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Erwin Amann is Professor, Department of Economics, University of Duisburg-Essen and Alex Possajennikov is Lecturer, School of Economics University of Nottingham

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Evolution in Symmetric Incomplete Information Games

Erwin Amann
Department of Economics
University of Duisburg-Essen
45117 Essen
Germany

Alex Possajennikov*
School of Economics
University of Nottingham
Nottingham NG7 2RD
Britain

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Abstract

The paper compares two models of evolution in symmetric two-player games with incomplete information. One model postulates that the type of a player is fixed, and evolution works within types. In the other model type-contingent strategies evolve. In the case of two types and two strategies it is shown that the stability properties of stationary states are the same under the two dynamics when payoffs do not depend on the type of the other player, but may differ when they do.

Keywords: incomplete information games, evolution, stability
JEL Code: C72

1 Introduction

The idea of the paper is to compare two models of evolution in symmetric incomplete information games. In a Bayesian game, the type of a player can either be determined once (like genetically programmed preferences) or be drawn randomly each time the player is called to play the game (think of a series of auctions where values for a given buyer are determined by the object sold and so appear random to the outside observer who does not know

*Email: Alex.Possajennikov@nottingham.ac.uk

the preferences of the buyer). If change in strategies can be modeled as a learning process, or as an evolutionary process, is the outcome different in the two situations?

It is well known that in terms of equilibria the two models are equivalent (this follows from the correspondence between mixed and behavior strategies in Kuhn, 1953). Following Harsanyi (1967-68), incomplete information about types of players can be modeled as imperfect information. Nature determines types of players and then the game is played. This game can be converted into a normal form, on which evolution can be analyzed by standard methods. This corresponds to the second model described above (types determined randomly each period, and type-contingent strategies evolve). The difference between models may come from the fact that when the type of a player is determined once and for all, he cannot switch between certain strategies in the normal form representation of the extensive form game with imperfect information. But is it important, i.e. does it lead to significant differences?

A literal interpretation of evolutionary models is usually one of a large population of individuals, each playing a pure strategy. Individuals are randomly matched to play a given game. We consider simultaneous move games where individuals know own type but not the type of the other player.

In the first model the population is divided into subpopulations corresponding to types. In each subpopulation there is an evolutionary process, but the fitness of a strategy depends also on what strategies other types are playing. The subpopulations may be of different sizes, which are given by the distribution of types in the population, but each individual is matched every time, either with a player of own type or with players of other types. The speed of evolution is the same in all subpopulations.

In the second model, each period players are divided into types according to the given distribution. If a player is more likely to be of a certain type, the type-contingent strategy for this type evolves faster than for other types. It turns out that the relative speed of evolution for type-contingent strategies does matter in some games.

Though the paper concerns mostly economic examples, in biological terms the difference between models may be seen as following. Being of different types is like having different genomes; if genomes are independent, the first model is more appropriate. If more sophisticated genes are considered, like genes for conditional behavior, the second model is more appropriate.

2 Models of Evolution in Symmetric Games with Incomplete Information

The underlying situation to be analyzed is as follows. Individuals in a large population are randomly matched to interact. The individuals can be of several types, which can be interpreted as having different preferences. The individuals know their own type but not the type of other players. They take decisions simultaneously and payoffs are realized.

One example of such a situation in economics is a sealed-bid auction. Bidders know their valuation, but not the valuation of other bidders. They submit bids simultaneously, and the rules of the auction determine who gets the object and how much each bidder pays. Further examples in economics are oligopoly with cost uncertainty, or bargaining with incomplete information.

Some situations in biology can be described by this model. Individuals in a species may be "strong" or "weak" but this is not shown by any external features. In an encounter, an individual has to choose certain action without knowing the type of the other individual.

We consider two-player games. We assume that the game is symmetric, i.e. both players in an interaction face the same situation. Let the finite set of types for each player be $T = \{t_1, \dots, t_n\}$, and let $\mu = \{\mu_1, \dots, \mu_n\}$ be a probability distribution on the set of types, with $\mu_i > 0, \sum_{i=1}^{n} \mu_i = 1$. Thus μ_i is the probability (or the belief) that the other player is of type t_i. We assume that the set of available strategies does not depend on the type of the player. Let this common set of strategies be $S = \{s_1, \dots, s_m\}$. If player 1 is of type t_i and chooses strategy s_k, and player 2 is of type t_j and chooses strategy s_l, the payoffs are $u_{t_i,t_j}(s_k, s_l), u_{t_j,t_i}(s_l, s_k)$. In the most general case the payoffs depend on own type, the type of the other player, and on strategies of the players. In most of economic examples, like the auction example, the payoffs do not depend on the valuation (i.e. type) of the other player, only on own valuation, and on strategies of both players. In the biological example, the payoffs may depend on the type ("strong" or "weak") of the other player, as well as on strategies of both players.

As mentioned in the introduction, the type of an individual can be either determined once before a series of interactions, or before each interaction. Correspondingly, one can formulate two models of evolution.

Evolution within types In the first model, the model of evolution within types, the population is divided into n subpopulations, one for each type. Within each subpopulation, there is an evolutionary process on strategies, and these processes are interdependent because payoff to a strategy for a given type depends on the strategies of other types. Let x_k^i be the proportion of individuals of type t_i that play strategy s_k. The expected payoff of such individuals is $u_{t_i}(s_k) = \sum_{j=1}^{n} \mu_j \sum_{l=1}^{m} x_l^j u_{t_i,t_j}(s_k, s_l)$. In a payoff monotone evolutionary dynamic (Weibull, 1995, Ch.4), the proportion of players of type t_i using strategy s_k grows relative to the proportion of players of type t_i using strategy s_l if $u_{t_i}(s_k) > u_{t_i}(s_l)$. We will work with one particular monotone dynamic, the replicator dynamic

$$\dot{x}_k^i = x_k^i(u_{t_i}(s_k) - u_{t_i}) \tag{1}$$

where $u_{t_i} = x_k^i \sum_{k=1}^{m} u_{t_i}(s_k)$ is the average payoff in the subpopulation of type t_i.

