Intersection Quasi-Orderings: An Alternative Proof*

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Abstract. This note provides an alternative proof of a known result on the existence of collections of orderings generating intersection quasi-orderings. Instead of proving the result from first principles, it is illustrated how it can be obtained by making use of an analogous relationship between partial orders and linear orders. *Journal of Economic Literature* Classification Nos.: C60, D11, D71.

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1 Introduction

In a recent paper, Donaldson and Weymark (1998) show that every quasi-ordering is the intersection of a collection of orderings (see also Duggan, 1997). This note provides an alternative proof of this result. Whereas the original proof proceeds by establishing the theorem directly and without reference to a related result for partial orders, the proof presented here makes use of the corresponding observation for partial orders and linear orders due to Dushnik and Miller (1941). As a consequence, only a few elementary additional steps are required in the proof. This methodology is also used in Richter's (1966) theorem on the existence of an ordering that rationalizes a choice correspondence satisfying the congruence axiom and in Fishburn's (1973) proof of a result by Hansson (1968). Hansson (1968) shows that a quasi-ordering can be extended to an ordering, and Fishburn (1973) provides a proof based on Szpilrajn's (1930) corresponding theorem on the extension of partial orders to linear orders.

In addition to providing an alternative proof of the specific theorem mentioned above, the major purpose of this note is to illustrate the general use of the methodology employed in relating properties of partial orders to analogous properties of quasi-orderings.

2 Definitions

Let M be a nonempty set. A binary relation B on M is a subset of $M \times M$. The relation B is

$$\begin{aligned}
\text{reflexive} &\Leftrightarrow (a,a) \in B \text{ for all } a \in M;\\
\text{asymmetric} &\Leftrightarrow [(a,b) \in B \Rightarrow (b,a) \notin B] \text{ for all } a,b \in M;\\
\text{transitive} &\Leftrightarrow [(a,b) \in B \text{ and } (b,c) \in B \Rightarrow (a,c) \in B] \text{ for all } a,b,c \in M;\\
\text{complete} &\Leftrightarrow [(a,b) \in B \text{ or } (b,a) \in B] \text{ for all } a,b \in M \text{ such that } a \neq b.
\end{aligned}$$

A *partial order* is an asymmetric and transitive binary relation, and a *linear order* is a complete partial order. A *quasi-ordering* is a reflexive and transitive binary relation, and an *ordering* is a complete quasi-ordering. The asymmetric factor A and the symmetric factor S of a binary relation B are defined by letting, for all $a, b \in M$,

$$\begin{aligned}
(a,b) \in A &\Leftrightarrow (a,b) \in B \text{ and } (b,a) \notin B;\\
(a,b) \in S &\Leftrightarrow (a,b) \in B \text{ and } (b,a) \in B.
\end{aligned}$$

A linear order B^* is a *linear extension* of a partial order B if and only if $B \subseteq B^*$. Analogously, an ordering B^* is an *ordering extension* of a quasi-ordering B if and only if $B \subseteq B^*$ and $A \subseteq A^*$.

3 Ordering Extensions

The following theorem regarding the extension of partial orders to linear orders is proven in Szpilrajn (1930).

Theorem 1 (Szpilrajn, 1930) *Every partial order has a linear extension.*

Hansson (1968) proves an analogous result for ordering extensions of quasi-orderings. While Hansson's proof proceeds directly and without reference to Szpilrajn's result, Fishburn's (1973, pp. 198–199) proof utilizes Theorem 1. As a consequence of employing this alternative proof technique, given the corresponding result for partial orders, the remainder of the proof can be obtained using elementary methods. Because parts of the proof of Theorem 2 are used in the following section, a proof analogous to Fishburn's is given explicitly.

Theorem 2 (Hansson, 1968; Fishburn, 1973) *Every quasi-ordering has an ordering extension.*

Proof. Let X be a nonempty set, and let R be a quasi-ordering on X. Let P and I denote the asymmetric and symmetric factors of R, respectively. For all $x \in X$, let $\mathcal{I}(x) = \{y \in X \mid (x, y) \in I\}$ denote the indifference class of R to which x belongs. The proof of the following lemma is straightforward and is thus omitted.

