbf{x}\right) = 0$ if $x_i = 0$. We assume throughout that all contestants are risk neutral so i's payoff is $Rp_i\left(\mathbf{x}\right) - x_i$. The same formula applies if the rent is divisible, provided we interpret p_i as i's share of the rent.

The contest is a simultaneous-move game and we focus on Nash equilibria. Note that any contestant has the option of making no outlay and receiving zero payoff irrespective of the actions of her opponents. So, equilibrium payoffs must be non-negative: $Rp_i\left(\widehat{\mathbf{x}}\right) - \widehat{x}_i \geq 0$ and summing this over n shows that aggregate expenditure cannot exceed the value of the rent at a pure-strategy Nash equilibrium. When mixed strategies are allowed, Baye et al.[2] show that aggregate expenditure may exceed the rent for some realizations of the uncertainty but not on average.

To obtain more results, we restrict attention to contest success function of the form[1]:

[1]This form of contest success function was introduced by Hirshleifer[7]. An axiomatic justification is given by Skaperdas[15].

$$p_i(\mathbf{x}) = \frac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)}.$$

The function f_i can be viewed as a transformation of contestant i's outlay, x_i, into input $y_i = f_i(x_i)$ and it is convenient to refer to f_i as a production function. We assume that the production technology is continuous, increasing and exhibits decreasing returns to scale. Without such an assumption, equilibria may not be unique, with resultant coordination problems, or may not even exist in pure strategies [5], [16]. It is also convenient to assume f_i can be differentiated.

A1. *For contestant i, f_i is continuous, strictly increasing, concave, differentiable for positive arguments and satisfies* $f_i(0) = 0$.

Under **A1** the derivative is non-increasing and it is natural to write $f_i'(0)$ for the least upper bound of $f_i'(x)$ in $x > 0$ where such a bound exists. Otherwise, we write $f_i'(0) = \infty$. The distinction between bounded and unbounded marginal product turns out to be a key determinant of rent dissipation. For the function $f_i(x) = a_i x^{r_i}$, we have $f_i'(0) = a_i$ if $r_i = 1$ and $f_i'(0) = \infty$ if $r_i < 1$.

It is often easier to work directly with inputs rather than outlays. Specifically, we let g_i denote the inverse function of f_i and rewrite contestant i's payoff function as

$$\pi_i(\mathbf{y}) = \frac{y_i}{Y} R - g_i(y_i),$$

where $Y = \sum_{j=1}^{n} y_j$. **A1** implies that g_i is continuous, strictly increasing, convex, differentiable for positive arguments and satisfies $g_i(0) = 0$. Since π_i is concave and differentiable in y_i, the first-order conditions are necessary and sufficient for a best response. Writing $\widetilde{Y}_i = Y - y_i$ for the aggregate input of contestant i's opponents, the first-order conditions for y_i to be a best response to $\widetilde{Y}_i$ are

$$\frac{\partial \pi_i}{\partial y_i} = \frac{\widetilde{Y}_i}{Y^2} R - g_i'(y_i) \leq 0 \tag{1}$$

with equality if $y_i > 0$. Our analysis relies heavily on the observation that, associated with each $Y > 0$ is a unique Nash equilibrium $\mathbf{y}$ satisfying $Y = \sum_{j=1}^{n} y_j$. (This is implicit in Proposition 2.1 below.) This allows us to define a *share function* $s_i(Y) = y_i/Y$ for each contestant i where y_i is the ith component of the associated equilibrium. For zero input by contestant i to be a best response (1) the inputs of her competitors, implies that $R/Y - g_i'(0) \leq 0$ which can be rewritten as $Y \geq Rf_i'(0)$, using $f_i' = 1/g_i'$. For such Y, $s_i(Y) = 0$. For $Y < Rf_i'(0)$, it follows from (1) that $s_i(Y)$ is the unique solution: σ of

$$(1-\sigma) R = Y g_i' (\sigma Y) . \tag{2}$$

For some production functions, it is possible to solve (2) to obtain an explicit form for the share function; two examples follow.

Example 2.1 *When $f_i(x) = a_i x$,*

$$s_i(Y) = \max \left\{ 1 - \frac{Y}{a_i R}, 0 \right\} . \tag{3}$$

Example 2.2 *When $f_i(x) = a_i \sqrt{x}$,*

$$s_i(Y) = \left(1 + \frac{2Y^2}{a_i^2 R} \right)^{-1} .$$

In general, analytic expressions for the share function will not be available, even for cases such as constant elasticity productions functions. Nevertheless, a number of useful properties of share functions can be derived from Assumption **A1**. The following proposition, proved in Section 8, summarizes the most basic results.

Proposition 2.1 *Suppose **A1** is satisfied for contestant i. Then that contestant has a well-defined and continuous share function for $Y > 0$ which has the following properties:*

1. *positive, differentiable and negative slope for $0 < Y < Rf_i'(0)$,*
2. *approaches unity as $Y \longrightarrow 0$,*
3. *is equal to zero if $Y \geq Rf_i'(0)$,*
4. *approaches zero as $Y \longrightarrow \infty$, if $f_i'(0) = \infty$.*

Note that there are two distinct types of share function. If the marginal product of contestant i is bounded, her share function decreases strictly up to $Y = Rf_i'(0)$ and is equal to zero for larger values. We refer to $Rf_i'(0)$ as the *dropout value* for contestant i. Example 2.1 illustrates such a share function with dropout value $a_i R$. Alternatively, the share function is strictly decreasing for all positive arguments and asymptotic to zero. Example 2.2 illustrates this case.

We use share functions because they allow us to avoid becoming embroiled in the complications of multi-dimensional best-response functions when characterizing Nash equilibria. The link between share functions and Nash equilibria is a consequence of the self consistency condition that requires shares to sum to one in equilibrium. For any $Y > 0$, we define the *aggregate share function*

$$S(Y) = \sum_{j=1}^{n} s_j(Y)$$

and $\widehat{Y}$ is a Nash equilibrium value of Y if and only if $S\left(\widehat{Y}\right) = 1$. The corresponding equilibrium strategy profile (in inputs), $\widehat{\mathbf{y}}$, satisfies $\widehat{y}_i = \widehat{Y} s_i\left(\widehat{Y}\right)$ for each i. Sometimes, we can apply this condition to obtain analytic expressions for equilibria.

Example 2.3 *Suppose there are two contestants with $f_i(x) = a_i x$ for $i = 1, 2$. Both contestants must supply positive input in equilibrium (or the value of the aggregate share function would be less than one) so, using (3),*

$$S\left(\widehat{Y}\right) = 1 - \frac{\widehat{Y}}{a_1 R} + 1 - \frac{\widehat{Y}}{a_2 R} = 1$$
$$\Longrightarrow \widehat{Y} = R\left(\frac{1}{a_1} + \frac{1}{a_2}\right)^{-1} \tag{4}$$

and $\widehat{\mathbf{y}} = (1/a_2, 1/a_1)\,\widehat{Y}^2/R$.

Example 2.4 *As a second example, suppose $f_i(x) = ax^r$ for all n contestants. Then, $\widehat{Y}$ is uniquely determined by the equation $s\left(\widehat{Y}\right) = 1/n$. Substituting in (2) and solving gives*

$$\widehat{Y} = na\left[\frac{(n-1)\,rR}{n^2}\right]^r \tag{5}$$

and $\widehat{y}_i = \widehat{Y}/n$.

In both these examples, there is a unique value of equilibrium aggregate input and therefore a unique equilibrium. This is a general result. Proposition 2.1 implies that the aggregate share function exceeds one for small enough Y, is less than one for large enough Y and is continuous. We can deduce that there is some value of Y at which the aggregate share function is equal to one and, since share functions are strictly decreasing where positive, this value is unique. Hence, there is a unique equilibrium. This provides an alternative proof of a recent result of Szidarovsky and Okuguchi[18].

