Efficient Solutions to Bargaining Problems with Uncertain Disagreement Points*

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Abstract. We consider a cooperative model of bargaining where the location of the disagreement point may be uncertain. Based on the maximin criterion, we formulate an ex ante efficiency condition and characterize the class of bargaining solutions satisfying this axiom. These solutions are generalizations of the monotone path solutions. Adding individual rationality yields a subclass of these solutions. By employing maximin efficiency and an invariance property that implies individual rationality, a new axiomatization of the monotone path solutions is obtained. Furthermore, we show that an efficiency axiom employing the maximax criterion leads to an impossibility result. *Journal of Economic Literature* Classification No.: C78.

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1 Introduction

The purpose of the cooperative model of bargaining as formulated by Nash (1950) is to recommend outcomes for bargaining situations involving two or more agents. A bargaining solution assigns a chosen utility vector to each bargaining problem represented by the corresponding set of feasible utility vectors and the disagreement point—the utility vector that results if the agents fail to reach an agreement.

Whereas most of the early contributions to the theory of bargaining (such as, for example, Nash, 1950, Kalai and Smorodinsky, 1975, and Kalai, 1977) focus on properties of bargaining solutions with respect to changes in the feasible set, there is now a substantial literature dealing with the role of the disagreement point in establishing a solution outcome. For example, Thomson (1987), Wakker (1987), Livne (1989), and Bossert (1994) analyze monotonicity properties of bargaining solutions with respect to changes in the disagreement point.

Properties of bargaining solutions that are motivated by the presence of uncertainty regarding the location of the disagreement point are examined, for instance, in Livne (1988), Chun (1989), Chun and Thomson (1990a,b,c), and Peters and van Damme (1991). These contributions analyze the consequences of imposing axioms such as disagreement point concavity and related conditions. Disagreement point concavity requires that agents weakly prefer to solve a contingent problem with an uncertain disagreement point immediately on the basis of their expected payoffs rather than waiting until the uncertainty is resolved. See, for example, Chun and Thomson (1990a,b) for details and Thomson (1994) for a survey and further references.

In this paper, we address the uncertainty issue with respect to the disagreement point from another angle. We define bargaining problems under uncertainty by specifying the disagreement points that could materialize in different states of the world with a fixed feasible set of utility vectors. We then impose an efficiency condition based on the maximin criterion and show that this axiom is satisfied only by a specific class of solutions that are generalizations of the monotone path solutions (see, for example, Thomson and Myerson, 1980). Furthermore, we illustrate how this class of solutions can be narrowed down by imposing individual rationality in addition to efficiency under uncertainy. Together with a strengthening of individual rationality, maximin efficiency is used to provide a characterization of the class of monotone path solutions.

The approach in this paper complements the one followed in Bossert, Nosal, and Sadanand (1996) and in Bossert and Peters (1998), where efficiency conditions with respect to various decision criteria are examined in bargaining models with uncertain feasible sets. Despite the conceptual similarity between those two approaches, some of the techniques employed here are quite different from those used in the above-mentioned earlier contributions, due to the restricted flexibility that obtains when feasible sets are assumed to be fixed. Because variations in the disagreement point appear to be easier to simulate in an experimental setting than (possibly very complex) changes in the feasible set of utility vectors, the model developed in this paper provides an interesting framework for empirical investigations in the analysis of bargaining situations. This is, of course, a feature shared by other contributions that focus on the effects of variations in the disagreement point with fixed feasible sets.

Section 2 introduces our basic notation and definitions. Efficient bargaining with respect to the maximin criterion for problems with uncertain disagreement points is analyzed in Section 3. To conclude the paper, a discussion of alternative decision criteria is provided in Section 4.

