Functional Equations and Population Ethics*

by

Charles Blackorby, Walter Bossert, and David Donaldson

September 1998; revised February 1999

Charles Blackorby: University of British Columbia and GREQAM

Walter Bossert: University of Nottingham

David Donaldson: University of British Columbia

* Financial support through a grant from the Social Sciences and Humanities Research Council of Canada is gratefully acknowledged.

Abstract

This paper illustrates the application of functional-equations results in population ethics. In an intertemporal framework, we provide characterizations of several classes of variable-population social orderings that may depend on individual lengths of life in addition to lifetime utilities. The generalized associativity equation turns out to play a major role in proving the results of this paper and in mathematical approaches to population ethics in general.

AMS (1991) Subject Classification: Primary 90A08. Secondary 39B22.

1. Introduction

Solutions to functional equations have become important tools in an increasing number of problems in social sciences such as economics and psychology.[1] Traditionally, issues involving social evaluation with a variable population have been analyzed mainly in a philosophical context—specifically, they have been a subject of investigation in ethics. Recently, formal social-choice theory has been applied to population problems, and mathematical techniques have become increasingly important. In this paper, we illustrate the use of the analysis of functional equations in population ethics.

Suppose a social ordering is used to rank different states of the world that may involve different populations and different population sizes, and intertemporal aspects are explicitly taken into consideration. We assume that, for each possible state of the world, the set of individuals alive, their birth dates, lengths of life, and lifetime utilities are known. Utilities are normalized so that a utility level of zero represents a neutral life—a life that is neither better nor worse than its alternative. These normalized utilities are assumed to be interpersonally comparable and numerically significant indicators of lifetime well-being.

In Blackorby, Bossert, and Donaldson [1995], we analyzed the consequences of a separability axiom that requires social evaluations in the present to be independent of the well-being of individuals whose lives are over in the states of the world to be compared, and we call this axiom independence of the utilities of the dead. In addition, some other axioms were employed, including the strong Pareto principle which implies, in the framework considered here, that the individual lifetime utilities contain enough information to rank all possible states of the world. Applying Gorman's [1968] result on overlapping sets of separable variables, the proof of which involves the solution of a generalized associativity equation (see Aczél [1966, p. 310]), we showed that the resulting social orderings are the critical-level generalized utilitarian orderings, which were first introduced in Blackorby and Donaldson [1984]. Critical-level generalized utilitarianism compares states of the world on the basis of the sum of the differences between the transformed utilities of those alive and a transformed critical level of lifetime utility. Critical-level utilitarianism is obtained in the special case where the transformation is the identity mapping. Applying a concave transformation generates inequality-averse orderings. The critical level is a parameter which represents an important ethical value judgment—adding, *ceteris paribus,* an individual with a lifetime utility equal to the critical level to an existing population leads to a state of the world that is indifferent to the original one. The critical level turns out to be independent of the utility levels and number of those alive in the state of the world under consideration, which is—in addition to the additive separability of the criterion used for fixed-population comparisons—another important consequence of independence of the utilities of the dead.

Although the critical-level generalized utilitarian principles have a strong ethical foundation (and our own position is that the strong Pareto principle should be respected), it is useful to explore the consequences of weakening strong Pareto in order to allow birth dates and lengths of life to matter in social evaluations. By considering those generalizations, we

[1] See Aczél [1987] for a discussion of various applications.

provide alternative social orderings that can be applied in circumstances where the strong Pareto principle is considered to be too restrictive in an intertemporal framework.

In Blackorby, Bossert, and Donaldson [1997], we examined the consequences of weakening strong Pareto to birth-date conditional strong Pareto. Together with independence of the utilities of the dead and other axioms, this condition leads to social orderings with a recursive structure and, with some additional axioms, ordering that are based on discounted utilities are obtained. In order to characterize geometric discounting in this framework, the solution to a Pexider equation (see Aczél [1966, p. 141]) resulting from a stationarity axiom is employed. See Blackorby, Bossert, and Donaldson [1997] for details.

In this paper, we provide an analysis that is dual to the one carried out in the above-described paper. Instead of birth-date conditional strong Pareto, we impose lifetime-conditional strong Pareto in order to allow for lengths of life to matter in establishing a social ordering. The main results of the paper provide characterizations of two classes of social orderings. The first of those is analogous to critical-level generalized utilitarianism, except that the transformations applied to individual utilities and the critical level may depend on the individuals' lengths of life. A subclass of those orderings results as a consequence of adding limited lifetime-dependence and strengthening one of the original axioms. The members of this class of orderings are generalizations of critical-level generalized utilitarianism which allow for the critical level to depend on the lifetime of an added individual.

2. Intertemporal Social Evaluation

The mathematical model employed in this paper is the one introduced in Blackorby, Bossert, and Donaldson [1995]. It provides a general framework for intertemporal social evaluation when populations can be different in different states of the world.