Corollary 2.2 (Szidarovsky and Okuguchi) *Contests in which* ***A1*** *is satisfied for all players have a unique Nash equilibrium.*

We are primarily interested in the *dissipation ratio*: $\rho = X/R$, where $X = \sum_{j=1}^{n} x_j$ is aggregate expenditure. The following formula allows us to deduce aggregate expenditure from aggregate input:

$$X = \sum_{j=1}^{n} g_j\left(y_j\right) = \sum_{j=1}^{n} g_j\left[Y s_j\left(Y\right)\right]. \tag{6}$$

Example 2.5 (Example 2.3 continued) *Applying (6) and using the formula (3), gives*

$$\widehat{X} = \sum_{j=1}^{2} \frac{\widehat{Y}}{a_j}\left(1 - \frac{\widehat{Y}}{a_j R}\right).$$

Using (4), we find

$$\widehat{\rho} = \frac{2a_1 a_2}{\left(a_1 + a_2\right)^2}.$$

This expression illustrates the role of asymmetry in reducing the proportion of the rent dissipated in rent seeking: $\widehat{\rho} \leq 1/2$ with equality if and only if $a_1 = a_2$. Indeed, the dissipation ratio can be arbitrarily small, if a_1 differs sufficiently from a_2 .

If all players have the same production function, (6) implies $\widehat{\rho} = n g_i\left(\widehat{Y}/n\right)$.

Example 2.6 (Example 2.4 continued) *If $f_i\left(x\right) = a x^r$ for all i where $r < 1$, we obtain*

$$\widehat{\rho} = \left(1 - \frac{1}{n}\right) r.$$

This formula exhibits both strategic interaction and technological effects. Finite n reduces the level of dissipation. However, even if there are many players the dissipation ratio is bounded away from unity.

Example 2.6 illustrates incomplete dissipation of the rent due to technological effects even in large contests. In the next section, we extend this result to large symmetric contests with general production functions.

3 Rent dissipation in large symmetric contests

In this and the next section, we address contests with free entry by investigating the limiting dissipation ratio as the number of contestants becomes large. We use a formal structure consisting of an infinite sequence, $\mathcal{S}$, of contestants each characterized by a production function. For any n, let $\mathcal{G}^n$ denote the contest played by the first n members of $\mathcal{S}$. We will write $X^n\left(Y^n\right)$ for the Nash equilibrium value of $X\left(Y\right)$ in $\mathcal{G}^n$, and examine the limit of the dissipation ratio ρ^n as $n \longrightarrow \infty$ where $\rho^n = X^n/R$. We have seen that this limit cannot exceed unity but it can be strictly less than one (Example 2.6).

We will write $\eta_i\left(x\right) = xf_i'\left(x\right)/f_i\left(x\right)$ for the elasticity of production of contestant i. The concavity of the production function implies that η_i is at most unity for positive input. However, it need not be a monotonic function and it is convenient to rule out pathological behavior such as multiple accumulation points as $x \longrightarrow 0$.

A2. *For contestant i, the elasticity $\eta_i\left(x\right)$ has a limit as x tends to zero through positive values. We will write $\eta_i\left(0\right)$ for $\lim_{x\longrightarrow o^+}\eta_i\left(x\right)$.*

Of course, for the constant elasticity production $f_i\left(x\right) = ax^r$ we have $\eta_i\left(0\right) = r \leq 1$. Note that, when the production function is linear the limiting elasticity is unity. The next lemma extends this conclusion to a much wider class of production functions. The proof is in Section 8.

Lemma 3.1 *Assume **A1** and that contestant i's marginal product is bounded above. Then **A2** holds with $\eta_i\left(0\right) = 1$.*

Note that the converse of this lemma is false. It is possible to construct production functions which satisfy **A1** and **A2** for which $f'\left(0\right) = \infty$, but $\eta_i\left(x\right) \longrightarrow 1$ as $x \longrightarrow 0$.

The main result of this section is that the dissipation ratio in large symmetric contests is equal to the limiting elasticity. To see this, consider a *symmetric sequence* $\mathcal{S}$ in which f_i is equal to f for all contestants. Wherever we can do so without confusion, we will omit the subscript, i. Aggregate input Y^n can be found from the requirement $ns\left(Y^n\right) = 1$. Substituting this into (2), shows that Y^n is the unique solution of

$$\left(n-1\right)R = nY^n g'\left(\frac{Y^n}{n}\right). \qquad (7)$$

Furthermore, with n identical contestants, (6) implies $Y^n = nf\left(X^n/n\right)$.

Substituting into (7) and using the fact that $g'(f(x)) = 1/f'(x)$, we obtain

$$\rho^n = \frac{X^n}{R} = \left(1 - \frac{1}{n}\right) \eta \left(\frac{X^n}{n}\right). \tag{8}$$

Since the aggregate outlay X^n can not exceed the value of the rent, $X^n/n \longrightarrow 0$ as $n \longrightarrow \infty$. Applying this limit in (8), gives the following result.

Theorem 3.2 *In a symmetric sequence $\mathcal{S}$ satisfying **A1** and **A2** for all contestants, $\rho_n \longrightarrow \eta(0)$ as $n \longrightarrow \infty$.*

Applying Lemma 3.1, gives the following corollary.

Corollary 3.3 *If the marginal product of all contestants in a large symmetric contest is bounded and **A1** is satisfied, the rent is almost fully dissipated.*

These results identify the boundedness or otherwise of production functions and, more particularly, the elasticity of production for small outlays as the determinant of the limiting dissipation ratio. For example, consider a symmetric sequence $\mathcal{S}$ in which production functions take the form $f(x) = a(x+k)^r - ak^r$. This satisfies **A1** if $a > 0$, $k \geq 0$ and $0 < r < 1$. We may deduce from the theorem, that if $k = 0$, the limiting dissipation ratio is r, whereas, for any $k > 0$, the limit is 1.

4 Large asymmetric contests

Example 2.5 demonstrated that heterogeneity amongst contestants can reduce dissipation. In this section, we examine whether such conclusions remain true in large contests. So, consider an *asymmetric sequence* $\mathcal{S}$ in which contestants come in T distinct types. Wherever it is necessary to distinguish a contestant of type $t = 1, \ldots, T$ from an individual contestant i, we enclose the subscript in parentheses to refer to the former. For example, $s_{(t)}$ is the share function of contestants of type t. We use $n_t(n)$ to denote the number of contestants of type t amongst the first n members of the sequence $\mathcal{S}$ and, wherever we can do so without ambiguity, omit the explicit dependence on n.

In light of Example 2.5, multiple types might be expected to reduce rent dissipation. This is indeed the case for small contests but, as the number of contestants increases one type will typically come to dominate the aggregate expenditure, effectively rendering the contest symmetric. The reduction in dissipation due to asymmetry disappears in the limit.

This is most easily seen when all contestants have bounded marginal product for then only one type will typically make a positive outlay once there are enough contestants of that type. For, it follows from Proposition 2.1 that the share functions of all contestants reach the axis. Suppose the contestants are arranged so that the dropout point of type T lies to the right of the dropout points of all other types. (This requires $f'_{(t)}(0) < f'_{(T)}(0)$ for all $t = 1, \ldots, T-1$.) If n_T is large enough, the aggregate share function will exceed unity at the dropout points of all other contestants. (We need $n_T \geq 1/s_{(T)}\left(Rf'_{(t)}(0)\right)$ for all $t \leq T-1$.) Hence, no other type will be active. If, further, there are infinitely many contestants of type T in $\mathcal{S}$, we may conclude from Theorem 3.2 that, in the limit, the whole of the rent is dissipated. We have established this result under the assumption that there is a unique type with the largest value of $f'_{(t)}(0)$. However, this is not essential. It is easy to see that the same result holds provided there are infinitely many contestants of at least one of the types with the joint highest value of $f'_{(t)}(0)$.

Theorem 4.1 *Suppose the production function of every contestant in the asymmetric sequence $\mathcal{S}$ satisfies **A1** and has bounded marginal product. If $f'_{(t)}(0) \leq f'_{(T)}(0)$ for all $t = 1, \ldots, T$ and there are infinitely many contestants of type T in $\mathcal{S}$, then $\rho^n \longrightarrow 1$ as $n \longrightarrow \infty$.*

Note that this result does not depend on the ordering of the contestants in $\mathcal{S}$. However, the assumption that the contestant with the largest dropout point, occurs infinitely often is essential to the result. To see this consider the following counter-example.