In the economic context, this model of evolution corresponds to a situation when the type of a player is determined before a series of interactions, for example, the cost structure of a duopolist before engaging in competition with other firms in several markets, or the type of a bargainer before several bargaining situations. The replicator dynamic can then be interpreted as the reduced form of a learning process given the type (e.g. the preferences) of an individual, as in Weibull (1995, Ch.4).

In the biological context, the model can be interpreted as follows. One gene determines the type of an individual, but not external features. The proportion of individuals with a given allele of this gene is fixed (it is possible to model the evolution of types also but this will not be pursued here). Another gene determines individual's strategy, and it is the proportions of alleles of this gene that evolve.

Evolution of type-contingent strategies The second model of evolution assumes that the type of an individual is determined randomly (according to distribution μ) before each interaction, and so each individual has type-contingent strategies "play s_k if type t_i". Let $x_{k_1 \ldots k_n}$ be the proportion of players that use type-contingent strategy "play s_{k_1} if type t_1, ... , play s_{k_n} if type t_n". The expected payoff of such an individual is $u(s_{k_1} \ldots s_{k_n}) = \sum_{i=1}^{n} \mu_i u_{t_i}(s_{k_i})$, where $u_{t_i}(s_{k_l})$ is the expected payoff when type t_i as given in the previous subsection, with $x_l^j = \sum_{k_j=l} x_{k_1 \ldots k_n}$ being the proportion of players that have "..., play s_{k_l} if type t_j, ..." in their strategy. Again, in a payoff monotone selection dynamic, the proportion of players using one strategy grows relative to the proportion of players using another strategy if $u(s_{k_1} \ldots s_{k_n}) > u(s_{l_1} \ldots s_{l_n})$. The replicator dynamic in this case is

$$\dot{x}_{k_1 \ldots k_n} = x_{k_1 \ldots k_n}(u(s_{k_1} \ldots s_{k_n}) - u) \tag{2}$$

where $u = \sum_{(s_1 \dots s_1)}^{(s_n \dots s_n)} x_{k_1 \dots k_n} u(s_{k_1} \dots s_{k_n})$ is the average payoff in the population.

This model of evolution, in the economic context, corresponds to a situation where the type of a player represents private information (rather than his preferences) that can change from interaction to interaction. Auction is a good example, since valuations can change from one auction to the next.

In biological context type-contingent strategies correspond to the genes that encode complicated instructions of playing conditional strategies. There is one population of individuals, and in each interaction an individual finds itself in a certain role, like "owner" or "intruder". The identification of roles, however, is imperfect, and so the individual does not know the role of the other player. Since each individual may find itself in each role, genes that encode conditional strategies evolve.

Basic differences and similarities The situation under the dynamic within types is described by n distributions on m strategies for each type, or by $n(m-1)$ independent variables. This description is equivalent to specifying a behavior strategy (Kuhn, 1953) for the game. Under the dynamic on type-contingent strategies the sutiation is given by one distribution on m^n strategies, i.e. by $m^n - 1$ variables. This description is equivalent to mixed strategies for the normal form of the game. For each mixed strategy there is a realization-equivalent behavior strategy, and for each fully mixed behavior strategy there are many realization-equivalent mixed strategies.

Equilibria of the game can be found by looking at either of the descriptions. It is usually easier to find equilibria through behavior strategies. Equilibria are stationary under either of the dynamics, but each equilibrium in fully mixed behavior strategies corresponds to a hyperplane of equilibria in mixed strategies. The stability properties of an equilibrium in behavior strategies can be compared with the stability properties of the corresponding set of equilibria in mixed strategies.

3 The Case of Two Types and Two Strategies

3.1 General Case

We restrict ourselves to the situations where there are only two types and two strategies that players can choose. Let the space of types be $T = \{t_1, t_2\}$, and the distribution is given by one parameter $0 < \mu < 1$, the probability of being type t_1 (thus the probability of being type t_2 is $1 - \mu$). Furthermore, let the set of strategies be $S = \{s_1, s_2\}$. The payoffs can be described by four matrices:

u_{t_1,t_1} :

	s_1	s_2
s_1	a_{11}	a_{12}
s_2	a_{21}	a_{22}

, u_{t_1,t_2} :

	s_1	s_2
s_1	c_{11}	c_{12}
s_2	c_{21}	c_{22}

u_{t_2,t_1} :

	s_1	s_2
s_1	d_{11}	d_{12}
s_2	d_{21}	d_{22}

, u_{t_2,t_2} :

	s_1	s_2
s_1	b_{11}	b_{12}
s_2	b_{21}	b_{22}

i.e. matrix $A = (a_{ij})$ gives the payoffs when player of type t_1 meet another player of type t_1, B is the payoff matrix when two players of type t_2 meet, C is the payoff matrix when type t_1 meets type t_2, and D is when type t_2 meets t_1.

Dynamic within types and equilibria Let x^1 be the distribution of strategies in the subpopulation of type t_1, $x^1 = \begin{bmatrix} x_1^1 \\ 1 - x_1^1 \end{bmatrix}$, and $x^2 = \begin{bmatrix} x_1^2 \\ 1 - x_1^2 \end{bmatrix}$ is the distribution of strategies for type t_2. Let $e_i \in \mathbb{R}^2$ be the unit vector with 1 on the i-th coordinate. Then the replicator dynamic within types is given by

$$\dot{x}_1^1 = x_1^1((e_1 - x^1) \cdot (\mu A x^1 + (1-\mu) C x^2)) \qquad (3)$$
$$\dot{x}_1^2 = x_1^2((e_1 - x^2) \cdot (\mu D x^1 + (1-\mu) B x^2)) \qquad (4)$$

Opening up the matrices leads to equations

$$\dot{x}_1^1 = x_1^1(1 - x_1^1)(\mu(a_1 - a_2)x_1^1 + (1-\mu)(c_1 - c_2)x_1^2 + \mu a_2 + (1-\mu)c_2) \qquad (5)$$
$$\dot{x}_1^2 = x_1^2(1 - x_1^2)(\mu(d_1 - d_2)x_1^1 + (1-\mu)(b_1 - b_2)x_1^2 + \mu d_2 + (1-\mu)b_2) \qquad (6)$$

where $a_1 = a_{11} - a_{21}, a_2 = a_{12} - a_{22}$ and b_i, c_i, d_i are defined analogously. The corner states, or the states on the boundary (where at least one $x_1^i = 0$ or 1) may be stationary under the dynamic because new strategies cannot appear, but not all of them are equilibria. For such a state to be an equilibrium, the last terms in the equations should not be pointing inside the state space of the dynamic, the unit square.