2 Preliminaries

Let $\mathbb{R}$ ($\mathbb{R}_+$, $\mathbb{R}_{++}$, $\mathbb{R}_{--}$) denote the set of all (nonnegative, positive, negative) real numbers. For a positive integer $n \geq 2$, $\mathbb{R}^n$ ($\mathbb{R}^n_+$, $\mathbb{R}^n_{++}$, $\mathbb{R}^n_{--}$) is the n-fold Cartesian product of $\mathbb{R}$ ($\mathbb{R}_+$, $\mathbb{R}_{++}$, $\mathbb{R}_{--}$). The origin of $\mathbb{R}^n$ is denoted by $\mathbf{0}$. The inner product of two vectors $x, y \in \mathbb{R}^n$ is denoted by $x \cdot y$. Our notation for vector inequalities is as follows. For all $x, y \in \mathbb{R}^n$, $x \geq y$ if $x_i \geq y_i$ for all $i = 1, \ldots, n$, and $x > y$ if $x_i > y_i$ for all $i = 1, \ldots, n$. A set $S \subset \mathbb{R}^n$ is strictly comprehensive if, for all $x \in S$ and all $y \in \mathbb{R}^n$ such that $x \geq y$ and $x \neq y$, there exists $z \in S$ such that $z > y$, and S is bounded from above if there exist $p \in \mathbb{R}^n_{++}$ and $\alpha \in \mathbb{R}$ such that $p \cdot x \leq \alpha$ for every $x \in S$. The interior of $S \subset \mathbb{R}^n$ is denoted by $I(S)$. For $x \in \mathbb{R}^n$ and $S \subset \mathbb{R}^n$, we define $x + S = \{y \in \mathbb{R}^n \mid \exists z \in S \text{ such that } x = y + z\}$.

Let $N = \{1, \ldots, n\}$ with $n \geq 2$ be the (fixed) set of agents involved in a bargaining problem. A set $S \subset \mathbb{R}^n$ is called admissible if it is closed, convex, strictly comprehensive and bounded from above. By Σ we denote the collection of all admissible sets. A pair (S, d) with $S \in \Sigma$ and $d \in I(S)$ is called a bargaining problem. The interpretation of a bargaining problem is that S represents the set of feasible utility vectors, and the point d is the disagreement point—the utility vector that results if the agents in N fail to reach an agreement. The collection of all bargaining problems is denoted by $\mathcal{B}$. A (bargaining) solution is a mapping $F: \mathcal{B} \to \mathbb{R}^n$ such that $F(S, d) \in S$ for all $(S, d) \in \mathcal{B}$.

Pareto optimality is a standard restriction imposed on bargaining solutions. This axiom requires that F should not select a utility vector in S that is dominated in the sense that all agents in N could be made better off by moving to another point in S. In order to define this condition formally, let $P(S) = \{x \in S \mid \nexists y \in S \setminus \{x\} \text{ such that } y \geq x\}$ for all $S \in \Sigma$. Note that, due to the strict comprehensiveness assumption, the strong and the weak Pareto sets coincide.

Pareto optimality: For all $(S, d) \in \mathcal{B}$, $F(S, d) \in P(S)$.

Another standard axiom is individual rationality. It requires that none of the agents is worse off at a solution than at the disagreement outcome. The set of individually rational bargaining outcomes for $(S, d) \in \mathcal{B}$ is given by $IR(S, d) := \{x \in S \mid x \geq d\}$.

Individual rationality: For all $(S, d) \in \mathcal{B}$, $F(S, d) \in IR(S, d)$.

Another property that is of importance in this paper is a domination axiom. It requires that, for a given feasible set S and two disagreement points d and e, there is a domination relationship between an agent's gains $F(S, d) - d$ and $F(S, e) - e$. For any two problems with the same feasible set of utility vectors S, all agents are at least as well-off in the solution outcome corresponding to the disagreement point d as they are in the solution outcome obtained for the disagreement point e, or vice versa. This axiom turns out to be implied by our efficiency requirement for bargaining problems with uncertain disagreement points.

Domination: For all $S \in \Sigma$, for all $d, e \in I(S)$,

$$F(S, d) - d \geq F(S, e) - e \;\text{ or }\; F(S, e) - e \geq F(S, d) - d.$$

The efficiency axiom that is of most importance in this paper is the ex ante efficiency property induced by the maximin criterion. In order to define this axiom, we use the following terminology. For $S \in \Sigma$, $d, e \in I(S)$, and $x, y \in S$, we say that the pair (x, y) min-dominates the pair $(F(S, d), F(S, e))$ if

$$\min\{x_i - d_i, y_i - e_i\} > \min\{F_i(S, d) - d_i, F_i(S, e) - e_i\}$$

for all $i \in N$.

