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Richard Cornes Jun-ichi Itaya

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Abstract

We extend the simple model of voluntary public good provision to allow for two or more public goods, and explore the new possibilities that arise in this setting. We show that, when there are many public goods, voluntary contribution equilibrium typically generates, not only too low a level of public good provision, but also the wrong mix of public goods. We also analyse the neutrality property in the more general setting, and extend a neutrality proposition of Bergstrom, Blume and Varian (1986).

Keywords: Public goods, Neutrality, Constrained Pareto efficiency

JEL classifications: D61, H41

1 Introduction

Economists' intuitions concerning voluntary public good provision have come mainly from analysis of the single public good model as set out, for example, by Cornes and Sandler [3] and Bergstrom, Blume and Varian [1]. The neutrality property, and the circumstances required for it to hold, are now well understood. In addition, the presumption of inefficiency of the voluntary contribution mechanism is well established, as is its specific interpretation as being the consequence of underprovision.

By contrast, there have been few explicit analyses of models with more than one public good. Notable exceptions are Kemp [6], Bergstrom, Blume and Varian [1] and Cornes and Schweinberger [4]. Kemp establishes a neutrality proposition on the assumption that all players are positive contributors to every public good. Bergstrom, Blume and Varian note that this assumption is problematic. They establish equilibrium existence in the presence of many public goods, and present a neutrality proposition. However, although they allow for the presence of noncontributors, their neutrality proposition invokes a rather restrictive assumption - one that can be relaxed, as we show below. Cornes and Schweinberger simultaneously develop a number of extensions of the basic model, making it a little difficult to pinpoint precisely what are the implications of assuming many public goods.

There are good reasons for wanting to explore models two or more public goods. There are many examples of several public goods being voluntarily and simultaneously supplied in the real world. National governments simultaneously contribute to many domestic and international public goods, local governments provide many local public goods, many individuals contribute voluntarily to several charitable causes, and so on. If we take public goods at all seriously, our models should be able to accommodate the possibility of more than one. Furthermore, not only do answers to existing questions change, but new questions arise as soon as a second public good is introduced.

In the standard model with a single public good, the possibility of corner solutions at which individuals choose not to contribute must be taken seriously. However, such a situation, though likely, is not generic. By contrast, if there is more than one public good, individuals will certainly be at corner solutions unless preferences are very special, even if the number of potential contributors is small. The presence of two or more public goods generates a second potential source of inefficiency associated with the voluntary contribution model, conceptually distinct from the widely accepted tendency towards underprovision. Not only will aggregate public good provision tend to be 'too low', but the mix of public goods will generally be 'wrong'. By this we simply mean that, starting from a Nash equilibrium, it will generally be possible to find a Pareto superior allocation without increasing the aggregate level of resources devoted to the provision of public goods above its equilibrium level. Finally, in the unlikely event that preferences and incomes conspire to generate an equilibrium at which all individuals are at interior solutions, individual contribution levels are indeterminate. Consequently, there is a coordination problem with respect to individual contribution levels.

2 Constrained Pareto Efficiency with Many Public Goods and Individuals.

There are n players, a single private good, and m public goods. The preferences of individual i are represented by the utility function

$$u_i = u_i\left(c_i, \mathbf{G}\right), \quad i = 1, 2, ..., n,$$

where $c_i \geqq 0$ is individual i's consumption of the private good, and $\mathbf{G}$ is the vector of total quantities of m public goods: $\mathbf{G} \equiv (G_1, G_2, ..., G_m) \in R_+^m$. $u_i(.)$ is strictly increasing in all arguments, strictly quasiconcave, and everywhere differentiable. All prices are unity, and an overall resource constraint requires the value of the bundle of private and public goods not to exceed some exogenously given level W:

$$\sum_{i=1}^{n} c_i + \sum_{j=1}^{m} G_j \leqq W.$$

2.1 Pareto efficiency

Consider the problem

$$\underset{\mathbf{c}\in\mathbf{R}_+^n, \mathbf{G}\in\mathbf{R}_+^m}{Maximise} \left\{ \sum_i \omega_i u_i\left(c_i, \mathbf{G}\right) \mid \sum_{i=1}^{n} c_i + \sum_{j=1}^{m} G_j \leqq W \right\}.$$

A Pareto efficient allocation satisfies the first-order conditions associated with this problem for some strictly positive set of weights $\omega_1, ..., \omega_n$. We confine attention to interior optima at which all private good consumption quantities are strictly positive[1]. The Lagrangean for this problem is

$$L = \sum_i \omega_i u_i\left(c_i, \mathbf{G}\right) - \lambda \left[\sum_{i=1}^{n} c_i + \sum_{j=1}^{m} G_j - W \right].$$

[1]Within the context of the model with a single public good, Campbell and Truchon [2] discuss the implications of Pareto efficiency with zero private good consumption levels.

The first-order conditions are

$$\begin{aligned}
\frac{\partial L}{\partial c_i} &= \omega_i u_{ic} - \lambda = 0, \\
\frac{\partial L}{\partial G_j} &= \sum_{i=1}^{n} \omega_i u_{ij} - \lambda = 0, \\
\frac{\partial L}{\partial \lambda} &= W - \sum_{i=1}^{n} c_i - \sum_{j=1}^{m} G_j = 0,
\end{aligned}$$

where $u_{ic} \equiv \frac{\partial u_i}{\partial c_i}$ and $u_{ij} \equiv \frac{\partial u_i}{\partial G_j}$. Eliminating the weights, we obtain a set of Samuelson conditions, one for each public good:

$$\sum_{i=1}^{n} \frac{u_{ij}}{u_{ic}} = 1, \qquad j = 1, ..., m.$$

Pareto efficiency requires the provision of each public good to be taken up to the point at which the social benefit from a further increment equals the social cost of that increment.

2.2 Constrained Pareto efficiency

In the economy with a single public good, comparison of Samuelson's condition with the Nash equilibrium conditions tells us that, in general, too few resources are devoted to public good provision at a voluntary contribution equilibrium. This remains the case in the presence of two or more public goods. However, we are interested in a further question: in the presence of more than one public good, what can we say about the **mix** of public goods? We will argue that, in general, Nash equilibrium also implies an inappropriate mix of public goods in the following sense. Starting from an equilibrium, it is generally possible to reduce output of one public good and increase that of another so as to generate a Pareto improvement, even without increasing the aggregate level of resources devoted to public good production. To address this issue, we introduce the notion of constrained Pareto efficiency.