It can be seen that apart from corner pure equilibria and boundary partially mixed equilibria, there can be a fully mixed interior equilibrium where lines $\mu(a_1 - a_2)x_1^1 + (1-\mu)(c_1 - c_2)x_1^2 + \mu a_2 + (1-\mu)c_2 = 0$ and $\mu(d_1 - d_2)x_1^1 + (1-\mu)(b_1 - b_2)x_1^2 + \mu d_2 + (1-\mu)b_2 = 0$ intersect, and in the non-generic case when these lines coincide there is a line of interior equilibria.

Depending on the payoffs there may be several situations involving pure, partially mixed, and fully mixed equilibria, several of which are illustrated on the examples below.

Example 1 *Hawk-dove game with incomplete information.*

Suppose individuals can be of two types, "strong" (s) and "weak" (w). In an interaction, they can either escalate (e) or retreat (r). In an interaction of individuals of the same type the game is the usual hawk-dove one. A strong type always wins a fight against a weak type when the conflict escalates, otherwise payoffs are as in the hawk-dove game:

same type :

	e	r
e	$\frac{1}{2}V_i - C_i$	V_i
r	0	$\frac{1}{2}V_i$

strong vs. weak :

	e	r
e	$V_s - C_s, -C_w$	$V_s, 0$
r	$0, V_w$	$\frac{1}{2}V_s, \frac{1}{2}V_w$

where V_s, V_w are the values of the contested resource for corresponding types, and C_s, C_w are the costs of the fight. We assume that $\frac{1}{2}V_i - C_i < 0, i = s, w$ and $C_s < C_w$. Let μ be the proportion of the strong type.

The change in x_1^1 vanishes when $\mu(-C_s)x_1^1 + (1-\mu)(\frac{1}{2}V_s - C_s)x_1^2 + \frac{1}{2}V_s = 0$ and the change in x_1^2 vanishes when $\mu(-C_w - \frac{1}{2}V_w)x_1^1 + (1-\mu)(-C_w)x_1^2 + \frac{1}{2}V_w = 0$. These two lines either coincide (in non-generic case $\frac{2C_s}{V_s} = \frac{2C_w+V_w}{V_w}$), or intersect outside the unit square. Thus there is no isolated interior equilibrium, and generically all equilibria are on the boundary. Figure 1 shows some possibilities. Note that it is possible to have equilibrium in which the weak type escalates and the strong one retreats, if the value of the resource is too low for the strong type.

Example 2 *Game with isolated interior equilibrium.*

Suppose there are two types. One type (c) wants to match the strategy of the opponent, while the other type (n) wants to play an action different from that of the opponent. Suppose that the payoffs are given by the following

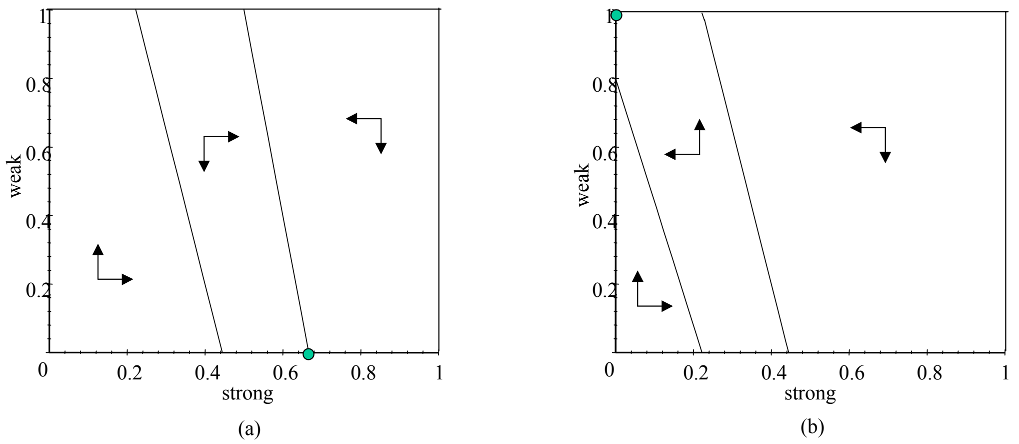

Figure 1: Phase diagram for Hawk-Dove game with $\mu = \frac{3}{4}, V_w = 1, C_s = \frac{3}{4}, C_w = 1$. In (a) $V_s = 1$, in (b) $V_s = \frac{1}{4}$.

matrices.

$u_{n,n}$:

	s_1	s_2
s_1	0	4
s_2	4	0

, $u_{n,c}$:

	s_1	s_2
s_1	0	3
s_2	3	0

$u_{c,n}$:

	s_1	s_2
s_1	3	0
s_2	0	3

, $u_{c,c}$:

	s_1	s_2
s_1	1	0
s_2	0	1

The dynamic is given by

$$\dot{x}_1^1 = x_1^1(1 - x_1^1)(-8\mu x_1^1 - 6(1-\mu)x_1^2 + 4\mu + 3(1-\mu)) \tag{7}$$
$$\dot{x}_1^2 = x_1^2(1 - x_1^2)(6\mu x_1^1 + 2(1-\mu)x_1^2 - 3\mu - (1-\mu)) \tag{8}$$

The point $x_1^1 = \frac{1}{2}, x_1^2 = \frac{1}{2}$ is an equilibrium for any μ. The dynamic rotates around this point, as illustrated in Figure 2.

