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Abstract

We exploit the aggregative structure of the public good model to provide a simple analysis of the voluntary contribution game. In contrast to the best response function approach, ours avoids the proliferation of dimensions as the number of players is increased, and can readily analyse games involving many heterogeneous players. We demonstrate the approach at work on the standard pure public economic model and show how it can analyse extensions of the basic model.

Key words: noncooperative games, public goods

JEL classifications: C72, H41

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1 Introduction

The simplicity of the pure public good model has long made it a favorite topic for students of public economics. Economists use it routinely to examine the inefficiency of decentralized resource allocation processes in the presence of externalities and to explore the properties of alternative mechanisms. Yet many existing analyses do not fully exploit its simplicity. The present paper develops an alternative way of analyzing the model that exploits its *aggregative* nature - that is, the fact that individuals care about, not an arbitrary vector of individual contributions, but a very specific aggregate, their unweighted sum. This feature has, of course, been noted by others - for example, Okuguchi [23], Corchon [9] and Cornes, Hartley and Sandler [12] suggest methods of analysis that can exploit this feature. This paper develops the approach exploited by Cornes, Hartley and Sandler, both simplifying it and also extending it to analyze comparative static questions. We feel our approach has several virtues. It permits a unified analysis of existence, uniqueness and comparative static properties of the voluntary contribution model without requiring the use of fixed point or other theorems in high dimensional spaces. Second, it also suggests a simple and revealing geometric representation. Finally, in addition to its transparency and simplicity, it offers a powerful and versatile tool of analysis for a wide range of games with aggregative structure. We demonstrate its simplicity by providing a unified account of the principal properties of the basic public good model, and we demonstrate its power and versatility by analyzing an extension of the model that accommodates unit costs that differ across contributors.

2 The Basic Pure Public Good Game

Our starting point is the basic voluntary contribution model set out in Cornes and Sandler ([13],[16]) and Bergstrom, Blume and Varian [4]. Under standard assumptions, this model possesses a unique Nash noncooperative equilibrium at which the pure public good will tend to be underprovided. The model possesses interesting and, at first sight, surprising comparative static properties. In particular, the neutrality proposition asserts that, if all contributors face the same unit cost of contributing to the public good, a redistribution of initial income between positive contributors will affect neither the equilibrium values of the total provision level Q nor the individual players' private good consumption and utility levels. This section introduces the basic model and exploits its aggregative structure to provide a novel and simple derivation of these and other familiar properties.

2.1 Assumptions of the model

There are n players. Player i's preferences are represented by the utility function

$$u_i = u_i(y_i, Q) \tag{1}$$

where y_i is the quantity of a private good and Q the total quantity of a pure public good. We impose the following assumptions:

A.1: Well-behaved individual preferences For all i, the function $u_i(.)$ is everywhere strictly increasing in both arguments[1].

A.2: Linear individual budget constraints Player i's budget constraint requires that

$$y_i + q_i \leqq m_i \tag{2}$$

where $q_i \geqq 0$ is her contribution to a pure public good and m_i is her income, which is assumed exogenous.

Since we assume the unit cost of contributing to be a constant and to be equal across players, there is no loss of generality in putting it equal to unity. Later, we extend the model and allow unit costs to vary across contributors.

We need to specify how individual contributions combine to determine the total quantity of the public good:

[1] The assumptions of quasiconcavity and differentiability, which are often invoked, may be dispensed with. The latter may be avoided by simply treating the individual's marginal rate of substitution as the slope of a supporting line. The positive properties of the model are essentially driven by assumption A.4, which itself may be regarded as the implication of appropriate restrictions on $u_i(.)$.

A.3: Additive Social Composition Function The total supply of the public good is the sum of all individual contributions:

$$Q = \sum_{j=1}^{n} q_j = q_i + Q_{-i} \tag{3}$$

where Q_{-i} is the sum of the contributions made by all players except i.

The budget constraint (2) may be written so as to incorporate the contributions of others explicitly as a component of player i's income endowment. Add the quantity Q_{-i} to both sides. This yields

$$y_i + Q \leqq m_i + Q_{-i}. \tag{4}$$

This requires that the value of the bundle consumed by i cannot exceed the value of her endowment point. This value is her "full income", $\phi_i \equiv m_i + Q_{-i}$. In addition, the player is restricted to allocations consistent with the condition that $y_i \leqq m_i$, reflecting the assumption that she cannot undo the contributions of others and transform them into units of the private consumption good.

Player i chooses nonnegative values of y_i and q_i to maximize utility subject to her budget constraint and the prevailing value of Q_{-i}. To any non-negative value of Q_{-i} there corresponds a unique utility-maximizing contribution level, $\widehat{q}_i$. By varying Q_{-i} parametrically, we generate her best response function, $\widehat{q}_i = b_i(Q_{-i})$. At a Nash equilibrium, every player's choice is a best response to the prevailing choices of all other players.

Figure 1 depicts an individual's preferences and constraint set. The values of Q_{-i} and m_i fix the endowment point E, and the constraint set is the area ODEF, where the slope of EF is minus one, reflecting the assumption that the marginal rate of transformation between q_i and y_i is unity. Strict quasiconcavity of $u_i(.)$ implies a unique utility-maximizing response, shown as the point of tangency T. The figure also shows the locus of tangencies traced out for a given value of m_i as Q_{-i} varies parametrically. The figure shows this locus to be everywhere upward-sloping. This reflects the following assumption that we impose on preferences:

A.4: Normality For every player i, both the private good and the public good are normal. That is, the locus of values of y_i and Q consistent with a given marginal rate of substitution has positive finite slope everywhere.

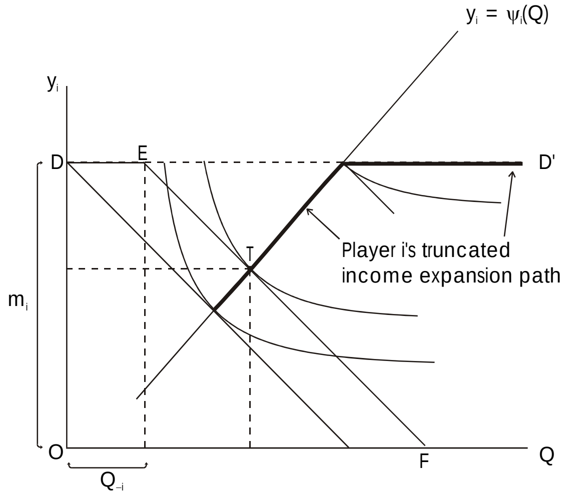

Figure 1: Player i's preferences and constraint set

2.2 The Replacement Function

2.2.1 The Individual Replacement Function

Consider the locus of allocations in (Q, y_i) space supported by the relative price of unity. Assumptions A.4 implies that, along this locus, the quantity of private good may be described by a single-valued function, $y_i = \psi_i(Q)$, which is everywhere strictly increasing. [To keep statements of results simple, we shall assume throughout that $\psi_i(Q) > 0$ at all allocations of interest].