The set of nonnegative (positive) integers is denoted by $\mathcal{Z}_+$ ($\mathcal{Z}_{++}$), and the set of all real numbers is $\mathcal{R}$. X is a set of social states of affairs, $N = \mathbf{N}(x)$ is the set of individuals alive in state $x \in X$, and $n = \mathbf{n}(x)$ is the number of individuals alive in x. If person i is alive in x, $s^i = S^i(x)$ is the period just before he or she is born and $l^i = L^i(x)$ is his or her lifetime. Therefore, person i is alive in periods $(s^i + 1, \ldots, s^i + l^i)$. The lifetime utility experienced by individual $i \in \mathbf{N}(x)$ in state x is $u^i = U^i(x)$. Lifetime utilities are assumed to be normalized so that a lifetime-utility level of zero represents neutrality. For an individual, a life, taken as a whole, is worth living if and only if lifetime utility is above neutrality. A fully informed, rational and selfish individual whose lifetime-utility level is below neutrality prefers not to have any of his or her experiences.[2] We assume that normalized lifetime utilities are numerically measurable and interpersonally fully comparable in order to allow for the largest possible class of social orderings.[3] Furthermore, we assume that no person can live more than $L \in \mathcal{Z}_{++}$ periods.

[2] See Broome [1993] for a discussion of neutrality and its normalization to zero.

[3] Other information assumptions could be used but would restrict the class of possible orderings. For a discussion of information assumptions in variable-population social-choice theory, see Blackorby, Bossert, and Donaldson [1999].

The set of individuals alive in state x, together with birth dates, lifetimes and lifetime utilities of those individuals, can be written as an alternative

$$A = \Big(N, (s^i, l^i, u^i)_{i \in N}\Big) = \mathbf{A}(x) = \Big(\mathbf{N}(x), \big(S^i(x), L^i(x), U^i(x)\big)_{i \in \mathbf{N}(x)}\Big). \tag{2.1}$$

The set $\mathcal{A}$ of admissible alternatives consists of all alternatives A in which N is nonempty and finite, $s^i \in \mathcal{Z}_+$, $l^i \in \{1, \ldots, L\}$, and $u^i \in \mathcal{R}$.[4] For notational convenience, we define $\mathcal{A}_n$ to be the set of all alternatives A in $\mathcal{A}$ such that $|N| = n$, and $\mathcal{A}_{N^n}$ is the set of all alternatives A such that $N = \{1, \ldots, n\}$. In addition, $\mathcal{A}^0_{N^n}$ is the set of alternatives A such that $N = \{1, \ldots, n\}$, $s^i = 0$, and $l^i = 1$ for all $i \in N$.

We assume that a single ordering $R \subseteq \mathcal{A} \times \mathcal{A}$ is sufficient to order X given $\mathbf{A}(x)$ for all x in X.[5] This is an information assumption—only the identities of those alive, their birth dates, lengths of life, and lifetime utility levels can influence the social ranking.

To conclude this section, we introduce critical-level generalized utilitarianism. R is a critical-level generalized utilitarian social ordering if there exist $\alpha \in \mathcal{R}$ and a continuous and increasing function $g: \mathcal{R} \longrightarrow \mathcal{R}$ with $g(0) = 0$ such that, for all $\overline{A}, \widehat{A} \in \mathcal{A}$,

$$\overline{A}\, R\, \widehat{A} \;\longleftrightarrow\; \sum_{i \in \overline{N}} [g(\bar{u}^i) - g(\alpha)] \ge \sum_{i \in \widehat{N}} [g(\hat{u}^i) - g(\alpha)]. \tag{2.2}$$

3. Axioms

In Blackorby, Bossert, and Donaldson [1995], it is assumed that R satisfies the strong Pareto condition, which implies that individual lifetime utilities are the only determinants of the ranking R. In this paper, we weaken this axiom in order to consider additional possibilities. Instead of strong Pareto, we require lifetime-conditional strong Pareto which allows, in addition to lifetime utilities, individual lengths of life to matter. This axiom is defined as follows.

Lifetime-Conditional Strong Pareto: *For all $\overline{A}, \widehat{A} \in \mathcal{A}$ with $\overline{N} = \widehat{N} = N$ and $\bar{l}^i = \hat{l}^i$ for all $i \in N$:*

(i) if, for all $i \in N$, $\bar{u}^i = \hat{u}^i$, then $\overline{A}\, I\, \widehat{A}$;

(ii) if, for all $i \in N$, $\bar{u}^i \ge \hat{u}^i$ with at least one strict inequality, then $\overline{A}\, P\, \widehat{A}$.

Lifetime-conditional strong Pareto applies to fixed populations only. To link alternatives with populations of the same size but with different individuals alive in each, we require the ordering R to be independent of the identities of the individuals alive. This requirement is captured in the following anonymity axiom.

[4] The null alternative $A_\emptyset$ in which no one is alive could be included in $\mathcal{A}$ without changing our results—see Blackorby, Bossert, and Donaldson [1995].

[5] An ordering is a reflexive, transitive, and complete binary relation. The symmetric and asymmetric factors of R are denoted by I and P, respectively.

