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Conjectural Variations in Aggregative Games: An Evolutionary Perspective

by

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Abstract

Suppose that in aggregative games, in which a player's payoff depends only on this player's strategy and on an aggregate of all players' strategies, the players are endowed with constant conjectures about the reaction of the aggregate to marginal changes in the player's strategy. The players play the equilibrium determined by their conjectures and equilibrium strategies determine the players' payoffs, which can be different for players with different conjectures. It is shown that with random matching in an infinite population, only consistent conjectures can be evolutionarily stable, where a conjecture is consistent if it is equal to the marginal change in the aggregate at equilibrium, determined by the players' actual best responses. In the finite population case in which relative payoffs matter, only zero conjectures representing aggregate-taking behavior can be evolutionarily stable. The results are illustrated with the examples of a linear-quadratic game (that includes a Cournot oligopoly) and a rent-seeking game.

Keywords: aggregative games, conjectural variations, evolutionary stability

JEL Codes: C72, D84

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1 Introduction

This paper shows that there exist evolutionary foundations behind certain conjectures in aggregative games. An aggregative game (see e.g. Corchón, 1994, and Cornes and Hartley, 2009, for a more recent treatment) is a game in which the payoff of a player depends on the player's own strategy and on an aggregate of the strategies of all players. For example, in the Cournot oligopoly, a firm's profit depends on the firm's own quantity and on the price, which is determined by the inverse demand function from the aggregate sum of all firms' quantities.

Conjectures, also called conjectural variations (see Figuieres et al., 2004, for a recent book-length discussion), describe players' beliefs, or expectations, about the reaction of other players to a change in a player's strategy. In the context of aggregative games, conjectures can be seen as beliefs about the reaction of the aggregate (see e.g. Kamien and Schwartz, 1983, and Sugden, 1985, for particular aggregative settings). Such beliefs determine players' equilibrium strategies. In equilibrium each player's strategy is a best response given the conjecture about the reaction of the aggregate to a deviation.

Equilibrium strategies lead to some payoffs of the players. Generally, players with different conjectures can get different payoffs. If there is an evolutionary process selecting players on the basis of these payoffs, then some conjectures would perform better than others. Potentially conjectures can be arbitrary; this paper shows that in well-behaved games the evolutionarily stable (for an infinite population and players randomly matched to play a given finite-player game) constant conjectures must be consistent at equilibrium: the beliefs about the reaction to an arbitrarily small deviation from equilibrium coincide with the marginal change in the aggregate, derived from the players' actual best response functions.[1] On the other hand, evolutionary stability for finite populations (where all players interact in the same game) selects zero conjectural variations: players believe that the aggregate does not change if their strategy changes. Such behavior is akin to the price-taking behavior in the standard perfect competition model in microeconomics, as noted in Fama and Laffer (1972) and Kamien and Schwartz (1983).

Players with consistent conjectures form correct expectations about the reaction of the aggregate and therefore it may appear intuitive that they have a higher payoff in evolution-

[1]Consistency of conjectures in various contexts was introduced and discussed in e.g. Laitner (1980), Bresnahan (1981), Perry (1982).

ary terms. Nevertheless, it is not obvious why it should be the case in a strategic context because other players may adjust their equilibrium behavior to the conjectures of all players. Having a consistent conjecture, however, does not have a detrimental strategic effect.

In a finite population, relative payoffs are important in evolutionary terms. If players are symmetric, the effect of the aggregate is the same on any two players and thus cancel out from the relative payoff evaluation. Players with zero conjecture indeed behave as if the aggregate does not change and therefore has no effect on payoffs, thus mimicking the condition for maximizing the relative payoff, leading to evolutionary stability of zero conjectures in this case.

The results generalize and combine several previous observations in the literature about evolutionary justifications of conjectures. Working with an infinite population, Dixon and Somma (2003) and Müller and Normann (2005) showed that consistent conjectures have evolutionary foundations in certain duopoly games, while Possajennikov (2009) generalized the result to arbitrary well-behaved two-player games. This paper extends the result to n-player aggregative games. On the other hand, for the finite population case Possajennikov (2003) provided evolutionary background for aggregate-taking behavior in aggregative games. In the context of conjectures, aggregate-taking behavior is equivalent to zero conjectural variation about the change in the aggregate.[2]

2 Aggregative Games and Conjectures

An aggregative game on the real line is given by $G = (\{1, \ldots, n\}, \{X_i\}_{i=1}^n, \{u_i\}_{i=1}^n)$, where $\{1, \ldots, n\}$ is the set of players, $X_i \subset \mathbb{R}$ is the strategy set of Player i and $u_i(x_i, X) : X_i \times \mathbb{R} \to \mathbb{R}$ is the payoff function of player i. Here, $X = f(x_1, \ldots, x_n) : X_1 \times \ldots \times X_n \to \mathbb{R}$ is the *aggregate* of players' strategies. The payoff of each player depends on the player's own strategy and this aggregate only: other players' strategies influence a player's payoff only through the aggregate.

It is assumed that the games are well behaved: the sets X_i are convex and the functions u_i and f are twice continuously differentiable.

A player's *conjecture* $r_i \in R_i$, where R_i is a convex subset of the real line $\mathbb{R}$, is a number representing the player's belief, or expectation, about the change in the aggregate

[2] Müller and Normann (2007) show that in a duopoly, finite population evolutionary stability indeed leads to a result that would be obtained with zero conjectures about the aggregate.

in response to a marginal change in the player's own strategy. If $B_i[Z]$ denotes the belief of Player i about the object Z, then $r_i = B_i\left[\frac{dX}{dx_i}\right]$. Note that the conjecture is about the total differential: the reactions of other players are incorporated in dX, and only the final change in X matters for the player's payoff. It is assumed that players entertain constant conjectures: r_i does not vary with $x_1, \ldots, x_n$. This assumption means that players are generally boundedly rational: $\frac{dX}{dx_i}$ derived from the players' reaction functions may vary with $x_1, \ldots, x_n$, although in some cases (as in the example in Section 4.1) it does not.[3]

Given the conjecture r_i, a player maximizes the payoff $u_i(x_i, X)$. If the solution of the player's maximization problem is interior, then the first-order condition $\frac{\partial u_i}{\partial x_i}(x_i, X) + \frac{\partial u_i}{\partial X}(x_i, X) \cdot B_i\left[\frac{dX}{dx_i}\right] = 0$ holds. Since the conjecture is constant, $r_i = B_i\left[\frac{dX}{dx_i}(x_1, \ldots, x_n)\right]$ for any $x_1, \ldots, x_n$, the condition is

$$F_i(x_i, X; r_i) := \frac{\partial u_i}{\partial x_i}(x_i, X) + \frac{\partial u_i}{\partial X}(x_i, X) \cdot r_i = 0. \tag{1}$$