A player's demand for the private good cannot exceed her money income. Her preferred private good consumption is therefore $\widehat{y}_i = \min\{m_i, \psi_i(Q)\}$. This function provides a simple way of describing our principal tool of analysis, the replacement function:

Definition 1 *Assumptions* A.1 - A.4 *are sufficient to imply the existence of a function, $r_i(Q, m_i)$, which uniquely determines player i's most preferred level of contribution as a function of money income and the observed total quantity of the public good:*

$$r_i(Q, m_i) = m_i - \widehat{y}_i = \max\{m_i - \psi_i(Q), 0\}.$$

The function $r_i(Q, m_i)$ is player i's replacement function.

The replacement function possesses a number of simple properties that follow directly from its definition and are useful in subsequent analysis. Denote by $\underline{Q}_i$ the quantity that is player i's best response when all other players' contributions are zero. At such an allocation, player i's contribution is the total provision level: $r_i\left(\underline{Q}_i, m_i\right) = \underline{Q}_i$. We call $\underline{Q}_i$ player i's **standalone value**, since it is the equilibrium level of her contribution when she is the sole contributor.

Now consider how the associated preferred value $\widehat{q}_i$ responds as Q changes. Consider two values of total provision, Q^0 and Q^1, such that $\underline{Q}_i \leqq Q^0 < Q^1$. Clearly, $\psi_i(Q^1) > \psi_i(Q^0)$, which in turn implies that $m_i - \psi_i(Q^1) < m_i - \psi_i(Q^0)$. Thus, we can infer the following proposition:

Proposition 2.1 *Consider any two values of total public good provision, Q^0 and Q^1, where $Q^1 > Q^0$. Then $r_i(Q^1, m_i) \leqq r_i(Q^0, m_i)$, with strict inequality holding if $\psi_i(Q^0) < m_i$.*

The following proposition summarizes the significant properties of $r_i(Q, m_i)$.

Proposition 2.2 *If assumptions* A.1, A.2, A.3 *and* A.4 *hold, player i has a replacement function $r_i(Q)$ with the following properties:*

1. $r_i(Q)$ is defined for all $Q \geqq \underline{Q}_i$, where $\underline{Q}_i$ [player i's **standalone value**] is the level of the public good that player i would contribute if she were the sole contributor.

2. $r_i\left(\underline{Q}_i\right) = \underline{Q}_i$

3. $r_i(Q)$ is continuous.

4. $r_i(Q)$ is everywhere nonincreasing[2].

Remark 1 *We call $r_i(Q)$ player i's replacement function for the following reason. Consider any $Q \geqq \underline{Q}_i$. Then there is a unique quantity $Z \in [0, Q]$ such that, if the amount Z were subtracted from the quantity Q, the player's best response to the remaining quantity would exactly replace the quantity removed, and $Z = r_i(Q)$.*

[2] In fact, wherever $r_i(Q)$ is strictly positive, it is also strictly decreasing in Q.

2.2.2 The Aggregate Replacement Function

We now define the **aggregate replacement function** of the game, $R(Q)$:

Definition 2 *The* aggregate replacement function of the game $R(Q)$ *is defined as*

$$R(Q) = \sum_{j=1}^{n} r_j(Q).$$

The properties of the individual replacement functions are either preserved or else are modified in very slight and straightforward ways by the operation of addition by which $R(Q)$ is generated from them. The following proposition summarizes the salient properties of $R(Q)$:

Proposition 2.3 *If assumptions* A.1 - A.4 *hold for all i, there is an aggregate replacement function,* $R(Q) \equiv \sum_j r_j(Q)$, *with the following properties:*

1. $R(Q)$ is defined for all $Q \geqq \max\left\{\underline{Q}_1, \underline{Q}_2, ..., \underline{Q}_n\right\}$
2. $R\left(\max\left\{\underline{Q}_1, \underline{Q}_2, ..., \underline{Q}_n\right\}\right) \geqq \max\left\{\underline{Q}_1, \underline{Q}_2, ..., \underline{Q}_n\right\}$
3. $R(Q)$ is continuous.
4. $R(Q)$ is everywhere nonincreasing[3].

Use of $R(Q)$ suggests a simple way of describing Nash equilibrium. A Nash equilibrium is an allocation at which every player is choosing her best response to the choices made by all other players. Clearly, the Nash equilibrium level of total provision, Q^*, must equal the sum of all best responses associated with the equilibrium allocation:

$$\widehat{q}_1 + \widehat{q}_2 + ... + \widehat{q}_n = Q^*.$$

We have just shown that each best response may be described by that player's replacement function. At a Nash equilibrium, therefore,

$$r_1(Q^*) + r_2(Q^*) + ... + r_n(Q^*) = Q^*.$$

To summarize, we have established the following characterization of Nash equilibrium:

[3] Again, wherever $R(Q)$ is strictly positive, it is strictly decreasing in Q.

Replacement function characterisation of Nash equilibrium *A Nash equilibrium is an allocation* $[r_1(Q^*), r_2(Q^*), r_3(Q^*), ..., r_n(Q^*), Q^*]$ *such that*

$$R(Q^*) \equiv \sum_{j=1}^{n} r_j(Q^*) = Q^*.$$

Remark 2 *This characterization does not require a proliferation of dimensions as the number of players increases. One simply adds together more functions, each defined on an interval of the real line.*

2.3 Nash Equilibrium: Existence and Uniqueness

Recall that a Nash equilibrium is an allocation at which $R(Q) = Q$. Geometrically, it is a point at which the graph of $R(Q)$ intersects the 45^o ray through the origin in $(Q, R(Q))$ space.

Referring back to **Proposition 2.4**, Property 1 identifies the domain on which $R(Q)$ is defined. Property 2 locates a value of Q in that domain for which $R(Q) \geqq Q$. Properties 3 and 4 guarantee the existence of a unique value, Q^*, at which $R(Q^*) = Q^*$. Thus, we can immediately draw the following inference:

Proposition 2.4 *There exists a unique Nash equilibrium in the pure public good game.*

Figure 2 illustrates this proposition. It depicts four individual replacement functions and the implied aggregate replacement function in a 4-player public good economy. The Nash equilibrium is the unique point of intersection between the graph of $R(Q)$ and the ray through the origin O with slope 1. Three observations are especially worth noting. First, existence and uniqueness are effectively established by a single elementary line of argument that exploits the continuity and monotonicity of the graph of $R(Q)$. Second, Properties 1- 4 of the aggregate replacement function are merely sufficient for the existence of a unique equilibrium. As we will indicate below, they are not necessary. Finally, the fact that the four players are heterogeneous in no way complicates any part of the analysis or exposition.