Definition 1 *The allocation* $(\mathbf{c}', \mathbf{G}') \in \mathbf{R}_+^n \times \mathbf{R}_+^m$ *is 'constrained Pareto efficient' if there does not exist another feasible allocation* $(\mathbf{c}, \mathbf{G}) \in \mathbf{R}_+^n \times \mathbf{R}_+^m$ *such that*

$$\sum_{j=1}^{m} G_j = \sum_{j=1}^{m} G_j'$$

and

$$u_i(c_i, \mathbf{G}) \geqq u_i(c_i', \mathbf{G}')$$

with at least one strict inequality[2]*.*

To characterize constrained Pareto efficient allocations in terms of the associated first-order conditions, we consider allocations consistent with the requirement that

$$\sum_{j=1}^{m} G_j = K,$$

where K is some exogenously fixed parameter. Within the set of feasible allocations in which the total amount of public good provision is constrained to equal K, a constrained Pareto efficient allocation satisfies the first-order conditions associated with the problem

$$\underset{\mathbf{c}\in\mathbf{R}_+^n, \mathbf{G}\in\mathbf{R}_+^m}{Maximise} \left\{ \sum_{i=1}^{n} \omega_i u_i(c_i, \mathbf{G}) \mid \sum_{j=1}^{m} G_j = K, \sum_{i=1}^{n} c_i = W - K \right\}.$$

The Lagrangean expression is

$$L(.) = \sum_{i=1}^{n} \omega_i u_i(c_i, \mathbf{G}) - \lambda \left[\sum_{j=1}^{m} G_j - K \right] - \mu \left[\sum_{i=1}^{n} c_i - W + K \right]. \quad (1)$$

Setting the relevant partial derivatives equal to zero, we obtain

$$\begin{aligned} \omega_i u_{ic} - \mu &= 0, \;\; i = 1, ..., n, \\ \sum_{i=1}^{n} \omega_i u_{ij} - \lambda &= 0, \;\; j = 1, ..., m, \end{aligned}$$

$$\Longrightarrow \sum_{i}^{n} \frac{u_{ij}}{u_{ic}} = \sum_{i}^{n} \frac{u_{ik}}{u_{ic}} = \frac{\lambda}{\mu}, \quad j, k = 1, ..., m; j \neq k. \quad (2)$$

Proposition 1 *Consider an economy with n individuals, m public goods and a single private good. Assume that all utility functions are everywhere strictly quasiconcave, strictly increasing and differentiable. Then, among allocations in which every individual consumes a strictly positive quantity of the private good, a necessary condition for constrained Pareto efficiency is that*

$$\sum_{i}^{n} \frac{u_{ij}}{u_{ic}} = \sum_{i}^{n} \frac{u_{ik}}{u_{ic}}, \quad j, k = 1, ..., m; j \neq k.$$

[2]The constraint on public goods is an equal value constraint. If the cost of a unit of public good j were p_j, the constraint would read $\sum_{j=1}^{m} p_j G_j = \sum_{j=1}^{m} p_j G_j'$.

Constrained Pareto efficiency is consistent with the aggregate level of resources to public good provision being 'too high' $\left(\frac{\lambda}{\mu} < 1\right)$ or 'too low' $\left(\frac{\lambda}{\mu} > 1\right)$, but insists that, whatever that level is, the sum of marginal benefits must be equal across all public goods. If this is not the case, it is possible to obtain a Pareto improvement at an unchanged value of $\sum_{j=1}^{m} G_j$.

Changing the public good mix alone is not by itself sufficient to realize Pareto improvement. It has to be accompanied by a redistribution of private good consumption. The possibility of such Pareto improvement, and the nature of the perturbation required to achieve it, may be clarified by using a theorem of the alternative. Mangasarian [7] provides a good exposition of such theorems. Kanbur and Myles [5] provide an economic application, and Myles [8] discusses their application to various economic problems. Starting from an arbitrary initial allocation, consider the effect on individuals' utility levels of a small perturbation:

$$\begin{pmatrix} du_1 \\ du_2 \\ \vdots \\ du_n \end{pmatrix} = \begin{pmatrix} u_{1c} & 0 & \cdots & 0 & u_{11} & \cdots & u_{1m} \\ 0 & u_{2c} & \cdots & 0 & u_{21} & \cdots & u_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nc} & u_{n1} & \cdots & u_{nm} \end{pmatrix} \begin{pmatrix} dc_1 \\ \vdots \\ dc_n \\ dG_1 \\ \vdots \\ dG_m \end{pmatrix}.$$

We want to restrict attention to allocations in which the total amount of public good provision is unchanged. Thus we require that

$$\sum_{j=1}^{m} dG_j = 0.$$

In view of the overall resource constraint, we also require that

$$\sum_{i=1}^{n} dc_i = 0.$$

Equations may be written in matrix form as

$$\begin{aligned} d\mathbf{u} &= \mathbf{A}d\mathbf{q}, \\ \mathbf{B}d\mathbf{q} &= 0, \\ \text{and } \mathbf{C}d\mathbf{q} &= 0, \end{aligned}$$

where

$$
\begin{aligned}
d\mathbf{u}^T &\equiv (du_1, du_2, ..., du_n), \\
d\mathbf{q}^T &\equiv (dc_1, ..., dc_n, dG_1, ..., dG_m), \\
\mathbf{A} &\equiv \begin{pmatrix} u_{1c} & 0 & \cdots & 0 & u_{11} & \cdots & u_{1m} \\ 0 & u_{2c} & \cdots & 0 & u_{21} & \cdots & u_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nc} & u_{n1} & \cdots & u_{nm} \end{pmatrix}, \\
\mathbf{B} &\equiv (1, 1, ..., 1, 0, 0, ..., 0), \\
\mathbf{C} &\equiv (0, 0, ..., 0, 1, 1, ..., 1),
\end{aligned}
$$

where the $(n+m)$-vector $\mathbf{B}$ contains n '1's and m 'zeroes', and the $(n+m)$-vector $\mathbf{C}$ contains n 'zeroes' and m '1's. Consider the following question. Starting from a Nash equilibrium, is there a perturbation that satisfies the required restrictions and permits a Pareto improvement? This is equivalent to the following question. Does there exist a vector $d\mathbf{q}$ such that

$$d\mathbf{u} = \mathbf{A}d\mathbf{q} > 0, \mathbf{B}d\mathbf{q} = 0 \text{ and } \mathbf{C}d\mathbf{q} = 0?$$

An answer to this question can be found by exploiting the following theorem of the alternative:

Theorem 1 *For three matrices* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$ *with the same number of columns, exactly one of the following holds:*

(i) there exists a vector $\mathbf{x}$ *such that* $\mathbf{Ax} \geq 0, \mathbf{Bx} \geqq 0, \mathbf{Cx} \geqq 0$*, or*

(ii) there exist three vectors $\mathbf{y}_1$, $\mathbf{y}_2$ *and* $\mathbf{y}_3$ *such that* $\mathbf{A}^T\mathbf{y}_1 + \mathbf{B}^T\mathbf{y}_2 + \mathbf{C}^T\mathbf{y}_3 = 0$, $\mathbf{y}_1 > 0$, $\mathbf{y}_2 \geq 0$ *and* $\mathbf{y}_3 \geq 0$.

The theorem implies that either there is a Pareto-improving perturbation [case (i)], or there is a set of strictly positive welfare weights on the households for which the allocation satisfies the first-order necessary conditions for maximizing the weighted sum of utilities [case (ii)]. In the latter case, the vectors $\mathbf{y}_2$ and $\mathbf{y}_3$ correspond to the Lagrangian multipliers associated with the constraints that restrict admissible perturbations.

To see this, note that the first-order conditions associated with (1) may be written in the form

$$\begin{pmatrix} u_{1c} & 0 & \cdots & 0 \\ 0 & u_{2c} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nc} \\ u_{11} & u_{12} & \cdots & u_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mn} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{pmatrix} + \begin{pmatrix} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mu + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \lambda = 0.$$

This condition, which characterizes a Pareto efficient allocation, precisely corresponds to case (ii) of the theorem, thus leading to equation (2). Hence, if it is not satisfied, we can be sure that there is a feasible Pareto improvement.