Suppose that all n players have some conjectures, given by $r_1, \ldots, r_n$. Denote the vector of conjectures by $r = (r_1, \ldots, r_n)$. The vector will sometimes be denoted by $r = (r_1, r_{-1}) = (r_i, r_{-i})$ to emphasize which player is being considered. If each player's solution of the payoff maximization problem is interior, then the equilibrium is characterized by the following equations:

$$\begin{aligned} F_1(x_1, X; r_1) &= 0 \\ &\cdots \\ F_n(x_n, X; r_n) &= 0 \\ X - f(x_1, \ldots, x_n) &= 0 \end{aligned} \tag{2}$$

The first n equations are the n first-order conditions for the maximization problems of the n players; the last equation is the equation defining the aggregate. The equations implicitly define equilibrium strategies $x_i^*(r_1, \ldots, r_n)$ and an equilibrium value of the aggregate $X^* = f(x_1^*, \ldots, x_n^*) = f(x_1^*(r), \ldots, x_n^*(r))$. In general an (interior) equilibrium may not exist or there may be multiple equilibria. It will be assumed in the sequel that the equilibrium selection $x_i^*(r_1, \ldots, r_n)$ exists and locally well behaved for the relevant values of conjectures.

[3] Focusing on constant conjectures restricts the dimensionality of the analysis and allows unambiguous selection in many settings. If conjectures vary with $x_1, \ldots, x_n$, then many choices of $x_1, \ldots, x_n$ can be justified with an appropriately constructed consistent conjectures (see Laitner, 1980, for such a result in a duopoly setting).

While conjectures seem to imply a dynamic process of reaction, the equilibrium concept has the following static interpretation. Suppose that currently players are playing or contemplating to play a certain strategy profile $(x_1, \ldots, x_n)$, which is common knowledge. Player i considering an (arbitrarily small) deviation from it believes that the other players would have time to react, leading to the change in the aggregate equal to r_i times the change in x_i before any payoffs accrue. The player believes that after these reactions, payoffs are realized. A necessary condition for the profile $(x_1, \ldots, x_n)$ to be an equilibrium in this setting is that no player expects to get a higher payoff after a marginal deviation and reactions to it, and that is what Equations (2) capture.

In a standard Nash equilibrium analysis, players choose best response keeping the actions of the other players fixed. In the current setting, this can be represented by $r_i = B_i \left[\frac{dX}{dx_i}\right] = \frac{\partial f}{\partial x_i}$. But the setting allows for many more conjectures, some of which turn out to be relevant in terms of evolution.

The *zero* conjectural variation $r_i = 0$ means that a player with such a conjecture has beliefs $B_i \left[\frac{dX}{dx_i}\right] = 0$. For such a player, a change in his or her own action does not affect the aggregate, which can be termed the *aggregate-taking* behavior (Possajennikov, 2003). Price-taking behavior in microeconomic competitive equilibrium models is an example of such a behavior.

To define a *consistent* conjecture, imagine that Player i's strategy can be varied freely while the remaining $n-1$ players behave optimally. Then there are n equations characterizing optimal choices:

$$\begin{aligned} F_j(x_j, X; r_j) &= 0, \; j \neq i \\ X - f(x_1, \ldots, x_n) &= 0 \end{aligned} \tag{3}$$

Treating x_i as a parameter and x_j, $j \neq i$ and X as variables, this system implicitly defines reaction functions $x_j^*(x_i)$ and $X^*(x_i) = f(x_1, x_2^*(x_1), \ldots, x_n^*(x_1))$. Again, it will be assumed that the reaction functions are well defined for the relevant values of conjectures. Then

Definition 1 *A conjecture r_i is* consistent *if $r_i = \frac{dX^*}{dx_i}(x_1^*, \ldots, x_n^*)$.*

The definition means that a conjecture is consistent if it, i.e. the belief about the change in the aggregate, is equal to the actual marginal change that would arise from optimal reactions of other players to a deviation by one player from the profile $(x_1^*, \ldots, x_n^*)$. This actual change is derived from the reaction functions of the players, which depend on the conjectures those players are holding.

3 Evolutionary Stability of Conjectures

3.1 Conjectures of Individual Players

Imagine that players are endowed with some conjectures. The previous section characterized the equilibrium strategies that the players use if the solution of their payoff maximization problems are interior. Substituting the solution into the payoff functions, the payoff of Player i is then $u_i(x_i^*(r), X^*(r))$.

Suppose that the players other than Player i have some fixed conjectures r_{-i}. Consider the following interpretation of the choice of conjecture for Player i. There is a large population of agents that can potentially be Player i. Each of the agents is endowed with a conjecture. Agents with different conjectures will get different payoff in the game if selected to play. An evolutionary interpretation is that the agents that get a higher payoff are more likely to survive or to reproduce. Thus,

Definition 2 *A conjecture r_i^{ES} is* evolutionarily stable *for Player i against given conjectures r_{-i} of the other players if $u_i(x^*(r_i^{ES}, r_{-i}), X^*(r_i^{ES}, r_{-i})) > u_i(x^*(r_i, r_{-i}), X^*(r_i, r_{-i}))$ for any $r_i \neq r_i^{ES}$.*

The definition requires that a player with the evolutionarily stable conjecture gets a higher payoff that a player with any other conjecture. If other conjectures could get the same payoff, there would be no selection pressure against them. The definition is adapted from the evolutionarily stable strategy (ESS) definition for asymmetric games (Selten, 1980).

The definition means that r_i^{ES} maximizes function $u_i(x^*(r_i, r_{-i}), X^*(r_i, r_{-i}))$ as a function of r_i. A necessary condition for an interior maximum is that $\frac{du_i}{dr_i} = 0$ at $r_i = r_i^{ES}$, or

$$\frac{\partial u_i}{\partial x_i}\frac{\partial x_i^*}{\partial r_i}(r_i, r_{-i}) + \frac{\partial u_i}{\partial X}\frac{dX^*}{dr_i}(r_i, r_{-i}) = 0 \tag{4}$$

at $r_i = r_i^{ES}$.

If $\frac{\partial u_i}{\partial X} \neq 0$ and $\frac{\partial x_i^*}{\partial r_i} \neq 0$, the left hand side can be rewritten as

$$\frac{\partial u_i/\partial x_i}{\partial u_i/\partial X} + \frac{dX^*/dr_i}{\partial x_i^*/\partial r_i} = 0.$$

From Equation (1), at the solution of Player i's maximization problem $-\frac{\partial u_i/\partial x_i}{\partial u_i/\partial X} = r_i$. Thus if r_i^{ES} is an evolutionarily stable conjecture, then $r_i^{ES} = \frac{dX^*/dr_i}{\partial x_i^*/\partial r_i}(r_i^{ES}, r_{-i})$.