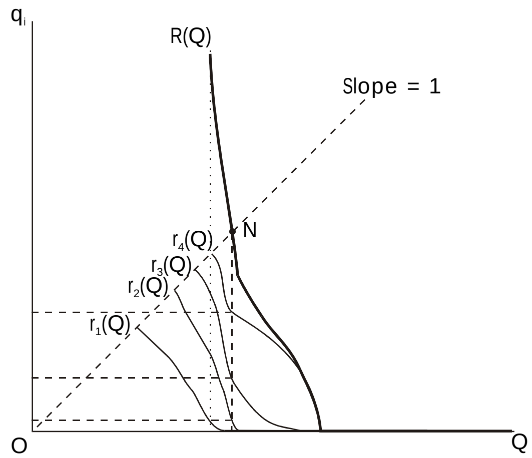

Figure 2: Individual and aggregate replacement functions in 4-player economy

2.4 Nash Equilibrium: Comparative Static Properties

Suppose we are initially at an equilibrium. Player i's replacement function may be written as $r(Q : \boldsymbol{\alpha})$, where $\boldsymbol{\alpha}$ is a vector of parameters whose values affect player i's replacement value. The effect of any shock on the equilibrium, represented as a change in $\boldsymbol{\alpha}$, may be thought of as shifting the graphs of individual replacement functions, and therefore the aggregate replacement function. The equilibrium value of Q rises, remains unchanged, or falls according to whether, at its initial equilibrium value, the aggregate replacement value rises, remains unchanged, or falls. Comparative static properties of the Nash equilibrium are conveniently tackled by first considering the comparative static properties of individual replacement functions.

2.4.1 Comparative statics of a player's replacement function

Corner solutions Figure 2 depicted an example in which every player's replacement function falls to zero at some finite value of Q. However, assumptions A.1 - A.4 do not necessarily imply that replacement functions have this property. For example, an income expansion path described by $y_i = \overline{y}_i - m_i/Q$, where $\overline{y}_i > 0$ is a parameter, is consistent with the assumptions. If $\overline{y}_i < m_i$ then, as Q grows, player i's private good consumption asymptotically approaches $\overline{y}_i < m_i$. Her replacement function therefore does not fall to zero, but converges to some positive value as Q increases.

We have not committed ourselves on whether or not there is a finite value of Q at which $r_i(Q) = 0$ for the simple reason that our analysis is not at all complicated, and our conclusions concerning existence and uniqueness are not affected, by our answer to this question. However, the existing literature typically assumes, either directly or indirectly, the existence of such a value. In what follows, we will adopt the following slightly stronger normality assumption[4]:

A.4*: Bounded normality For every player i, there is a finite value of Q, $\overline{Q}_i$, such that $\psi_i\left(\overline{Q}_i\right) = m_i$.

We will call the quantity $\overline{Q}_i$ player i's **dropout value**, since it is the value of total provision at which, as Q increases, she drops out of the set of positive contributors: $r_i(Q) = 0$ for all $Q > \overline{Q}_i$. Let the equilibrium level of aggregate provision be Q^N. Then any player whose dropout value falls short of Q^N will be a noncontributor at that equilibrium[5].

The response to a change in income Consider an exogenous change in player i's money income, and suppose that player i is a positive contributor both before and after the income shock. The graphical depiction should make clear how our conclusions are modified if, at some stage, the player becomes constrained by the requirement that $q_i \geqq 0$. Since we are interested in the effects of changes in income, we interpret the α_i's as players' income levels: $r_i(.) = r_i(Q, m_i)$ and $R(Q, \mathtt{m})$ where $\mathtt{m} \equiv (m_1, m_2, ..., m_n)$. Now hold Q fixed and consider the effect of a potentially discrete change in i's income on her replacement value under the assumption that both before and after the change player i is a positive contributor. Geometrically, we are interested in the vertical shift in the graph of $r_i(.)$ against Q. Under our assumptions, there is a one-to-one mapping between player i's preferred level of Q and her full income, ϕ_i. Thus, if player i is choosing optimally, an unchanged value of Q implies that her full income is unchanged. But if full income and Q are

[4] Andreoni [1] makes an assumption that has essentially the same implication as A.4*.

[5] Andreoni and McGuire [2] note the significance of players' dropout values in determining who are in the set of positive contributors at a given equilibrium. More recently, McGuire and coauthors have exploited this magnitude in further exploration of the characteristics of equilibrium - see [21] and [22].

both unchanged, $\widehat{y}_i$ must also be unchanged, since $\widehat{y}_i + Q = \phi_i$. But we know that

$$\widehat{y}_i = m_i - r_i(Q, m_i).$$

Suppose income is initially m_i^0 and suppose that it now changes to m_i^1 while Q is held fixed at, say $\widetilde{Q}$. Then we can infer that

$$\widehat{y}_i^1 - \widehat{y}_i^0 = 0 = m_i^1 - r_i\left(\widetilde{Q}, m_i^1\right) - \left[m_i^0 - r_i\left(\widetilde{Q}, m_i^0\right)\right]. \tag{5}$$

Rearrangement of (5) yields the following proposition:

Proposition 2.5 *In the basic pure public good model, let player i be a positive contributor both before and after an exogenous change in money income from m_i^0 to m_i^1. Then, at unchanged $\widetilde{Q}$, the change in her replacement value is*

$$r_i\left(\widetilde{Q}, m_i^1\right) - r_i\left(\widetilde{Q}, m_i^0\right) = m_i^1 - m_i^0.$$

The utility of player i can be written as $u_i(\min\{\psi_i(Q), m_i\}, Q)$. For any player who contributes at equilibrium, $\min\{\psi_i(Q), m_i\} = \psi_i(Q)$. This immediately implies the following result:

Proposition 2.6 *If players i and j have identical preferences but possibly different incomes, and if they both make positive contributions at equilibrium, then both will enjoy the same utility level.*

The response to a change in Q We already know that an increase in Q by itself implies a fall in a player's replacement value if that value is initially strictly positive. However, it is useful to obtain a more precise statement of this response and to relate it to more familiar terms. Write the demand function for the public good of a positive contributor as $\widehat{Q}_i(\phi_i) = \widehat{Q}_i(m_i + Q_{-i})$. The derivative $\widehat{Q}_i'$ is player i's **marginal propensity to contribute** to the public good.