To highlight graphically our general statement, consider an example with two individuals and two public goods. In this case, we have

$$\begin{aligned} du_1 &= u_{1c}dc_1 + u_{11}dG_1 + u_{12}dG_2, \\ du_2 &= u_{2c}dc_2 + u_{21}dG_1 + u_{22}dG_2, \\ dc_1 + dc_2 &= 0, \\ dG_1 + dG_2 &= 0. \end{aligned}$$

Now consider perturbations consistent with $du_1 = 0$. Substitution and a little rearrangement yields

$$\left[\frac{dc_1}{dG_1}\right]_{du_1=0} = \frac{u_{12}}{u_{1c}} - \frac{u_{11}}{u_{1c}}. \tag{3}$$

Similarly,

$$\left[\frac{dc_1}{dG_1}\right]_{du_2=0} = \frac{u_{21}}{u_{2c}} - \frac{u_{22}}{u_{2c}}. \tag{4}$$

Pareto improvement is possible if and only if, in the neighborhood of the initial allocation,

$$\left[\frac{dc_1}{dG_1}\right]_{du_1=0} \neq \left[\frac{dc_1}{dG_1}\right]_{du_2=0}. \tag{5}$$

Using (5) and rearranging, Pareto improvement is possible if and only if

$$\frac{u_{11}}{u_{1c}} + \frac{u_{21}}{u_{2c}} \neq \frac{u_{12}}{u_{1c}} + \frac{u_{22}}{u_{2c}},$$

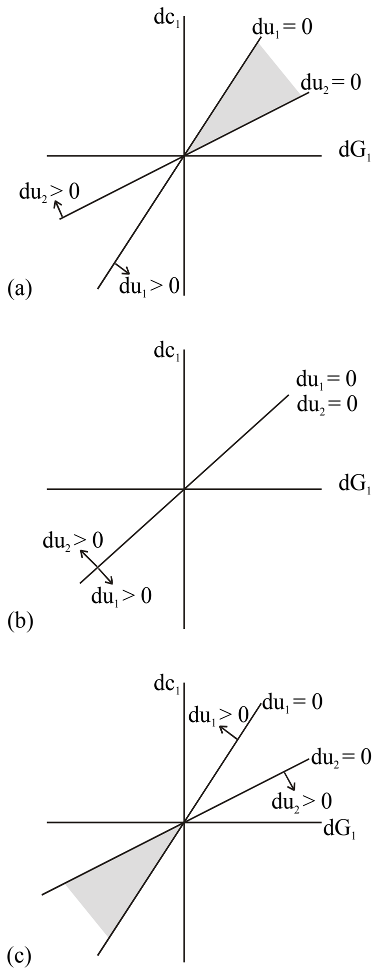


Figure 1:

-that is, if and only if (2) is violated.

Figure 1 shows three possible situations[3]. In Panels (a) and (c), inequality (5) is satisfied.

In the situation depicted by Panel (a), it is possible to make both individuals better off by an increase in G_1 accompanied by an increase in c_1. By itself, an increase in G_1 helps individual 1 and hurts 2. A transfer of private good consumption is necessary, and also sufficient, to compensate individual 2 for her utility reduction while allowing individual 1 to enjoy at least her

[3]Figure 1 does not exhaust all possible ways of drawing the diagram. The lines that correspond to zero utility change for one or other of the individuals may be downward-sloping. However, this does not affect our principal conclusions.

initial utility level. In the situation depicted by Panel (c), a reduction in G_1 helps individual 2 but hurts 1. In this case, it must be accompanied by a transfer of private consumption from individual 2 to 1 in order to generate a Pareto improvement. Finally, Panel (b) depicts a situation in which no Pareto improvement is possible, which corresponds to case (ii) of theorem 1. We will argue below that one cannot generally expect the situation depicted in Panel (b) to arise at a Nash equilibrium - though possible, it is most unlikely. In general, a reallocation of expenditure between the public goods, accompanied by an appropriate redistribution of the resources available for private consumption, is sufficient to permit a Pareto improving perturbation from an existing equilibrium allocation.

3 Equilibrium and Efficiency in a 2×2 Model

3.1 The model

This section works through a specific example to show that, in a perfectly standard setting, one cannot generally expect a Nash equilibrium to be constrained Pareto efficient. There are two individuals and two public goods, for which reason we will call this the 2×2 model. Preferences of individual i are represented by the Cobb-Douglas utility function

$$u_i\left(c_i, X, Y\right) = c_i X^{\alpha_i} Y^{\beta_i}, \tag{6}$$

where c_i is a private good, X and Y are the total quantities of two public goods, and α_i, $\beta_i > 0$. Note that, as the value of any quantity approaches zero, the associated marginal utility approaches infinity. This ensures that, at any equilibrium, all private good quantities and total public good provision levels are strictly positive. Furthermore, at such an allocation, the marginal utility of each good is strictly positive. In particular, we can be sure that, at equilibrium, $c_i > 0$ for all i and that, in the neighborhood of an equilibrium, utility functions are strictly increasing in all arguments. They are also strictly quasiconcave, and everywhere continuously differentiable. Finally, Cobb-Douglas utility functions are weakly separable. As a consequence, individual i's indifference map in (X, Y) space can be drawn independently of the precise realized value of c_i and is homothetic. These features greatly simplify subsequent discussion without making preferences unusual or idiosyncratic in any relevant respect. We claim that the conclusions that we draw from this example are generic.

We assume throughout that the prices of all goods are unity. Individual

i's budget constraint is

$$c_i + x_i + y_i = w_i, \tag{7}$$

where x_i and y_i are, respectively, individual i's contributions to X and Y, and w_i is her exogenous income.

The assumption of separability that is reflected in (6) implies that each individual's preferences over X and Y can be defined, and the associated indifference map drawn, independently of the precise value of private good consumption. Homotheticity of the subutility function in (X, Y) space implies that, for each individual, the locus of points at which the marginal rate of substitution between X and Y equals their relative price of unity is a straight line through the origin. In general, the two individuals' expansion paths in (X,Y) space will have different slopes[4]. The separability and homotheticity that are built into our example imply that each individual i's most preferred ratio, $(Y/X)_i^*$, at the given prices is unique. We will call this ratio individual i's "ideal ratio." Since the two individuals consume the same quantities of the two public goods, it follows that at any allocation and, *a fortiori*, at any Nash equilibrium, their marginal rates of substitution between X and Y must differ if $\alpha_1/\beta_1 \neq \alpha_2/\beta_2$. Panel (a) of Figure 2 shows a situation in which $\alpha_1/\beta_1 > \alpha_2/\beta_2$, and in Panel (b) $\alpha_1/\beta_1 = \alpha_2/\beta_2$. If one were to imagine the parameters α_i and β_i being randomly drawn from a continuum set of those parameters, we would not typically expect the situation depicted in Panel (b) to arise - that is to say, it is non-generic. Situation (a) therefore seems empirically the more plausible one to consider.

3.2 Equilibrium when $\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$

If $\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$, then the two individuals share a common ideal ratio at the existing prices. The equilibrium ratio of total public good levels must then equal the common ideal ratio. Suppose this were not the case - for example, suppose that $\frac{Y^N}{X^N} > \left(\frac{Y}{X}\right)_1^* = \left(\frac{Y}{X}\right)_2^*$. Consequently, at the prevailing equilibrium, at least one of the individuals, say i, must be choosing contributions such that $\frac{y_i}{x_i} > \left(\frac{Y}{X}\right)_1^*$. This cannot be a Nash equilibrium, since given the prevailing contribution of the other, individual i can do better by consuming an unchanged level of c_i, while also contributing less to Y and more to X.