Speaking somewhat loosely in mathematical terms, if one treats $dr_i = \partial r_i$ (since r_i is an independent variable, its total and partial differentials are equal) as a small change in r_i, one can cancel it from the above expression. Then $r_i^{ES} = \frac{dX^*}{dx_i^*}$. Recalling that a conjecture is consistent if $r_i = \frac{dX^*}{dx_i}(x_1^*, \dots, x_n^*)$, if a conjecture is evolutionarily stable, then it has to be consistent.

Using implicit function theorems, the intuition can be made precise. To simplify notation, consider $i = 1$ (the reasoning for the other players is analogous). From the system of equations (2)

$$\begin{array}{ccccccccccccc}
\frac{\partial F_1}{\partial x_1}\frac{\partial x_1^*}{\partial r_1} & + & 0 & + & \dots & + & 0 & + & \frac{\partial F_1}{\partial X}\frac{dX^*}{dr_1} & + & \frac{\partial F_1}{\partial r_1} & = & 0 \\
0 & + & \frac{\partial F_2}{\partial x_2}\frac{\partial x_2^*}{\partial r_1} & + & \dots & + & 0 & + & \frac{\partial F_2}{\partial X}\frac{dX^*}{dr_1} & & & = & 0 \\
\dots & & \dots & & \dots & & \dots & & \dots & & & & \dots \\
0 & + & 0 & + & \dots & + & \frac{\partial F_n}{\partial x_n}\frac{\partial x_n^*}{\partial r_1} & + & \frac{\partial F_n}{\partial X}\frac{dX^*}{dr_1} & & & = & 0 \\
-\frac{\partial f}{\partial x_1}\frac{\partial x_1^*}{\partial r_1} & + & -\frac{\partial f}{\partial x_2}\frac{\partial x_2^*}{\partial r_1} & + & \dots & + & -\frac{\partial f}{\partial x_n}\frac{\partial x_n^*}{\partial r_1} & + & \frac{dX^*}{dr_1} & & & = & 0
\end{array}$$

Let

$$A = \begin{pmatrix}
\frac{\partial F_1}{\partial x_1} & 0 & \dots & 0 & \frac{\partial F_1}{\partial X} \\
0 & \frac{\partial F_2}{\partial x_2} & \dots & 0 & \frac{\partial F_2}{\partial X} \\
\vdots & \dots & \ddots & \dots & \vdots \\
0 & 0 & \dots & \frac{\partial F_n}{\partial x_n} & \frac{\partial F_n}{\partial X} \\
-\frac{\partial f}{\partial x_1} & -\frac{\partial f}{\partial x_2} & \dots & -\frac{\partial f}{\partial x_n} & 1
\end{pmatrix}.$$

Then $|A| = \prod_{i=1}^n \frac{\partial F_i}{\partial x_i} + \sum_{i=1}^n \frac{\partial f}{\partial x_i}\frac{\partial F_i}{\partial X}\prod_{j \neq i}\frac{\partial F_j}{\partial x_j}$.

Suppose that $|A| \neq 0$. By Cramer's rule,

$$\frac{\partial x_1^*}{\partial r_1} = \frac{1}{|A|}\begin{vmatrix}
-\frac{\partial F_1}{\partial r_1} & 0 & \dots & 0 & \frac{\partial F_1}{\partial X} \\
0 & \frac{\partial F_2}{\partial x_2} & \dots & 0 & \frac{\partial F_2}{\partial X} \\
\vdots & \dots & \ddots & \dots & \vdots \\
0 & 0 & \dots & \frac{\partial F_n}{\partial x_n} & \frac{\partial F_n}{\partial X} \\
0 & -\frac{\partial f}{\partial x_2} & \dots & -\frac{\partial f}{\partial x_n} & 1
\end{vmatrix},$$

or $\frac{\partial x_1^*}{\partial r_1} = \frac{1}{|A|}\left(-\frac{\partial F_1}{\partial r_1}\right)|A_{-1}|$, where

$$A_{-1} = \begin{pmatrix} \frac{\partial F_2}{\partial x_2} & \dots & 0 & \frac{\partial F_2}{\partial X} \\ \dots & \ddots & \dots & \vdots \\ 0 & \dots & \frac{\partial F_n}{\partial x_n} & \frac{\partial F_n}{\partial X} \\ -\frac{\partial f}{\partial x_2} & \dots & -\frac{\partial f}{\partial x_n} & 1 \end{pmatrix}$$

and thus $|A_{-1}| = \prod_{i=2}^{n} \frac{\partial F_i}{\partial x_i} + \sum_{i=2}^{n} \frac{\partial f}{\partial x_i}\frac{\partial F_i}{\partial X}\prod_{j\neq i, j\neq 1}\frac{\partial F_j}{\partial x_j}$. Also,

$$\frac{dX^*}{dr_1} = \frac{1}{|A|}\begin{vmatrix} \frac{\partial F_1}{\partial x_1} & 0 & \dots & 0 & -\frac{\partial F_1}{\partial r_1} \\ 0 & \frac{\partial F_2}{\partial x_2} & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{\partial F_n}{\partial x_n} & 0 \\ -\frac{\partial f}{\partial x_1} & -\frac{\partial f}{\partial x_2} & \dots & -\frac{\partial f}{\partial x_n} & 0 \end{vmatrix},$$

or $\frac{dX^*}{dr_1} = -\frac{1}{|A|}\frac{\partial F_1}{\partial r_1}\frac{\partial f}{\partial x_1}\prod_{j\neq 1}\frac{\partial F_j}{\partial x_j}$. Therefore,

$$\frac{dX^*/dr_1}{\partial x_1^*/\partial r_1} = \frac{-\frac{\partial F_1}{\partial r_1}\frac{\partial f}{\partial x_1}\prod_{j\neq 1}\frac{\partial F_j}{\partial x_j}}{\left(-\frac{\partial F_1}{\partial r_1}\right)\left(\prod_{i=2}^{n}\frac{\partial F_i}{\partial x_i} + \sum_{i=2}^{n}\frac{\partial f}{\partial x_i}\frac{\partial F_i}{\partial X}\prod_{j\neq i, j\neq 1}\frac{\partial F_j}{\partial x_j}\right)},$$

or $\frac{dX^*/dr_1}{\partial x_1^*/\partial r_1} = \frac{1}{|A_{-1}|}\frac{\partial f}{\partial x_1}\prod_{j\neq 1}\frac{\partial F_j}{\partial x_j}$, provided that $|A_{-1}| \neq 0$ (from Equation (1) $\frac{\partial F_1}{\partial r_1} = \frac{\partial u_1}{\partial X}$ and thus $\frac{\partial F_1}{\partial r_1} \neq 0$).

