Price-Independent Welfare Prescriptions and Population Size*

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Abstract

This note investigates the extension of Roberts' price-independent welfare prescriptions to alternatives in which population size and composition can vary. We show that ethically unsatisfactory orderings result. Suppose that a single person is to be added to a population that is unaffected in utility terms. Either all such additions must be regarded as bad or some expansions in which the added person's life is not worth living must be ranked as social improvements.

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Price-Independent Welfare Prescriptions and Population Size

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This note investigates the extension of price-independent welfare prescriptions (Roberts [1980a]) to different populations sizes. Price independence requires that there exist a single ordering of nominal incomes that is consistent with a Bergson-Samuelson social-welfare ordering for each price vector. Such orderings can be used to justify inequality measures based on nominal incomes and are also needed for consistency of cost-benefit tests (Blackorby and Donaldson [1984a], Roberts [1980a]). Cost-benefit analysis must often deal with projects that change the number of people in the population. This is most obviously the case for population-control projects, but population changes are also the result of public-health and infrastructure projects.

Given welfarism, policies can be evaluated by means of the corresponding *alternatives*. An alternative A is described by a set of people who are alive and a corresponding vector of utilities, and we write

$$A = \Big(N, \{u^i\}_{i \in N}\Big) \tag{1}$$

where N is the set of named individuals alive with $|N|$ finite and u^i is person i's utility level for all $i \in N$. To allow for the largest class of Bergson-Samuelson social-welfare orderings, we assume that utilities are numerically measurable and fully interpersonally comparable.[1] $\mathcal{A}$ is the set of all possible alternatives,[2] and R is an ordering (a reflexive, complete, and transitive binary relation) of $\mathcal{A}$. $\bar{A}\, R\, \hat{A}$ means that alternative $\bar{A}$ is socially at least as good as alternative $\hat{A}$, and strict preference (P) and indifference (I) are defined in the usual way. We assume that R satisfies the *strong Pareto* condition.

Strong Pareto: *For all $\bar{A}, \hat{A} \in \mathcal{A}$ such that $\bar{N} = \hat{N} = N$:*

(i) if $\bar{u}^i = \hat{u}^i$ for all $i \in N$, then $\bar{A}\, I\, \hat{A}$;

(ii) if $\bar{u}^i \geq \hat{u}^i$ for all $i \in N$ with at least one strict inequality, then $\bar{A}\, P\, \hat{A}$.

[1] See Blackorby, Donaldson, and Weymark [1984], Bossert [1991, 1997], d'Aspremont and Gevers [1977], Roberts [1980b,c], and Sen [1974, 1977] for discussions of information assumptions in fixed-population social choice. Blackorby, Bossert, and Donaldson [1996] examine information invariance in a variable-population framework.

[2] $\mathcal{A}$ may or may not contain the *null alternative* in which is no one is alive. Whether or not this alternative is included has no consequences on our results.

$\mathcal{R}$ ($\mathcal{R}_+$, $\mathcal{R}_{++}$) is the set of (non-negative, positive) real numbers, and $\mathcal{Z}_{++}$ is the set of positive integers. Person i's direct utility function is $U^i: \mathcal{R}^m_+ \longmapsto \mathcal{R}$ and his or her indirect utility function is $V^i: \mathcal{R}^m_{++} \times \mathcal{R}_+ \longmapsto \mathcal{R}$, where $m \in \mathcal{Z}_{++}$ and $m \geq 2$. That is,

$$V^i(\mathbf{p}, y^i) = \max_{\mathbf{x}^i} \left\{ U(\mathbf{x}^i) \mid \mathbf{p} \cdot \mathbf{x}^i \leq y^i \right\} \tag{2}$$

where $\mathbf{x}^i \in \mathcal{R}^m_+$ is person i's consumption vector, $\mathbf{p} \in \mathcal{R}^m_{++}$ is a price vector (faced by all individuals who are alive)[3] and y^i is person i's income, wealth, or lifetime consumption. We assume that each U^i is such that a solution to the maximization problem in (2) exists for all $\mathbf{p} \in \mathcal{R}^m_{++}$ and all $y^i \in \mathcal{R}_+$. Furthermore, U^i is assumed to be monotonic which guarantees that V^i is increasing in y^i.