Now consider a change in the contributions of others. If player i is a positive contributor, her preferred quantity of the public good may be written either as $\widehat{Q}_i(\phi_i)$ or, using the replacement function, as $Q_{-i} + r_i(Q, m_i)$. At an equilibrium, therefore, we may write

$$\widehat{Q}_i(m_i + Q_{-i}) = Q_{-i} + r_i\left(\widehat{Q}_i(m_i + Q_{-i}), m_i\right). \tag{6}$$

Now consider a change in Q_{-i}, and suppose that i is a positive contributor both before and after the change. Differentiating (6),

$$\widehat{Q}_i' = 1 + r_{iQ}\widehat{Q}_i' \tag{7}$$

where $r_{iQ} \equiv \frac{\partial r_i\left(\widehat{Q}_i(m_i+Q_{-i}),m_i\right)}{\partial Q}$. Rearranging,

$$r_{iQ} = \frac{\widehat{Q}_i' - 1}{\widehat{Q}_i'}. \tag{8}$$

This result provides a useful link with the more familiar marginal propensity to contribute, which we exploit below.

2.4.2 Comparative statics of equilibrium provision

The comparative static properties of the individual replacement functions lead directly to a number of interesting properties of equilibrium provision. The first is the celebrated neutrality property:

Corollary 1 (Neutrality) *Any redistribution of income between a set of positive contributors that leaves that set unchanged also leaves the equilibrium allocation unchanged.*

Proof. The value of the aggregate replacement function may be expressed as

$$R(Q) = \sum_{i\in C} m_i - \sum_{i\in C} \psi_i(Q) \tag{9}$$

where C denotes the set of positive contributors. Thus, if Q^* is a Nash equilibrium, any redistribution of initial incomes that leaves both the aggregate income, and the set, of contributors unchanged leaves 9 unaffected. Q^* therefore remains the equilibrium value. ■

The neutrality property implies that, in a well-defined sense, the set of positive contributors who face the same unit cost of public good provision behaves like a single individual. If attention is confined to income distributions that are consistent with a given set of positive contributors, then the aggregate replacement function associated with that set depends upon just two arguments: the total income of all contributors, and the value of Q:

Corollary 2 *For all income distributions consistent with a given set C of players being the positive contributors to the public good in equilibrium,* $R(.) = R(Q, M_C)$, *where* $M_C = \sum_{j \in C} m_j$.

Consider the response of total equilibrium public good provision to a change in the total income received by the set of contributors. We assume throughout that the set of contributors is unchanged. At a Nash equilibrium,

$$R(Q) = \sum_{j \in C} r_j(Q, m_j) = Q. \tag{10}$$

Now suppose that the contributors' income levels change. At the new equilibrium, it remains the case that the sum of replacement values equals the total provision. Differentiating (10),

$$\sum_{j \in C} r_{jQ}(.)\, dQ + \sum_{j \in C} r_{jm}(.)\, dm_j = dQ \tag{11}$$

We have already shown that $r_{jm}(.) = 1$ for all $j \in C$. Writing $M_C \equiv \sum_{j \in C} m_j$, equation (11) becomes

$$\sum_{j \in C} r_{jQ}(.)\, dQ + dM_C = dQ$$

or

$$\frac{dQ}{dM_C} = \frac{1}{1 - \sum_{j \in C} r_{jQ}(.)}. \tag{12}$$

Substituting from 8,

$$\frac{dQ^N}{dM_C} = \frac{1}{1 + \sum_{j \in C} \left(\frac{1}{\hat{Q}'_j} - 1 \right)} = \frac{1}{1 - n_C + \sum_{j \in C} \left(\frac{1}{\hat{Q}'_j} \right)}.$$

To summarize,

Proposition 2.7 *Let the aggregate income of the set of contributors change by an amount dM_C, and assume that the set of positive contributors is unchanged. Then the response of the equilibrium provision of the public good is given by*

$$\frac{dQ^N}{dM_C} = \frac{1}{1 - n_C + \sum_{j \in C} \left(1/\widehat{Q}'_j\right)}. \tag{13}$$

This is precisely the result obtained by Cornes and Sandler (2000). To get a feeling for the magnitude of this response, suppose that contributors are identical, with $\widehat{Q}'_j = \widehat{Q}'$ for all $j \in C$. Then (13) may be written as

$$\frac{dQ^N}{dM_C} = \frac{\widehat{Q}'}{\widehat{Q}' + \left(1 - \widehat{Q}'\right) n}. \tag{14}$$

Normality implies that $0 < \left(1 - \widehat{Q}'\right) < 1$. If we slightly strengthen this condition slightly by supposing that $\left(1 - \widehat{Q}'\right)$ is bounded away from zero - that is, $\left(1 - \widehat{Q}'\right) \geqq \epsilon > 0$ - then (14) implies that

$$\lim_{n \to \infty} \frac{dQ^N}{dM_C} = 0.$$

For example, suppose that each has a constant marginal propensity to contribute of $\widehat{Q}' = 0.5$. Inspection of (14) shows that the aggregate equilibrium response in an n-player game is $dQ^N/dM_C = 1/\left(\frac{1}{\widehat{Q}'} - 1\right) n_C + 1 = 1/\left(n_C + 1\right)$. If $n = 10$, $dQ^N/dM_C = 1/11$. If $n_C = 100$, $dQ^N/dM = 1/101$. For a given common value of the individual marginal propensity, the magnitude of the aggregate propensity falls rapidly as n_C increases.

A further implication of (13) is worth noting. Consider an equilibrium in which the existing contributors are not identical. Denote by $\widehat{Q}'_{\min}$ the lowest of the individual marginal propensities to contribute, and suppose for simplicity that there is just one player whose marginal propensity to contribute takes this value. Then

$$\frac{dQ^N}{dM} = \frac{1}{1 + \sum_j^n \left(\frac{1}{\widehat{Q}'} - 1\right)} = \frac{1}{\frac{1}{\widehat{Q}'_{\min}} + \sum^n_{\substack{j=1 \\ j \neq \min}} \left(\frac{1}{\widehat{Q}'_j} - 1\right)}. \tag{15}$$

Normality implies that each player's marginal propensity to contribute is less than one, which in turn implies that the summation term must be positive. (15) implies that, at any equilibrium allocation, the aggregate response dQ/dM is less than the smallest individual response $\widehat{Q}'_{\mathsf{min}}$. Not only does the interaction between players' responses dampen the response of aggregate provision to any change in the income of the set of positive contributors. In addition, the presence of just one contributor with a low propensity to contribute is enough to place a precise upper bound on the aggregate propensity to contribute of a given set of positive contributors.