The reaction of the aggregate to independent changes in the strategy of Player 1 is found from system (3). Using the implicit function theorem, $\frac{dX^*}{dx_1}$ can be found from

$$\begin{array}{ccccccccccc} \frac{\partial F_2}{\partial x_2}\frac{\partial x_2^*}{\partial x_1} & + & \dots & + & 0 & + & \frac{\partial F_2}{\partial X}\frac{dX^*}{dx_1} & & & = & 0 \\ \dots & & \dots & & \dots & & \dots & & & & \dots \\ 0 & + & \dots & + & \frac{\partial F_n}{\partial x_n}\frac{\partial x_n^*}{\partial x_1} & + & \frac{\partial F_n}{\partial X}\frac{dX^*}{dx_1} & & & = & 0 \\ -\frac{\partial f}{\partial x_2}\frac{\partial x_2^*}{\partial x_1} & + & \dots & + & -\frac{\partial f}{\partial x_n}\frac{\partial x_n^*}{\partial x_1} & + & \frac{dX^*}{dx_1} & + & -\frac{\partial f}{\partial x_1} & = & 0 \end{array}$$

Then

$$\frac{dX^*}{dx_1} = \frac{1}{|A_{-1}|}\begin{vmatrix} \frac{\partial F_2}{\partial x_2} & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \frac{\partial F_n}{\partial x_n} & 0 \\ -\frac{\partial f}{\partial x_2} & \dots & -\frac{\partial f}{\partial x_n} & \frac{\partial f}{\partial x_1} \end{vmatrix},$$

or $\frac{dX^*}{dx_1} = \frac{1}{|A_{-1}|} \frac{\partial f}{\partial x_1} \prod_{i \neq 1} \frac{\partial F_i}{\partial x_i}$.

Therefore it holds that $\frac{dX^*}{dx_1} = \frac{dX^*/dr_1}{\partial x_1^*/\partial r_1}$, if the regularity conditions allowing that the solutions of the systems above are satisfied. Thus if r_1^{ES} is an evolutionarily stable conjecture, then $r_1^{ES} = -\frac{\partial u_1/\partial x_1}{\partial u_1/\partial X} = \frac{dX^*/dr_1}{\partial x_1^*/\partial r_1} = \frac{dX^*}{dx_1}$, i.e. r_1^{ES} is consistent. Since the index 1 was used only to make the notation simpler, the result holds for any player.

Proposition 1 *Suppose that players' conjectures are given by* $r = (r_1, \ldots, r_n)$*,* r_i *is interior in* R_i*. Under the regularity conditions that*

(i) There exist a solution $x_i^*(r), X^*(r)$ *of system (2);*

(ii) $\frac{\partial u_i}{\partial X} \neq 0$ *at* $x_i^*(r), X^*(r)$*;*

(iii) $|A| = \prod_{i=1}^n \frac{\partial F_i}{\partial x_i} + \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial F_i}{\partial X} \prod_{j \neq i} \frac{\partial F_j}{\partial x_j} \neq 0$ *at* r *and* $x_i^*(r), X^*(r)$*;*

(iv) $|A_{-i}| = \prod_{j \neq i} \frac{\partial F_j}{\partial x_j} + \sum_{j \neq i} \frac{\partial f}{\partial x_j} \frac{\partial F_j}{\partial X} \prod_{k \neq i, k \neq j} \frac{\partial F_k}{\partial x_k} \neq 0$ *at* r *and* $x_i^*(r), X^*(r)$*;*

if r_i *is an evolutionarily stable conjecture, then* r_i *is consistent.*

Thus only consistent conjectures can generally be evolutionarily stable. Note that only the necessary first-order condition was used; it may happen that such a condition is not sufficient to find the maximum of a function. Appropriate assumptions on the concavity of functions can ensure that a consistent conjecture is indeed evolutionarily stable. Examples in Section 4 demonstrate that consistent conjectures indeed can be evolutionarily stable.

If the choice of conjectures is interpreted as a concious choice of a player instead of the product of evolution, then the result means that only choosing a consistent conjecture can be an interior best response of a player to a given vector of conjectures of the other players in aggregative games.[4] This holds for any conjectures of the other players and therefore only consistent conjectures can be an equilibrium of the game of choosing conjectures, or, equivalently, be evolutionarily stable if the conjectures are selected by evolution.

While the result may appear intuitive (players with consistent conjectures correctly anticipate the reaction of the aggregate), one needs to keep in mind that it is not straightforward. Players with given conjectures get payoffs via strategies used in equilibrium of the

[4]Dixon and Somma (2003) note in the linear-quadratic Cournot duopoly context that the best response conjecture of one firm to any given conjecture of the other firm equals the actual slope of that other firm's reaction function.

game with these conjectures. Because of possible strategic reactions of other players, it is not obvious why payoffs in such an equilibrium should be higher with consistent conjectures than with other conjectures. Even consistent conjectures are not necessarily the most 'rational' ones from other viewpoints (see e.g. Makowski, 1987). As discussed in Section 2 and illustrated in the examples in Section 4, in particular they do not necessarily lead to the standard Nash equilibrium concept. For example, in a linear Cournot duopoly, a zero conjecture about the reaction of the other firm is not consistent because the slope of the reaction function is non-zero at equilibrium. Similarly, in a linear Cournot oligopoly a consistent conjecture about the aggregate production is smaller than unity thus making the equilibrium with consistent and evolutionarily stable conjectures different from the Nash equilibrium (where the aggregate production is expected to change exactly by the change in the firm's production level).

3.2 Symmetric Games and Infinite and Finite Populations of Players

An aggregative game is *symmetric* if the strategy spaces are the same for all players, $X_i = X_j$ for all i, j, the aggregate is symmetric, $X(\pi(x_1), \ldots, \pi(x_n)) = X(x_1, \ldots, x_n)$ for any permutation π, and the payoff functions are symmetric, $u_i(x_i, X) = u_j(x_j, X)$ if $x_i = x_j$, for any i, j. Suppose further that the conjecture spaces are the same for all players, $R_i = R_j$ for all i, j. If the situation is symmetric, one can suppress the indices of players and consider only one generic player, say Player 1. In a symmetric game, one can compare payoffs of different players and thus one can consider one population of players for the evolutionary analysis.

Consider an infinite population of players randomly matched to play an n-player aggregative game G. In a symmetric situation, a slight strengthening of the Maynard Smith and Price (1973) definition of evolutionary stability, adapted to the current model, is

Definition 3 *A conjecture r^{ES} is* evolutionarily stable *if $r_i = r^{ES}$ for all i and it holds that $u_1(x^*(r_1^{ES}, r_{-1}^{ES}), X^*(r_1^{ES}, r_{-1}^{ES})) > u_1(x^*(r_1, r_{-1}^{ES}), X^*(r_1, r_{-1}^{ES}))$ for any $r_1 \neq r^{ES}$.*

The condition of the definition requires r^{ES} to be the best choice if the other players use r^{ES}. Then Equation (4) again provides the necessary condition for a conjecture to be evolutionarily stable. Therefore the analysis of the previous subsection still applies:

Proposition 2 *Under the regularity conditions of Proposition 1, if r^{ES} is an interior evolutionarily stable conjecture in a symmetric aggregative game with random matching in infinite population, then r^{ES} is consistent.*

If the population is finite and all n players participate in the same interaction, Schaffer (1988) proposed a *finite population evolutionary stability* concept, based on relative payoffs. Adapted to the present context of symmetric aggregative games,

Definition 4 *A conjecture r^{fES} is* finite-population evolutionarily stable *(fES) if $r_i = r^{fES}$ for all i and it holds that $u_1(x_1^*(r_{-n}^{fES}), X^*(r_{-n}^{fES})) > u_1(x_1^*(r_{-1}^{fES}), X^*(r_{-1}^{fES}))$, where $r_{-n}^{fES} = (r^{fES}, r^{fES}, \dots, r)$ and $r_{-1}^{fES} = (r, r^{fES}, \dots, r^{fES})$, for any $r \neq r^{fES}$.*

The idea of the definition is that if any one player has a conjecture r different from r^{fES}, then that player will get a lower payoff than the players who have conjecture r^{fES} in an equilibrium of the game with 1 player having conjecture r and $n-1$ players having conjecture r^{fES}.