Following standard practice in the population-ethics literature, we introduce the utility level that represents *neutrality,* and call it u_ν.[4] If utility is above neutrality, life is worth living, and if it is below, life is not worth living—a rational, fully informed, self-interested person would prefer not to have his or her experiences. We assume that neutrality requires a *positive* level of income for every price vector[5] and that such a level of income exists. That is, for each $i \in \mathcal{Z}_{++}$ and each $\mathbf{p} \in \mathcal{R}^m_{++}$, there exists $s^i \in \mathcal{R}_{++}$ with

$$V^i(\mathbf{p}, s^i) = u_\nu. \tag{3}$$

Consequently,

$$V^i(\mathbf{p}, y^i) < u_\nu \text{ for all } y^i < s^i \tag{4}$$

because U^i is monotonic and, therefore, V^i is increasing in y^i.

Income-alternatives correspond to utility alternatives and we write

$$A_y = \left(N, \left\{y^i\right\}_{i \in N}\right). \tag{5}$$

$\mathcal{A}_y$ is the set of all possible income-alternatives. Price-independent welfarism requires that there exist an ordering R_y of $\mathcal{A}_y$ such that

$$\left(\bar{N}, \left\{\bar{y}^i\right\}_{i \in \bar{N}}\right) R_y \left(\hat{N}, \left\{\hat{y}^i\right\}_{i \in \hat{N}}\right) \iff \left(\bar{N}, \left\{V^i(\mathbf{p}, \bar{y}^i)\right\}_{i \in \bar{N}}\right) R \left(\hat{N}, \left\{V^i(\mathbf{p}, \hat{y}^i)\right\}_{i \in \hat{N}}\right) \tag{6}$$

for all $\bar{A}_y$, $\hat{A}_y \in \mathcal{A}_y$ and all $\mathbf{p} \in \mathcal{R}^m_{++}$. Because R satisfies strong Pareto, price independence implies that R_y must satisfy the analogous condition applied to incomes.

[3] See Slivinsky [1983] for a discussion of fixed-population price independence in situations where different individuals may face different prices.

[4] See Broome [1993]. Neutrality is usually normalized to zero. For an introduction to population ethics, see Blackorby, Bossert, and Donaldson [1995, 1997a,b], Blackorby and Donaldson [1984b], Bossert [1990], Broome [1992], Hammond [1988], Heyd [1992], Hurka [1982, 1983], McMahan [1981], Narveson [1967], Parfit [1976, 1982, 1984], and Sen [1991].

[5] The alternative to our assumption would imply the implausible claim that any positive level of consumption, no matter how small, would make life worth living.

Lemma 1 shows that the ordering R_y satisfies *extended homotheticity:* common scaling of the incomes in any two income-alternatives preserves their ranking. This is a straightforward generalization of Roberts' [1980a] corresponding fixed-population result, stated in his Proposition 3.

Lemma 1: *Price-independent welfarism implies that R_y satisfies extended homotheticity. That is,*

$$\left(\bar{N}, \{\bar{y}^i\}_{i\in\bar{N}}\right) R_y \left(\hat{N}, \{\hat{y}^i\}_{i\in\hat{N}}\right) \iff \left(\bar{N}, \{\lambda\bar{y}^i\}_{i\in\bar{N}}\right) R_y \left(\hat{N}, \{\lambda\hat{y}^i\}_{i\in\hat{N}}\right) \tag{7}$$

for all $\bar{A}_y$, $\hat{A}_y \in \mathcal{A}_y$ and all $\lambda \in \mathcal{R}_{++}$.