Anonymity: *For all* $\overline{A}, \widehat{A} \in \mathcal{A}$ *with* $\bar{n} = \hat{n}$*, if there exists a bijection* $\rho : \overline{N} \longrightarrow \widehat{N}$ *such that* $\bar{s}^{\bar{\imath}} = \hat{s}^{\rho(\bar{\imath})}$*,* $\bar{l}^{\bar{\imath}} = \hat{l}^{\rho(\bar{\imath})}$*, and* $\bar{u}^{\bar{\imath}} = \hat{u}^{\rho(\bar{\imath})}$ *for all* $\bar{\imath} \in \overline{N}$*, then* $\overline{A}\, I\, \widehat{A}$*.*

Furthermore, we assume that the fixed-population orderings implied by R are continuous.

Continuity: *For all* $\widetilde{N} \neq \emptyset$*, the restriction of the ordering* R *to* $\{A \in \mathcal{A} \mid N = \widetilde{N}\}$ *is continuous in lifetime utilities.*

The following axiom requires that individuals whose lives are over do not affect the social ranking of alternatives. We need some additional notation in order to define it formally. For $A \in \mathcal{A}$ and $t \in \mathcal{Z}_{++}$, let $B_t(A) := \{i \in N \mid s^i + 1 < t\}$ and $D_t(A) := \{i \in N \mid s^i + l^i < t\}$.

Independence of the Utilities of the Dead: *For all* $\overline{A}, \widehat{A} \in \mathcal{A}$*, for all* $t \in \mathcal{Z}_{++}$*, for all* $N_t \subseteq \overline{N} \cap \widehat{N}$ *such that* $N_t \neq \overline{N}$ *and* $N_t \neq \widehat{N}$*, if*

$$B_t(\overline{A}) = D_t(\overline{A}) = D_t(\widehat{A}) = B_t(\widehat{A}) = N_t, \tag{3.1}$$

$(\bar{s}^i, \bar{l}^i, \bar{u}^i) = (\hat{s}^i, \hat{l}^i, \hat{u}^i)$ *for all* $i \in N_t$*, and the alternatives* $\overline{A}_t$ *and* $\widehat{A}_t$ *are obtained by removing all individuals in* N_t *from* $\overline{A}$ *and* $\widehat{A}$*, then*

$$\overline{A}\, R\, \widehat{A} \longleftrightarrow \overline{A}_t\, R\, \widehat{A}_t. \tag{3.2}$$

In order to rule out degenerate cases, we assume the existence of some critical levels of lifetime utility. A critical level for a given alternative, a given birth date, and a given lifetime is a level of utility such that if an individual experiencing the critical level of utility with the given birth date and lifetime is added to the alternative in question and no one else is affected by this population augmentation, then the new alternative is indifferent to the original one. We use two variants of an axiom requiring the existence of such critical levels.

Weak Existence of Critical Levels: *There exist* $A_0 \in \mathcal{A}$*,* $j \in \mathcal{Z}_{++} \setminus N_0$*,* $\bar{s}_0 \in \mathcal{Z}_+$*,* $\bar{l}_0 \in \{1, \ldots L\}$*,* $c_0 \in \mathcal{R}$ *such that* $\widetilde{A}\, I\, A_0$*, where* $\widetilde{N} = N_0 \cup \{j\}$*,* $(\tilde{s}^i, \tilde{l}^i, \tilde{u}^i) = (s_0^i, l_0^i, u_0^i)$ *for all* $i \in N_0$*,* $\tilde{s}^j = \bar{s}_0$*,* $\tilde{l}^j = \bar{l}_0$*, and* $\tilde{u}^j = c_0$*.*

Existence of Critical Levels: *There exists* $A_0 \in \mathcal{A}$ *such that, for all* $l \in \{1, \ldots, L\}$*, there exist* $j \in \mathcal{Z}_{++} \setminus N_0$*,* $\bar{s}_0 \in \mathcal{Z}_+$*,* $c_0 \in \mathcal{R}$ *such that* $\widetilde{A}\, I\, A_0$*, where* $\widetilde{N} = N_0 \cup \{j\}$*,* $(\tilde{s}^i, \tilde{l}^i, \tilde{u}^i) = (s_0^i, l_0^i, u_0^i)$ *for all* $i \in N_0$*,* $\tilde{s}^j = \bar{s}_0$*,* $\tilde{l}^j = l$*, and* $\tilde{u}^j = c_0$*.*

The following axiom ensures that there are nontrivial trade-offs between length of life and lifetime utility.

Individual Lifetime Substitution Principle: *For all* $j \in \mathcal{Z}_{++}$*, there exists a function* $\Lambda^j : \mathcal{Z}_+ \times \{1, \ldots, L\} \times \mathcal{R} \longrightarrow \mathcal{R}$ *such that, for all* $A \in \mathcal{A}$ *with* $j \in N$*,* $\widetilde{A}\, I\, A$*, where* $\widetilde{N} = N$*,* $(\tilde{s}^i, \tilde{l}^i, \tilde{u}^i) = (s^i, l^i, u^i)$ *for all* $i \in N \setminus \{j\}$*,* $\tilde{s}^j = s^j$*,* $\tilde{l}^j = 1$*, and* $\tilde{u}^j = \Lambda^j(s^j, l^j, u^j)$*.*

Finally, we introduce an axiom that limits the influence on lifetimes on the social ranking. It requires that, for fixed-population comparisons, lifetimes can only count to the extent that it does not matter who has which lifetime.