Example 4.1 *There are $n-1$ contestants with $f(x) = a_1 x$ and one contestant with $f(x) = a_2 x$, where $a_2 > a_1$. Contestants with production function $a_1 x$ must participate, or only one contestant would be active which is impossible in equilibrium. Since the share function for $f(x) = a_2 x$ has dropout point $a_2 R$ which lies to the right of the dropout point for $f(x) = a_1 x$, the contestant with $f(x) = a_2 x$ also participates. Hence, all contestants are active and Y^n satisfies*

$$(n-1)\left(1 - \frac{Y^n}{a_1 R}\right) + 1 - \frac{Y^n}{a_2 R} = 1,$$

using the formula (3) for the share function. Solving, we find

$$Y^n = (n-1) R \left(\frac{n-1}{a_1} + \frac{1}{a_2}\right).$$

Applying (6),

$$\begin{aligned}\rho_n &= (n-1)\left(1-\frac{Y^n}{a_1R}\right)\frac{Y^n}{a_1R}+\left(1-\frac{Y^n}{a_2R}\right)\frac{Y^n}{a_2R} \\ &= \left[\frac{n-1}{a_2}\right]\left[\frac{2-n}{a_2}+\frac{2(n-1)}{a_1}\right]\left[\frac{1}{a_2}+\frac{n-1}{a_1}\right]^{-2} \\ &\longrightarrow \left(2-\frac{a_1}{a_2}\right)\frac{a_1}{a_2} \text{ as } n \longrightarrow \infty.\end{aligned}$$

By suitable choice of a_1/a_2 the proportion of the rent dissipated can be made arbitrarily small even for large contests.

Theorem 4.1 exploited boundedness of the marginal product to deduce that inefficient types will be inactive for large n, allowing us to use results from large symmetric contests. By contrast, if the marginal products of all types are unbounded, all contestants will supply positive input in $\mathcal{G}^n$ for any n, since their share function is always positive (Proposition 2.1). However, we can still obtain a generalization of Theorem 3.2 for asymmetric contests by showing that, although the aggregate input of inefficient types is positive, it becomes a vanishingly small proportion of the aggregate input of the efficient type as $n \longrightarrow \infty$. To establish these results requires an extension of the theory of share functions developed in Section 2.

First note that, if all contestants have unbounded marginal product, then $Y^n \longrightarrow \infty$ as $n \longrightarrow \infty$. This is because the value of the aggregate share function of $\mathcal{G}^n$ at any Y exceeds unity for large enough n and therefore $Y^n > Y$, by Proposition 2.1. If $y^n_{(t)}[\, x^n_{(t)}]$ denotes the input[expenditure] of contestants of type t in $\mathcal{G}^n$, we have $y^n_{(t)} = s_{(t)}(Y^n)\, Y^n = f_{(t)}\left(x^n_{(t)}\right)$. The absence of boundary solutions for such contestants allows us to apply the interior first-order conditions (2) to deduce that

$$\left[1-s_{(t)}(Y^n)\right] s_{(t)}(Y^n)\, R = y^n_{(t)} g_i'\left(y^n_{(t)}\right). \tag{9}$$

Since $s_{(t)}(Y) \longrightarrow 0$ as $Y \longrightarrow \infty$ (Proposition 2.1), $y^n_{(t)}$ and therefore $x^n_{(t)}$ vanishes for large n. For future use, we summarize these observations in the following Lemma.

Lemma 4.2 *Suppose the production function of every contestant in the asymmetric sequence $\mathcal{S}$ satisfies* ***A1*** *and has unbounded marginal product. Then, as $n \longrightarrow \infty$, (i) $Y^n \longrightarrow \infty$, (ii) $y^n_{(t)} \longrightarrow 0$, (iii) $x^n_{(t)} \longrightarrow 0$.*

When there are many contestants, aggregate input becomes large and the analysis rests on the asymptotic properties of share functions. Example

2.2 shows that, if $f_i(x)$ is proportional to $\sqrt{x}$, the share function is asymptotically proportional to Y^{-2}. Under the following technical refinement of Assumption **A2**, we can generalize this result.
A2*. *For contestant i, **A2** holds and furthermore, the elasticity $\eta_i(x)$ has a right-sided derivative at the origin.*

The importance of this assumption is that it allows us to deduce that share functions are proportional to $Y^{-\alpha_t}$ for large Y, where

$$\alpha_t = \left[1 - \eta_{(t)}(0)\right]^{-1}. \tag{10}$$

Formally, we have the following limit, proved in Section 8.

Lemma 4.3 *Suppose **A1** and **A2*** hold for contestants of type t and $\eta_{(t)}(0) < 1$. Then $Y^{\alpha_t} s_{(t)}(Y) \longrightarrow A_t(> 0)$ as $Y \longrightarrow \infty$.*

To apply this result, we start by deriving a useful formula for the dissipation ratio when all contestants have unbounded marginal product. Using the fact that $f_i' = 1/g_i'$, we can deduce from (9) that

$$x_{(t)}^n = \eta_{(t)}\left(x_{(t)}^n\right)\left[1 - s_{(t)}(Y^n)\right] s_{(t)}(Y^n) R.$$

This gives the desired formula

$$\rho^n = \sum_{t=1}^{T} \eta_{(t)}\left(x_{(t)}^n\right) n_t(n) s_{(t)}(Y^n)\left[1 - s_{(t)}(Y^n)\right]. \tag{11}$$

We can use Lemma 4.3 to deduce the limiting value of ρ^n from (11) when $\eta_{(T)}(0) < \eta_{(t)}(0)$ for all $t \neq T$. These inequalities imply that, if $t \neq T$, we have $\alpha_T < \alpha_t$ and the lemma allows us to deduce that

$$Y^{\alpha_T} s_{(t)}(Y) \longrightarrow 0 \text{ as } Y \longrightarrow \infty.$$

The equilibrium condition for Y^n can be written

$$\sum_{t=1}^{T} n_t(n) s_{(t)}(Y^n) = 1.$$

Hence, the aggregate share of contestants of type T approaches unity as $n \longrightarrow \infty$:

$$n_T(n) s_{(T)}(Y^n) = \frac{[n_T(n)/n][Y^n]^{\alpha_T} s_{(T)}(Y^n)}{\sum_{t=1}^{T}[n_t(n)/n][Y^n]^{\alpha_T} s_{(t)}(Y^n)} \longrightarrow \frac{A_T}{A_T} = 1,$$

assuming that $n_T(n)/n$ has a positive limit. Of course the aggregate share of other types vanishes in the limit: $n_t s_{(t)}(Y^n) \longrightarrow 0$ for $t \neq T$. Applying these limits and Lemma 4.2(iii) in (11) allows us to deduce that $\rho^n \longrightarrow \eta_{(T)}(0)$ as $n \longrightarrow \infty$. Although we have assumed that there is a unique type which maximizes $\eta_{(t)}(0)$, it is straightforward to extend the argument to the case of multiple maximizers, proving the following result.

Theorem 4.4 *Suppose the production function of every contestant in the asymmetric sequence $\mathcal{S}$ satisfies **A1** and **A2*** and has unbounded marginal product. Then*

$$\rho^n \longrightarrow \min_{t=1,\ldots,T} \eta_{(t)}(0) \quad \textit{as } n \longrightarrow \infty,$$

provided the limiting proportion in $\mathcal{S}$ of at least one of the types achieving the minimum exceeds zero.

Theorem 4.4 extends readily to the case when there is a mixture of types, some with bounded and others with unbounded marginal product. Provided at least one of the types with unbounded marginal product has positive limiting proportion in $\mathcal{S}$ and n is large enough, the aggregate share function at the dropout points of all types with bounded marginal product will exceed unity. This means that such types will be inactive in $\mathcal{G}^n$ and Theorem 4.4 remains valid *mutatis mutandis.*

The main theme of this section is that the extra reduction in rent dissipation caused by asymmetry in small contests is typically eliminated in large contests. Provided the most efficient types are sufficiently richly represented in the sequence of contestants there are only two possibilities. All participants are of one or more of the most efficient types, or all contestants participate but the aggregate input of the most efficient types swamps the aggregate input of the inefficient types. Even so, unless the marginal product of all types is bounded, the rent will be incompletely dissipated in the limit. In the next section, we show that if the number of contestants is made endogenous by imposing a positive entry fee, the remainder of the rent will be dissipated in entry fees provided these are small enough.