The Jacobian of the dynamic at equilibrium is $J = \begin{bmatrix} -2\mu & -\frac{3}{2}(1-\mu) \\ \frac{3}{2}\mu & \frac{1}{2}(1-\mu) \end{bmatrix}$ with $\det J = \mu(1-\mu)\frac{5}{4} > 0$ and $trJ = \frac{1}{2} - \frac{5}{2}\mu$. The equilibrium is asymptotically stable for $\mu > \frac{1}{5}$, is a center surrounded by periodic orbits for $\mu = \frac{1}{5}$, and unstable for $\mu < \frac{1}{5}$.

Dynamic of type-contingent strategies The normal form of the game with 2 types and 2 strategies is a 4×4 matrix M. The elements of M can be computed explicitly from matrices A, B, C, D and μ, but for the dynamic it

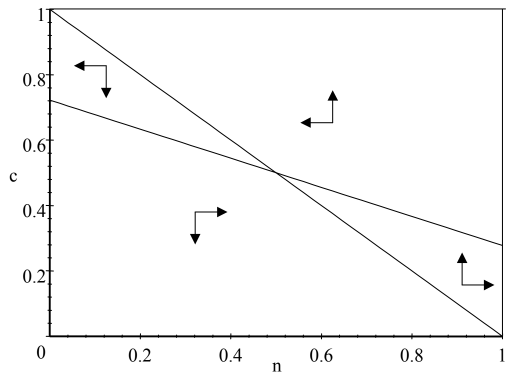

Figure 2: Phase diagram for the game in Example 2 with $\mu = \frac{1}{4}$.

is sufficient to specify the expected payoff of a mixed strategy $x \in \Delta^4$ against another strategy y. In our case it is given by

$$x \cdot My = \mu(x^1 \cdot (\mu Ay^1 + (1-\mu)Cy^2)) + (1-\mu)(x^2 \cdot (\mu Dy^1 + (1-\mu)By^2))$$

where x^1, y^1 are behavior strategies of type t_1 derived from mixed strategies x, y (and correspondingly x^2, y^2 are behavior strategies of type t_2). If $x_{s_i s_j}$ is denoted for brevity as x_{ij}, then $x^1 = \begin{bmatrix} x_{11} + x_{12} \\ x_{21} + x_{22} \end{bmatrix}$ and $x^2 = \begin{bmatrix} x_{11} + x_{21} \\ x_{12} + x_{22} \end{bmatrix}$. The replicator dynamic of type-contingent strategies is

$$\dot{x}_{ij} = x_{ij}[(f_{ij} - x) \cdot Mx] \tag{9}$$

where f_{ij} is the 4-dimensional unit vector with 1 on the place corresponding to x_{ij} and 0 on other places.

For each interior equilibrium (x_1^1, x_1^2) of the dynamic within types there is a set of equilibria $(x_{11}, x_1^1 - x_{11}, x_1^2 - x_{11}, 1 + x_{11} - x_1^1 - x_1^2)$, parametrized by $x_{11} \in (\max\{0, x_1^1 + x_1^2 - 1\}, \min\{x_1^1, x_1^2\})$, of the dynamic of type-contingent strategies.

Following Gaunersdorfer et al. (1991), in the dynamic on the normal form $\left(\frac{x_{ik}}{x_{il}}\right)^{\cdot} = \frac{x_{ik}}{x_{il}}((f_{ik} - f_{il}) \cdot Mx) = \frac{x_{ik}}{x_{il}}[(e_k - e_l) \cdot (\mu Dx^1 + (1-\mu)Bx^2)]$. Then $\left(\frac{x_{ik}}{x_{il}}\frac{x_{jl}}{x_{jk}}\right)^{\cdot} = \frac{x_{ik}}{x_{il}}\frac{x_{jl}}{x_{jk}}[(e_k - e_l) \cdot (\mu Dx^1 + (1-\mu)Bx^2)] + \frac{x_{ik}}{x_{il}}\frac{x_{jl}}{x_{jk}}[(e_l - e_k) \cdot (\mu Dx^1 + (1-\mu)Bx^2)] = 0$, i.e. $\frac{x_{ik}}{x_{il}}\frac{x_{jl}}{x_{jk}}$ is invariant. Manifolds $W_K = \{x : x_{11}x_{22} = Kx_{12}x_{21}\}$ are invariant for any $K > 0$, i.e. trajectories that start on a given manifold stay on this manifold.

Since $x_{11} + x_{12} + x_{21} + x_{22} = 1$ and $x_{11}x_{22} = Kx_{12}x_{21}$, for given K there are only two independent variables. The dynamic foliates into a set of 2-dimensional dynamics, one for each invariant manifold. All manifolds have the same borders $x_1^1 = 0, x_1^1 = 1, x_1^2 = 0, x_1^2 = 1$. For an interior equilibrium (x_1^1, x_1^2) each invariant manifold contains exactly one corresponding equilibrium in the normal form space.

On each manifold the dynamic of behavior strategies x_1^1, x_1^2 can be analyzed. Since $x_1^1 = x_{11} + x_{12}, x_1^2 = x_{11} + x_{21}$ the dynamic of behavior strategies induced by the dynamic of mixed strategies in the normal form is

$$\dot{x}_1^1 = (x_{11}(f_{11} - x) + x_{12}(f_{12} - x)) \cdot Mx \qquad (10)$$

$$\dot{x}_1^2 = (x_{11}(f_{11} - x) + x_{21}(f_{21} - x)) \cdot Mx \qquad (11)$$

Opening up these equations leads to

$$\begin{aligned}
\dot{x}_1^1 &= \mu[x_1^1(1 - x_1^1)(\mu a x_1^1 + (1-\mu)c x_1^2 + \mu a_2 + (1-\mu)c_2)] + \\
&\quad (1-\mu)[(x_{11}x_{22} - x_{12}x_{21})(\mu d x_1^1 + (1-\mu)b x_1^2 + \mu d_2 + (1-\mu)b_2)] \\
\dot{x}_1^2 &= \mu[(x_{11}x_{22} - x_{12}x_{21})(\mu a x_1^1 + (1-\mu)c x_1^2 + \mu a_2 + (1-\mu)c_2)] + \\
&\quad (1-\mu)[x_1^2(1 - x_1^2)(\mu d x_1^1 + (1-\mu)b x_1^2 + \mu d_2 + (1-\mu)b_2)]
\end{aligned}$$

where $a = a_1 - a_2$ and analogously for b, c, d.