It immediately follows that, if $\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$, the individuals will have the same

[4]These assumptions rule out the possibility that the two individuals' expansion paths intersect each other. Consequently, the results we obtain here will be valid over the whole (X,Y) space.

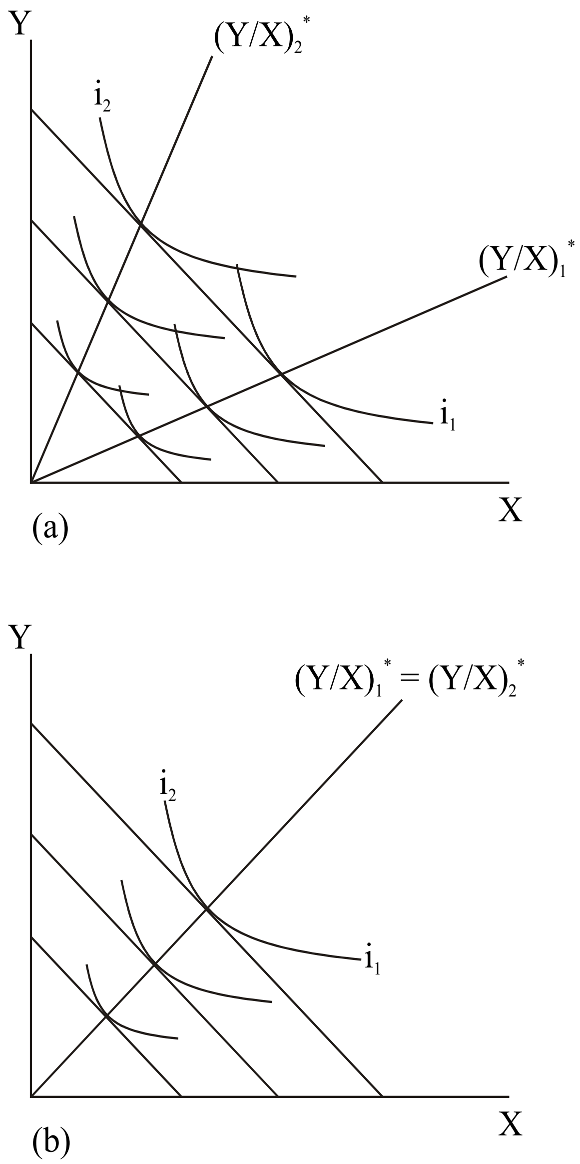


Figure 2:

marginal rates of substitution at equilibrium:

$$\frac{\partial u_1/\partial X}{\partial u_1/\partial c_1} = \frac{\partial u_1/\partial Y}{\partial u_1/\partial c_1} = 1 = \frac{\partial u_2/\partial X}{\partial u_2/\partial c_2} = \frac{\partial u_2/\partial Y}{\partial u_2/\partial c_2}$$

$$\Longrightarrow \frac{\partial u_1/\partial X}{\partial u_1/\partial c_1} + \frac{\partial u_2/\partial X}{\partial u_2/\partial c_2} = \frac{\partial u_1/\partial Y}{\partial u_1/\partial c_1} + \frac{\partial u_2/\partial Y}{\partial u_2/\partial c_2}.$$

The first-order condition for constrained Pareto efficiency, (2), is satisfied. A notable feature of this situation is that, even though the equilibrium levels of X and Y may be uniquely determined, individual contributions are indeterminate, so that the individuals face a coordination problem with respect to their individual contribution levels. If $(x_1^*, x_2^*, y_1^*, y_2^*)$ is an equilibrium vector of contributions, so too is any vector $(x_1^* + \Delta, x_2^* - \Delta, y_1^* - \Delta, y_2^* + \Delta)$, where Δ may take any positive or negative value consistent with both individuals' budget constraints being satisfied and contribution levels being nonnegative.

Although such an equilibrium is constrained Pareto efficient, it is unlikely to arise for the reason we have given. The probability that $\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2}$ can plausibly be argued to be zero. We therefore turn to what we regard as the more likely situation in which $\frac{\alpha_1}{\beta_1} \neq \frac{\alpha_2}{\beta_2}$.Without loss of generality, we focus on the situation in which $\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2}$ and therefore $\left(\frac{Y}{X}\right)_1^* < \left(\frac{Y}{X}\right)_2^*$.

3.3 Equilibrium when $\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2}$

If $\frac{\alpha_1}{\beta_1} > \frac{\alpha_2}{\beta_2}$, then $\left(\frac{Y}{X}\right)_2^* > \left(\frac{Y}{X}\right)_1^*$. No allocation, and therefore no equilibrium allocation, equates the marginal rates of substitution across individuals. Moreover, a Nash equilibrium can never imply equilibrium values of X and Y outside the cone spanned by the lines $\left(\frac{Y}{X}\right)_2^*$ and $\left(\frac{Y}{X}\right)_1^*$ in Figure 2(a). Denote the Nash equilibrium quantities by X^N and Y^N. For the moment, we leave to one side the precise location of the Nash equilibrium. Our argument to this point implies that there are three possibilities to consider:

1. $Y^N/X^N = \left(\frac{Y}{X}\right)_1^*$,
2. $\left(\frac{Y}{X}\right)_1^* < Y^N/X^N < \left(\frac{Y}{X}\right)_2^*$,
3. $Y^N/X^N = \left(\frac{Y}{X}\right)_2^*$.

Consider each of these in turn:

Case 1: $Y^N/X^N = \left(\frac{Y}{X}\right)_1^*$: If the prevailing equilibrium ratio equals individual 1's ideal ratio, then at that allocation $\frac{\partial u_1/\partial Y}{\partial u_1/\partial X} = 1$. This is consistent with individual 1 contributing to both public goods. However, $\frac{\partial u_2/\partial Y}{\partial u_2/\partial X} > 1$, which implies that individual 2 certainly does not contribute to X. At most, he contributes to Y alone. Thus, in the present case we can conclude that **either** individual 1 contributes to both goods and individual 1 to Y alone, **or** individual 1 contributes to both goods and individual 2 to neither.

Now consider the second possibility:

Case 2: $\left(\frac{Y}{X}\right)_2^* > Y^N/X^N > \left(\frac{Y}{X}\right)_1^*$: If the equilibrium allocation lies strictly between the two individuals' ideal ratios, then we must have $\frac{\partial u_2/\partial Y}{\partial u_2/\partial X} > 1 > \frac{\partial u_1/\partial Y}{\partial u_1/\partial X}$. Individual 1 contributes only to X, and individual 2 contributes only to Y.

Case 3: $Y^N/X^N = \left(\frac{Y}{X}\right)_2^*$: Analysis of case 3 follows the same lines as that of Case 1.

We need to be more explicit about the precise location of the Nash equilibrium. To do so, we consider a numerical example.