Maximin efficiency: For all $S \in \Sigma$, for all $d, e \in I(S)$, there exist no $x, y \in S$ such that (x, y) min-dominates $(F(S, d), F(S, e))$.

In the larger part of this paper we study the consequences of this condition for the solution F. The relevance of this axiom is in its interpretation in terms of the agents' assessment of uncertain bargaining outcomes. Suppose that the feasible set S is known but the disagreement point is uncertain: all that is known to the agents is that the disagreement point is one of the two vectors d and e. Consequenctly, if the solution F is to be employed, agent $i \in N$'s gain will be $F_i(S,d) - d_i$ or $F_i(S,e) - e_i$, and the above criterion assumes that he (pessimistically) evaluates this uncertain gain by taking the minimum of these two numbers. In that case, ex ante efficiency of F in the presence of this disagreement point uncertainty means that F satisfies maximin efficiency.

The optimistic counterpart of maximin efficiency is maximax efficiency. In that case, an agent uses her highest possible utility to evaluate contingent bargaining outcomes under disagreement point uncertainty. Analogously to min-domination, max-dominmation is defined as follows. For $S \in \Sigma$, $d, e \in I(S)$, and $x, y \in S$, we say that the pair (x, y) max-dominates the pair $(F(S,d), F(S,e))$ if

$$\max\{x_i - d_i, y_i - e_i\} > \max\{F_i(S,d) - d_i, F_i(S,e) - e_i\}$$

for all $i \in N$. The corresponding efficiency property is maximax efficiency.

Maximax efficiency: For all $S \in \Sigma$, for all $d, e \in I(S)$, there exist no $x, y \in S$ such that (x, y) max-dominates $(F(S,d), F(S,e))$.

The consequences of imposing this alternative decision criterion will be discussed briefly in Section 4.

3 Maximin efficiency and monotone path solutions

We begin our investigation of the consequences of maximin efficiency by showing that maximin efficiency is equivalent to the conjunction of Pareto optimality and domination. This observation is employed in the proof of the main characterization result in this section and is stated in the following lemma.

Lemma 1 *A solution F is maximin efficient if, and only if, F is Pareto optimal and satisfies domination.*

Proof Assume that F is maximin efficient. We first show Pareto optimality and then domination.

Suppose, to the contrary, that F is not Pareto optimal. Then there is an $(S,d) \in \mathcal{B}$ and an $x \in S$ with $x > F(S,d)$. Thus, obviously, the pair (x, x) min-dominates $(F(S,d), F(S,d))$, and we have a violation of maximin efficiency.

Next suppose, again to the contrary, that F does not satisfy domination. Then there exist $S \in \Sigma$, $d, e \in I(S)$, and $i, j \in N$ with

$$F_i(S,d) - d_i > F_i(S,e) - e_i \quad \text{and} \quad F_j(S,d) - d_j < F_j(S,e) - e_j.$$

Let $N_d \subset N$ be the subset of agents $k \in N$ for which $F_k(S,d) - d_k > F_k(S,e) - e_k$ and let $N_e \subset N$ be the subset of agents $k \in N$ for which $F_k(S,d) - d_k < F_k(S,e) - e_k$. By definition, $i \in N_d$ and $j \in N_e$. By the strict comprehensiveness of S, we can find an $x \in S$ by perturbing $F(S,d)$ and a $y \in S$ by perturbing $F(S,e)$ such that the following inequalities are satisfied:

$$\begin{array}{ll} \text{for all } k \in N_d, & F_k(S,d) - d_k > x_k - d_k > y_k - e_k > F_k(S,e) - e_k; \\ \text{for all } k \in N_e, & F_k(S,e) - e_k > y_k - e_k > x_k - d_k > F_k(S,d) - d_k; \\ \text{for all } k \in N \setminus (N_d \cup N_e), & x_k > F_k(S,d) \text{ and } y_k > F_k(S,e). \end{array}$$

Then, by construction, (x,y) dominates $(F(S,d), F(S,e))$, contradicting maximin efficiency.