When $x_{11}x_{22} = x_{12}x_{21}$ the equations reduce to

$$\begin{aligned}
\dot{x}_1^1 &= \mu[x_1^1(1 - x_1^1)(\mu(a_1 - a_2)x_1^1 + (1-\mu)(c_1 - c_2)x_1^2 + \mu a_2 + (1-\mu)c_2)] \\
\dot{x}_1^2 &= (1-\mu)[x_1^2(1 - x_1^2)(\mu(d_1 - d_2)x_1^1 + (1-\mu)(b_1 - b_2)x_1^2 + \mu d_2 + (1-\mu)b_2)]
\end{aligned}$$

The difference of these equations from the equations of the dynamic within types is only in the multiplicative terms μ and $1 - \mu$. When the dynamic within types involves a fully mixed equilibrium, the stability properties of this equilibrium may change depending on these multiplicative terms. If $\mu \neq \frac{1}{2}$, the fully mixed equilibrium can have different stability properties even in the dynamic within types and the dynamic on manifold W_1.

Example 3 *Example 2 continued.*

With the multiplicative terms, the Jacobian of the dynamic in the example is $J' = \begin{bmatrix} -2\mu^2 & -\frac{3}{2}\mu(1-\mu) \\ \frac{3}{2}\mu(1-\mu) & \frac{1}{2}(1-\mu)^2 \end{bmatrix}$. Then $\det J' = \mu^2(1-\mu)^2\frac{5}{4} > 0$, and $trJ' = -2\mu^2 + \frac{1}{2}(1-\mu)^2 = \frac{1}{2}(-3\mu^2 - 2\mu + 1)$. It holds that $trJ' > 0$ when $\mu < \frac{1}{3}$, $trJ' = 0$ when $\mu = \frac{1}{3}$ and $trJ' < 0$ when $\mu > \frac{1}{3}$. Therefore, for $\mu \in \left(\frac{1}{5}, \frac{1}{3}\right)$ the equilibrium $\left(\frac{1}{2}, \frac{1}{2}\right)$ is asymptotically stable under the dynamic within types but unstable under the dynamic on the normal form on W_1.

Claim 1 *With two types and two strategies, it is possible that the equilibria of the game have different stability properties under the dynamic within types and under the dynamic of type-contingent strategies.*

Intuitively, even on manifold W_1 the change in stability comes from the fact that with type-contingent strategies, strategy for the more frequent type evolves faster than for the less frequent type, while in the dynamic within types the speed of evolution is the same for both types. For the orbits that spiral around the fully mixed equilibrium, this change in the relative speed is important.

If other manifolds $W_K, K \neq 1$ are taken into account, the set of interior equilibria is less likely to be stable, as every equilibrium in it has to be stable on the corresponding manifold.

On the first sight it seems to be more diffucult to have stability of interior equilibria (as a set) in the dynamic on the normal form, since equilibria have to be stable on every manifold. However, the following example demonstrates that a set of equilibria can be stable in the dynamic of type-contingent strategies while unstable in the dynamic within types.

Example 4 *Stability in type-contingent strategies and instability in the dynamic within types.*

Consider the game given by the following matrices.

$u_{n,n}$:

	s_1	s_2
s_1	0	1
s_2	1	0

, $u_{n,c}$:

	s_1	s_2
s_1	0	3
s_2	3	0

$u_{c,n}$:

	s_1	s_2
s_1	3	0
s_2	0	3

, $u_{c,c}$:

	s_1	s_2
s_1	4	0
s_2	0	4

and $\mu = \frac{3}{4}$. The dynamic within types is

$$\dot{x}_1^1 = x_1^1(1 - x_1^1)(-\frac{3}{2}x_1^1 - \frac{3}{2}x_1^2 + \frac{3}{2}) \quad (12)$$

$$\dot{x}_1^2 = x_1^2(1 - x_1^2)(\frac{9}{2}x_1^1 + 2x_1^2 - \frac{13}{4}) \quad (13)$$

The Jacobian of the dynamic at equilibrium $\left(\frac{1}{2}, \frac{1}{2}\right)$ is $J = \begin{bmatrix} -\frac{3}{8} & -\frac{3}{8} \\ \frac{9}{16} & \frac{1}{2} \end{bmatrix}$. Since $\det J = \frac{3}{128} > 0$ and $trJ = \frac{1}{8} > 0$, the equilibrium is unstable.

The dynamic of type-contingent strategies in the interior foliates into two-dimensional dynamics

$$\dot{x}_1^1 = \frac{3}{4}x_1^1(1-x_1^1)(-\frac{3}{2}x_1^1 - \frac{3}{2}x_1^2 + \frac{3}{2}) + \frac{1}{4}(x_{11} - x_1^1 x_1^2)(\frac{9}{2}x_1^1 + 2x_1^2 - \frac{13}{4})$$
$$\dot{x}_1^2 = \frac{3}{4}(x_{11} - x_1^1 x_1^2)(-\frac{3}{2}x_1^1 - \frac{3}{2}x_1^2 + \frac{3}{2}) + \frac{1}{4}x_1^2(1-x_1^2)(\frac{9}{2}x_1^1 + 2x_1^2 - \frac{13}{4})$$

on the invariant manifolds. Since $x_1^1 = x_{11} + x_{12}, x_1^2 = x_{11} + x_{21}, x_{11} + x_{12} + x_{21} + x_{22} = 1$ and $x_{11}x_{22} = Kx_{12}x_{21}$, x_{11} can be found from the equation $(1-K)x_{11}^2 + [1-(1-K)(x_1^1 + x_1^2)]x_{11} - Kx_1^1 x_1^2 = 0$. The point $\left(\frac{1}{2}, \frac{1}{2}\right)$ is equilibrium on every manifold, and at it $x_{11} = \frac{\sqrt{K}}{2(\sqrt{K}+1)}$. The Jacobian of the dynamic at equilibrium is $J' = \frac{1}{32(\sqrt{K}+1)}\begin{bmatrix} -18 & -5\sqrt{K} - 13 \\ 18 & -5\sqrt{K} + 13 \end{bmatrix}$. Since $\det J' = \left(\frac{1}{32(\sqrt{K}+1)}\right)^2 \cdot 180\sqrt{K} > 0$ and $tr J' = \frac{1}{32(\sqrt{K}+1)}(-5 - 5\sqrt{K}) < 0$ for any K, the equilibria are asymptotically stable on all manifolds W_K. They are also stable on the boundary.