Example 1 *The individuals' preferences are represented by the Cobb-Douglas utility functions:*

$$\begin{aligned} u_1 &= c_1 X^2 Y, \\ u_2 &= c_2 X Y^2. \end{aligned}$$

Aggregate income in the economy is unity:

$$w_1 + w_2 = 1.$$

Tedious but elementary manipulations reveal that, depending on the precise initial distribution of the aggregate income, Nash equilibrium may fall into any one of 5 regimes, according to the pattern of individual contributions. The following table summarizes the regimes:

Regime		x_1^N	y_1^N	x_2^N	y_2^N
I	$w_1 \leqq \frac{1}{9}$	0	0	$\frac{w_2}{4}$	$\frac{w_2}{2}$
II	$\frac{1}{9} < w_1 \leqq \frac{1}{3}$	$\frac{(8w_1 - w_2)}{9}$	0	$\frac{3w_2 - 6w_1}{9}$	$\frac{4(w_1+w_2)}{9}$
III	$\frac{1}{3} < w_1 \leqq \frac{2}{3}$	$\frac{2w_1}{3}$	0	0	$\frac{2w_2}{3}$
IV	$\frac{2}{3} < w_1 \leqq \frac{8}{9}$	$\frac{4(w_1+w_2)}{9}$	$\frac{3w_1 - 6w_2}{9}$	0	$\frac{(8w_2 - w_1)}{9}$
V	$w_1 > \frac{8}{9}$	$\frac{w_1}{2}$	$\frac{w_1}{4}$	0	0

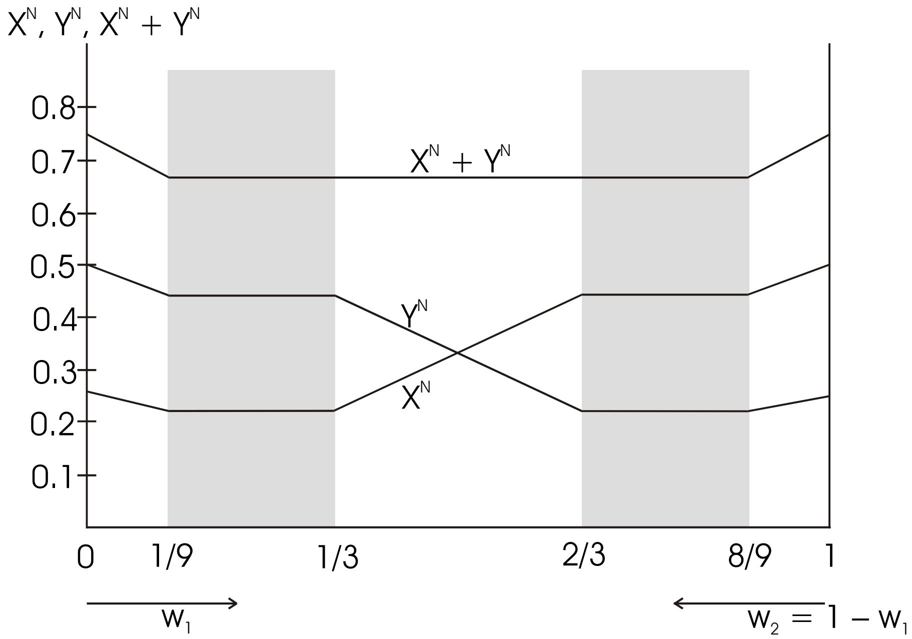


Figure 3:

Table 1.

A significant feature of the table is the following: in Regimes II and IV, the aggregate quantities of the two public goods depend only on aggregate income, not on its precise distribution:

$$\frac{1}{9} < w_1 \leqq \frac{1}{3} \Longrightarrow X^N = \frac{2(w_1+w_2)}{9}, Y^N = \frac{4(w_1+w_2)}{9},$$
$$\frac{2}{3} < w_1 \leqq \frac{1}{9} \Longrightarrow X^N = \frac{4(w_1+w_2)}{9}, Y^N = \frac{2(w_1+w_2)}{9}.$$

Figure 3 illustrates the behavior of equilibrium provision of the public goods as the income distribution varies.

It may be confirmed that

$$\frac{1}{9} < w_1 \leqq \frac{1}{3} \Longrightarrow \frac{u_{2X}}{u_{2c}} = \frac{c_2}{X} = 1 > \frac{1}{4} = \frac{c_1}{Y} = \frac{u_{1Y}}{u_{1c}}.$$

Similarly,

$$\frac{2}{3} < w_1 \leqq \frac{8}{9} \Longrightarrow \frac{u_{2X}}{u_{2c}} = \frac{c_2}{X} = \frac{1}{4} < 1 = \frac{c_1}{Y} = \frac{u_{1Y}}{u_{1c}}.$$

Thus income distributions within these ranges certainly do not generate constrained Pareto efficient equilibria. In fact, in this example there are just three constrained Pareto efficient equilibrium allocations. Two occur at extreme distributions: $w_1 = 0, w_2 = 1$ and $w_1 = 1, w_2 = 0$. These are constrained Pareto efficient for a familiar reason: if one individual enjoys all the income, there is no way to compensate her for any perturbation away from her preferred public good mix, since the other individual's endowment is zero. The third possibility is more interesting, and arises when the initial distribution is equal: $w_1 = w_2 = 1/2$.

The source of the constrained inefficiency is readily explained. Suppose that, in our numerical example, $w_1 = 2/3$ and $w_2 = 1/3$. Recall that the constrained Pareto efficiency requirement is

$$\frac{\partial u_1/\partial X}{\partial u_1/\partial c_1} + \frac{\partial u_2/\partial X}{\partial u_2/\partial c_2} = \frac{\partial u_1/\partial Y}{\partial u_1/\partial c_1} + \frac{\partial u_2/\partial Y}{\partial u_2/\partial c_2}.$$

At the Nash equilibrium associated with the chosen income distribution, we know that $\frac{\partial u_1/\partial X}{\partial u_1/\partial c_1} = \frac{\partial u_1/\partial Y}{\partial u_1/\partial c_1} = \frac{\partial u_2/\partial Y}{\partial u_2/\partial c_2} = 1$. Thus constrained efficiency requires that

$$\frac{\partial u_2/\partial X}{\partial u_2/\partial c_2} = 1.$$

However, this cannot be the case at this allocation, which implies a ratio X/Y that is far from individual 2's ideal ratio. Intuitively, a reduction in X , accompanied by an increase in Y such that $X + Y$ is constant, has only a second-order adverse effect on individual 1's utility. However, it has a first-order beneficial effect on individual 2's utility. There is a transfer of private good consumption from individual 2 to 1 which, if it were to accompany the substitution of Y for X, would enable both to be better off than at the initial equilibrium.

Starting at $(w_1, w_2) = (2/3, 1/3)$, consider the path traced out by the resulting equilibria as individual 1's initial income falls and that of individual 2 rises. The associated equilibrium involves less X and more Y, while $X+Y$ remains constant (until we reach the point where $(w_1, w_2) = (1/3, 2/3)$). As we move along the path traced out by the resulting equilibria, it remains the case that $\frac{\partial u_1/\partial X}{\partial u_1/\partial c_1} = \frac{\partial u_2/\partial Y}{\partial u_2/\partial c_2} = 1$. However, $\frac{\partial u_1/\partial Y}{\partial u_1/\partial c_1}$ falls and $\frac{\partial u_2/\partial X}{\partial u_2/\partial c_2}$ rises. Thus there is one, and only one, Nash equilibrium on the path that is constrained Pareto efficient. This is the symmetric allocation at which $(X, Y) = (1/3, 1/3)$, achieved from an initial income distribution of $(w_1, w_2) = (1/2, 1/2)$.