Limited Lifetime-Dependence: *For all $\overline{A}, \widehat{A} \in \mathcal{A}$ with $\overline{N} = \widehat{N} = N$ and $(\bar{s}^i, \bar{u}^i) = (\hat{s}^i, \hat{u}^i)$ for all $i \in N$, if there exists a bijection $\pi \colon N \longmapsto N$ such that $\bar{l}^i = \hat{l}^{\pi(i)}$ for all $i \in N$, then $\overline{A}\, I\, \widehat{A}$.*

4. Lifetime-Dependent Population Principles

As a preliminary result, we show that the conjunction of independence of the utilities of the dead and lifetime-conditional strong Pareto implies a stronger separability axiom, which is defined as follows.

Birth-Date Independent Separability: *For all $\overline{A}, \widehat{A} \in \mathcal{A}$, for all $N_* \subseteq \overline{N} \cap \widehat{N}$ such that $N_* \neq \overline{N}$ and $N_* \neq \widehat{N}$, if $(\bar{l}^i, \bar{u}^i) = (\hat{l}^i, \hat{u}^i)$ for all $i \in N_*$ and the alternatives $\overline{A}_*$ and $\widehat{A}_*$ are obtained by removing all individuals in N_* from $\overline{A}$ and $\widehat{A}$, then*

$$\overline{A}\, R\, \widehat{A} \longleftrightarrow \overline{A}_*\, R\, \widehat{A}_*. \tag{4.1}$$

We obtain

Lemma 1: *If R satisfies lifetime-conditional strong Pareto and independence of the utilities of the dead, then R satisfies birth-date independent separability.*

Proof: Let $\overline{A}, \widehat{A} \in \mathcal{A}$ and $N_* \subseteq \overline{N} \cap \widehat{N}$ be such that $N_* \neq \overline{N}$, $N_* \neq \widehat{N}$, and $(\bar{l}^i, \bar{u}^i) = (\hat{l}^i, \hat{u}^i)$ for all $i \in N_*$. Let $\overline{A}', \widehat{A}' \in \mathcal{A}$ be such that $\overline{N}' = \overline{N}$ and $(\bar{l}'^i, \bar{u}'^i) = (\bar{l}^i, \bar{u}^i)$ for all $i \in \overline{N}$, $\widehat{N}' = \widehat{N}$ and $(\hat{l}'^i, \hat{u}'^i) = (\hat{l}^i, \hat{u}^i)$ for all $i \in \widehat{N}$, $\bar{s}'^i = 0$ for all $i \in \overline{N} \cap N_*$, $\hat{s}'^i = 0$ for all $i \in \widehat{N} \cap N_*$, $\bar{s}'^i = L$ for all $i \in \overline{N} \setminus N_*$, and $\hat{s}'^i = L$ for all $i \in \widehat{N} \setminus N_*$. It follows that $B_{L+1}(\overline{A}') = B_{L+1}(\widehat{A}') = D_{L+1}(\overline{A}') = D_{L+1}(\widehat{A}') = N_*$ and $(\bar{s}'^i, \bar{l}'^i, \bar{u}'^i) = (\hat{s}'^i, \hat{l}'^i, \hat{u}'^i)$ for all $i \in N_*$. By lifetime-conditional strong Pareto, $\overline{A}'\, I\, \overline{A}$ and $\widehat{A}'\, I\, \widehat{A}$. Therefore,

$$\overline{A}\, R\, \widehat{A} \longleftrightarrow \overline{A}'\, R\, \widehat{A}'. \tag{4.2}$$

Removing all individuals in N_* from $\overline{A}'$ and $\widehat{A}'$ results in the alternatives $\overline{A}_*$ and $\widehat{A}_*$, and independence of the utilities of the dead implies $\overline{A}'\, R\, \widehat{A}'$ if and only if $\overline{A}_*\, R\, \widehat{A}_*$. Together with (4.2), it follows that $\overline{A}\, R\, \widehat{A}$ if and only if $\overline{A}_*\, R\, \widehat{A}_*$. ■

The following lemma shows that some of our axioms impose an additive structure on the restrictions of R to alternatives with a fixed population size.