3.2 Payoffs Do Not Depend on the Type of the Other Player

Let us return to our economic examples. In an auction the payoff of a player depends on own valuation but not on the valuation of the other player (but, of course, it depends on the bids of both players). In duopoly, profit depends on own cost but not on the cost of the other firm. Thus, payoffs to a player do not depend on the type of the other player. In terms of the previous section, matrices A and C are the same, as well as matrices B and D.

Dynamic within types Equations (5) and (6) reduce to

$$\dot{x}_1^1 = x_1^1(1-x_1^1)((\mu x_1^1 + (1-\mu)x_1^2)(a_1 - a_2) + a_2) \quad (14)$$
$$\dot{x}_1^2 = x_1^2(1-x_1^2)((\mu x_1^1 + (1-\mu)x_1^2)(b_1 - b_2) + b_2) \quad (15)$$

The lines where $\dot{x}_1^1 = 0$ and $\dot{x}_1^2 = 0$ have the same slope $-\frac{\mu}{1-\mu}$, therefore they either do not intersect or coincide. In the case when the lines do not intersect there is no interior equilibrium. All equilibria are either pure or partially mixed. In the case when the lines coincide there is a continuum (a line) of fully mixed equilibria. In either case there is no isolated interior equilibrium, so the dynamic cannot rotate around a point.

Proposition 1 *When payoffs do not depend on the type of the other player there is no isolated interior equilibrium.*

Dynamic of type-contingent strategies The reduction of the dynamic of type-contingent strategies to the 2-dimensional space in this case leads to

$$\begin{aligned}\dot{x}_1^1 &= \mu x_1^1(1-x_1^1)[(\mu x_1^1+(1-\mu)x_1^2)(a_1-a_2)+a_2]+\\ &\quad (1-\mu)(x_{11}x_{22}-x_{12}x_{21})[(\mu x_1^1+(1-\mu)x_1^2)(b_1-b_2)+b_2]\\ \dot{x}_1^2 &= \mu(x_{11}x_{22}-x_{12}x_{21})[(\mu x_1^1+(1-\mu)x_1^2)(a_1-a_2)+a_2]+\\ &\quad (1-\mu)x_1^2(1-x_1^2)[(\mu x_1^1+(1-\mu)x_1^2)(b_1-b_2)+b_2]\end{aligned}$$

Again, the reduction of the dynamic to manifold $W_1=\{x : x_{11}x_{22}=x_{12}x_{21}\}$ differs from the dynamic within types only by the multiplicative terms μ and $1-\mu$. In the generic case, when there is no interior equilibrium, the term is not important as the dynamic cannot cycle or spiral. In the non-generic case of the line of equilibria in the interior it may be important, as a Lyapunov stable point on a line may become unstable (or the other way round) after the relative speed of changes in variables alter.

Example 5 *Relative speed matters in non-generic case.*

Suppose that the payoffs are given by matrices

u_n :

	s_1	s_2
s_1	0	4
s_2	4	0

, u_c :

	s_1	s_2
s_1	1	0
s_2	0	1

where u_n gives the payoff of type n independently of the type of the other player, and similarly for type u_c. Suppose $\mu=\frac{1}{4}$. Then the dynamic within types is

$$\begin{aligned}\dot{x}_1^1 &= x_1^1(1-x_1^1)(-4)(\frac{1}{2}(x_1^1+3x_1^2)-1)\\ \dot{x}_1^2 &= x_1^2(1-x_1^2)(\frac{1}{2}(x_1^1+3x_1^2)-1)\end{aligned}$$

The line of equilibria for the dynamic is $x_1^1+3x_1^2=2$. Multiplying the first equation by $\frac{1}{4x_1^1(1-x_1^1)}$, the second by $\frac{1}{x_1^2(1-x_1^2)}$, and adding them up, we obtain $\dot{x}_1^1\frac{1}{4x_1^1(1-x_1^1)}+\dot{x}_1^2\frac{1}{x_1^2(1-x_1^2)}=0$. It follows that $\frac{d}{dt}\left(\ln\left(\left(\frac{x_1^1}{1-x_1^1}\right)^{1/4}\frac{x_1^2}{1-x_1^2}\right)\right)=0$, or that $\left(\frac{x_1^1}{1-x_1^1}\right)^{1/4}\frac{x_1^2}{1-x_1^2}$ is invariant in the dynamic. Figure 3(a) shows the isoscales. Since the slope of them is less than the slope of the equilibrium line, some equilibria in the middle of the interval are Lyapunov stable.

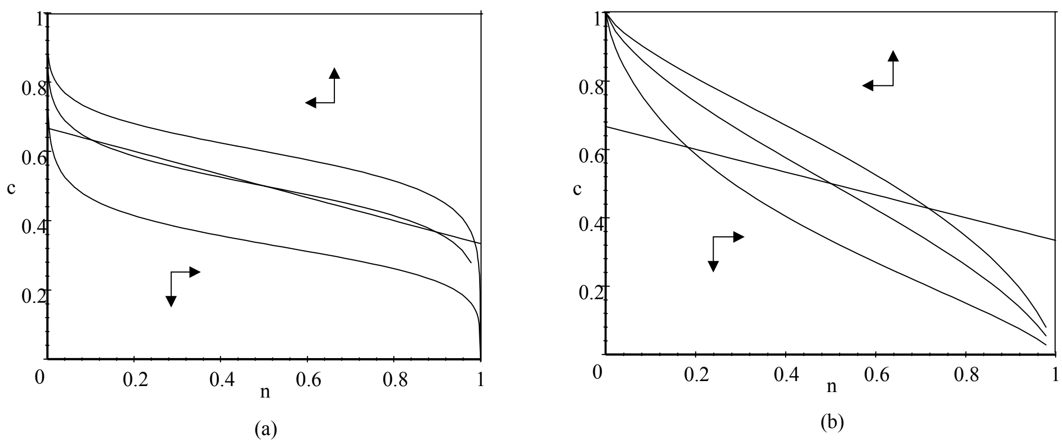

Figure 3: Line of equilibria and isoscales

The dynamic of type-contingent strategies on W_1 is

$$\begin{aligned}\dot{x}_1^1 &= x_1^1(1-x_1^1)(-1)(\frac{1}{2}(x_1^1+3x_1^2)-1) \\ \dot{x}_1^2 &= x_1^2(1-x_1^2)\frac{3}{4}(\frac{1}{2}(x_1^1+3x_1^2)-1)\end{aligned}$$