We can summarize the conclusions of this section in the following remarks:

Remark 1 *In the 2-individual Cobb-Douglas example, if $\frac{\alpha_1}{\beta_1} \neq \frac{\alpha_2}{\beta_2}$, then the individuals' ideal ratios of public goods must differ, and there is no Nash equilibrium at which both individuals contribute to both public goods.*

Remark 2 *In the 2-individual Cobb-Douglas example, if* $\frac{\alpha_1}{\beta_1} \neq \frac{\alpha_2}{\beta_2}$*, then there is only one interior initial income distribution for which the associated Nash equilibrium is constrained Pareto efficient.*

4 Equilibrium and Neutrality in a 2×2 Model

We turn now to the matter of neutrality. Figure 3 shows two shaded regions, corresponding to regimes II and IV, in which income transfers do not affect the real equilibrium. This neutrality property does not depend on the Cobb-Douglas for its validity, and may be demonstrated using the line of argument employed in Cornes and Sandler [3].

Note that in Regime II both individuals make positive contributions to public good X - we will say that, in that regime, they "share an interest" in X. Slightly more formally, and more generally,

Definition 2 *Individuals* i *and* j *share an interest in public good* k *at an allocation if, at that allocation, both individuals' marginal rates of substitution between that public good and the private good equal the relative price,* p_k/p_c*, where* p_k *represents the price of public good* k*.*

Similarly, in Regime IV they share an interest in Y. Suppose that the initial income distribution places the individuals somewhere in Regime II. Now transfer an amount Δ from individual 1 and give it to individual 2.

Denote the initial equilibrium allocation by the quantities

$$\left(c_1^0, c_2^0, x_1^0, x_2^0, y_1^0, y_2^0\right)$$

where, by assumption, $y_1^0 = 0$. Such an allocation implies satisfaction of both individuals' budget constraints with equality:

$$c_i^0 + x_i^0 + y_i^0 = w_i, i = 1, 2.$$

After the transfer, consider the following allocation:

$$\left(c_1^0, c_2^0, x_1^0 - \Delta, x_2^0 + \Delta, y_1^0, y_2^0\right).$$

If $\Delta \leqq x_1^0$, such an allocation is feasible, and satisfies both individuals' budget constraints with equality:

$$\begin{aligned} c_1^0 + \left(x_1^0 - \Delta\right) + y_1^0 &= w_1 - \Delta, \\ c_2^0 + \left(x_2^0 + \Delta\right) + y_2^0 &= w_2 + \Delta. \end{aligned}$$

Moreover, each individual is still enjoying the same real consumption bundle, since neither of the private consumption levels changed, nor have either of the total provision levels of the public goods. Since relative costs are, by assumption, constant, this allocation remains a Nash equilibrium. The inequality $\Delta \leqq x_1^0$ ensures that the new allocation remains within Regime II.

Thus, although in general we cannot have both individuals contributing to both goods, the following proposition establishes the continued relevance of the neutrality proposition:

Proposition 2 *In the 2×2 model, a shared interest in just one public good is sufficient to generate neutrality.*

In Regime II, the two individuals are linked by their common interest in X, and in Regime IV by their common interest in Y.

5 Constrained Pareto Efficiency and Neutrality in a 2×3 Model

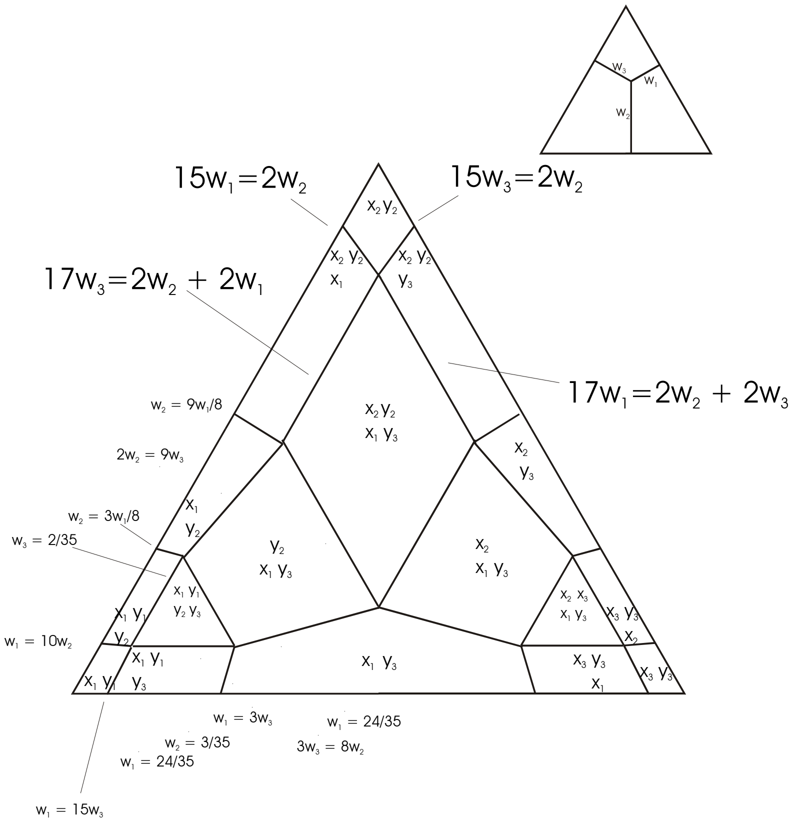
Allowing a third individual not only enables us to check the robustness of the constrained Pareto efficiency and neutrality properties of the two-individual game, but also uncovers an additional possibility, that of 'partial neutrality'. We continue to assume that there are two public goods, but we add one more individual. We call this the 2×3 model. Assume that the individuals can be strictly ranked according to their ideal ratios at the prevailing prices:

$$\left(\frac{Y}{X}\right)_1^* > \left(\frac{Y}{X}\right)_2^* > \left(\frac{Y}{X}\right)_3^* .$$

Again, a specific Cobb-Douglas example enables us to locate various regimes, distinguished by the patterns of positive and zero contributions. Let the three individuals have the following preferences:

$$\begin{aligned} u_1 &= c_1 X^3 Y, \\ u_2 &= c_2 X^2 Y^2, \\ u_3 &= c_3 X Y^3. \end{aligned}$$

We continue to suppose that the total income of the group is unity, and that unit prices are all unity. Figure 4 depicts all possible income distributions with the help of the 2-dimensional simplex - see the inset for clarification of our convention for depicting initial incomes.