Lemma 2: *If R satisfies lifetime-conditional strong Pareto, anonymity, continuity, independence of the utilities of the dead, and the individual lifetime substitution principle, then there exists a function $h: \{1,\ldots,L\} \times \mathcal{R} \longrightarrow \mathcal{R}$, continuous and increasing in its second argument with $h(1,0)=0$ such that, for all $n \geq 3$ and for all $\overline{A}, \widehat{A} \in \mathcal{A}_n$,*

$$\overline{A} \, R \, \widehat{A} \longleftrightarrow \sum_{i \in \overline{N}} h(\bar{l}^i, \bar{u}^i) \geq \sum_{i \in \widehat{N}} h(\hat{l}^i, \hat{u}^i). \tag{4.3}$$

Proof: Let $j \in \mathcal{Z}_{++}$ and let Λ^j be as in the definition of the individual lifetime substitution principle. By lifetime-conditional strong Pareto, $\Lambda^j(s^j, l^j, u^j) = \Lambda^j(0, l^j, u^j)$ for all $A \in \mathcal{A}$ such that $j \in N$. By anonymity, the functions Λ^j can be chosen to be identical for all $j \in \mathcal{Z}_{++}$, and we can therefore define $\Lambda: \{1,\ldots,L\} \times \mathcal{R} \longrightarrow \mathcal{R}$ by letting $\Lambda(l,u) := \Lambda^j(0,l,u)$ for all $j \in \mathcal{Z}_{++}$, for all $l \in \{1,\ldots,L\}$, and for all $u \in \mathcal{R}$.

Let $n \in \mathcal{Z}_{++}$. Applying Debreu's [1954] representation theorem, continuity implies that there exists a continuous function $f^n: \mathcal{R}^n \longrightarrow \mathcal{R}$ such that, for all $\overline{A}, \widehat{A} \in \mathcal{A}^0_{N^n}$,

$$\overline{A} \, R \, \widehat{A} \longleftrightarrow f^n(\bar{u}^1, \ldots, \bar{u}^n) \geq f^n(\hat{u}^1, \ldots, \hat{u}^n). \tag{4.4}$$

By lifetime-conditional strong Pareto, f^n is increasing in all arguments, and by anonymity, f^n is a symmetric function. Now let $n \geq 3$ and $\overline{A}, \widehat{A} \in \mathcal{A}^0_{N^n}$. Furthermore, let $N_* \subset \{1,\ldots,n\}$ and $m = |N \setminus N_*|$. Birth-date independent separability (see Lemma 1) implies

$$f^n(\bar{u}^1, \ldots, \bar{u}^n) \geq f^n(\hat{u}^1, \ldots, \hat{u}^n) \longleftrightarrow f^m((\bar{u}^i)_{i \in N \setminus N_*}) \geq f^m((\hat{u}^i)_{i \in N \setminus N_*}). \tag{4.5}$$

Therefore, $N \setminus N_*$ is separable from its complement in f^n for any choice of N_*. Gorman's [1968] theorem on overlapping separable sets of variables, which can be proven by invoking the solution to the generalized associativity equation (see Aczél [1966, p. 312] and Blackorby, Primont, and Russell [1978, p. 127]), implies that f^n is additively separable. That is, there exist continuous and increasing functions $F^n: \mathcal{R} \longrightarrow \mathcal{R}$ and $\bar{g}^i_n: \mathcal{R} \longrightarrow \mathcal{R}$ for all $i \in \{1,\ldots,n\}$ such that

$$f^n(u^1, \ldots, u^n) = F^n\Big(\sum_{i=1}^n \bar{g}^i_n(u^i)\Big) \tag{4.6}$$

for all $(u^1, \ldots, u^n) \in \mathcal{R}^n$. Because f^n is symmetric, each $\bar{g}^i_n$ can be chosen to be independent of i and we define $\bar{g}_n := \bar{g}^i_n$ for all $i \in \{1,\ldots,n\}$. Because (4.5) must be satisfied for all $n \geq 3$ and all $m < n$, $\bar{g}_n$ and F^n can be chosen independently of n, and we write $\bar{g} := \bar{g}_n$ and $F := F^n$ for all $n \geq 3$. Defining $g: \mathcal{R} \longrightarrow \mathcal{R}$ by setting $g(u) := \bar{g}(u) - \bar{g}(0)$, it follows that g is continuous, increasing, and satisfies $g(0) = 0$. Furthermore, for all $\overline{A}, \widehat{A} \in \mathcal{A}^0_{N^n}$,

$$\begin{aligned}\overline{A} \, R \, \widehat{A} &\longleftrightarrow F\Big(\sum_{i=1}^{n} \bar{g}(\bar{u}^i)\Big) \geq F\Big(\sum_{i=1}^{n} \bar{g}(\hat{u}^i)\Big) \\ &\longleftrightarrow \sum_{i=1}^{n} \bar{g}(\bar{u}^i) \geq \sum_{i=1}^{n} \bar{g}(\hat{u}^i) \\ &\longleftrightarrow \sum_{i=1}^{n} g(\bar{u}^i) \geq \sum_{i=1}^{n} g(\hat{u}^i).\end{aligned} \tag{4.7}$$

Now let $\overline{A}, \widehat{A} \in \mathcal{A}_{N^n}$ with $n \geq 3$. Let $\overline{A}', \widehat{A}' \in \mathcal{A}_{N^n}$ be such that $\bar{s}'^i = \hat{s}'^i = 0$, $\bar{l}'^i = \bar{l}^i$, $\hat{l}'^i = \hat{l}^i$, $\bar{u}'^i = \bar{u}^i$, and $\hat{u}'^i = \hat{u}^i$ for all $i \in \{1, \ldots, n\}$. By lifetime-conditional strong Pareto,

$$\overline{A}' \, I \, \overline{A} \text{ and } \widehat{A}' \, I \, \widehat{A}. \tag{4.8}$$