Lemma 1 *For all $x, y \in X$,*

$$\mathcal{I}(x) = \mathcal{I}(y) \;\Leftrightarrow\; (x, y) \in I.$$

Let $\mathcal{X} = \{\mathcal{I}(x) \mid x \in X\}$ be the set of all indifference classes of R. $\mathcal{X}$ is nonempty because X is nonempty and R is reflexive. Now define the relation $\mathbf{P}$ on $\mathcal{X}$ as follows. For all $\mathbf{x}, \mathbf{y} \in \mathcal{X}$,

$$(\mathbf{x}, \mathbf{y}) \in \mathbf{P} \;\Leftrightarrow\; \text{there exists } (x, y) \in P \text{ such that } [\mathbf{x} = \mathcal{I}(x) \text{ and } \mathbf{y} = \mathcal{I}(y)]. \tag{1}$$

We obtain

Lemma 2 *The relation* $\mathbf{P}$ *is a partial order.*

Proof of Lemma 2. To prove that $\mathbf{P}$ is asymmetric, suppose that, by way of contradiction, $(\mathbf{x}, \mathbf{y}) \in \mathbf{P}$ and $(\mathbf{y}, \mathbf{x}) \in \mathbf{P}$ for some $\mathbf{x}, \mathbf{y} \in \mathcal{X}$. Therefore, there exist $(x, y), (y', x') \in P$ such that $\mathbf{x} = \mathcal{I}(x)$, $\mathbf{y} = \mathcal{I}(y)$, $\mathbf{y} = \mathcal{I}(y')$, and $\mathbf{x} = \mathcal{I}(x')$. Hence, $\mathcal{I}(x) = \mathcal{I}(x')$ and $\mathcal{I}(y) = \mathcal{I}(y')$, and Lemma 1 implies $(x, x') \in I$ and $(y, y') \in I$. By transitivity and $(x, y) \in P$, we obtain $(x', y') \in P$, contradicting $(y', x') \in P$.

Now let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathcal{X}$ be such that $(\mathbf{x}, \mathbf{y}), (\mathbf{y}, \mathbf{z}) \in \mathbf{P}$. By definition, there exist $(x, y), (y', z) \in P$ such that $\mathbf{x} = \mathcal{I}(x)$, $\mathbf{y} = \mathcal{I}(y)$, $\mathbf{y} = \mathcal{I}(y')$, and $\mathbf{z} = \mathcal{I}(z)$. Therefore, $\mathcal{I}(y) = \mathcal{I}(y')$ and, by Lemma 1, $(y, y') \in I$. Because R is transitive, $(x, z) \in P$. Hence, there exists $(x, z) \in P$ such that $\mathbf{x} = \mathcal{I}(x)$ and $\mathbf{z} = \mathcal{I}(z)$, which implies $(\mathbf{x}, \mathbf{z}) \in \mathbf{P}$. Therefore, $\mathbf{P}$ is transitive. ■

Now the proof of Theorem 2 is continued. By Theorem 1, $\mathbf{P}$ has a linear extension $\mathbf{P}^*$. Define the relation R^* on X by letting, for all $x, y \in X$,

$$(x, y) \in R^* \;\Leftrightarrow\; (x, y) \in I \text{ or } (\mathcal{I}(x), \mathcal{I}(y)) \in \mathbf{P}^*.$$

To complete the proof, it is sufficient to prove that R^* is an ordering extension of R. That R^* is an ordering is straightforward to verify.

Let $(x, y) \in R$. If $(x, y) \in I$, $(x, y) \in R^*$ follows immediately. If $(x, y) \in P$, it follows that $(\mathcal{I}(x), \mathcal{I}(y)) \in \mathbf{P}$ and, because $\mathbf{P}^*$ is a linear extension of $\mathbf{P}$, $(\mathcal{I}(x), \mathcal{I}(y)) \in \mathbf{P}^*$. Hence, $(x, y) \in R^*$.