Proof: For any $\bar{A}_y$, $\hat{A}_y \in \mathcal{A}_y$, $\lambda \in \mathcal{R}_{++}$,

$$\begin{aligned}
&\left(\bar{N}, \{\bar{y}^i\}_{i\in\bar{N}}\right) R_y \left(\hat{N}, \{\hat{y}^i\}_{i\in\hat{N}}\right)\\
\iff &\left(\bar{N}, \{V^i(\mathbf{p}, \bar{y}^i)\}_{i\in\bar{N}}\right) R \left(\hat{N}, \{V^i(\mathbf{p}, \hat{y}^i)\}_{i\in\hat{N}}\right)\\
\iff &\left(\bar{N}, \{V^i(\lambda\mathbf{p}, \lambda\bar{y}^i)\}_{i\in\bar{N}}\right) R \left(\hat{N}, \{V^i(\lambda\mathbf{p}, \lambda\hat{y}^i)\}_{i\in\hat{N}}\right)\\
\iff &\left(\bar{N}, \{\lambda\bar{y}^i\}_{i\in\bar{N}}\right) R_y \left(\hat{N}, \{\lambda\hat{y}^i\}_{i\in\hat{N}}\right),
\end{aligned} \tag{8}$$

where the third line of (8) follows from homogeneity of degree zero of the V^i and the second and fourth lines follow from price-independent welfarism.

■

Suppose that a single individual is to be added to an alternative without affecting the utilities of any of the existing people. A critical level of utility for alternative A and individual $j \notin N$ is the utility level which, if experienced by j, would make the two alternatives equally good. If the critical level is u_c, then

$$\left(N \cup \{j\}, \left(\{u^i\}_{i\in N}, u_c\right)\right) I\ A. \tag{9}$$

If u_c exists, it must be unique because of strong Pareto, and it may depend on any or all of the identities of the existing people (N), the utility levels of the existing people ($\{u^i\}_{i\in N}$), and the identity of the added person (j).[6]

If a person-j critical level of utility exists for A, then price independence implies

$$\left(N \cup \{j\}, \left(\{y^i\}_{i\in N}, y_c\right)\right) I_y\ A_y, \tag{10}$$

[6] Except for Theorem 2, strong Pareto is only used to guarantee the uniqueness of critical levels.

where $V^i(\mathbf{p}, y^i) = u^i$ for all $i \in N$, and $V^j(\mathbf{p}, y_c) = u_c$ for all $\mathbf{p} \in \mathcal{R}^m_{++}$. y_c is a person-j critical level of income for the income-alternative A_y. Because the reverse implication is also true, a critical level of utility exists for A and j if and only if a critical level of income exists for A_y and j. Both critical levels must be unique because of strong Pareto. We write the person-j critical level of income for A_y as

$$y_c = C^N_j\big(\{y^i\}_{i \in N}\big). \tag{11}$$

Lemma 1 implies that

$$\begin{aligned} &\Big(N \cup \{j\}, \Big(\big\{y^i\big\}_{i \in N}, y_c\Big)\Big) \; I_y \; \Big(N, \Big(\big\{y^i\big\}_{i \in N}\Big)\Big) \\ \iff &\Big(N \cup \{j\}, \Big(\big\{\lambda y^i\big\}_{i \in N}, \lambda y_c\Big)\Big) \; I_y \; \Big(N, \Big(\big\{\lambda y^i\big\}_{i \in N}\Big)\Big) \end{aligned} \tag{12}$$

for all $\lambda \in \mathcal{R}_{++}$. This implies that if the person-j critical level of income exists for an income-alternative, it exists when incomes are scaled; in addition, the critical level is scaled by the same factor. This observation is sufficient for Corollary 1.