By repeated application of the individual lifetime substitution principle, there exist $\overline{A}'', \widehat{A}'' \in \mathcal{A}_{N^n}$ such that $\bar{s}''^i = \hat{s}''^i = 0$, $\bar{l}''^i = \hat{l}''^i = 1$, $\bar{u}''^i = \Lambda(\bar{l}'^i, \bar{u}'^i)$, $\hat{u}''^i = \Lambda(\hat{l}'^i, \hat{u}'^i)$ for all $i \in \{1, \ldots, n\}$, and $\overline{A}'' \, I \, \overline{A}'$ and $\widehat{A}'' \, I \, \widehat{A}'$. Together with (4.8), this implies

$$\overline{A} \, R \, \widehat{A} \longleftrightarrow \overline{A}'' \, R \, \widehat{A}''. \tag{4.9}$$

By definition, $\overline{A}'', \widehat{A}'' \in \mathcal{A}^0_{N^n}$. Using (4.7),

$$\begin{aligned}\overline{A}'' \, R \, \widehat{A}'' &\longleftrightarrow \sum_{i=1}^{n} g(\bar{u}^i) \geq \sum_{i=1}^{n} g(\hat{u}^i) \\ &\longleftrightarrow \sum_{i=1}^{n} g\big(\Lambda(\bar{l}^i, \bar{u}^i)\big) \geq \sum_{i=1}^{n} g\big(\Lambda(\hat{l}^i, \hat{u}^i)\big).\end{aligned} \tag{4.10}$$

Defining $h := g \circ \Lambda$, it follows that h is continuous and increasing in its second argument, and $h(1, 0) = 0$. Using (4.9) and (4.10), it follows that

$$\overline{A} \, R \, \widehat{A} \longleftrightarrow \sum_{i=1}^{n} h(\bar{l}^i, \bar{u}^i) \geq \sum_{i=1}^{n} h(\hat{l}^i, \hat{u}^i). \tag{4.11}$$

Finally, anonymity implies that we have

$$\overline{A} \, R \, \widehat{A} \longleftrightarrow \sum_{i \in \overline{N}} h(\bar{l}^i, \bar{u}^i) \geq \sum_{i \in \widehat{N}} h(\hat{l}^i, \hat{u}^i) \tag{4.12}$$

for all $\overline{A}, \widehat{A} \in \mathcal{A}_n$ with $n \geq 3$. ■

The following theorem provides a characterization of a class of orderings which generalize the critical-level generalized utilitarian orderings by allowing lifetimes to matter in social evaluations.

Theorem 1: *R satisfies lifetime-conditional strong Pareto, anonymity, continuity, independence of the utilities of the dead, weak existence of critical levels, and the individual lifetime substitution principle if and only if there exist $\bar{\alpha} \in \mathcal{R}$ and $h\colon \{1,\ldots,L\}\times\mathcal{R} \longrightarrow \mathcal{R}$ continuous and increasing in its second argument with $h(1,0)=0$ such that, for all $\overline{A},\widehat{A} \in \mathcal{A}$,*

$$\overline{A}\,R\,\widehat{A} \longleftrightarrow \sum_{i\in\overline{N}} \Big[h(\bar{l}^i,\bar{u}^i) - h(1,\bar{\alpha})\Big] \geq \sum_{i\in\widehat{N}} \Big[h(\hat{l}^i,\hat{u}^i) - h(1,\bar{\alpha})\Big]. \tag{4.13}$$

Proof: The sufficiency part of the theorem is straightforward to establish. By weak existence of critical levels, there exist $A_0 \in \mathcal{A}$, $j \in \mathcal{Z}_{++} \setminus N_0$, $\bar{s}_0 \in \mathcal{Z}_+$, $\bar{l}_0 \in \{1,\ldots L\}$, $c_0 \in \mathcal{R}$ such that $\widetilde{A}\,I\,A_0$, where $\widetilde{N} = N_0 \cup \{j\}$, $(\tilde{s}^i,\tilde{l}^i,\tilde{u}^i) = (s_0^i,l_0^i,u_0^i)$ for all $i \in N_0$, $\tilde{s}^j = \bar{s}_0$, $\tilde{l}^j = \bar{l}_0$, and $\tilde{u}^j = c_0$. By the individual lifetime substitution principle, $\widetilde{A}\,I\,A'$, where A' is defined by letting $N' = \widetilde{N}$, $s'^i = \tilde{s}^i$, $l'^i = \tilde{l}^i$, $u'^i = \tilde{u}^i$ for all $i \in N_0$, and $s'^j = \bar{s}_0$, $l'^j = 1$, $u'^j = \Lambda(\tilde{l}^j, c_0)$. By transitivity,