5 Symmetric contests with small costs of entry

In the contests considered so far, the number of participants is exogenous. This finesses the problem of explaining why some potential contestants choose to participate in the contest whilst others do not. Endogenizing the number of contestants was first addressed soon after the Tullock contest was introduced. For example, Corcoran[4] used a free-entry/zero-profit condition to obtain a formula for the number of contestants but this only gave a finite answer for non-concave production functions. (He was primarily interested in $f_i(x) = x^r$ with $r > 1$.) Higgins et al.[6] also used a zero-profit condition together with a positive and non-refundable entry fee but employed a non-standard form of (symmetric) contest with the property that expenditure was independent of the number of participants. Michaels[9] included entry costs but focused on design issues rather than limiting dissipation. Schoonbeek and Kooreman [14] restricted entry by requiring a minimum level of input from any contestant who actively participated. They restricted attention to two players.

In this and the next section, we study a simple model with entry costs. As in the preceding sections, we start with a sequence $\mathcal{S}$ of potential contestants, who, in this section, are assumed to have the same production function. We modify the previous model by requiring participants in the contest to pay a non-refundable entry fee of $\kappa > 0$. The decision to enter is taken during the preliminary stages of the game after which there is a final stage consisting of a simultaneous-move contest played by those potential contestants who chose to pay the entry fee.

Formally, we consider a nested set of sequential games, one for each $\kappa > 0$, and write $\mathcal{G}(\kappa)$ for the game associated with cost κ. In this game the set of players consists of the first $M(\kappa)$ potential contestants from $\mathcal{S}$ where $M(\kappa)$ is be "large enough" in a sense to be discussed below. Each player must decide whether or not to enter and pay the entry fee. These decisions are taken in the order determined by $\mathcal{S}$ and players are assumed to be aware of the decision made by their predecessors in the sequence. The game concludes with a contest played by the entrants, in which participant i chooses outlay x_i and receives payoff $Rp_i(\mathbf{x}) - x_i - \kappa$ where $\mathbf{x}$ is the vector of outlays. Non-entrants receive a payoff of zero. We will discuss some modifications of the timing in this model in the concluding section.

By Theorem 2.2, the contest played at the final stage will have a unique Nash equilibrium in pure strategies. Since all decisions made prior to the final stage are binary, $\mathcal{G}(\kappa)$ will have a subgame perfect equilibrium in pure strategies which can be found by backwards induction. Unless there are ties in the value of the pay-offs, this equilibrium will be unique. For ease of exposition, we resolve ties by forcing players indifferent between entering and not entering to choose the former. Under this restriction $\mathcal{G}(\kappa)$ has a unique equilibrium. (But our results are essentially unchanged if the restriction is lifted.) On the equilibrium path, entrants always precede those potential contestants who choose not to enter. This early-mover advantage is a consequence of the ordering of entry decisions together with the fact that payoffs depend only on the number of entrants and not on their position in $\mathcal{S}$. If there are $\widehat{n}$ entrants on the equilibrium path, the payoff in a contest with $\widehat{n}$ contestants must be at least κ. Hence, payoffs, net of entry cost, in the subgame which starts after $\widehat{n}$ potential contestants have chosen to enter must be non-negative. Since staying out has a net cost of zero, an application of backwards induction shows that the first $\widehat{n}$ contestants will choose to enter and therefore lie on the equilibrium path.

When considering total expenditure on rent-seeking, entry fees must be taken into account. If X is aggregate input and there are n entrants, total expenditure is $X + n\kappa$. In equilibrium, this can never exceed the value of the rent. To see this, consider an entrant i who subsequently chooses outlay $\widehat{x}_i$. At the time of entry, that contestant had the option of staying out of the contest and receiving a payoff of 0. Since they chose entry the payoff from entry must be non-negative: $Rp_i(\widehat{\mathbf{x}}) - \widehat{x}_i - \kappa \geq 0$. The result follows by summing over all entrants.

Lemma 5.1 *Suppose that on the equilibrium path of a subgame perfect equilibrium of $\mathcal{G}(\kappa)$ the first $\widehat{n}$ players enter and aggregate expenditure in the final stage is $\widehat{X}$. Then, $\widehat{X} + \widehat{n}\kappa \leq R$.*

We wish the number of entrants to be determined purely by the relative levels of payoffs and entry fees and not subject to any exogenous constraints. So, we assume that there are enough potential contestants to ensure that some will choose not to enter. By the preceding lemma, a sufficient condition is that $M(\kappa) \geq (R/\kappa) + 1$ for all $\kappa > 0$. For the rest of this section, we will write $n(\kappa)$ for the number of entrants on the equilibrium path in $\mathcal{G}(\kappa)$ and $X(\kappa)$ for the aggregate output in a contest with $n(\kappa)$ entrants. To address the case of free entry, we will investigate the dissipation ratio $\rho(\kappa) = [X(\kappa) + \kappa n(\kappa)]/R$ as $\kappa \longrightarrow 0$. We now show that, under suitable assumptions, $\rho(\kappa)$ approaches unity as κ approaches zero.

To complete our analysis of subgame perfect equilibria, we need to characterize $n(\kappa)$. To do this, it is useful to view i's payoff in a contest as a function of aggregate equilibrium input:

$$\widetilde{\pi}_i(Y) = s_i(Y) R - g_i(Y s_i(Y)). \tag{12}$$

The value of Y will typically alter as a result of a change in i's competitive environment, for example a change in a rival's payoff function or an additional contestant. To examine the effect of such a change on i, we need to sign the slope of $\widetilde{\pi}_i$. The next proposition, proved in Section 8, does this.

Proposition 5.2 *Suppose **A1** is satisfied for contestant i and $Y > 0$ satisfies $s_i(Y) > 0$. Then $\widetilde{\pi}_i'(Y) < 0$.*

Since adding another contestant shifts the aggregate share function upwards, equilibrium input increases and the proposition implies that the payoff of a currently active contestant falls. When all contestants are identical and X^n denotes the aggregate outlay with n contestants, we conclude that, payoffs: $(R - X^n)/n$ are strictly decreasing in n. It follows that there is a unique n satisfying

$$\frac{R}{n} - \frac{X^n}{n} - \kappa \geq 0 > \frac{R}{n+1} - \frac{X^{n+1}}{n+1} - \kappa. \tag{13}$$

We claim that $n(\kappa)$ is this value of n. To see this note that the left-hand inequality means that each of the first $n(\kappa)$ players receives a non-negative payoff and therefore prefers to enter, given her predecessors have done so. The right-hand inequality ensures that player $n(\kappa)+1$ and all her successors do better to stay out, once $n(\kappa)$ players have entered, than to enter the contest.

We now consider what happens as the entry cost decreases to zero. In equilibrium, the number of entrants approaches infinity as the entry cost goes to zero, since payoffs in symmetric contests approach zero as the number of contestants becomes infinite. Hence, the total entry cost paid by all participants, the product of these two quantities, may have a positive limit. From Lemma 5.1, this limit can never exceed the undissipated portion of the rent. When the production function has bounded marginal product, the whole rent is dissipated in large games (Corollary 3.3) and therefore for small entry fees. In this case the total entry cost vanishes for small entry costs and the results for large simultaneous-move contests are an effective guide to contests with free entry. However, when the marginal product is unbounded, incomplete rent dissipation leaves room for a positive limiting value of $\kappa n(\kappa)$ and we will now show that this results in complete dissipation of the remaining rent.