Multiplying the first equation by $\frac{3}{4x_1^1(1-x_1^1)}$, the second equation by $\frac{1}{x_1^2(1-x_1^2)}$, and adding them up, we obtain $\dot{x}_1^1\frac{3}{4x_1^1(1-x_1^1)}+\dot{x}_1^2\frac{1}{x_1^2(1-x_1^2)}=0$. It follows that $\frac{d}{dt}\left(\ln\left(\left(\frac{x_1^1}{1-x_1^1}\right)^{3/4}\frac{x_1^2}{1-x_1^2}\right)\right)=0$, or that $\left(\frac{x_1^1}{1-x_1^1}\right)^{3/4}\frac{x_1^2}{1-x_1^2}$ is invariant in the dynamic. Drawing the isoscales of this expression gives the picture in Figure 3(b). All equilibria on the line are unstable.

The dynamic of type-contingent strategies on W_1 is equivalent to the dynamic within types in the sense that both $\dot{x}_1^1$ and $\dot{x}_1^2$ always have the same sign under both dynamics. On other invariant manifolds $W_K, K \neq 1$ the directions of the dynamics may differ as $\dot{x}_1^i$ can have different signs under the two dynamics. However, in the generic case all equilibria are on the boundary, and it can be shown that the stability properties of them are preserved.

Proposition 2 *Generically, stability properties of equilibria do not differ under the two dynamics in the case of two types and two strategies when payoffs do not depend on the type of the other player.*

Proof. See Appendix. ■

4 Relationship to the Literature and Conclusion

The games in this paper are a special case of population games as asymmetric animal conflicts described in Selten (1980). The formulation of Selten's model is more general as it allows players to receive correlated signals, and so meet only own type, or only other types (though Selten himself analyses only the latter case). As a way of generalization, the probability of meeting a particular type may be made dependent on own type of the player.

In the literature, explicit dynamic evolutionary models were analyzed for population games with perfectly correlated signals. The case with two strategies and two types when only players of the same type meet is considered in Cressman et al. (2000). It is shown that the two dynamics have the same dynamic stability properties, because rotating around an interior equilibrium is not possible with only two strategies. However, with more than two strategies this does not hold anymore, as shown in Chamberland and Cressman (2000), where an example with three strategies is given, with an equilibrium that is asymptotically stable in the dynamic within types but not in the dynamic on the normal form.

The opposite case when only players of different types meet is analyzed in Gaunersdorfer et al. (1991). Even with two types and two strategies stability properties of stationary states may differ under the two dynamics. An equilibrium that is Lyapunov stable in the dynamic within types may correspond to a set of equilibria on the normal form that is unstable because equilibria on some $W_K, K \neq 1$ are unstable. In the present paper the opposite may also happen: an equilibrium that is unstable in the dynamic within types can be stable in the dynamic on the normal form as Example 4 shows.

In the other papers there was no need to distinguish whether payoffs depend on the type of the other player or not, as a type could meet only one other type (itself or the other type). This paper makes the distinction, showing that it is important as the dynamics generically have the same stability properties if payoffs do not depend on the type of the other player, but can have different properties if the payoffs do depend on the type of the other player. The relative speed of evolution for the two types matters when there is an interior stationary state and the dynamics rotate around it.

The other papers consider mostly biological applications. If one takes the view that economic situations are described by incomplete information games, and learning can be modeled by the replicator dynamic, this paper provides a basis for analysis of possible outcomes in economic situations. It shows that it is important to specify how exactly learning is modeled in the case when payoffs depend on the type of the other player, but it is not important when payoffs are independent of that type.

As we have shown, even with just two strategies the dynamics can have quite different properties in the case when payoffs depend also on the type of the other player. With more than two strategies, even in the case when payoffs do not depend on that type, it seems generically possible to have an equilibrium partially mixed for both types. We conjecture that its stability properties may then differ between the two dynamics.

A Proof of Proposition 2

The dynamic within types is given by

$$\begin{aligned}
\dot{x}_1^1 &= x_1^1(1-x_1^1)((\mu x_1^1+(1-\mu)x_1^2)(a_1-a_2)+a_2) = f_1(x_1^1,x_1^2)\\
\dot{x}_1^2 &= x_1^2(1-x_1^2)((\mu x_1^1+(1-\mu)x_1^2)(b_1-b_2)+b_2) = f_2(x_1^1,x_1^2)
\end{aligned}$$

and the dynamic on the normal form by

$$\begin{aligned}
\dot{x}_{11} &= x_{11}[\mu(1-x_1^1)[(\mu x_1^1+(1-\mu)x_1^2)(a_1-a_2)+a_2]+\\
&\quad (1-\mu)(1-x_1^2)[(\mu x_1^1+(1-\mu)x_1^2)(b_1-b_2)+b_2]] = g_{11}(x_{11},x_{12},x_{21})\\
\dot{x}_{12} &= x_{12}[\mu(1-x_1^1)[(\mu x_1^1+(1-\mu)x_1^2)(a_1-a_2)+a_2]+\\
&\quad (1-\mu)(-x_1^2)[(\mu x_1^1+(1-\mu)x_1^2)(b_1-b_2)+b_2]] = g_{12}(x_{11},x_{12},x_{21})\\
\dot{x}_{21} &= x_{21}[\mu(-x_1^1)[(\mu x_1^1+(1-\mu)x_1^2)(a_1-a_2)+a_2]+\\
&\quad (1-\mu)(1-x_1^2)[(\mu x_1^1+(1-\mu)x_1^2)(b_1-b_2)+b_2]] = g_{21}(x_{11},x_{12},x_{21})
\end{aligned}$$