$$\overline{A}\,I\,A'. \tag{4.14}$$

Define $\bar{\alpha} := \Lambda(\tilde{l}^j, c_0)$, and let $A \in \mathcal{A}$. By anonymity, we can, without loss of generality, assume that $N \cap N_0 = \emptyset$ and $j \notin N$. Using (4.14) and birth-date independent separability (see Lemma 1), it follows that

$$\Big(N_0 \cup N, \big((s_0^i,l_0^i,u_0^i)_{i\in N_0}, (s^i,l^i,u^i)_{i\in N}\big)\Big) \tag{4.15}$$

is indifferent to

$$\Big(N_0 \cup N \cup \{j\}, \big((s_0^i,l_0^i,u_0^i)_{i\in N_0}, (s^i,l^i,u^i)_{i\in N}, (\bar{s}_0,1,\bar{\alpha})\big)\Big). \tag{4.16}$$

Again using birth-date independent separability,

$$A = \Big(N, \big((s^i,l^i,u^i)_{i\in N}\big)\Big)\;I\;\Big(N \cup \{j\}, \big((s^i,l^i,u^i)_{i\in N}, (\bar{s}_0,1,\bar{\alpha})\big)\Big). \tag{4.17}$$

Therefore, adding individual j with a lifetime utility of $\bar{\alpha}$ and a lifetime of one to any alternative is a matter of indifference and, thus, $\bar{\alpha}$ is a lifetime-one critical level for any $A \in \mathcal{A}$.

Next, we extend the additive-separability result of Lemma 2 to fixed-population comparisons with population sizes one and two. Consider $\overline{A},\widehat{A} \in \mathcal{A}_1$. By anonymity, we can without loss of generality assume $\overline{N} = \widehat{N} = \{i\}$, and we immediately obtain

$$\begin{aligned}
\overline{A}\,R\,\widehat{A} &\longleftrightarrow \Lambda(\bar{l}^i,\bar{u}^i) \geq \Lambda(\hat{l}^i,\hat{u}^i)\\
&\longleftrightarrow g\big(\Lambda(\bar{l}^i,\bar{u}^i)\big) \geq g\big(\Lambda(\hat{l}^i,\hat{u}^i)\big)\\
&\longleftrightarrow h(\bar{l}^i,\bar{u}^i) \geq h(\hat{l}^i,\hat{u}^i).
\end{aligned} \tag{4.18}$$

Now let $\overline{A},\widehat{A} \in \mathcal{A}_2$. Let $j \in \mathcal{Z}_{++} \setminus (\overline{N} \cup \widehat{N})$, and define $\overline{A}'$ and $\widehat{A}'$ by adding j with an arbitrary birth date, lifetime $l^j = 1$, and lifetime utility $u^j = \bar{\alpha}$ to $\bar{A}$ and $\widehat{A}$, respectively.

By anonymity and the observation that $\bar{\alpha}$ is a lifetime-one critical level for all alternatives, we obtain $\overline{A}\, I\, \overline{A}'$ and $\widehat{A}\, I\, \widehat{A}'$ and, thus, $\overline{A}\, R\, \widehat{A}$ if and only if $\bar{A}'\, R\, \widehat{A}'$. By Lemma 2,

$$\overline{A}'\, R\, \widehat{A}' \longleftrightarrow \sum_{i\in\overline{N}} h(\bar{l}^i, \bar{u}^i) + h(1, c_0) \geq \sum_{i\in\widehat{N}} h(\hat{l}^i, \hat{u}^i) + h(1, c_0) \tag{4.19}$$

and, therefore,

$$\overline{A}\, R\, \widehat{A} \longleftrightarrow \sum_{i\in\overline{N}} h(\bar{l}^i, \bar{u}^i) \geq \sum_{i\in\widehat{N}} h(\hat{l}^i, \hat{u}^i). \tag{4.20}$$

To complete the proof, suppose finally that $\overline{A}, \widehat{A} \in \mathcal{A}$ and $\bar{n} \neq \hat{n}$. Without loss of generality, let $\bar{n} > \hat{n}$. Define $\widehat{A}'$ by adding $(\bar{n} - \hat{n})$ individuals to $\widehat{N}$, each with an arbitrary birth date, a lifetime of one period, and a lifetime utility of $\bar{\alpha}$. By definition of a critical level, $\widehat{A}'\, I\, \widehat{A}$ and, therefore, Lemma 2 (or the above argument if $\bar{n} = 2$) implies

$$\begin{aligned} \overline{A}\, R\, \widehat{A} &\longleftrightarrow \overline{A}\, R\, \widehat{A}' \\ &\longleftrightarrow \sum_{i\in\overline{N}} h(\bar{l}^i, \bar{u}^i) \geq \sum_{i\in\widehat{N}} h(\hat{l}^i, \hat{u}^i) + (\bar{n} - \hat{n}) h(1, \bar{\alpha}) \\ &\longleftrightarrow \sum_{i\in\overline{N}} \left[h(\bar{l}^i, \bar{u}^i) - h(1, \bar{\alpha})\right] \geq \sum_{i\in\widehat{N}} \left[h(\hat{l}^i, \hat{u}^i) - h(1, \bar{\alpha})\right]. \blacksquare \end{aligned} \tag{4.21}$$

If weak existence of critical levels is strengthened to existence of critical levels and limited lifetime dependence is added, the resulting class of orderings can be narrowed down further.