This suggests that a public good economy with a large number of potential contributors displays an 'approximate neutrality property' in the following sense. Suppose that players differ from one another with respect to both preferences and income. One can imagine drawing individuals from a joint distribution of preferences and income. For any given individual, specification of the preference map and income level determines that player's dropout value. Now order the individuals according to their dropout values. Denote by type-H the type with the highest dropout value. Suppose that there is a finite number of types. As more individuals are drawn from the joint probability distribution, the number of type-H's increases. There will be a number of type-H's, say n_H^* - such that, for any $n_H \geqq n_H^*$, the resulting equilibrium involves only this type making a positive contribution. Although, as Andreoni ([1]) shows, the proportion of the total population that makes a positive contribution may be vanishingly small, it may yet be the case that the absolute number of type-H's - each of whom is a positive contributor - is large. If it is large enough, the aggregate marginal propensity to contribute may be vanishingly small. If this is the case, then redistribution amongst any types will have little effect on the equilibrium aggregate level of provision.

What are the **normative** implications of a redistribution of initial incomes in this model? We have shown that redistributions of initial income among positive contributors change nothing. Redistributions among noncontributors benefit the recipients and hurt the donors, leaving the utilities of all others unchanged. But what about redistributions from noncontributors to contributors? Cornes and Sandler [17] show that, even when every individual faces the same unit cost of contribution to the public good, such transfers can lead to a new Nash equilibrium that Pareto-dominates the equilibrium associated with the initial income distribution. This is easily shown in a simple two-type economy. Consider an equilibrium of a public goods economy at which there are n_N noncontributors and n_C positive contributors. The utility of a typical noncontributor is

$$u_N = u_N(y_N, Q) = u_N(m_N, Q)$$

Now suppose that the same amount of income is taken from each noncontributor and given to a positive contributor. To keep the exposition simple, assume the set of contributors is unchanged at the new equilibrium. Let the total extra income received by all contributors be dM_C. Each noncontributor loses an amount of income $dm_N = -dM_C/n_N$.

The change in utility of a typical noncontributor is

$$\begin{aligned} du_N &= \frac{\partial u_N(m_N, Q)}{\partial y_N} dm_N + \frac{\partial u_N(m_N, Q)}{\partial Q} dQ \\ &= \frac{\partial u_N(m_N, Q)}{\partial y_N} [\nu_N dQ + dm_N] \\ &= \frac{\partial u_N(m_N, Q)}{\partial y_N} [\nu_N dQ - dM_C/n_N] \end{aligned}$$

where $\nu_N \equiv \frac{\partial u_N(m_N,Q)/\partial Q}{\partial u_N(m_N,Q)/\partial y_N}$ is the noncontributor's marginal valuation of the public good. The fact that an individual is choosing not to contribute implies that, at equilibrium, $\nu_N < c$. This is consistent with her placing a strictly positive valuation on the public good. The typical noncontributor will be better off if, in the course of adjustment to the new equilibrium, $\nu_N dQ - dM_C/n_N > 0$.

To determine whether noncontributors are made better off, we need to determine the endogenous response of total provision. We already know that

$$dQ = \left\{ \frac{\widehat{Q}'}{\widehat{Q}' + \left(1 - \widehat{Q}'\right) n_C} \right\} dM_C \tag{16}$$

where $\widehat{Q}'$ is the marginal propensity to contribute of the typical contributor. Substituting into (16),

$$du_N = \frac{\partial u_N(m_N, Q)}{\partial y_N} \left\{ \nu_N \left(\frac{\widehat{Q}'}{\widehat{Q}' + \left(1 - \widehat{Q}'\right) n_C} \right) - 1/n_N \right\} dM_C$$

The right hand side is positive if the expression in square brackets is positive - that is, if

$$\frac{n_N \widehat{Q}' \nu_N - \widehat{Q}' - \left(1 - \widehat{Q}'\right) n_C}{\left(\widehat{Q}' + \left(1 - \widehat{Q}'\right) n_C\right) n_C} > 0.$$

The denominator of this expression is positive. Therefore the utility of a noncontributor rises if the numerator is positive. Rearranging, this requires that

$$\widehat{Q}' > \frac{n_C}{n_N \nu_N + n_C - 1}.$$

This makes sense. If n_N is large, each noncontributor is only one of many who are giving up income, and the gain in the aggregate income of contributors may be significant by comparison. Furthermore, the greater is $\widehat{Q}'$, the greater is the additional public good provision purchased by a given transfer of income.

3 Varying Unit Costs across Contributors

There has recently been a lot of interest in applying the public good framework to settings in which it is natural to allow unit costs to differ across potential contributors - see, for example, the discussions of regional and global public goods by Arce and Sandler[3] and Sandler [24], and the various contributions in [19]. The replacement function approach accommodates this extension very easily. Throughout this section, we write player i's budget constraint as

$$a_i y_i + c_i q_i \leqq m_i,$$

where the unit cost parameters a_i and c_i may vary across players. We leave the reader to confirm that, for a given vector of initial incomes, this slight modification does not affect the listed properties of $r_i(Q)$. Consequently, the essential properties of the individual replacement functions are unaffected, and the argument that establishes existence and uniqueness of a Nash equilibrium goes through exactly as before. This extension does not complicate subsequent comparative static analysis. However, it does change some of our conclusions in interesting ways. In particular, the neutrality property associated with income redistribution amongst contributors no longer holds. Equilibria resulting from different distributions of a given aggregate initial endowment among a group of positive contributors may be capable of being Pareto-ranked, and a reduction in contributor i's unit cost may benefit every individual except i[6].