Note that the inequality (13) can be rearranged as

$$0 \leq 1 - \frac{X^{n(\kappa)} + \kappa n(\kappa)}{R} < \frac{X^{n(\kappa)+1} - X^{n(\kappa)} + \kappa}{R} \tag{14}$$

for all $\kappa > 0$. Under **A2**, Theorem 3.2 implies that $X^n \longrightarrow \eta(0) R$, as $n \longrightarrow \infty$. It follows that the right-hand side of (14) approaches zero as $\kappa \longrightarrow 0$, proving that the whole rent is dissipated in the limit.

Theorem 5.3 *In a symmetric sequence $\mathcal{S}$ satisfying **A1** and **A2** for all contestants, $\rho(\kappa) \longrightarrow 1$ as $\kappa \longrightarrow 0$.*

Comparing this theorem with Theorem 3.2 indicates that the many-player limit may be a misleading guide to contests with free entry when marginal products are unbounded.

Our focus on small entry costs can be viewed as a way of abstracting away from strategic effects. However, accounting for entry costs increases rent dissipation even when these costs are not small despite the restricted number of contestants and the opportunity for strategic play that this will entail. Indeed, when there are few entrants, we would expect high entry costs relative to the rent. For example, consider Example 2.4, in which payoffs are

$$\pi^n = \frac{1}{n}(1-r)R + \frac{1}{n^2}rR.$$

Now introduce an entry cost $\kappa = \pi^n - \varepsilon$, where ε is small and positive. Then, there will be n entrants in equilibrium and the proportion of rent dissipated will be $1-\varepsilon$. For example, in the case of linear production, if κ is just less than $R/4$, nearly all the rent is dissipated even though only two contestants are observed competing for it.

Even if κ is not so finely tuned, entry costs can dissipate a significant proportion of the rent. The right-hand side of inequality (13) says that $\pi^{n(\kappa)+1}$ is a lower bound on κ. Hence, if $n(\kappa) = n$, using the result from Example 2.6,

$$\begin{aligned} X^n + n\kappa &> \left(1 - \frac{1}{n}\right) rR + n\pi^{n+1} \\ &= \frac{n}{n+1}R + \frac{n^2-n-1}{n(n+1)^2}rR. \end{aligned}$$

Consider the case of four observed contestants. If $r = 1/2$, the proportion of the rent dissipated rises from 37.5%, when entry costs are left out of account, to between 85.5% and 100% if they are included. When the production function is linear, the dissipation ratio of 75% when entry costs are ignored increases to at least 91% with their inclusion.

6 Asymmetric contests with entry costs

In this section, we extend the model of the previous section to the case where potential contestants differ. So, we suppose that members of the sequence of potential contestants $\mathcal{S}$ come in T types each of which occurs infinitely often in $\mathcal{S}$ and is characterized by a distinct production function. As we wish to allow the cost of entry to differ amongst types, we assume entrants of type t pay a fee of $\lambda_t \kappa$, where $\lambda_t > 0$. As in the previous section, we write $\mathcal{G}(\kappa)$ for the game in which the first $M(\kappa)$ players from $\mathcal{S}$ choose sequentially whether to enter the final simultaneous-move contest. It is easy to extend Lemma 5.1 to the asymmetric case: subgame perfect equilibria of $\mathcal{G}(\kappa)$ cannot exhibit over-dissipation even when entry fees are taken into account. We choose $M(\kappa)$ to ensure that there is a potential contestant i such that the successors in $\mathcal{S}$ of i (a) choose not to enter on the equilibrium path of $\mathcal{G}(\kappa)$ and (b) contain at least one player of each type.

The argument used in the preceding section to demonstrate early-mover advantage are readily extended to multiple types. In particular, for each t, entrants of type t always precede potential contestants of type t on the equilibrium path of the subgame perfect equilibrium of $\mathcal{G}(\kappa)$. Hence, equilibrium outcome is completely determined once the number of entrants of each type is specified. Typically, when the sequence $\mathcal{S}$ is asymmetric these numbers will depend on the precise ordering of types in $\mathcal{S}$. However, for all small enough κ this complication disappears. We will show that, typically, only one type will enter once κ falls below a threshold value which is independent of the ordering of types in $\mathcal{S}$. This permits us to use the results from the previous section to analyze the limiting value of the dissipation ratio as $\kappa \longrightarrow 0$.

We first consider the case in which all potential contestants have bounded marginal revenue and suppose that

$$f'_{(t)}(0) \leq f'_{(T-1)}(0) < f'_{(T)}(0) \text{ for all } t = 1, \ldots, T-2. \tag{15}$$

Let $\overline{Y}$ be the type-$(T-1)$ dropout point: $\overline{Y} = R f'_{(T-1)}(0)$, let N be the smallest integer satisfying $N \geq 1/s_{(T)}\left(\overline{Y}\right)$ and let Y^* satisfy $s_{(T)}(Y^*) = 1/N$. Finally, define $\overline{\kappa} = \widetilde{\pi}_{(T)}(Y^*)/\lambda_T > 0$. [Recall the definition of $\widetilde{\pi}_i$ in (12).] Consider those potential contestants choosing to enter on the equilibrium path in the subgame perfect equilibrium of $\mathcal{G}(\kappa)$. We claim that, if $\kappa < \overline{\kappa}$, all such entrants are of type T.

To justify the claim, suppose, to the contrary that there is an entrant of type t on the equilibrium path. To make entry worthwhile such a contestant must be active in the final contest. This requires that the equilibrium value of aggregate input must be less than its dropout point and therefore less than $\overline{Y}$, by (15) the largest dropout point other than for type T. This can only happen if the number of entrants of type T does not exceed $N-1$. Let i be the last potential contestant of type T in $\mathcal{S}$ who plays in $\mathcal{G}(\kappa)$. By construction, i does not enter in equilibrium and we establish a contradiction by showing that i would be better off changing their decision. Were i to enter, the equilibrium aggregate input in the final contest would rise to Y', say. Then, either $Y' \leq \overline{Y} < Y^*$, or $Y' > \overline{Y}$. In the latter case, all contestants in the final contest would be of type T and $s_{(T)}(Y') \geq 1/N$ since there would be at most N contestants. In either case, i's payoff net of entry cost would be at least $\widetilde{\pi}_i(Y^*) - \kappa\lambda_T > 0$ which exceeds the payoff of zero from not entering. Hence, only players of type T will enter. This argument is readily extended to the case when $f'_{(t)}(0)$ does not have a unique maximizer and we summarize these conclusions in the following proposition.

Proposition 6.1 *Suppose there are T types with production functions satisfying **A1** and $f'_{(t)}(0) < \infty$ for all t. Then there exists $\overline{\kappa} > 0$ such that, for any asymmetric sequence $\mathcal{S}$ of these types, all entrants on the equilibrium path in any subgame perfect equilibrium of $\mathcal{G}(\kappa)$ is of type T if $0 < \kappa < \overline{\kappa}$.*

Corollary 3.3 allows us to deduce that the whole rent will be dissipated in the limit: $\rho(\kappa) \longrightarrow 1$ as $\kappa \longrightarrow 0$. When marginal products are bounded, the total expenditure on entry fees vanishes in the limit, as in the symmetric case.

When marginal products are unbounded, a more subtle argument is needed, but we are still able to show that, once the entry fee is small enough, the equilibrium will involve entrants of a single type and will therefore be independent of the ordering of $\mathcal{S}$. These conclusions apply the following necessary conditions for a subgame perfect equilibrium of $\mathcal{G}(\kappa)$ in which there are n_t entrants of type t on the equilibrium path.