Theorem 2: *R satisfies lifetime-conditional strong Pareto, anonymity, continuity, independence of the utilities of the dead, existence of critical levels, the individual lifetime substitution principle, and limited lifetime dependence if and only if there exist $\alpha\colon \{1, \ldots, L\} \longrightarrow \mathcal{R}$ and $g\colon \mathcal{R} \longrightarrow \mathcal{R}$ continuous and increasing with $g(0) = 0$ such that, for all $\overline{A}, \widehat{A} \in \mathcal{A}$,*

$$\overline{A}\, R\, \widehat{A} \longleftrightarrow \sum_{i\in\overline{N}} \Big[g\big(\bar{u}^i\big) - g\big(\alpha(\bar{l}^i)\big)\Big] \geq \sum_{i\in\widehat{N}} \Big[g\big(\hat{u}^i\big) - g\big(\alpha(\hat{l}^i)\big)\Big]. \tag{4.22}$$

Proof: Again, it is straightforward to establish sufficiency. By existence of critical levels, there exists an alternative $A_0 \in \mathcal{A}$ such that, for all $l \in \{1, \ldots, L\}$, a lifetime-l critical level must exist for A_0. Using the same reasoning as in the proof of Theorem 1, this critical level must be the same for all alternatives as a consequence of birth-date independent separability. Because of lifetime-conditional strong Pareto, this critical level is unique for each $l \in \{1, \ldots, L\}$ and, therefore, we can write it as a function $\alpha\colon \{1, \ldots, L\} \longrightarrow \mathcal{R}$. Because all axioms used in Theorem 1 are satisfied, (4.13) holds with $\bar{\alpha} = \alpha(1)$. Let $A \in \mathcal{A}$ be such that $N = \{1, 2\}$. By limited lifetime dependence, it follows that h must satisfy the functional equation

$$h(l, u) = h(l, u) + h(1, 0) = h(1, u) + h(l, 0) \tag{4.23}$$

for all $l \in \{1, \ldots, L\}$ and all $u \in \mathcal{R}$. Letting $g(u) := h(1, u)$ for all $u \in \mathcal{R}$ and $\hat{g}(l) := h(l, 0)$ for all $l \in \{1, \ldots, L\}$, it follows that

$$h(l, u) = g(u) - \hat{g}(l) \tag{4.24}$$

for all $u \in \mathcal{R}$ and all $l \in \{1, \ldots, L\}$. By definition of a critical level,

$$h\big(l, \alpha(l)\big) - h\big(1, \alpha(1)\big) = 0 \tag{4.25}$$

for all $l \in \{1, \ldots, L\}$, and (4.24) implies

$$g\big(\alpha(l)\big) - \hat{g}\big(l\big) = g\big(\alpha(1)\big) - \hat{g}\big(1\big) \tag{4.26}$$

for all $l \in \{1, \ldots, L\}$. Substituting (4.24) and (4.26) into (4.13) completes the proof. ■

5. Conclusion

This paper provides an illustration of the use of functional equations in population ethics. In particular, we characterize classes of lifetime-dependent social orderings. Therefore, the paper complements earlier work by exploring the consequences of relaxing the strong Pareto principle so as to allow for lifetimes to matter.

REFERENCES

Aczél, J., 1966, *Lectures on Functional Equations and Their Applications,* Academic Press, New York.

Aczél, J., 1987, *A Short Course on Functional Equations: Based upon Recent Applications to the Social and Behavioral Sciences,* Reidel, Dordrecht.

Blackorby, C., W. Bossert, and D. Donaldson, 1995, Intertemporal population ethics: critical-level utilitarian principles, *Econometrica* **63**, 1303–1320.

Blackorby, C., W. Bossert, and D. Donaldson, 1997, Birth-date dependent population ethics: critical-level principles, *Journal of Economic Theory* **77**, 260–284.

Blackorby, C., W. Bossert, and D. Donaldson, 1999, Information invariance in variable-population social-choice problems, *International Economic Review,* forthcoming.

Blackorby, C. and D. Donaldson, 1984, Social criteria for evaluating population change, *Journal of Public Economics* **25**, 13–33.

Blackorby, C., D. Primont, and R. Russell, 1978, *Duality, Separability and Functional Structure: Theory and Economic Applications,* North-Holland/American Elsevier, Amsterdam/New York.

Broome, J., 1993, Goodness is reducible to betterness: the evil of death is the value of life, in *The Good and the Economical: Ethical Choices in Economics and Management,* P. Koslowski, ed., Springer-Verlag, Berlin, 69–83.

Debreu, G., 1954, Representation of a preference ordering by a numerical function, in *Decision Processes,* R.M. Thrall, C.H. Coombs, and R.L. Davis. eds., Wiley, New York, 159–166.

Gorman, W.M., 1968, The structure of utility functions, *Review of Economic Studies* **32**, 369–390.