Schaffer (1988) shows that a fES strategy maximizes *relative payoff*, represented by the difference $u_i(x_i, x_{-i}) - u_j(x_j, x_{-j})$ (in symmetric games, it does not matter which players i, j are considered). In the current context, an fES conjecture r^{fES} is a solution of the problem

$$\max_{r_1} u_1(x_1^*(r_1, r_{-1}^{fES}), X^*(r_1, r_{-1}^{fES})) - u_j(x_j^*(r_1, r_{-1}^{fES}), X^*(r_1, r_{-1}^{fES}))$$

for $j \neq 1$. The first-order condition of this maximization problem is

$$\frac{\partial u_1}{\partial x_1}\frac{\partial x_1^*}{\partial r_1} + \frac{\partial u_1}{\partial X}\frac{dX^*}{dr_1} - \frac{\partial u_j}{\partial x_j}\frac{\partial x_j^*}{\partial r_1} - \frac{\partial u_j}{\partial X}\frac{dX^*}{dr_1} = 0. \tag{5}$$

If r^{fES} is the solution of the maximization problem, then the above condition is satisfied at $r_1 = r^{fES}$. Suppose that if conjectures are the same for all players, then a symmetric equilibrium x^* is played (since the game is symmetric), where $x_i^* = x_j^*$. Then at equilibrium $\frac{\partial u_1}{\partial x_1} = \frac{\partial u_j}{\partial x_j}$ and $\frac{\partial u_1}{\partial X} = \frac{\partial u_j}{\partial X}$. Equation (5) reduces to

$$\frac{\partial u_1}{\partial x_1}\left(\frac{dx_1^*}{dr_1} - \frac{dx_j^*}{dr_1}\right) = 0.$$

Consider the last term in the expression. For $\frac{dx_1^*}{dr_1} - \frac{dx_j^*}{dr_1} = 0$ to hold, both Players 1 and j should react in the same way to a change in conjecture of Player 1. Recall that

$\frac{dx_1^*}{dr_1} = \frac{1}{|A|}\left(-\frac{\partial F_1}{\partial r_1}\right)\left(\prod_{i=2}^n \frac{\partial F_i}{\partial x_i} + \sum_{i=2}^n \frac{\partial f}{\partial x_i}\frac{\partial F_i}{\partial X}\prod_{j\neq i, j\neq 1}\frac{\partial F_j}{\partial x_j}\right)$. From system (2),

$$\frac{\partial x_n^*}{\partial r_1} = \frac{1}{|A|}\begin{vmatrix} \frac{\partial F_1}{\partial x_1} & 0 & \dots & -\frac{\partial F_1}{\partial r_1} & \frac{\partial F_1}{\partial X} \\ 0 & \frac{\partial F_2}{\partial x_2} & \dots & 0 & \frac{\partial F_2}{\partial X} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \frac{\partial F_n}{\partial X} \\ -\frac{\partial f}{\partial x_1} & -\frac{\partial f}{\partial x_2} & \dots & 0 & 1 \end{vmatrix},$$

or $\frac{\partial x_n^*}{\partial r_1} = \frac{1}{|A|}\frac{\partial F_1}{\partial r_1}\frac{\partial f}{\partial x_1}\frac{\partial F_n}{\partial X}\prod_{j\neq n, j\neq 1}\frac{\partial F_j}{\partial x_j}$.

In the symmetric case, indices n and any $j \neq 1$ are interchangeable. Simplifying the expressions above,

$$\frac{dx_1^*}{dr_1} - \frac{dx_j^*}{dr_1} = \frac{-\partial F_1/\partial r_1 \cdot (\partial F_1/\partial x_1)^{n-2}}{|A|}\left(\frac{\partial F_1}{\partial x_1} + n\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial X}\right).$$

If $\frac{dx_1^*}{dr_1} - \frac{dx_j^*}{dr_1} \neq 0$, then the necessary condition for finite population evolutionary stability is $\frac{\partial u_1}{\partial x_1} = 0$. But recall that a necessary condition for a player to maximize the payoff in the game is $\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial X} r_1 = 0$. If $\frac{\partial u_1}{\partial X} \neq 0$, then the only way to satisfy the two conditions is $r_1 = 0$.

Proposition 3 *Suppose that players' conjectures in a symmetric aggregative game are given by $r = (r_1, \dots, r_n)$, r_i is interior in R_i, $r_i = r_j$ for all i, j. Under symmetric equilibrium selection, the regularity conditions of Proposition 1 and*

- *(v)* $\frac{\partial F_i}{\partial x_i} \neq 0$ *at* r *and* $x_i^*(r), X^*(r)$;
- *(vi)* $\frac{\partial F_i}{\partial x_i} + n\frac{\partial f}{\partial x_i}\frac{\partial F_i}{\partial X} \neq 0$ *at* r *and* $x_i^*(r), X^*(r)$,

if r_i is a finite population evolutionarily stable conjecture, then $r_i = 0$.

The result is related to the result in Possajennikov (2003), where the coincidence of the first-order conditions for the aggregate-taking and the finite-population evolutionarily stable behaviors is shown. In the current setting, zero conjectural variation $r_i = 0$ means aggregate-taking behavior, i.e. Player i believes that the aggregate does not change if the player changes his or her strategy. The finite-population evolutionarily stable conjecture r^{fES} is a way of committing to the finite-population evolutionarily stable behavior. That $r^{fES} = 0$ is then another manifestation of the connection between aggregate-taking and finite-population evolutionarily stable behaviors.