Condition 6.2 *For each type t, if $n_t > 0$ then*

$$\widetilde{\pi}_{(t)}(Y) = Rs_{(t)}(Y) - g_{(t)}\left[Ys_{(t)}(Y)\right] \geq \kappa\lambda_t, \tag{16}$$

where $Y > 0$ is the unique solution of

$$\sum_{u=1}^{T} n_u s_{(u)}(Y) = 1. \tag{17}$$

For each type t,

$$\widetilde{\pi}_{(t)}(Y'_t) < \kappa\lambda_t. \tag{18}$$

where Y'_t is the unique solution of

$$s_{(t)}(Y'_t) + \sum_{u=1}^{T} n_u s_{(u)}(Y'_t) = 1. \tag{19}$$

The inequality (16) reflects the equilibrium requirement that potential contestants who choose to enter on the equilibrium path must receive a non-negative payoff, net of entry cost, in the final contest played on this path. Note that Y in (17) is the value of aggregate input in this contest. For type t, (18) expresses the equilibrium requirement that were player i, the final potential contestant of type t (in the ordering of $\mathcal{S}$), to enter, her payoff in the resulting contest would fall short of the entry cost. Note that no successor of i chooses to enter on the equilibrium path. It follows from Proposition 5.2[2] that, in the subgame which commences with i entering the contest, the payoffs to all other types of contestant in any final contest in this subgame are lower than those in the game played at the conclusion of the equilibrium path. Hence, no potential contestant other than i chooses to enter on the equilibrium path of this subgame, which means that the aggregate input in the contest played in equilibrium in this subgame is given by (19) so that $\widetilde{\pi}_{(t)}\left(Y_t'\right)$ is i's payoff in this contest.

The next proposition shows that for small enough entry costs only the most efficient types, as measured by the limiting elasticity of production at the origin, will choose to enter. The proof, which is rather intricate because of the necessity to deal with all orderings of $\mathcal{S}$, utilizes further asymptotic properties of share functions and of payoffs. Details may be found in Section 8.

Proposition 6.3 *Suppose there are T types with production functions satisfying **A1** and **A2*** and $\eta_T(0) < \eta_t(0) < 1$ for all $t = 1, \ldots, T-1$. Then there exists $\overline{\kappa} > 0$ such that, for any asymmetric sequence of these types, if $n_1, \ldots, n_T$ satisfy Condition 6.2 with $0 < \kappa < \overline{\kappa}$, then $n_1 = \cdots = n_{T-1} = 0$.*

Note that, unlike the case of bounded marginal costs, a positive entry fee is essential for this result. Even in a large game, if there is no entry fee, all types will 'enter' in the sense of making a positive outlay. However, we also need the entry fee to be small to 'smooth out' the details of the entry process and, in particular, to obtain a result independent of the ordering of potential contestants.

Proposition 6.3 requires every type to have limiting elasticities less than one. However, it can be extended to games with a mixture of types: potential contestants with both bounded and unbounded marginal products. This is a consequence of the fact that share functions of contestants with bounded marginal products eventually reach the axis. Once there are enough entrants with unbounded marginal product, the aggregate share value exceeds one at

[2]The entry of i raises the aggregate share function so that aggregate input rises and payoffs fall by the proposition.

the dropout values of all contestants with bounded marginal product. The arguments above extend readily to show that the number of entrants with unbounded marginal product is unbounded as the entry fee approaches zero. Hence, contestants with bounded marginal product will be inactive in the final-stage contest and therefore will not enter in the initial stages. We can use this to extend the previous result in the following corollary which also exploits the necessity of Condition 6.2 for equilibrium.

Corollary 6.4 *Suppose there are T types with production functions satisfying **A1** and for any type t with unbounded marginal product, **A2*** holds with $\eta_T(0) < \eta_t(0) < 1$. Then there exists $\overline{\kappa} > 0$ such that, for any asymmetric sequence $\mathcal{S}$ of these types, only potential contestants of type T enter in any subgame perfect equilibrium of $\mathcal{G}(\kappa)$ for all $\kappa \in (0, \overline{\kappa})$.*

The main application of this Corollary is that we are left with entrants of a single type once the entry fee becomes sufficiently small. We may then use the results obtained for symmetric contests to deduce that the full expenditure on rent-seeking, including entry fees, exhausts the rent when the entry fee is small.

Theorem 6.5 *Suppose there are T types with production functions satisfying **A1** and for any type t with unbounded marginal product, **A2*** holds with $\eta_T(0) < \eta_t(0) < 1$. Then, for any asymmetric sequence $\mathcal{S}$, $\rho(\kappa) \longrightarrow 1$ as $\kappa \longrightarrow 0$.*

Theorems 6.5 and 5.3 exhibit full dissipation in the limit for the specific model of entry embodied in the sequential game $\mathcal{G}(\kappa)$ (though, in the asymmetric case, this conclusion does not depend on the ordering of types in $\mathcal{S}$) but our results are robust to some changes in the way in which entry is modelled. Proposition 6.3, used to prove Theorem 5.3, implies full limiting dissipation in any entry process in which Condition 6.2 is necessary for equilibrium. For example, we could allow entry decisions to be made simultaneously by replacing $\mathcal{G}(\kappa)$ with a two stage game, $\mathcal{G}^*(\kappa)$, in which, in the first stage, players must decide whether to enter without knowing the decisions taken by her competitors. This makes the first stage a simultaneous-move game and the second and final stage is the same as $\mathcal{G}(\kappa)$: entrants play a final contest. Pay-offs are the same as $\mathcal{G}(\kappa)$.

Unlike $\mathcal{G}(\kappa)$ itself, $\mathcal{G}^*(\kappa)$ has multiple equilibria. For example, even in the symmetric case, the set of entrants in any subgame perfect equilibrium can be replaced by any other set of contestants of the same cardinality. There will also be equilibria with mixed strategies in the first stage. Indeed, the only symmetric equilibrium will be in mixed strategies but since such solutions are controversial in contest theory (and indeed in many other applications [17]), we do not discuss them further here. With so many equilibria the players face extreme problems of coordination. Nevertheless, if we assume that players have some mechanism available to help them coordinate on a single pure-strategy subgame perfect equilibrium, this must have the property that the payoff to entrants into the final stage receive a payoff no smaller than the entry cost whereas the payoff of non-entrants, were they to enter, would fall short of the cost of entry. This shows says that Condition 6.2 is necessary for $\mathcal{G}^*(\kappa)$ and the principal theorems in this and the previous section remain valid for pure-strategy solutions of $\mathcal{G}^*(\kappa)$. The ordering of players in $\mathcal{G}(\kappa)$ serves to select which players enter the contest rather than the numbers of each type who enter, the latter being independent of the ordering for small enough κ. Indeed, the ordering can be viewed as a selection mechanism for resolving the coordination problem in $\mathcal{G}^*(\kappa)$.

7 Conclusions

We have investigated both symmetric and asymmetric contests in which the number of players is specified exogenously and is large or is endogenously determined by an entry process with a small cost of entry. We have shown that when production functions have bounded marginal product the whole value of the rent is dissipated in expenditures in the contest. This holds even for asymmetric examples provided that the most efficient types occur infinitely often in the sequence of potential contestants. We have also demonstrated that the remaining undissipated portion of the value of the rent can be expended in entry costs when the entry process is formally modelled and entry costs are small, a result which is independent of the ordering of the types of potential contestant or some details of the entry process.

8 Appendix: Proofs

Proof of Proposition 2.1. The existence of a share function is established for $0 < Y < Rf_i'(0)$, by observing that (2) has a unique solution $\sigma \in (0,1)$. This follows since the left hand side decreases from R to 0 and, under Assumption **A1**, the right hand side increases from $Yg_i'(0)\,[< R]$ to $Yg_i'(Y)\,[> 0]$ as σ goes from 0 to 1. Differentiability of the solution with respect to Y is a consequence of the implicit function theorem. When the share function is positive we can differentiate the first order conditions (2) and rearrange the result to give

$$s_i'(Y) = -\frac{-g_i'[Ys_i(Y)] - Ys_i(Y)\, g_i''[Ys_i(Y)]}{R + Y^2 g_i''[Ys_i(Y)]}.$$

Since $g_i'(y) > 0$ and $g_i''(y) \geq 0$ for $y > 0$ by **A1,** we deduce that $s_i'(Y) < 0$, proving the first part of the Lemma.

If $Y \longrightarrow 0$, the right hand side of (2) goes to zero which implies that $\sigma \longrightarrow 1$, establishing Part 2

Part 3 follows from the first order conditions in Section 2. Note that, if $Y \geq Rf_i'(0)$, then $\sigma = 0$ satisfies these conditions. Furthermore, there can be no positive solution since now the left hand side of (2) is less than R for positive σ, whereas the right hand side is at least $Yg_i'(0)\,[\geq R]$ since g_i' is a non-decreasing function. We may conclude that (2) has no positive solution for such Y and, further, that $\sigma = 0$ is a solution if and only if $Yg_i'(0) = R$ i.e. $Y = Rf_i'(0)$. This shows that the share function is continuous at $Y = Rf_i'(0)$ when $f_i'(0)$ is finite, completing the proof of continuity in Part 1 and justifying the assumption that s_i is well-defined.