Now assume that F is Pareto efficient and satisfies domination. Suppose, contrary to what we want to prove, that F does not satisfy maximin efficiency. Then there are $S \in \Sigma$, $d, e \in I(S)$, and $x, y \in S$ such that the pair (x,y) min-dominates the pair $(F(S,d), F(S,e))$. By domination, without loss of generality, $F_i(S,d) - d_i \leq F_i(S,e) - e_i$ for all $i \in N$. Then, for all $i \in N$,

$$x_i - d_i \geq \min\{x_i - d_i, y_i - e_i\} > \min\{F_i(S,d) - d_i, F_i(S,e) - e_i\} = F_i(S,d) - d_i.$$

This implies $F(S,d) \notin P(S)$, a violation of Pareto optimality. □

Now we turn to a characterization of the class of all solutions satisfying maximin efficiency. This class is a generalization of the class of monotone path solutions (see, for example, Thomson and Myerson, 1980). In order to introduce those solutions formally, we need some further definitions.

A monotone path is a subset X of $\mathbb{R}^n \setminus \mathbb{R}^n_{--}$ such that, for all $x, y \in X$, $x \geq y$ or $y \geq x$. We say that a monotone path X is compatible with an admissible set $S \in \Sigma$ if, for all $d \in I(S)$, $(d + X) \cap P(S) \neq \emptyset$. Observe that by the monotonicity property of a monotone path the latter set can contain at most one point.

The solution F is a generalized monotone path solution if for every $S \in \Sigma$, there is a monotone path Φ^S compatible with S such that, for every $(S,d) \in \mathcal{B}$, we have $\{F(S,d)\} = (d + \Phi^S) \cap P(S)$. In that case, we say that F is generated by the collection $\{\Phi^S \mid S \in \Sigma\}$. F is an individually rational generalized monotone path solution if F is a generalized monotone path solution generated by a collection $\{\Phi^S \mid S \in \Sigma\}$ such that $\Phi^S \subset I\!R^n_+$ for all $S \in \Sigma$. Finally, F is a monotone path solution if F is an individually rational generalized monotone path solution such that $\Phi^S = \Phi^T$ for all $S, T \in \Sigma$.

In geometrical terms, given a monotone path Φ^S compatible with $S \in \Sigma$, the corresponding solution outcome for $(S, d) \in \mathcal{B}$ is obtained by translating Φ^S using the translation vector d and intersecting the resulting set with the Pareto frontier of S. The class of generalized monotone path solutions—unlike the class of monotone path solutions discussed in the earlier literature—contains solutions that are not individually rational. Individual rationality is ensured if all monotone paths Φ^S are such that they do not contain any nonpositive points as in the definition of the individually rational generalized monotone path solutions. A graphical illustration of the construction of some monotone path solutions is provided in Figure 1 (the individually rational case) and in Figure 2 (without individual rationality).

[**Figures 1 and 2 about here**]

The compatibility requirement defined above is important to ensure that the solution generated by a collection of monotone paths is well-defined. Figure 3 illustrates an example where the monotone path $X = \{(t, t-2) \mid t \in (1, \infty)\}$ is compatible with a set $S \in \Sigma$, whereas Figure 4 shows that the same path may be incompatible with other admissible sets such as S'.

[**Figures 3 and 4 about here**]

It is easy to verify that a monotone path compatible with an admissible set S is connected and unbounded from above. However, it may or may not be bounded from below. For example, the monotone path $X = \{(t, t-1) \mid t \in I\!R_{++}\}$ is compatible with all $S \in \Sigma$ and bounded from below, whereas the path $X' = \{(t, \ln(t)) \mid t \in I\!R_{++}\}$ is compatible with all admissible feasible sets as well but X' obviously is not bounded from below. See Figures 5 and 6 for illustrations.

[**Figures 5 and 6 about here**]

We now obtain the following characterization of the class of generalized monotone path solutions which, to the best of our knowledge, has not been discussed and axiomatized in the earlier literature.

Theorem 1 *A solution F is maximin efficient if, and only if, it is a generalized monotone path solution.*

Proof It is easy to see that a generalized monotone path solution is Pareto optimal and satisfies domination. Therefore the if-part of the theorem follows by Lemma 1.