4 Examples

4.1 Linear-quadratic aggregative games and Cournot oligopoly

Consider the following game with n players. Player i chooses $x_i \in X_i \subset \mathbb{R}_+$. Suppose that the payoff function is

$$u_i(x_i, X) = ax_i - \frac{c}{2}x_i^2 + bx_iX,$$

where $X = x_1 + \ldots + x_n$ is the aggregate, $a, c > 0$ and $b \neq 0$. A typical economic example of an aggregative game, the Cournot oligopoly, is an example of such a game for $b < 0$. In the linear-quadratic Cournot oligopoly players are firms, they choose production quantities q_i, the total production is $Q = q_1 + \ldots + q_n$, the inverse demand function is $P(Q) = a + bQ$ and cost functions are $C(q_i) = \frac{c}{2}q_i^2$. Dixon and Somma (2003) and Müller and Normann (2005) analyzed the evolutionary stability of consistent conjectures in such a case for $b < 0$ and $n = 2$.

Suppose that the players have some conjectures $r = (r_1, \ldots, r_n)$. The necessary conditions for profit maximization at an interior solution for each player is

$$a - cx_i + br_ix_i + bX = 0,\ i = 1, \ldots, n.$$

Suppose that x_1 is allowed to vary freely, while $x_2, \ldots, x_n$ are chosen optimally. Then from the above equations

$$\begin{array}{ccccccccc}
(br_2 - c + b)x_2 & + & bx_3 & + & \ldots & + & bx_n & = & -a - bx_1 \\
bx_2 & + & (br_3 - c + b)x_3 & + & \ldots & + & bx_n & = & -a - bx_1 \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots \\
bx_2 & + & bx_3 & + & \ldots & + & (br_n - c + b)x_n & = & -a - bx_1
\end{array}$$

Let

$$A = \begin{pmatrix} br_2 - c + b & b & \ldots & b \\ b & br_3 - c + b & \ldots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \ldots & br_n - c + b \end{pmatrix}$$

and let A_i be the matrix obtained from matrix A by substituting the column vector $(-a - bx_1)\,(11\ldots1)'$ of the right-hand sides of the equations in place of column i of matrix A. Then $x_i^* = \frac{|A_i|}{|A|}$ and $X^* = x_1 + \sum_{i=2}^n x_i^* = x_1 + \frac{1}{|A|}\sum_{i=2}^n |A_i| = x_1 + \frac{1}{|A|}(-a - bx_1)\sum_{i=2}^n |\hat{A}_i|$, where $\hat{A}_i$ is the matrix obtained from matrix A_i by having column $(-a - bx_1)(11\dots1)'$ replaced by $(11\dots1)'$. Since neither $|A|$ nor $|\hat{A}_i|$ depend on x_1, the relationship between X^* and x_1 is linear, thus the constant conjectures about $\frac{dX^*}{dx_1}$ are justified in this setting at an interior equilibrium.

If the players other than Player 1 all have the same conjecture $r = \frac{dX}{dx_i}$, the necessary conditions for profit maximization for each of the players are $a - cx_i + brx_i + bX = 0$. The second-order conditions $-c + 2br < 0$ are satisfied for $r < \frac{c}{2b}$ if $b > 0$ and for $r > \frac{c}{2b}$ if $b < 0$. Adding up the first-order conditions for the $n-1$ players other than Player 1 gives $(n-1)a - c(X - x_1) + br(X - x_1) + (n-1)bX = 0$. Therefore $X^* = \frac{c-br}{c-br-(n-1)b}x_1 + \frac{(n-1)a}{c-br-(n-1)b}$ and $\frac{dX^*}{dx_1} = \frac{c-br}{c-br-(n-1)b}$. A symmetric consistent conjecture is then characterized by

$$r = \frac{c - br}{c - br - (n-1)b}. \tag{6}$$

If $b > 0$, the quadratic equation can have zero, one, or two solutions satisfying the second-order condition constraint $r < \frac{c}{2b}$ depending on the parameter values. The case $b < 0$ is simpler. If $b < 0$, Equation (6) has one root r_C between 0 and 1 because at $r = 0$ the right hand side is $\frac{c}{c-(n-1)b} > 0$, and at $r = 1$ the right hand side is $\frac{c-b}{c-b-(n-1)b} < 1$. The other root is $r < \frac{c}{b} < \frac{c}{2b}$, violating the second-order condition for the maximization problem. Thus for any $n > 1$ there is a unique symmetric consistent $r_C \in (0,1)$.[5]

The condition for r to be consistent can be rewritten as $cr_C - br_C^2 - (n-1)br_C - c + br_C = 0$. Then $\frac{dr_C}{dn} = -\frac{-br_C}{c-2br_C-(n-2)b} < 0$ for $r_C \in (0,1)$. The symmetric consistent conjecture decreases as n increases. As $n \to \infty$, then $r_C \to 0$. With infinitely many players, the consistent conjecture is that one player has no influence on the aggregate.

If all players have the same conjecture r, the equilibrium is characterized by n first-order conditions $a - cx_i + bx_i r + bX = 0$. Adding them up gives $na - cX + brX + nbX = 0$, or $X = \frac{na}{c-br-nb}$. The standard Nash equilibrium is obtained by taking conjectures $r = 1$, since then $\frac{dX}{dx_i} = 1$. In the context of the Cournot oligopoly x_i are production quantity choices. Then the total quantity X with consistent conjectures $r_C < 1$ for all players for a given n is larger than the total quantity for Nash conjectures $r = 1$.

If all players have consistent conjectures, $X = \frac{na}{c-br_C-nb}$ and $\frac{dX}{dn} = \frac{a(c-br_C+nb\cdot dr_C/dn)}{(c-br_C-nb)^2}$. Since $\frac{dr_C}{dn} < 0$, then $\frac{dX}{dn} > 0$. In the Cournot oligopoly setting, quantity monotonically increases with n if conjectures adjust to be consistent.

[5] This corresponds to the negative consistent conjecture $\delta = \frac{d(\sum_{j\neq i} x_j)}{dx_i} = r_i - 1$ for negatively sloped demand functions in Perry (1982).

If conjectures are $r = 0$, then $X = \frac{na}{c-nb}$. Again, if x_i are quantity choices and $P = a+bX$ is the demand function, then this is the perfectly competitive outcome arising from price-taking behavior. Since marginal cost is cx_i, the individual supply is $P = cx_i$, or $x_i = P/c$. The total supply is $X_S = \frac{n}{c}P$ and the inverse supply is $P = \frac{c}{n}X_S$. Equating demand and supply gives $\frac{c}{n}X = a + bX$, or $X = \frac{na}{c-nb}$. Thus zero conjectures about the aggregate, which characterize aggregate-taking behavior, mean competitive (Walrasian) equilibrium for any number n of firms in this setting.

Proposition 2 shows that only a consistent conjecture r_C can be evolutionarily stable in an infinite population of players randomly matched to play the n-player game. To check whether the consistent conjecture is indeed evolutionarily stable in the current setting with $b < 0$, consider the maximization problem

$$\max_{r_1} \pi_1 = ax_1^*(r_1, r_{-1}) - \frac{c}{2}(x_1^*(r_1, r_{-1}))^2 + bx_1^*(r_1, r_{-1})X^*(r_1, r_{-1}), \tag{7}$$

where the conjectures of $n-1$ players other than Player 1 are $r_{-1} = (r_C, \ldots, r_C)$.