If $f_i'(0) = \infty$, (2) holds for all positive Y and we can deduce that $s_i(Y)$ satisfies the inequality $g_i'(s_i(Y)) \leq R/Y$. Since the only zero of $g_i'(0)$ is the origin, we may conclude that $s_i(Y) \longrightarrow 0$ as $Y \longrightarrow \infty$ and the proof is completed. ■

Proof of Lemma 3.1. Under the hypothesis of the proposition, let μ_i denote the least upper bound of $f_i'(x)$. Since f_i is concave, for any $\delta > 0$ there is $\varepsilon > 0$ such that

$$\mu_i - \frac{\mu_i \delta}{2} \leq f_i'(x) \leq \mu_i \text{ for } 0 < x < \varepsilon.$$

Since $f_i(0) = 0$, we can integrate this result from 0 to x to obtain

$$\left(\mu_i - \frac{\mu_i \delta}{2}\right) x \leq f_i(x) \leq \mu_i x \text{ for } 0 < x < \varepsilon$$

and deduce that

$$1 - \frac{\delta}{2} \leq \eta_i(x) \leq \left(1 - \frac{\delta}{2}\right)^{-1} \text{ for } 0 < x < \varepsilon.$$

If δ is chosen to satisfy $0 < \delta < 1$, then $(1 - \delta/2)^{-1} < 1 + \delta$ so that $|\eta_i(x) - 1| < \delta$ for $0 < x < \varepsilon$. This establishes the lemma. ■

To derive the asymptotic form for share functions as $Y \longrightarrow \infty$ it is useful to start with a lemma relating inverse share functions to elasticity.

Lemma 8.1 *Suppose* ***A1*** *and* ***A2**** *hold for player* i. *Then*

$$y^{-1/\eta_i(0)} g_i(y) \longrightarrow K_i > 0 \text{ as } y \longrightarrow 0^+, \tag{20}$$
$$y^{1-1/\eta_i(0)} g_i'(y) \longrightarrow K_i/\eta_i(0) > 0 \text{ as } y \longrightarrow 0^+. \tag{21}$$

Proof. Defining $\xi_i(y) = [\eta_i(g_i(y))]^{-1}$ for $y > 0$ and $\xi_i(0) = [\eta_i(0)]^{-1}$, we have

$$\xi_i(y) = \frac{y}{g_i}\frac{dg_i}{dy}$$

and we can solve this equation for g_i:

$$g_i(y) = g_i(1)\exp\left\{\int_1^y \frac{\xi_i(z)}{z}dz\right\}. \tag{22}$$

Now note that

$$y^{-\xi_i(0)} = \exp\left\{\int_1^y \frac{-\xi_i(0)}{z}dz\right\}, \tag{23}$$

so that

$$y^{-\xi_i(0)} g_i(y) = g_i(1)\exp\left\{\int_1^y \frac{\xi_i(z)-\xi_i(0)}{z}dz\right\}. \tag{24}$$

Under Assumption **A2(ii)** $\eta_i(x)$ has a right-sided derivative at the origin and the same is true of $\xi_i(x)$. Therefore the integrand in (24) is a continuous function for $0 \le z \le 1$, if it takes the value of the derivative at $z = 0$. Hence the right-hand side of (24) is well-defined at $y = 0$ (taking the value K_i, say) and (20) follows.

A similar argument proves (21). Differentiating (22) and using (23) gives

$$y^{1-\xi_i(0)} g_i'(y) = g_i(1)\,\xi_i(y)\exp\left\{\int_1^y \frac{\xi_i(z)-\xi_i(0)}{z}dz\right\}. \tag{25}$$

The limit (21) follows by a similar argument to that establishing (20). ■

Completion of proof of Lemma 4.3. We can write the interior first order conditions (2), for contestants of type t, as

$$\left[1 - s_{(t)}(Y)\right] s_{(t)}(Y)\,R = y g_{(t)}'(y) \tag{26}$$

where $y = Y s_{(t)}(Y)$. Proposition 2.1 states that $s_{(t)}(Y) \longrightarrow 0$ as $Y \longrightarrow \infty$ and (26) implies $y \longrightarrow 0$. Equation (26) can be arranged as

$$s_{(t)}(Y)\,Y^{-\vartheta/\eta_{(t)}(0)} = \left\{R^{-1}\left[1 - s_{(t)}(Y)\right]^{-1} y^{1-1/\eta_{(t)}(0)} g_{(t)}'(y)\right\}^{\vartheta},$$

where $\vartheta = \eta_{(t)}(0) / \left[\eta_{(t)}(0) - 1\right]$. It follows from (21) that $s_{(t)}(Y)\, Y^{-\vartheta/\eta_{(t)}(0)}$ has a finite limit as $Y \longrightarrow \infty$. The lemma is a consequence of the fact that $-\vartheta/\eta_{(t)}(0) = \alpha_t$. ■

Proof of Lemma 5.2. From Proposition 2.1, $s_{(t)}$ is differentiable where positive, so

$$\frac{d\widetilde{\pi}_i}{dY} = Rs_i(Y) - \{s_i(Y) + Ys_i'(Y)\}\, g_i'[Ys_i(Y)].$$

Multiplying by Y, substituting from the first order conditions (2) and simplifying yields

$$\frac{Y}{R}\frac{d\widetilde{\pi}_i}{dY} = Ys_i(Y)\, s_i'(Y) - s_i(Y)\{1 - s_i(Y)\}.$$

By Proposition 2.1 and $0 < s_{(t)} < 1$ the right hand side is negative. ■

To prove Proposition 6.3, we first show that the type-t payoff function $\widetilde{\pi}_{(t)}(Y)$ decreases with Y and has the same asymptotic form as the type-t share function.

Lemma 8.2 *Suppose **A1** and **A2*** hold for players of type t and $\eta_{(t)}(0) < 1$, then*

$$Y^{\alpha_t}\widetilde{\pi}_{(t)}(Y) \longrightarrow RA_t\left[1 - \eta_{(t)}(0)\right] \text{ as } Y \longrightarrow \infty,$$

where A_t is defined in Lemma 4.3.

Proof. We start by using the first order conditions to establish a relationship between the limiting constants K_i in Lemma 8.1 and A_i in Lemma 4.3 for players of type t. These conditions can be rewritten, for type t, in the form

$$R\left[1 - s_{(t)}(Y)\right] = \left[Y^{\alpha_i} s_{(t)}(Y)\right]^{\left[1-\eta_{(t)}(0)\right]/\eta_{(t)}(0)} y_{(t)}^{1-1/\eta_{(t)}(0)} g_{(t)}'\left(y_{(t)}\right) \tag{27}$$

where $y_{(t)} = Ys_{(t)}(Y)$. By Lemma 4.3, $y \longrightarrow 0$ as $Y \longrightarrow \infty$. Taking the limit in Lemmas 8.1 and 4.3 and rearranging the result gives

$$K_t = A_t^{1-1/\eta_{(t)}(0)} R\eta_{(t)}(0). \tag{28}$$

From the formula for $\widetilde{\pi}_{(t)}$, we have, after some manipulation,

$$Y^{\alpha_t}\widetilde{\pi}_{(t)}(Y) = RY^{\alpha_t} s_{(t)}(Y) - \left[Y^{\alpha_t} s_{(t)}(Y)\right]^{1/\eta_{(t)}(0)} y^{-1/\eta_{(t)}(0)} g_{(t)}(y).$$

The proof is completed by taking the limit, applying the results of Lemmas 8.1 and 4.3, and using (28) to rearrange the result. ■

Proposition 13, applied to players of type t, implies that there is a threshold $\kappa_t^* > 0$ such that $\widetilde{\pi}_{(t)}(Y) = \lambda_t \kappa$ has a unique solution Y for all $\kappa < \kappa_t^*$. We will write $Y_t(\kappa)$ for this solution. The next result records an obvious implication of Lemma 8.2.