For the only-if part assume that F is maximin efficient. Hence, by Lemma 1, F is Pareto optimal and satisfies domination. Fix $S \in \Sigma$, and define

$$\Phi^S := \{x \in I\!R^n \mid x = F(S,d) - d \text{ for some } d \in I(S)\}.$$

By Pareto optimality, $x \notin I\!R^n_{--}$ for all $x \in \Phi^S$, and by domination, $x \leq y$ or $x \geq y$ for all $x, y \in \Phi^S$. So Φ^S is a monotone path. It follows directly that Φ^S is compatible with S and that, in particular, $\{F(S,d)\} = (d + \Phi^S) \cap P(S)$ for every disagreement point $d \in I(S)$. $\square$

By adding individual rationality to maximin efficiency, a characterization of the individually rational generalized monotone path solutions is obtained. Note that this class is still considerably larger than the class of monotone path solutions themselves because no restrictions are imposed on the relationship between the monotone paths for different admissible sets. The proof of this characterization result is immediate and is therefore omitted.

Theorem 2 *A solution F is maximin efficient and individually rational if, and only if, it is an individually rational generalized monotone path solution.*

We now add a stronger axiom than individual rationality to maximin efficiency in order to obtain a characterization of the monotone path solutions. This axiom requires that a solution is invariant with respect to translations that leave the set of the agents' gains over the disagreement outcome of a problem unchanged.

IR-invariance: For all $(S,d), (T,e) \in \mathcal{B}$, if $IR(S-d, \mathbf{0}) = IR(T-e, \mathbf{0})$, then $F(S,d) = F(T,e)$.

IR-invariance combines a translation invariance property with an independence of non-individually rational alternatives property. See, for example, Peters (1992) for a discussion of axioms of that nature. As a preliminary result, we state the following lemma.

Lemma 2 *If a solution F is Pareto optimal and IR-invariant, then F is individually rational.*

Proof Let $(S,d) \in \mathcal{B}$ and let $x \in P(S) \setminus IR(S,d)$. By strict comprehensiveness, there exists $(T,d) \in \mathcal{B}$ with $IR(T,d) = IR(S,d)$ and $x \notin T$. Then IR-invariance implies that $x \neq F(S,d)$. Hence, F is individually rational. □

The next theorem shows that, in combination with maximin efficiency, IR-invariance can be used to characterize the monotone path solutions. This characterization differs from those that can be found in the earlier literature in that it is largely based on a disagreement point axiom—changes in the feasible set only enter through the IR-invariance axiom.

Theorem 3 *A solution F is maximin efficient and IR-invariant if, and only if, it is a monotone path solution.*

Proof The if-part is obvious. For the only-if part let F be a solution satisfying maximin efficiency and IR-invariance. By Theorem 2 and Lemma 2, F is an individually rational generalized monotone path solution. We have to show that all the paths involved in generating this solution are identical.

As a first step, consider two feasible sets $V, W \in \Sigma$ whose boundaries $P(V)$ and $P(W)$ are hyperplanes. Let Φ^V and Φ^W denote the corresponding monotone paths according to F, and take an arbitrary real number $t > 0$ and points $v \in I(V)$ and $w \in I(W)$ such that $F(V,v) - v$ is the point of Φ^V with sum of the coordinates equal to t, and $F(W,w) - w$ is the point of Φ^W with sum of the coordinates equal to t. Next, let $d, e \in I(V \cap W)$ be such that $IR(V \cap W - d, \mathbf{0}) = IR(V - v, \mathbf{0})$ and $IR(V \cap W - e, \mathbf{0}) = IR(W - w, \mathbf{0})$. IR-invariance applied twice then yields that, up to the point with sum of the coordinates equal to t, the monotone paths Φ^V and Φ^W must coincide because they both coincide with $\Phi^{V \cap W}$ up to t. Since t was chosen arbitrarily, we conclude that $\Phi^V = \Phi^W$. Therefore, the monotone paths corresponding to the values of the solution F for admissible sets determined by hyperplanes are identical.

Finally, let $(S,d) \in \mathcal{B}$ be arbitrary. In view of the preceding step, it is sufficient to prove that $\{F(S,d)\} = (d + \Phi^V) \cap P(S)$ for some admissible set $V \in \Sigma$ determined by an arbitrary hyperplane. Let $t > 0$ be equal to the sum of the coordinates of $F(S,d) - d$. By the strict comprehensiveness of S, it is possible to find such a set $V \in \Sigma$ and a disagreement point $v \in I(V \cap S)$ such that $IR(S,d) = IR(S \cap V, d)$, $IR(V,v) = IR(S \cap V, v)$, and $F(V,v) - v$ has sum of the coordinates equal to t. The desired result now follows by applying IR-invariance twice. □

4 Alternative decision criteria

As is the case for solutions to bargaining problems under uncertainty regarding the feasible set (see Bossert, Nosal, and Sadanand, 1996), applying the maximax decision criterion leads to rather undesirable conclusions. In particular, if efficiency with respect to the maximin criterion is replaced with maximax efficiency, we obtain an impossibility result.