Finally, let $(x, y) \in P$. Therefore, $(x, y) \in R$ and, by the above argument, $(x, y) \in R^*$. Because $(x, y) \notin I$, the definition of R^* implies $(\mathcal{I}(x), \mathcal{I}(y)) \in \mathbf{P}^*$. Because $\mathbf{P}^*$ is asymmetric, $(\mathcal{I}(y), \mathcal{I}(x)) \notin \mathbf{P}^*$ and, because $(y, x) \notin I$, it follows that $(y, x) \notin R^*$. Hence, $(x, y) \in P^*$, which completes the proof. ■

4 Intersection Quasi-Orderings

Among other results, Dushnik and Miller (1941) show that every partial order is the intersection of a collection of linear orders.

Theorem 3 (Dushnik and Miller, 1941) *Every partial order is the intersection of a collection of linear orders.*

Donaldson and Weymark (1998) prove an analogous theorem for intersection quasi-orderings. Their proof is based on a result by Suzumura (1976), which shows that transitivity can be weakened in Hansson's extension theorem (Theorem 2 above). Again, an alternative proof can be given by appealing to the corresponding theorem for partial orders and establishing the existence of a collection of orderings generating a quasi-ordering in a few simple steps.

Theorem 4 (Donaldson and Weymark, 1998) *Every quasi-ordering is the intersection of a collection of orderings.*

Proof. Let X be a nonempty set, and let R be a quasi-ordering on X with asymmetric factor P and symmetric factor I. Define $\mathcal{I}(x)$ for all $x \in X$ and $\mathcal{X}$ as in the proof of Theorem 2, and let $\mathbf{P}$ be the binary relation on $\mathcal{X}$ defined by (1). By Lemma 2, $\mathbf{P}$ is a partial order. By Theorem 3, $\mathbf{P}$ is the intersection of a collection $\mathcal{P}$ of linear orders on $\mathcal{X}$. For every $\mathbf{P}^* \in \mathcal{P}$, define the relation $R^*(\mathbf{P}^*)$ on X by letting, for all $x, y \in X$,

$$(x, y) \in R^*(\mathbf{P}^*) \;\Leftrightarrow\; (x, y) \in I \text{ or } (\mathcal{I}(x), \mathcal{I}(y)) \in \mathbf{P}^*.$$

Let $\mathcal{R}$ be the collection of relations $\{R^*(\mathbf{P}^*) \mid \mathbf{P}^* \in \mathcal{P}\}$. By Theorem 2, $R^*(\mathbf{P}^*)$ is an ordering extension of R for all $\mathbf{P}^* \in \mathcal{P}$. To complete the proof, it is sufficient to show that R is the intersection of all orderings in $\mathcal{R}$.

Suppose $(x, y) \in R$. By Theorem 2, $(x, y) \in R^*(\mathbf{P}^*)$ for all $\mathbf{P}^* \in \mathcal{P}$. Therefore, $(x, y) \in \cap_{\mathbf{P}^* \in \mathcal{P}} R^*(\mathbf{P}^*)$.

Now suppose $(x, y) \in \cap_{\mathbf{P}^* \in \mathcal{P}} R^*(\mathbf{P}^*)$. This implies that, for all $\mathbf{P}^* \in \mathcal{P}$,

$$(x, y) \in I \text{ or } (\mathcal{I}(x), \mathcal{I}(y)) \in \mathbf{P}^*. \tag{2}$$

If $(x, y) \in I$, $(x, y) \in R$ follows immediately. If $(x, y) \notin I$, (2) implies $(\mathcal{I}(x), \mathcal{I}(y)) \in \mathbf{P}^*$ for all $\mathbf{P}^* \in \mathcal{P}$. Because $\mathbf{P}$ is the intersection of all orderings in $\mathcal{P}$, this implies $(\mathcal{I}(x), \mathcal{I}(y)) \in \mathbf{P}$. By definition of $\mathbf{P}$, there exists $(x', y') \in P$ such that $\mathcal{I}(x) = \mathcal{I}(x')$ and $\mathcal{I}(y) = \mathcal{I}(y')$. Lemma 1 implies $(x, x') \in I$ and $(y, y') \in I$ and, by transitivity, we obtain $(x, y) \in P$ and hence $(x, y) \in R$. ■

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