15w1=2w2
15w3=2w2
17w3=2w2 + 2w1
17w1=2w2 + 2w3
w2 = 9w1/8
2w2 = 9w3
w2 = 3w1/8
w3 = 2/35
w1 = 10w2
w1 = 3w3
w2 = 3/35
w1 = 24/35
w1 = 24/35
3w3 = 8w2
w1 = 15w3

Figure 4:

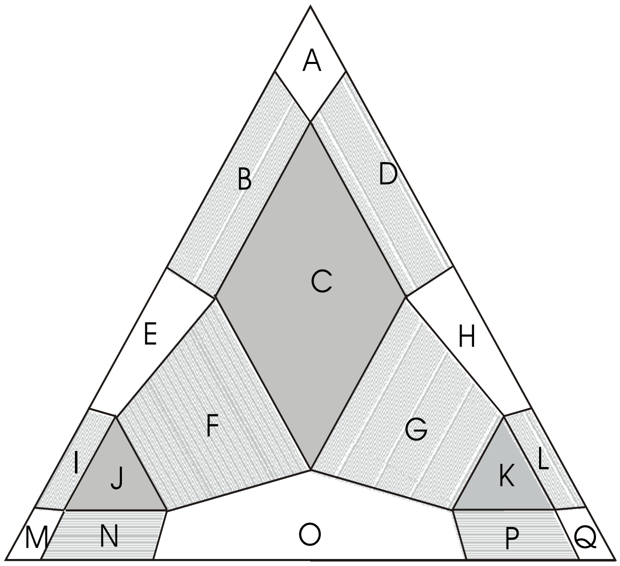


Figure 5:

Each point in the simplex represents an income distribution, the individuals' incomes being the lengths of that point's perpendiculars to the three sides. Some tedious calculations allow us to identify the 17 regimes that exist in this example. Figure 4 shows the 17 regimes. In each, the listed variables x_i, y_i are the contribution levels that are positive in that regime. The equations are those of the boundaries between the regimes. Figure 5 reproduces the regimes and introduces our labelling conventions. The shading will be explained below.

5.1 Pareto (In)efficiency in a 2×3 model

Continuing with our 3-individual example, we now show that the set of initial income distributions that lead to constrained Pareto efficient equilibria is the set represented by (i) all the points within regime C, (ii) those points within Regime A at which $w_1 = w_3$, and (iii) those points within regime O at which $w_1 = w_3$.

First, consider Regime C. Within this regime, we know that the following conditions hold:

$$\frac{\partial u_1/\partial X}{\partial u_1/\partial c_1} = \frac{\partial u_2/\partial X}{\partial u_2/\partial c_2} = \frac{\partial u_2/\partial Y}{\partial u_2/\partial c_2} = \frac{\partial u_3/\partial Y}{\partial u_3/\partial c_3} = 1. \tag{8}$$

We also know that

$$\begin{aligned} 3c_1 &= x_1 + x_2 = X, \\ 2c_2 &= x_1 + x_2 = X, \\ 2c_2 &= y_2 + y_3 = Y, \\ 3c_3 &= y_2 + y_3 = Y, \\ x_3 &= 0, \\ y_1 &= 0. \end{aligned} \tag{9}$$

Constrained Pareto efficiency requires that

$$\frac{\partial u_1/\partial X}{\partial u_1/\partial c_1} + \frac{\partial u_2/\partial X}{\partial u_2/\partial c_2} + \frac{\partial u_3/\partial X}{\partial u_3/\partial c_3} = \frac{\partial u_1/\partial Y}{\partial u_1/\partial c_1} + \frac{\partial u_2/\partial Y}{\partial u_2/\partial c_2} + \frac{\partial u_3/\partial Y}{\partial u_3/\partial c_3}$$

or, in view of (8),

$$\frac{\partial u_3/\partial X}{\partial u_3/\partial c_3} = \frac{\partial u_1/\partial Y}{\partial u_1/\partial c_1}.$$

Differentiating the utility functions, this requires that

$$\frac{c_3}{X} = \frac{c_1}{Y},$$

which is clearly implied by (9) . Thus, all allocations in Regime C are constrained Pareto efficient.

Now consider Regime A. The pattern of positive contributions implies that, at any allocation in A,

$$\begin{aligned} \tfrac{\partial u_2/\partial X}{\partial u_2/\partial c_2} &= 1, \\ \tfrac{\partial u_2/\partial Y}{\partial u_2/\partial c_2} &= 1, \\ 2c_2 = x_2 = X &= \tfrac{2w_2}{5}, \\ 2c_2 = y_2 = Y &= \tfrac{2w_2}{5}, \\ x_1 &= 0, \\ x_3 &= 0, \\ y_1 &= 0, \\ y_3 &= 0, \\ c_1 &= w_1, \\ c_3 &= w_3. \end{aligned} \tag{10}$$

Differentiating the utility functions, the requirement for constrained Pareto efficiency becomes

$$\frac{3c_1}{X} + \frac{c_3}{X} = \frac{c_1}{Y} + \frac{3c_3}{Y}.$$

Substituting from the system (10), constrained Pareto efficiency requires that

$$\frac{3w_1}{2w_2/5} + \frac{w_3}{2w_2/5} = \frac{w_1}{2w_2/5} + \frac{3w_3}{2w_2/5},$$

which in turn requires that

$$w_1 = w_3.$$

A careful auditing of each regime reveals that only those income distributions in C, together with all other income distributions in which $w_1 = w_3$, yield constrained Pareto efficient equilibrium outcomes.

5.2 Neutrality in a 2×3 model

Consider first Regime C. Here individuals 1 and 2 share an interest in X, and individuals 2 and 3 share an interest in Y. Our earlier reasoning can be used to show that the Cobb-Douglas assumption may be dropped without damaging the neutrality property. Start from any income distribution consistent with Regime C. Then that remains an equilibrium after any set of transfers that keeps the distribution within Regime C. The same observation holds for any pair of income distributions that lie within Regime J, and for any pair within K.

Now consider Regimes B, G and I. Within each of these, individuals 1 and 2 share an interest in either X or Y. However, individuals 2 and 3 have no shared interest. Within each of these Regimes, there is a 'partial neutrality' property. Suppose we start at an equilibrium within Regime B. After any transfer between the two individuals with a shared interest that remains within Regime B, that allocation remains an equilibrium. However, any net transfer between individual 3 and either of the other individuals will lead to a different equilibrium. The income transfer must be in the direction of the hatching if it is to produce an unchanged equilibrium. Again, both the initial and the final income distributions must lie within the same Regime. A shift from a point in B to a point in G will not lead to neutrality.

In the same way, areas D, F and L represent regimes within which there is partial neutrality with respect to transfers between individuals 2 and 3, but not involving individual 1. Finally, areas N and P represent regimes for which there is partial neutrality between individuals 1 and 3.

6 Extending to an $n \times m$ Model

Although our exposition has relied on specific examples, we claim that the major qualitative conclusions that we have drawn hold generally. The main role of our restrictions on preferences is to limit the number of possible regimes - that is, patterns of positive contributions across individuals - and to facilitate the computation of these regimes from knowledge of the preferences and endowments.

For example, relaxation of homotheticity immediately destroys our ability to rank individuals globally according to their ideal ratios. The situation is analogous to factor intensity rankings of industries in international trade theory. Our 2×2 example has 5 possible regimes. More general preferences would permit four additional possibilities. In one, individual 1 contributes to both goods and individual 2 to X alone. In another, both contribute to both goods. In the third, individual contributes to both and individual 1 to Y alone. Finally, it is possible that individual 1 contributes solely to Y and individual 2 solely to X.

There seems little to be gained by exploring these complications further. It is sufficient to observe that allowing for more general preferences greatly complicates the mapping from sets of initial income distributions to regimes. More significant is the observation that, at any observed equilibrium, we know the pattern of observed contributions. We know that if individual i is contributing to public good j, then $\frac{\partial u_i/\partial G_j}{\partial u_i/\partial c_i} = 1$. On the other hand, if individual i is not contributing to good k then, in general, $\frac{\partial u_i/\partial G_k}{\partial u_i/\partial c_i} < 1$[5].

We have already introduced the idea of two individuals sharing an interest in a public good. We now introduce the idea of individuals being linked.