3.1 Replacement functions with arbitrary unit costs

If unit costs are allowed to take arbitrary positive values, the function that describes player i's expansion path is written $\psi_i\left(Q, \frac{c_i}{a_i}\right)$, her consumption of the private good is $\widehat{y}_i\left(Q, m_i, a_i, c_i\right) \equiv \min\left\{\psi_i\left(Q, \frac{c_i}{a_i}\right), \frac{m_i}{a_i}\right\}$, and her realized utility level is

$$u_i\left(\widehat{y}_i, Q\right) = u_i\left(\min\left\{\psi_i\left(Q, \frac{c_i}{a_i}\right), \frac{m_i}{a_i}\right\}, Q\right). \tag{17}$$

Before considering the effects of changes in unit costs, we should draw attention to the following direct, but perhaps surprising, implication of (17) :

Proposition 3.1 *Let players i and j be positive contributors in equilibrium. Suppose further that they have identical preferences, but differ with respect to their unit costs and income levels. Then at equilibrium,*

$$\frac{c_i}{a_i} > \frac{c_j}{a_j} \Longrightarrow u\left(\psi_i\left(Q, \frac{c_i}{a_i}\right), Q\right) > u\left(\psi_j\left(Q, \frac{c_j}{a_j}\right), Q\right).$$

This is an immediate consequence of the fact that $\psi_i\left(Q, \frac{c}{a}\right)$ is a strictly increasing function of c/a. In short, higher cost contributors are better off than otherwise identical lower cost contributors. It does not pay to have a comparative advantage as a producer of the public good. Note, too, that the

[6] A number of writers have recently analysed comparative static possibilities of the kind considered in this section. See, for example, Konrad [20], Buchholz and Konrad [6] and Ihori [18] for analyses of games in which income is transferred at stage one in anticipation of a static public good provision game at stage two. Buchholz, Konrad and Lommerud [7] allow the provision game itself to be sequential, and Buchholz and Peters [8] modify the model to allow income to be endogenously determined. Boadway and Hayashi [5] analyze a model in which unit cost differences across contributors arise in a rather different context.

values of individual incomes play no explicit role in this result. Of course, their values will determine whether, indeed, individuals i and j are contributors. But the fates of positive contributors are locked together, independently of the precise values of m_i and m_j.

3.2 Comparative statics of income changes

Denote by the set C the set of positive contributors at the Nash equilibrium of a public good economy. Consider the equilibrium response by contributor i to a change in her income. Proposition 2.5 is modified as follows:

Proposition 3.2 *Let player i be a positive contributor both before and after an exogenous change in money income from m_i^0 to m_i^1. Then, at unchanged $\widetilde{Q}$, the change in her replacement value is*

$$r_i\left(\widetilde{Q}, m_i^1\right) - r_i\left(\widetilde{Q}, m_i^0\right) = \frac{m_i^1 - m_i^0}{c_i}.$$

Starting at a Nash equilibrium, consider the effect on equilibrium of income redistribution among contributors. Assume that the set of positive contributors is not changed by the redistribution. At the initial equilibrium provision level, Q^*, the value of the aggregate replacement function rises, stays unchanged, or falls according to whether

$$\left(\sum_{j\in C}\left[r_j\left(Q, m_j^1, a_j, c_j\right) - r_j\left(Q, m_j^0, a_i, c_i\right)\right] =\right)\sum_{j\in C}\frac{\left(m_j^1 - m_j^0\right)}{c_j} >, = \text{ or } < 0.$$

For a given set of incomes, the aggregate replacement function is nonincreasing in Q. Therefore the following corollary of proposition 3.2 holds.

Corollary 3 *If an income redistribution leaves the set of positive contributors unchanged, then aggregate equilibrium provision rises, remains unchanged or falls according to whether* $\sum_{j\in C}\frac{\left(m_j^1 - m_j^0\right)}{c_j} >, =$ *or* < 0.

For example, redistribution from contributor A to contributor B increases equilibrium provision if $c_A > c_B$. Redistribution from a higher to a lower cost contributor enhances efficiency, and the efficiency gain is partly taken through an increase in the provision of the public good.

Not only does such a redistribution increase equilibrium public good provision - it is also Pareto-improving. The reasoning is simple. Each individual's preference map in (y, Q) space is fixed throughout the present thought experiment. Under the normality assumption, if each individual is enjoying a higher level of total public good provision after the redistribution, she must have moved upwards and to the right along her income expansion path. Hence, her consumption of the private good is higher, and so must be her utility. In short,

Corollary 4 *If an income redistribution leaves the set of positive contributors unchanged, the new equilibrium is Pareto superior to, identical with, or Pareto inferior to the initial equilibrium according to whether* $\sum_{j \in C} \frac{\left(m_j^1 - m_j^0\right)}{c_j} >, = $ *or* < 0*.*

The following implication of differences in the unit cost of public good provision across contributors is worth noting. A redistribution may lead to an increase in public good provision even though it reduces the aggregate income of contributors. This follows from the simple observation that the inequalities $\sum_{j \in C} \left(m_j^1 - m_j^0\right) < 0$ [a reduction in the aggregate income of contributors] and $\sum_{j \in C} \frac{\left(m_j^1 - m_j^0\right)}{c_j} > 0$ [a Pareto improving change in contributors' incomes] are perfectly consistent with one another if unit costs vary across individuals. Conversely, of course, an increase in the aggregate income of contributors is consistent with a reduction in the equilibrium level of provision.

4 Comparative statics of unit cost changes

The possibility of differing unit cost levels across contributors invites a production theoretic interpretation of the model. It is then natural to consider the implications of changes in the unit costs of contributors, which may be interpreted as reflecting exogenous changes in the technology available to them.

4.1 A change in c_i alone

Let player i be a positive contributor at equilibrium. Now consider equilibrium responses to a discrete fall in her unit cost as a public good provider from its initial value of c_i^0 to c_i^1.

$$c_i^1 < c_i^0 \Longrightarrow \psi_i\left(Q, \frac{c_i^1}{a_i}\right) < \psi_i\left(Q, \frac{c_i^0}{a_i}\right)$$
$$\therefore \quad r_i\left(Q, m_i, a_i, c_i^1\right) > r_i\left(Q, m_i, a_i, c_i^0\right).$$

Evaluated at the initial Nash equilibrium, Q^{N_0}, the aggregate replacement value must now therefore exceed the equilibrium value. Therefore the equilibrium value of public good provision must rise: $Q^{N_1} > Q^{N_0}$.

The rise in equilibrium value of public good provision has an unambiguously beneficial effect for all players other than player i:

Proposition 4.1 *A reduction in c_i raises the equilibrium utility levels of all players other than i.*

However, the effect on the utility of contributor i herself is ambiguous. On the one hand, the increase in equilibrium provision is beneficial. However, for any given level of provision, player i finds herself contributing a higher equilibrium share:

Proposition 4.2 *A reduction in c_i may either raise or reduce i's equilibrium level of utility.*

The only player not guaranteed an increase in utility is the one who enjoyed the exogenous reduction in unit cost as a contributor!