The equilibrium values of X^* and x_1^* are found from the conditions $a - cx_1^* + br_1x_1^* + bX^* = 0$ and $n-1$ conditions $a - cx_i^* + bx_i^*r_C + bX^* = 0$ for $i \neq 1$. The latter conditions add up to $(n-1)a - c(X^* - x_1^*) + br_C(X^* - x_1^*) - (n-1)bX^* = 0$, or $(n-1)a - (c - br_C - (n-1)b)X^* + (c - br_C)x_1^* = 0$. Thus,

$$\begin{array}{rcrcrcl} b\dfrac{dX^*}{dr_1} & + & -(c-br_1)\dfrac{dx_1^*}{dr_1} & + & bx_1^* & = & 0 \\ -(c - br_C - (n-1)b)\dfrac{dX^*}{dr_1} & + & (c-br_C)\dfrac{dx_1^*}{dr_1} & & & = & 0 \end{array}$$

Let $A = b(c - br_C) - (c - br_1)(c - br_C - (n-1)b)$. Then $\frac{dX^*}{dr_1} = \frac{1}{A}(-bx_1^*)(c - br_C)$ and $\frac{dx_1^*}{dr_1} = \frac{1}{A}(-bx_1^*)(c - br_C - (n-1)b)$.

Maximizing function (7), the first order condition is $\frac{d\pi_1}{dr_1} = a\frac{dx_1^*}{dr_1} - cx_1^*\frac{dx_1^*}{dr_1} + b(X^*\frac{dx_1^*}{dr_1} + x_1^*\frac{dX^*}{dr_1}) = 0$, or $\frac{d\pi_1}{dr_1} = bx_1^*\left(\frac{dX^*}{dr_1} - r_1\frac{dx_1^*}{dr_1}\right) = 0$. If $x_1^* \neq 0$, the only solution of this equation is if $r_1(c - br_C - (n-1)b) = c - br_C$, which means that $r_1 = r_C$.

The second-order condition for maximization is $\frac{d^2\pi}{dr_1^2} = b\frac{dx_1^*}{dr_1}(\frac{dX^*}{dr_1} - r_1\frac{dx_1^*}{dr_1}) + bx_1^*(\frac{d^2X^*}{dr_1^2} - \frac{dx_1^*}{dr_1} - r_1\frac{d^2x_1^*}{dr_1^2})$. At $r_1 = r_C$, the first term is zero. Also, since $\frac{d^2X^*}{dr_1^2} = -b(c - br_C)\frac{d(x_1^*/A)}{dr_1}$ and $\frac{d^2x_1^*}{dr_1^2} = -b(c - br_C - (n-1)b)\frac{d(x_1^*/A}{dr_1}$, $\frac{d^2X^*}{dr_1^2} - r_1\frac{d^2x_1^*}{dr_1^2} = 0$ at $r_1 = r_C$ because $c - br_C = r_C(c - br_C - (n-1)b)$. Thus $\frac{d^2\pi}{dr_1^2} = -bx_1^*\frac{dx_1^*}{dr_1}$. Since at $r_1 = r_C$, $\frac{dx_1^*}{dr_1} = \frac{-bx_1^*}{2br_C - c} < 0$, the second-order condition is satisfied.

For finite populations, only the conjecture $r = 0$ can be evolutionarily stable. Consider the maximization problem

$$\max_{r_1}(a + bX^*(r_1, r_{-1}))(x_1^*(r_1, r_{-1}) - x_n^*(r_1, r_{-1})) - \frac{c}{2}((x_1^*(r_1, r_{-1}))^2 - (x_n^*(r_1, r_{-1}))^2) \quad (8)$$

where the conjecture of $n-1$ firms is $r_{-1} = (0, \ldots, 0)$.

The equilibrium values of X^*, x_1^* and x_i^* are found from the conditions $a + bX^* - (c - br_1)x_1^* = 0$ and $n-1$ conditions $a + bX^* - cx_i^* = 0$ for $i \neq 1$. The last conditions add up to $(n-1)a + (n-1)bX^* - c(X^* - x_1^*) = 0$, or $(n-1)a - (c - (n-1)b)X^* + cx_1^* = 0$. From these two conditions,

$$\begin{array}{rcccccl} b\dfrac{dX^*}{dr_1} & + & -(c - br_1)\dfrac{dx_1^*}{dr_1} & + & bx_1^* & = & 0 \\ -(c-(n-1)b)\dfrac{dX^*}{dr_1} & + & c\dfrac{dx_1^*}{dr_1} & & & = & 0 \end{array}$$

Let $A = bc - (c - br_1)(c - (n-1)b)$. Since $r_1 > \frac{c}{2b} > \frac{c}{b}$, $A < 0$. Then $\frac{dX^*}{dr_1} = \frac{1}{A}(-bx_1^*)c < 0$ and $\frac{dx_1^*}{dr_1} = \frac{1}{A}(-bq_1^*)(c-(n-1)b) < 0$ at interior q_1^*. Note also that since $x_n^* = \frac{1}{n-1}(X^* - x_1^*)$, $\frac{dx_n^*}{dr_1} = \frac{1}{A}(-b^2x_1^*) > 0$.

Maximizing function (8), the first order condition is $(a + bX^*)(\frac{dx_1^*}{dr_1} - \frac{dx_n^*}{dr_1}) + b\frac{dX^*}{dr_1}(x_1^* - x_n^*) - cx_1^*\frac{dx_1^*}{dr_1} + cx_n^*\frac{dx_n^*}{dr_1} = 0$. From the equilibrium equations, $a + bX^* - cx_1^* = -br_1x_1^*$ and $a + bX^* - cx_n^* = 0$ thus the condition becomes $b\frac{dX^*}{dr_1}(x_1^* - x_n^*) - bx_1^*r_1\frac{dx_1^*}{dr_1} = 0$. If $r_1 > 0$, then $q_1^* < q_n^*$ and thus the first term on the left-hand side is negative, and so is the second term. Similarly, if $r_1 < 0$, then $q_1^* > q_n^*$ and thus the first term is positive, and so is the second term. Thus the only solution of the first order condition is $r_1 = 0$. Note also that the sign of the left-hand side for $r_1 < 0$ and for $r_1 > 0$ imply that the function is indeed maximized at $r_1 = 0$.

The results can be summarized as

Proposition 4 *In the linear-quadratic game of this section with $b < 0$, which includes Cournot oligopoly,*

(i) the consistent conjecture $r_C \in (0,1)$ satisfying $r_C = \frac{c - br_C}{c - br_C - (n-1)b}$ is the unique evolutionarily stable conjecture for infinite population;

(ii) the aggregate-taking conjecture $r = 0$ is the unique finite population evolutionarily stable conjecture.