Lemma 8.3 *Suppose **A1** and **A2*** hold for players of type t, then $Y_t(\kappa) \longrightarrow \infty$ as $\kappa \longrightarrow 0$.*

Lemmas 4.3 and 8.2 allow us to relate $Y_t(\kappa)$ with type-t share functions.

Lemma 8.4 *Suppose **A1** and **A2*** hold for players of types t and u and $\eta_t(0) < 1$ and $\eta_u(0) < 1$. Then $\kappa^{-\alpha_u/\alpha_t} s_{(u)}(Y_t(\kappa))$ has a finite and positive limit as $\kappa \longrightarrow 0$.*

Proof. Writing $B_t = RA_t\left[1 - \eta_{(t)}(0)\right] > 0$, where A_t is defined as in Lemma 4.3, the required limit is $C = A_u B_t^{-\alpha_u/\alpha_t}$. Let $\varepsilon > 0$. We shall show that $\kappa^{-\alpha_u/\alpha_t} s_{(u)}(Y_t(\kappa))$ is within ε of C for all small enough κ.

As a preliminary observation, note that

$$\frac{A_u + z}{(B_t - z)^{\alpha_u/\alpha_t}} \longrightarrow C \text{ as } z \longrightarrow 0$$

and it follows that, there is an $\varepsilon' > 0$ such that

$$C - \varepsilon < \frac{A_u - \varepsilon'}{(B_t + \varepsilon')^{\alpha_u/\alpha_t}} < \frac{A_u + \varepsilon'}{(B_t - \varepsilon')^{\alpha_u/\alpha_t}} < C + \varepsilon. \tag{29}$$

By Lemma 8.2, there is a Y_1 such that

$$\left|Y^{\alpha_t}\widetilde{\pi}_{(t)}(Y) - B_t\right| < \varepsilon' \text{ for all } Y > Y_1.$$

For all $\kappa < \widetilde{\kappa}_1$ (say), we have $Y_t(\kappa) > Y_1$ by Lemma 8.3 and therefore, recalling $\widetilde{\pi}_{(t)}(Y_t(\kappa)) = \kappa$,

$$\frac{\kappa}{B_t + \varepsilon'} < [Y_t(\kappa)]^{-\alpha_t} < \frac{\kappa}{B_t - \varepsilon'} \tag{30}$$

$$\Longrightarrow \left(\frac{\kappa}{B_t + \varepsilon'}\right)^{\alpha_u/\alpha_t} < [Y_t(\kappa)]^{-\alpha_u} < \left(\frac{\kappa}{B_t - \varepsilon'}\right)^{\alpha_u/\alpha_t}.$$

Applying Lemma 4.3 to contestants of type u, there is a Y_2 such that

$$\left|Y^{\alpha_u} s_{(u)}(Y) - A_u\right| < \varepsilon' \text{ for all } Y > Y_2.$$

For all $\kappa < \widetilde{\kappa}_2$ (say), we have $Y_t(\kappa) > Y_2$ and therefore

$$[Y_t(\kappa)]^{-\alpha_u}(A_u - \varepsilon') < s_{(u)}(Y_t(\kappa)) < [Y_t(\kappa)]^{-\alpha_u}(A_u + \varepsilon'). \tag{31}$$

If $\kappa < \min\{\widetilde{\kappa}_1, \widetilde{\kappa}_2\}$, we can combine (30) and (31) to obtain

$$\frac{A_u - \varepsilon'}{(B_t + \varepsilon')^{\alpha_u/\alpha_t}} < \kappa^{-\alpha_u/\alpha_t} s_{(u)}(Y_t(\kappa)) < \frac{A_u + \varepsilon'}{(B_t - \varepsilon')^{\alpha_u/\alpha_t}},$$

and comparison with (29) completes the proof. ■

The following corollary is particularly useful.

Corollary 8.5 *If $\alpha_T < \alpha_t$ then, for $u = 1, \ldots, T$,*

$$\frac{s_{(u)}\left(Y_T\left(\kappa\right)\right)}{s_{(u)}\left(Y_t\left(\kappa\right)\right)} \longrightarrow 0 \text{ as } \kappa \longrightarrow 0.$$

Proof. Apply Lemma 8.4 to the right hand side of:

$$\frac{s_{(u)}\left(Y_T\left(\kappa\right)\right)}{s_{(u)}\left(Y_t\left(\kappa\right)\right)} = \frac{\kappa^{-\alpha_u/\alpha_T} s_{(u)}\left(Y_T\left(\kappa\right)\right)}{\kappa^{-\alpha_u/\alpha_t} s_{(u)}\left(Y_t\left(\kappa\right)\right)} \kappa^{\theta},$$

where

$$\theta = \alpha_u \left(\frac{1}{\alpha_T} - \frac{1}{\alpha_t}\right) > 0.$$

■

We have now assembled the machinery required to prove Proposition 6.3.

Completion of proof of Proposition 6.3.

We consider entry costs satisfying

$$\kappa < \overline{\kappa}_1 = \min_{t=1,\ldots,T} \{\kappa_t^*\}$$

so that $Y_t(\kappa)$ is defined for all t. Applying the second part of Condition 6.2 to type T: if

$$s_{(T)}\left(Y_T'\right) + \sum_{u=1}^{T} n_u s_{(u)}\left(Y_T'\right) = 1, \tag{32}$$

then $\widetilde{\pi}_{(T)}\left(Y_T'\right) < \kappa\lambda_T = \widetilde{\pi}_{(T)}\left(Y_T\left(\kappa\right)\right)$. It follows from Proposition 5.2 that $Y_T' > Y_T(\kappa)$ and hence, by Proposition 2.1, $s_{(t)}\left(Y_T'\right) \leq s_{(t)}\left(Y_T\left(\kappa\right)\right)$ for all t, with strict inequality if $s_{(t)}\left(Y_T\left(\kappa\right)\right) > 0$. We may deduce from (32) that

$$s_{(T)}\left(Y_T\left(\kappa\right)\right) + \sum_{u=1}^{T} n_u s_{(u)}\left(Y_T\left(\kappa\right)\right) > 1. \tag{33}$$

Note that (10) implies that $\alpha_t > \alpha_T$ for $t = 1, \ldots, T-1$. It follows from Corollary 8.5 that there is a $\overline{\kappa}_2 \in (0, \overline{\kappa}_1]$ such that, if $0 < \kappa < \overline{\kappa}_2$,

$$2 s_{(u)}\left(Y_T\left(\kappa\right)\right) < s_{(u)}\left(Y_t\left(\kappa\right)\right) \tag{34}$$

for all $t \neq T$ and all $u = 1, \ldots, T$. Lemma 8.3 and Proposition 2.1 imply that there is a $\overline{\kappa}_3 \in (0, \overline{\kappa}_1]$ such that, if $0 < \kappa < \overline{\kappa}_3$,

$$2 s_{(T)}\left(Y_T\left(\kappa\right)\right) < 1. \tag{35}$$

Multiplying (33) by 2 and rearranging gives

$$\begin{aligned} 1 &< 2s_{(T)}\left(Y_T\left(\kappa\right)\right) - 1 + 2\sum_{u=1}^{T} n_u s_{(u)}\left(Y_T\left(\kappa\right)\right) \\ &< 2\sum_{u=1}^{T} n_u s_{(u)}\left(Y_t\left(\kappa\right)\right), \end{aligned}$$

for $t = 1, \ldots, T-1$. The second inequality exploits (34) and (35) and implies that, if Y satisfies (17) in Condition 6.2, then $Y > Y_t(\kappa)$ and holds provided $0 < \kappa < \overline{\kappa} = \min\{\overline{\kappa}_2, \overline{\kappa}_3\}$. Hence, $\widetilde{\pi}_{(t)}(Y) < \widetilde{\pi}_{(t)}(Y_t(\kappa)) = \kappa\lambda_t$ which conflicts with (16). Condition 6.2 can only be satisfied if $n_t = 0$, completing the proof. ■

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