Definition 3 *Individuals h and h+k are linked at an equilibrium if there is a set of public goods [labelled $G_1, G_2, ..., G_k$] and a set of individuals [labelled h, h+1, ...,h+k] such that, at that equilibrium,*

- *Individual h shares an interest with individual h+1 in public good G_1,*
- *Individual h+1 shares an interest with individual h+2 in public good G_2,*
- *...,*
- *Individual h+k-1 shares an interest with individual h+k in public good G_k.*

[5]There is, of course, the possibility that an individual is 'on the margin', contributing nothing because, at the existing allocation, her appropriate marginal valuation precisely equals the public good's unit cost when her contribution is zero. This implies that the allocation is on the boundary between two regimes, a possibility that we henceforth ignore on the grounds that it is a zero probability event.

Clearly, if individuals h and $h+k$ are linked, so too are any two individuals who belong to the chain that links them. Now consider a Nash equilibrium in an economy consisting of many individuals, a single private good, and many public goods. Suppose that individuals h and $h+k$ are linked. There is no loss of generality in supposing that they are linked through individuals $h+1$, $h+2$, ..., $h+k-1$, and by public goods 1, 2, ..., k. We therefore have the following chain of relationships consisting of their budget constraints at equilibrium:

$$c_h^0 + g_{h1}^0 = w_h,$$

$$c_{h+j}^0 + g_{h+j\ j}^0 + g_{h+j\ j+1}^0 = w_{h+j} \quad j = 1, 2, ..., k-1,$$

$$c_{h+k}^0 + g_{h+k\ k}^0 = w_{h+k},$$

where $g_{h+j\ j}^0$ stands for individual $h+j$'s contribution to public good j at the initial equilibrium, and we assume that all the individual contribution levels are strictly positive. Now suppose that there is a redistribution of income from individual h to individual $h+k$, so that their incomes become $w_h - \Delta$ and $w_{h+k} + \Delta$ respectively. Now consider the allocation, in which the quantities are indicated by the superscript 1, characterized by the following:

$$\left(c_h^1, g_{h\ 1}^1\right) = \left(c_h^0, g_{h\ 1}^0 - \Delta\right),$$

$$\left(c_{h+j}^1, g_{h+j\ j}^1, g_{h+j\ j+1}^1\right) = \left(c_{h+j}^0, g_{h+j\ j}^0 + \Delta, g_{h+j\ j+1}^0 - \Delta\right) \quad j = 1, 2, ..., k-1,$$

$$\left(c_{h+k}^1, g_{h+k\ k}^1\right) = \left(c_{h+k}^0, g_{h+k\ k}^0 + \Delta\right).$$

If $\Delta \leq \min\left\{g_{h\ 1}^0, g_{h+1\ 2}^0, ..., g_{h+k-1\ k}^0\right\}$, the new allocation is feasible in the sense that the budget constraint of every individual remains satisfied with equality. Since each of the individuals involved is enjoying an unchanged consumption of the private good (i.e. $c_i^0 = c_i^1$, $i = h, h+1, ..., h+k$), and an unchanged aggregate level of all public goods (i.e. $G_j^0 = G_j^1$, $j = 1, 2, ..., k$), and since each has unchanged preferences, the first-order conditions associated with each player's maximization problem remain satisfied, and the allocation therefore remains a Nash equilibrium.

Proposition 3 (Partial neutrality) *A pure income redistribution amongst a set of linked individuals that maintains the link between them has no effect on the original equilibrium allocation.*

This is a more general statement than is Theorem 7 in Bergstrom, Blume and Varian [1]. They consider three groups of individuals - those who contribute 'only to G', those who contribute 'to G and H', and those who contribute 'only to H'. The statement of their theorem requires that the wealth transfer leaves unchanged the total wealth of each of these three groups. However, a net transfer that reduces the wealth of the first group and increases that of the last group is also neutral provided that the initial members of the three groups remain linked[6].

Finally, we draw attention to another new feature that may characterize equilibrium in the presence of more than one public good. Suppose that we observe an equilibrium in which two individuals share an interest in each of two goods. Suppose, for example, that for individuals i and j and public goods k and ℓ, the individual contributions $g^0_{i\,k}$, $g^0_{i\,\ell}$, $g^0_{j\,k}$ and $g^0_{j\,\ell}$ are all strictly positive at an observed equilibrium. Starting from the same initial income distribution, consider any other allocation at which the contribution levels are $g^0_{i\,k}+\Delta$, $g^0_{i\,\ell}-\Delta$, $g^0_{j\,k}-\Delta$ and $g^0_{j\,\ell}+\Delta$. If $\Delta \leq \min\left\{g^0_{i\,\ell}, g^0_{j\,k}\right\}$, such an allocation is feasible. It gives each individual her original consumption bundle and the same levels of public goods as before, satisfies each individual's first-order conditions, and is therefore a Nash equilibrium. Thus we can conclude that

Proposition 4 *Even if, starting from a given vector of incomes, there is a unique vector of Nash equilibrium level of public good provision, nevertheless, if two individuals share an interest in the same two public goods, there is a coordination problem with respect to those individuals' contributions to the goods in which they share an interest.*

7 Conclusions

Although a number of papers formally incorporate the possibility of many public goods, little attention has been paid to the new questions that one

[6]To see this, suppose there are just three individuals. The initial Nash equilibrium is $\left(G^0, H^0\right)$. Suppose individual GONLY, who contributes only to G, loses income Δw_{GONLY}. HONLY's income rises by the same amount. Such a transfer is not allowed by Bergstrom, Blume and Varian. Now consider the allocation in which $\Delta g_{GONLY} = \Delta w_{GONLY}$, $\Delta h_{HONLY} = \Delta w_{HONLY} = -\Delta w_{GONLY}$, and $(\Delta g_{BOTH}, \Delta g_{BOTH}) = (-\Delta w_{GONLY}, -\Delta h_{HONLY})$. A quick check shows that, at this new equilibrium, $G^1 = G^0 + \Delta g_{GONLY} + \Delta g_{BOTH} = G^0$, $H^1 = H^0 + \Delta h_{HONLY} + \Delta h_{BOTH} = H^0$, and all three individuals enjoy an unchanged level of private good consumption. It is also an equilibrium. Thus a net transfer between two linked individuals, provided it does not destroy the chain that links them, preserves the initial allocation as an equilibrium.

can ask of equilibrium in a world of many public goods. We have shown that the presence of 2 or more public goods introduces a new aspect to the consideration of the inefficiency of equilibrium. Not only may too few resources be devoted to public good production, but their mix may be inefficient in a sense that we have made precise.

We have not explored the policy implications of the multiple public good model extensively. It may be observed that there are many public goods in the real world, some of which are supplied exclusively by governments, some of which are supplied by private agents, and some by both. This observation provokes the following two questions, one of which is a positive question, the other normative. Why do governments choose to supply such particular goods? Which public goods should be supplied by government? The literature on private provision of public goods continues to ask the question of how much governments should provide a public good, based on a single public good model. Our model provides a promising analytical vehicle for further exploring these important and hitherto overlooked issues.

Although we have assumed that all public goods are supplied by summation technology, it is natural to assume a situation where the public goods are supplied by different technologies such as best-shot or weaker- or weakest-link. The analysis of this paper is just a first step towards addressing these more interesting and important issues in a multiple public good setting.

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