4.2 A numerical example

The following extended example illustrates clearly some of the comparative static possibilities opened up by allowing parametric variation in c_i, and helps to emphasise our claim that surprising responses are quite likely. Each of n players has the Cobb-Douglas utility function $u(y, Q) = yQ$, and the budget constraint of i is $y_i + c_i q_i = 1$. Player i's replacement function is readily shown to be

$$q_i = r_i(Q : c_i) = \max\left\{\frac{1}{c_i} - Q, 0\right\}.$$

If all n players contribute at equilibrium, the equilibrium level of provision is

$$Q = \frac{1}{n+1} \sum_{j=1}^{n} \left(\frac{1}{c_j} \right)$$

and the equilibrium level of utility enjoyed by contributor i is

$$\begin{aligned} u_i &= (1 - c_i q_i) Q = \left\{ 1 - c_i \left(\frac{1}{c_i} - Q \right) \right\} Q = c_i Q^2 \\ &= c_i \left\{ \frac{1}{n+1} \sum_{j=1}^{n} \left(\frac{1}{c_j} \right) \right\}^2 . \end{aligned}$$

To begin, suppose that there are two players. Let the unit cost of contributor 2 be unity, and consider how contributor 1's equilibrium utility varies with her unit cost. Substituting the appropriate values of n and c_2, we get

If $c_1 \leqq 0.5$, only player 1 contributes, and $u_1 = y_1 q_1 = \frac{1}{2}\frac{1}{2c_1} = \frac{1}{4c_1}$.

If $0.5 < c_1 < 2$, both contribute, and $u_1 = \frac{1}{9}\frac{(c_1+1)^2}{c_1}$.

If $c_1 \geqq 2$, player 1 does not contribute, and $u_1 = m_1 Q = 1 \times \frac{1}{2} = \frac{1}{2}$.

The graph of this function is shown in Figure 4. For all $c_1 < 0.5$, contributor 1 is the sole contributor. For $0.5 < c_1 < 2$, all contribute. If $c_1 \geqq 2$, contributor 1 will not find it worthwhile to contribute in equilibrium. Observe that, up to the point where the unit costs of the individuals are equal, an increase in 1's unit cost reduces her equilibrium utility. Thereafter, further increases in her unit cost benefit her up to the point where she drops out. The symmetric situation, in which, $c_1 = c_2$, is precisely the point at which the effect of contributor 1's cost on her own equilibrium utility is of second order - her equilibrium utility is minimized.

If the example is now extended to allow for more than two players, the picture changes slightly. Suppose there are 4 players, and that $c_2 = c_3 = c_4 = 1$. Figure 5 shows how player 1's equilibrium utility varies with her unit

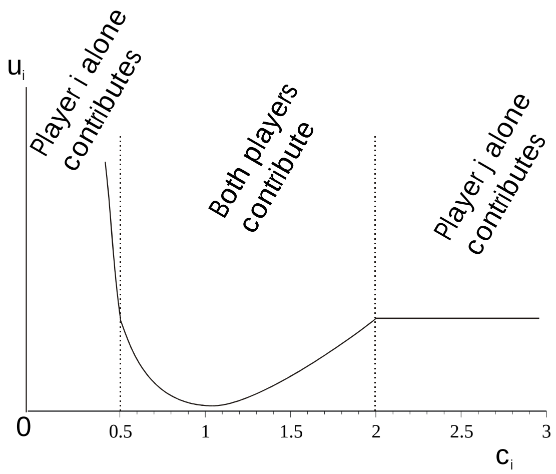

Figure 3: Equilibrium utility graphed against unit cost in 2-person economy with Cobb-Douglas preferences

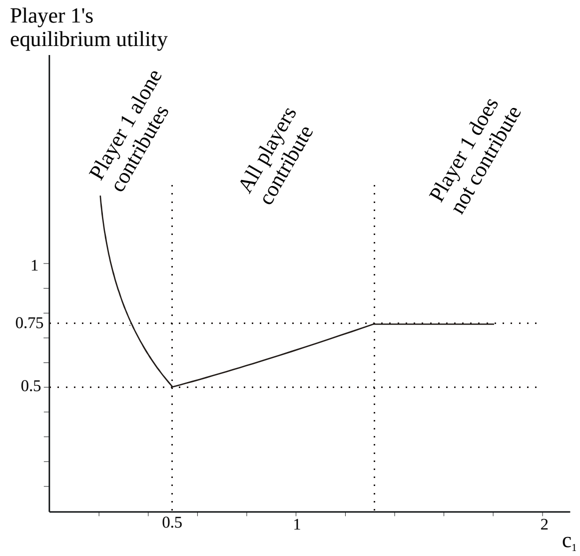

Figure 4: Equilibrium utility graphed against unit cost in 4-person economy with Cobb-Douglas preferences.

cost. For sufficiently low values of c_1, player 1 is the sole contributor, and increases in her unit cost lower her equilibrium utility. At $c_1 = 0.5$, the other players enter, and from that point onwards, increases in c_1 have a first-order negative effect on her equilibrium utility. At $c_1 = 4/3$, contributor 1 drops out, and from that point onwards further increases in c_1 have no effect on her equilibrium utility. It is striking that, throughout the range of values for which all contribute, the effect of changes in c_1 on the equilibrium level of u_1 is 'perverse'. This has interesting implications for any extension of this model that allows players to take steps to enhance, or lower, their productivity as contributors prior to playing the contribution game.

4.3 An equal proportional change in all unit cost levels.

Write player i's unit cost as λc_i. Then an equal proportional change in each player's unit cost is modelled as a change in λ. Player i's replacement function may be written as

$$r_i(Q, m_i, a_i, \lambda c_i) = \frac{m_i - a_i \widehat{y}_i}{\lambda c_i} = Max \left\{ \frac{m_i - a_i \psi_i \left(Q, \frac{\lambda c_i}{a_i}\right)}{\lambda c_i}, 0 \right\}.$$

A reduction in λ shifts every contributor's replacement function upward, thereby increasing the equilibrium level of provision. By itself, the increase in Q increases the utility of each. However, we cannot conclude that every player is made better off. Recall player i's utility function:

$$u_i(\widehat{y}_i, Q) = u_i\left(\min\left\{\psi_i\left(Q, \frac{\lambda c_i}{a_i}\right), \frac{m_i}{a_i}\right\}, Q\right).$$

Player i enjoys a higher equilibrium level of Q. However, for any given level of Q, she also experiences a lower level of private good consumption. This effect may dominate her utility response. Another way to understand this possibility is to observe that, in the move to the new equilibrium, the total contributions made by players other than player i may fall. This harmful change in i's endowment may outweigh the benefit of her own relative price effect.