Corollary 1: *If price-independent welfarism is satisfied and the person-j critical level of income $C^N_j\big(\{y^i\}_{i \in N}\big)$ exists for $A_y = \big(N, \big(\{y^i\}_{i \in N}\big)\big)$ and for $j \notin N$, then, for all $\lambda \in \mathcal{R}_{++}$, the person-$j$ critical level exists for the alternative $\big(N, \big(\{\lambda y^i\}_{i \in N}\big)\big)$, and*

$$C^N_j\big(\{\lambda y^i\}_{i \in N}\big) = \lambda C^N_j\big(\{y^i\}_{i \in N}\big). \tag{13}$$

If all critical levels exist, the functions $\{C^N_j\}$ are defined for all $\{y^i\}_{i \in N}$, and are homogeneous of degree one. Theorem 1 demonstrates that, in this case, there must be critical levels of utility that are below neutrality.

Theorem 1: *If price-independent welfarism holds and all critical levels exist, there exist critical levels of utility that are below neutrality.*

Proof: For any alternative $A_y \in \mathcal{A}_y$, let $y_c = C^N_j\big(\{y^i\}_{i \in N}\big)$ be the person-j critical level, where $j \notin N$. Now consider the alternative $A^\lambda_y = \big(N, \big(\{\lambda y^i\}_{i \in N}\big)\big)$. Its critical level is λy_c by Corollary 1. Consequently, for any $\mathbf{p} \in \mathcal{R}^m_{++}$, the person-$j$ critical level of utility for the alternative $A^\lambda = \big(N, \big(\{V^i(\mathbf{p}, \lambda y^i)\}_{i \in N}\big)\big)$ must be $V^j(\mathbf{p}, \lambda y_c)$. λ can be chosen so that λy_c is arbitrarily close to zero, with

$$V^j(\mathbf{p}, \lambda y_c) < u_\nu, \tag{14}$$

and the critical level is below neutrality.

■

An example of a price-independent welfare prescription is given by the utility functions $u^i = V^i(\mathbf{p}, y^i) = a(\mathbf{p})y^i$ for all $i \in \mathcal{Z}_{++}$, and a principle that requires

$$\bar{A}\; R\; \hat{A} \iff \sum_{i \in \bar{N}} \bar{u}^i \geq \sum_{i \in \hat{N}} \hat{u}^i, \tag{15}$$

so that

$$\bar{A}_y\; R_y\; \hat{A}_y \iff \sum_{i \in \bar{N}} \bar{y}^i \geq \sum_{i \in \hat{N}} \hat{y}^i. \tag{16}$$

This principle is not the same as classical utilitarianism, which uses the value function of (15) only when utilities are normalized so that zero represents neutrality. In this example, all critical levels of income and utility are zero and neutrality is above zero. Consequently, all critical levels are below neutrality.

Another example is provided by

$$\bar{A}\; R\; \hat{A} \iff \frac{1}{|\bar{N}|} \sum_{i \in \bar{N}} \bar{u}^i \geq \frac{1}{|\hat{N}|} \sum_{i \in \hat{N}} \hat{u}^i \tag{17}$$

and by the utility functions used in the first example. This is the average-utilitarian principle and

$$\bar{A}_y\; R_y\; \hat{A}_y \iff \frac{1}{|\bar{N}|} \sum_{i \in \bar{N}} \bar{y}^i \geq \frac{1}{|\hat{N}|} \sum_{i \in \hat{N}} \hat{y}^i. \tag{18}$$

In this example, critical levels are average utility and average income. Consequently, for low enough levels of average utility, critical levels of utility are below neutrality.

Although critical levels exist for all commonly used population principles, it is possible that critical levels may not exist for some or all alternatives and added individuals. Given strong Pareto, three possibilities exist. Defining

$$A_+ = \Big(N \cup \{j\}, \big(\{u^i\}_{i \in N}, u^j\big)\Big), \tag{19}$$

the first case has $A_+ \, P \, A$ for all u^j in the image of U^j—the expanded population is always regarded as better. This is the strong pro-natalist position and it favours the creation of people whose lives would be below neutrality. The second case is strongly anti-natalist, with $A \, P \, A_+$ for all possible u^j. The third case is slightly more complex. It allows $A_+ \, P \, A$ for some values of u^j and $A \, P \, A_+$ for the rest. Because of strong Pareto, all values of u^j for which $A_+ \, P \, A$ must be greater than all the values for which $A \, P \, A_+$.