Theorem 4 *There exists no maximax efficient solution.*

Proof Let F be a solution, and consider any $(S,d) \in \mathcal{B}$ and $i, j \in N$ with $i \neq j$. Because S is strictly comprehensive, there exist $x, y \in S$ such that $x_i < F_i(S,d)$, $x_k > F_k(S,d)$ for all $k \in N \setminus \{i\}$, $y_j < F_j(S,d)$, and $y_k > F_k(S,d)$ for all $k \in N \setminus \{j\}$. Then (x,y) max-dominates $(F(S,d), F(S,d))$, which proves that F cannot be maximax efficient. □

In order to avoid this impossibility result, the above maximax efficiency requirement could be weakened by requiring x and y to be in the individually rational portion of (S,d) and (S,e), respectively. In that case, dictatorial solutions (among others) become available.

Another alternative to the maximin criterion is to use minimax regret. As in Bossert and Peters (1998), this leads, in the individually rational case, to solutions that are dual to those described in Theorems 2 and 3, where monotone paths originating from the utopia point (the vector of maximal payoffs of the agents within the individually rational portion of a problem) rather than the disagreement point are used. Of course, these solutions are not well-defined without individual rationality because utopia points do not exist in that case. See Bossert and Peters (1998) for a formal definition and a discussion of minimax regret in the context of bargaining problems.

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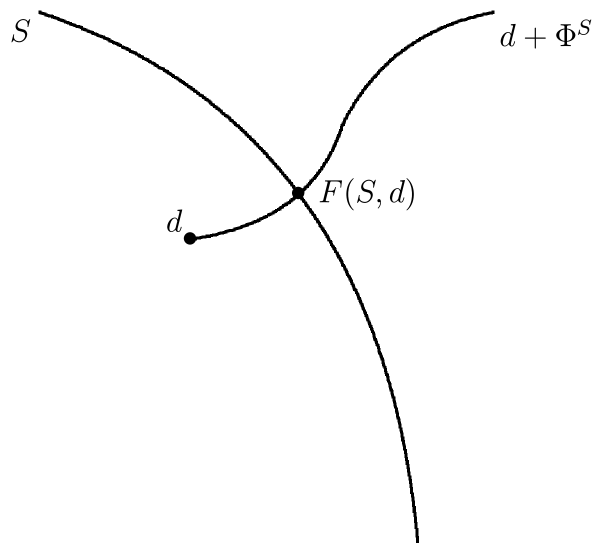