Proposition 4.3 *An equal proportional reduction in all contributors' unit costs will raise the equilibrium level of total provision. However, it may or may not raise the equilibrium level of utility enjoyed by contributor i.*[7]

If players are identical in every respect, so that the game is symmetric, the equilibrium response of the representative contributor's utility to a change in λ can be signed. In this case, every player's replacement function shifts upwards in response to a reduction in λ. Hence total provision rises. So to must each player's contribution at the new equilibrium. Therefore each player experiences both an increase in her endowment of contributions by others and also a beneficial relative price effect.

Proposition 4.4 *If all contributors are identical, an equal reduction in every contributor's unit cost raises both the equilibrium total provision of public good and the equilibrium level of utility enjoyed by the representative contributor.*

This last proposition was obtained by Cornes and Sandler ([15]).

4.4 Variations in a across players

We now analyse the implications of changes in the parameter a_i. Suppose that player i is a positive contributor, and consider an exogenous reduction in a_i - that is, technical progress for i as a private good generator. At unchanged level of Q, the response of i's replacement value is given by

$$r_i(.) = \frac{m_i - a_i\psi_i(.)}{c_i}. \tag{18}$$

A reduction in a_i works through two channels. It leads to an increase in the value of $\psi_i(.)$: at the unchanged value of Q, player i substitutes into

[7]Here is an example of a situation in which an equal proportional reduction in every contributor's unit cost may make a contributor worse off. There are two individuals, whose utility functions are $u_1 = y_1^{3/4} + Q^{3/4}$ and $u_2 = \mathrm{Min}\{y_2, Q\}$. Initially, $a_1 = a_2 = c_1 = c_2 = m_1 = m_2 = 1$. It may be confirmed that, in equilibrium, $(y_1, y_2, Q, u_1, u_2) = (0.67, 0.67, 0.67, 1.47, 0.67)$. Now suppose that $c_1 = c_2 = 0.9$, while all other parameter values remain unchanged. The new equilibrium values are $(y_1, y_2, Q, u_1, u_2) = (0.46, 0.76, 0.78, 1.44, 0.78)$. Player 1, with the high elasticity of substitution between the two goods, has responded by contributing significantly more to the public good, and is worse off. Player 2, by contrast, is now contributing less, and is better off.

private good consumption. However, inspection of (18) reveals that $\psi_i(.)$ is multiplied by a_i, which is now lower. The net outcome is therefore ambiguous. Whether the reduction in a_i shifts the replacement function up, down, or leaves it unchanged depends on the strength of the substitution response to the relative price change. Consider the partial derivative of $r_i(.)$ with respect to a_i. From (18),

$$\frac{\partial r_i(.)}{\partial a_i} = -\frac{a_i \frac{\partial \psi_i(.)}{\partial a_i} + \psi_i(.)}{c_i}.$$

Thus $\frac{\partial r_i(.)}{\partial a_i} <, =$ or > 0 according to whether $\frac{a_i}{\psi_i(.)} \frac{\partial \psi_i(.)}{\partial a_i} >, =$ or < -1. The term $\frac{a_i}{\psi_i(.)} \frac{\partial \psi_i(.)}{\partial a_i}$ is the price elasticity of demand for the private good given that Q is held constant.

An example may help to give some idea of the likely magnitude of this elasticity . Let player i's preferences be represented by the CES utility function:

$$u_i(y_i, Q) = (y_i^\nu + Q^\nu)^{1/\nu}.$$

It may be shown that the replacement function is

$$\widehat{q}_i = r_i(Q, m_i, a_i, c_i) = \frac{m_i - a_i^{\frac{\nu}{\nu-1}} Q}{c_i}.$$

Inspection of this expression reveals that the sign of the response of $r_i(.)$ to a change in a_i depends in a simple way on the value of ν. Three situations may be identified:

- $0 < \nu < 1$: preferences are convex, with elasticity of substitution greater than one: $a_i^1 < a_i^0 \Longrightarrow r_i(Q, m_i, a_i^1, c_i) < r_i(Q, m_i, a_i^0, c_i)$.
- $\nu \to 0$: preferences are Cobb-Douglas, with elasticity of substitution equal to one: $a_i^1 < a_i^0 \Longrightarrow r_i(Q, m_i, a_i^1, c_i) = r_i(Q, m_i, a_i^0, c_i)$.
- $\infty < \nu < 0$: preferences are convex, with elasticity of substitution less than one: $a_i^1 < a_i^0 \Longrightarrow r_i(Q, m_i, a_i^1, c_i) > r_i(Q, m_i, a_i^0, c_i)$.

More general preferences do not map so neatly into this behavioral taxonomy. However, we can draw the following agnostic conclusion:

	q_i^N	Q^N	u_i^N	u_j^N, $j \neq i$
$r_i(Q, m_i, a_i^1, c_i) < r_i(Q, m_i, a_i^0, c_i)$	$\downarrow$	$\downarrow$	?	$\downarrow$
$r_i(Q, m_i, a_i^1, c_i) = r_i(Q, m_i, a_i^0, c_i)$	-	-	$\uparrow$	-
$r_i(Q, m_i, a_i^1, c_i) > r_i(Q, m_i, a_i^0, c_i)$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$

Table 1: Equilibrium responses to a change in i's unit cost of private good

Proposition 4.5 *At a given value of Q, contributor i's replacement value may increase, remain unchanged, or fall in response to a reduction in a_i.*

Although it is agnostic, this proposition is significant because the equilibrium responses depend on which of the three situations holds. Table 1 summarises the most interesting equilibrium responses to a fall in a_i[a dash means that the variable does not change its value].

In short, the equilibrium responses to a change in a_i depend critically on those demand factors that determine the direction in which the change shifts player i's replacement function.

5 Concluding Comments

Aggregative games can be analyzed by conditioning the choices of individual players on the sufficient statistic that appears as an argument in the payoff of each. In contrast to the best response function, which conditions player i's choice on the choices of all players excluding player i, this simple trick avoids the unwelcome proliferation of dimensions as the number of heterogeneous players increases. It thereby permits a simple analysis of asymmetric aggregative games even with many players. It also lends itself to an elementary and revealing geometric representation.

Our discussion of these functions and their application to public good and sharing games has hardly scratched the surface of a potentially significant range of applications. Aggregative games lie at the heart of many other economic models, such as Cournot oligopoly, contest theory and cost and surplus sharing models. The use of replacement functions offers the prospect of further insights in these and in many other applications. We have elsewhere extended our approach to model problems - such as cost and surplus sharing, and Tullock contests - in which 'best response' and 'replacement' functions fail to be monotonic[8]. We believe that replacement functions and their extension to share functions are tools that merit serious attention as the most natural tools for analyzing aggregative games.

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