The first is clearly unsatisfactory: it regards the *ceteris paribus* addition of people below neutrality as good. The second regards population expansion as bad no matter how well off the added person is.

Price-independent welfarism has implications even when critical levels do not exist. Theorem 2 indicates that price-independent welfare prescriptions must either be strongly anti-natalist or must regard the addition of some person below neutrality to a utility-unaffected population as a social improvement.

Theorem 2: *If price-independent welfarism holds, then either*

(i) there exist an alternative $A \in \mathcal{A}$, an individual $j \notin N$, and a utility level u^j such that u^j is below neutrality ($u^j < u_\nu$) and

$$\left(N \cup \{j\}, \left(\left\{u^i\right\}_{i \in N}, u^j\right)\right) P\ A \tag{20}$$

or

(ii) for all $A \in \mathcal{A}$, all $j \notin N$, and all u^j,

$$A\ P \left(N \cup \{j\}, \left(\left\{u^i\right\}_{i \in N}, u^j\right)\right). \tag{21}$$

Proof: Clearly, (i) and (ii) are mutually exclusive. Suppose that (ii) does not hold. Then there exist $A \in \mathcal{A}$, $j \notin N$, $\mathbf{p} \in \mathcal{R}^m_{++}$, $\{y^i\}_{i \in N}$, $\overset{\circ}{y}^j \in \mathcal{R}_+$ such that

$$\left(N \cup \{j\}, \left(\left\{V^i(\mathbf{p}, y^i)\right\}_{i \in N}, V^j(\mathbf{p}, \overset{\circ}{y}^j)\right)\right) R \left(N, \left(\left\{V^i(\mathbf{p}, y^i)\right\}_{i \in N}\right)\right). \tag{22}$$

By monotonicity and strong Pareto,

$$\left(N \cup \{j\}, \left(\left\{V^i(\mathbf{p}, y^i)\right\}_{i \in N}, V^j(\mathbf{p}, y^j)\right)\right) P \left(N, \left(\left\{V^i(\mathbf{p}, y^i)\right\}_{i \in N}\right)\right) \tag{23}$$

for all $y^j > \overset{\circ}{y}^j$. Let y^j be any income level with that property. Consequently, by price-independent welfarism,

$$\left(N \cup \{j\}, \left(\left\{y^i\right\}_{i \in N}, y^j\right)\right) P_y \left(N, \left(\left\{y^i\right\}_{i \in N}\right)\right). \tag{24}$$

By extended homotheticity (see Lemma 1),

$$\left(N \cup \{j\}, \left(\left\{\lambda y^i\right\}_{i \in N}, \lambda y^j\right)\right) P_y \left(N, \left(\left\{\lambda y^i\right\}_{i \in N}\right)\right) \tag{25}$$

for all $\lambda \in \mathcal{R}_{++}$. Using price-independent welfarism again,

$$\left(N \cup \{j\}, \left(\left\{V^i(\mathbf{p}, \lambda y^i)\right\}_{i \in N}, V^j(\mathbf{p}, \lambda y^j)\right)\right) P \left(N, \left(\left\{V^i(\mathbf{p}, \lambda y^i)\right\}_{i \in N}\right)\right) \tag{26}$$

for all $\mathbf{p} \in \mathcal{R}^m_{++}$ and all $\lambda \in \mathcal{R}_{++}$. λ can be chosen to be small enough so that

$$V^j(\mathbf{p}, \lambda y^j) < u_\nu, \tag{27}$$

and the theorem is proved.

■

The results of our theorems lead us to conclude that sensible price-independent welfare prescriptions are not possible when population size and composition can change across alternatives. If critical levels exist, some of them must be below neutrality and the principles must judge the *ceteris paribus* addition of at least some people whose lives are not worth living to be a good thing. If critical levels do not exist, either the principle must possess the same ethically unsatisfactory property or *all* additions to a utility-unaffected population must be regarded as bad.

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