4.2 Rent-seeking game

Consider the following rent-seeking game. Players choose expenditures $x_i \in \mathbb{R}_+$. The probability that Player i wins a prize of value $V > 0$ is $\frac{x_i}{\sum_{j=1}^n x_j}$. The payoff of Player i is

$$u_i(x_i, X) = \frac{x_i}{X}V - x_i,$$

where $X = x_1 + \ldots + x_n$ is the aggregate. If $X = 0$, let $u_i = 1/n$.

An interior solution of the maximization problem of Player i with conjecture r_i is characterized by $\frac{1}{X}V - 1 - \frac{x_i}{X^2}V \cdot B\left[\frac{dX}{dx_i}\right] = 0$, or

$$\frac{X - x_i r_i}{X^2}V - 1 = 0.$$

The second order condition $-\frac{(X-x_i r_i)2Xr_i}{X^4}V < 0$ is satisfied locally at the solution of the first-order condition for $r_i > 0$. The first-order condition can be rewritten as $XV - x_i r_i V - X^2 = 0$. If $X_{-i} = X - x_i$, the equation becomes $-x_i^2 + (V - r_i V - 2X_{-i})x_i + X_{-i}(V - X_{-i}) = 0$. If $V - X_{-i} > 0$, then there is unique root of this equation on $\mathbb{R}_+$, therefore the solutions of the first-order condition is an optimal choice for Player i.

If $r_i = 0$, then the first order condition becomes $V/X - 1 = 0$, or $x_i^* = V - X_{-i}$. If $x_i < x_i^*$, then the left-hand side of the first order condition equation is positive, and if $x_i > x_i^*$, the left-hand side is negative. Thus the solution of the first order condition is an optimal choice in this case as well.

Consider the $n-1$ players other than Player 1 having conjecture r. Adding up the $n-1$ first order conditions, $(n-1)XV - rV(X - x_1) - (n-1)X^2 = 0$. Then $\frac{dX}{dx_1} = -\frac{rV}{(n-1)V - rV - 2(n-1)X}$.

Suppose that all players have the same conjecture. Adding up the n first-order conditions leads to $nXV - rXV - nX^2 = 0$, or $X = \frac{n-r}{n}V$ (assuming $r < n$ to have the interior X). The consistency condition for r is $r = -\frac{rV}{(n-1)V - rV - 2(n-1)(n-r)V/n}$, or

$$r = -\frac{r}{n - 1 - r - 2(n-1)\frac{n-r}{n}}.$$

The condition further simplifies to $n - 1 - r - 2(n-1)\frac{n-r}{n} = -1$, or $(n-r)(2-n) = 0$.

If $n = 2$, then the condition is satisfied for any r and thus any conjecture r is consistent.[6] In Possajennikov (2009) it is shown that any $r \in (0, 2)$ is then evolutionarily stable. If $n \neq 2$,

[6] Perry (1982) notes this result in the equivalent (in terms of payoffs) Cournot duopoly game with unit-iso-elastic inverse demand function $P(Q) = aQ^{-1}$ and constant marginal cost.

then only $r = n$ can be consistent. However, then $X = 0$ which is not an interior solution and thus the analysis is not applicable to it since consistency is not well defined at the boundary. Nevertheless, it can be said that no conjecture leading to an interior solution can be evolutionarily stable for infinite population because no such conjecture is consistent.

To analyze the evolutionary stability in finite population, recall that only conjecture $r = 0$ can be evolutionarily stable. Consider the maximization problem

$$\max_{r_1} \frac{x_1^*(r_1, r_{-1})}{X^*(r_1, r_{-1})} V - x_1^*(r_1, r_{-1}) - \frac{x_n^*(r_1, r_{-1})}{X^*(r_1, r_{-1})} V - x_n^*(r_1, r_{-1}) \tag{9}$$

where the conjectures of $n-1$ players $r_{-1} = (0, \ldots, 0)$.

The equilibrium values of X^*, x_1^* and x_i^* are found from the conditions $X^*V - x_1^* r_1 V - (X^*)^2 = 0$ and $n-1$ conditions $X^*V - (X^*)^2 = 0$ for $i \neq 1$. From the latter conditions $X^* = V$ as the unique interior solution. Then the first condition becomes $-x_1^* r_1 V = 0$.

If $r_1 > 0$, then the equilibrium $x_1^* = 0$ and Player 1 gets 0 payoff. If $r_1 = 0$, then the equilibrium x_1^* can be positive but since $X^* = V$, players with positive x_j also have 0 payoff. Therefore $r = (0, \ldots, 0)$, although not evolutionarily stable, can be considered *neutrally stable*: a player with any other conjecture $r_i > 0$ does not get a higher payoff than the player with $r_j = 0$.

Proposition 5 *In the symmetric rent-seeking game with probability of winning* $\frac{x_i}{X}$*,*

(i) if $n = 2$*, then any conjecture* $r \in (0, n)$ *is consistent and evolutionarily stable for infinite population; if* $n > 2$*, then no conjecture leading to an interior solution is consistent or evolutionarily stable;*

(ii) For finite population, no conjecture is evolutionarily stable; the aggregate-taking conjecture $r = 0$ *is the unique neutrally stable conjecture for finite population.*

5 Conclusion

In this paper I consider conjectural variations for aggregative games as beliefs about the change in the aggregate in response to a (marginal) change in a player's strategy. I show that if players are endowed with such conjectures, play an equilibrium of the game with these conjectures and obtain some payoff in this equilibrium, then only consistent conjectures can be evolutionarily stable in infinite population. This means that a player with a different conjecture would get a lower payoff than a player with a consistent conjecture if players are randomly matched in groups of n to play the game. An alternative interpretation is that if the choice of conjecture was a conscious decision, then choosing a consistent conjecture is a best response against any conjectures of the other players.

In a finite population, zero conjectures are evolutionarily stable. Zero conjecture here means that a player does not expect the aggregate to change in response to a change in the player's own strategy. Such a conjecture commits the player to choose a behavior that maximizes the difference between the player's payoffs and the payoff of any other player (in a symmetric game with the other players using the same strategy, it does not matter which other player is considered). This is because in this situation the influence on player's own payoff and on another player's payoff through the aggregate cancel out, thus if the player is concerned about relative payoffs, the change in the aggregate can be ignored.

The results are illustrated on the examples of aggregative game settings that include Cournot oligopoly and rent-seeking games. In the linear-quadratic Cournot oligopoly the consistent conjecture leads to more production than the Nash equilibrium conjecture, which implies zero response of the other players. This consistent conjecture is shown to be evolutionarily stable. The zero conjecture leading to the aggregate-taking behavior (which means price-taking behavior in this context) is finite population evolutionarily stable. In rent-seeking games no conjecture leading to well-defined interior behavior is consistent if the number of players $n > 2$, while if $n = 2$ any conjecture $r \in (0, n)$ is consistent and evolutionarily stable. Aggregate-taking behavior implied by the finite population (neutrally) evolutionarily stable conjecture lead to full dissipation of the rent for any n.

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