or by

$$\begin{aligned}
\dot{x}_1^1 &= \mu x_1^1(1-x_1^1)[(\mu x_1^1+(1-\mu)x_1^2)(a_1-a_2)+a_2]+\\
&\quad (1-\mu)(x_{11}x_{22}-x_{12}x_{21})[(\mu x_1^1+(1-\mu)x_1^2)(b_1-b_2)+b_2]\\
\dot{x}_1^2 &= \mu(x_{11}x_{22}-x_{12}x_{21})[(\mu x_1^1+(1-\mu)x_1^2)(a_1-a_2)+a_2]+\\
&\quad (1-\mu)x_1^2(1-x_1^2)[(\mu x_1^1+(1-\mu)x_1^2)(b_1-b_2)+b_2]
\end{aligned}$$

where $x_1^1 = x_{11}+x_{12}, x_1^2 = x_{11}+x_{21}, x_{11}+x_{12}+x_{21}+x_{22} = 1$, and $x_{11}x_{22} = Kx_{12}x_{21}$ in the interior of the state space.

Interior stationary states in the dynamic within types exist only in the non-generic case $b_1a_2 = a_1b_2$. Let $(\mu x_1^1+(1-\mu)x_1^2)(a_1-a_2)+a_2 = a, (\mu x_1^1+(1-\mu)x_1^2)(b_1-b_2)+b_2 = b$, generically $a \neq b$. In the normal form dynamic $\dot{x}_1^1 = 0 \Rightarrow \mu x_1^1(1-x_1^1)a = -(1-\mu)(x_{11}x_{22}-x_{12}x_{21})b \Rightarrow \dot{x}_1^2 = \frac{(1-\mu)b}{x_1^1(1-x_1^1)}(x_1^2(1-x_1^2)x_1^1(1-x_1^1)-(x_{11}x_{22}-x_{12}x_{21})^2)$. Since $x_1^2(1-x_1^2)x_1^1(1-x_1^1)-(x_{11}x_{22}-$

$x_{12}x_{21})^2 = (1-x_1^1)x_{12}x_{11} + x_1^1 x_{21}x_{22}$, in the interior of the normal form space it is non-zero. It may be zero in the interior of x_1^1, x_1^2 space when $x_{11}, x_{22} = 0$ or $x_{12}, x_{21} = 0$. Analogously, $\dot{x}_1^2 = 0$ implies $\dot{x}_1^1 \neq 0$ in the interior of the normal form space. Therefore, there are no interior stationary states for the normal form dynamic in the generic case either.

On the boundary of x_1^1, x_1^2 space either $x_1^i = 0$ or $x_1^i = 1$. In either case $x_{11}x_{22} - x_{12}x_{21} = 0$, and the directions of the dynamics coincide. In particular, a boundary state that is stationary in the dynamic within types is stationary in the normal form dynamic.

The second dynamic can have more stationary states than the first one (e.g. when $a_1 = 2, a_2 = -1, b_1 = 1, b_2 = -2, \mu = \frac{1}{2}, x_{11} = x_{22} = \frac{1}{2}, x_{12} = x_{21} = 0$ then $x_1^1 = x_1^2 = \frac{1}{2}$ is a stationary state, while in the first dynamic it is not stationary). Each such state is boundary in the dynamic on the normal form but interior in the space x_1^1, x_1^2.

Stationary states that are not equilibria are unstable. Such states are stationary because the best response strategy for at least one of the types is absent. If this strategy appears, its proportion will grow, so such states are not stable in the dynamic within types. They are also not stable on W_1 since signs of $\dot{x}_1^1$ and $\dot{x}_1^2$ are preserved there. Therefore they are not stable in the dynamic on the normal form. States that are stationary in the normal form dynamic but not in the dynamic within types are unstable by this argument as they are not equilibria.

If a stationary state is a strict equilibrium, then it is asymptotically stable in both dynamics, since the equilibrium strategy is the unique best response to all small perturbations of itself.

Consider then a boundary stationary state that is an equilibrium, which may be pure or partially mixed, and which is not strict. In the generic case, such an equilibrium is partially mixed, and only one of the two types is indifferent in equilibrium. Let the equilibrium be on $x_1^1 = 0$, let s_2 be the unique best response for type t_1, and let type t_2 be indifferent in equilibrium (the reasoning for an equilibrium on other boundaries is analogous). This implies that $\dot{x}_1^1 < 0$ in a neighborhood of equilibrium, or that $(\mu x_1^1 + (1-\mu)x_1^2)(a_1 - a_2) + a_2 < 0$. Also, $(\mu x_1^1 + (1-\mu)x_1^2)(b_1 - b_2) + b_2 = 0$ at equilibrium.

The eigenvalues of the Jacobian of the dynamic within types are $(\mu x_1^1 + (1-\mu)x_1^2)(a_1 - a_2) + a_2 < 0$ and $x_1^2(1-x_1^2)(1-\mu)(b_1 - b_2)$. Thus in the generic case the equilibrium is stable iff $b_1 - b_2 < 0$ and unstable iff $b_1 - b_2 > 0$. ($b_1 - b_2 = 0$ is not a generic case).

The eigenvalues of the Jacobian of the dynamic on the normal form at equilibrium are $\mu[(\mu x_1^1 + (1-\mu)x_1^2)(a_1 - a_2) + a_2] < 0, \mu[(\mu x_1^1 + (1-\mu)x_1^2)(a_1 - a_2) + a_2] < 0$, and $x_{21}(1-x_{21})(1-\mu)^2(b_1 - b_2)$. If $b_1 - b_2 < 0$ they are all negative, and the equilibrium is stable, while if $b_1 - b_2 > 0$, the equilibrium is unstable. This is the same condition as for the stability of equilibrium in the dynamic within types, thus stability properties of equilibria in these two dynamics coincide.

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