Figure 1: An individually rational generalized monotone path solution

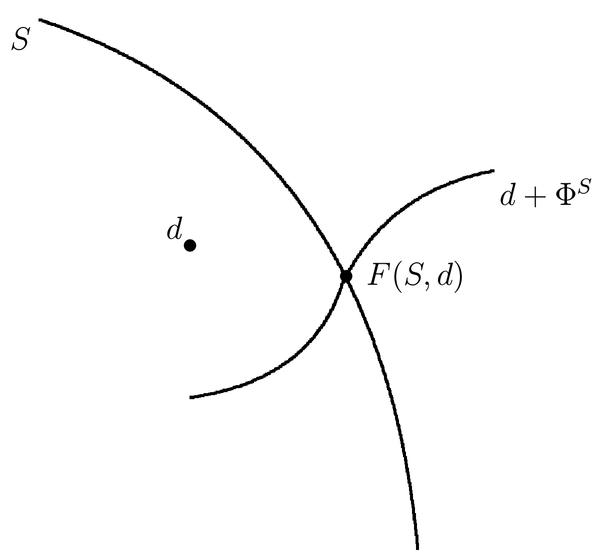

Figure 2: A nonindividually rational generalized monotone path solution

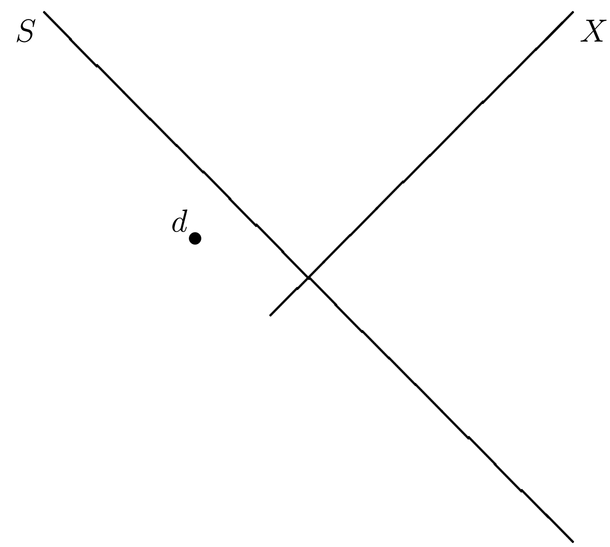

Figure 3: Compatibility

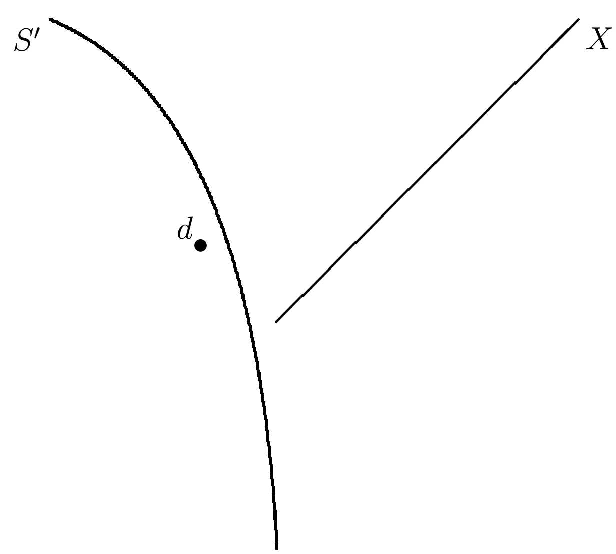

Figure 4: Incompatibility

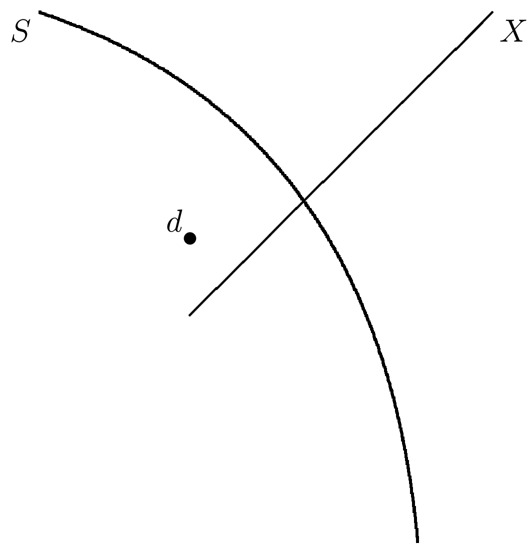

Figure 5: A path that is bounded from below

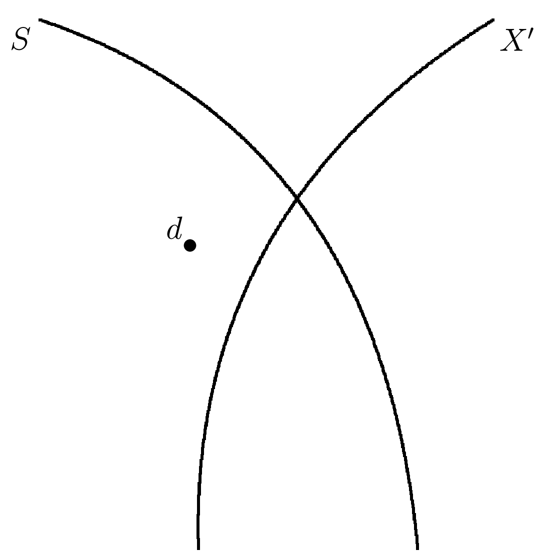

Figure 6: A path that is not bounded from below