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A Stochastic Generalization of the Revealed Preference Approach to the Theory of Consumers' Behavior[•]

by

Taradas Bandyopadhyay
Department of Economics, University of California, Riverside CA 92521, USA.

Indraneel Dasgupta
School of Economics, University of Nottingham, Nottingham, NG7 2RD, UK.

and

Prasanta K. Pattanaik
Department of Economics, University of California, Riverside CA 92521, USA.

Address for Correspondence:

Prasanta K. Pattanaik
Department of Economics
University of California, Riverside, CA 92521, USA.
Ph: (909) 787-5037, ext. 1592; Fax: (909) 787-5685.
E-mail: ppat@ucrac1.ucr.edu

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Abstract

We generalize the stochastic revealed preference approach to consumers' behavior introduced by Bandyopadhyay, Dasgupta and Pattanaik (1999). We identify a restriction on stochastic demand behavior that we term "Stochastic Substitutability" (SS). The Weak Axiom of Stochastic Revealed Preference (WASRP) introduced by them implies stochastic substitutability, a result which generalizes Samuelson's Fundamental Substitution Theorem. The classical substitution theorem and the stochastic non-positivity of the own substitution effect follow from this result. If the consumer spends her entire wealth with probability one, then WASRP is equivalent to SS. The deterministic recoverability result also follows from our equivalence relationship.

KEYWORDS: Stochastic Choice, Weak Axiom of Stochastic Revealed Preference,
Stochastic Substitutability, Stochastic Substitution Theorem.

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1. Introduction

The classical revealed preference approach to the theory of consumers' behavior assumes a deterministic demand function. However, experimental evidence suggests that the choices of individuals may not always be open to explanation in terms of a deterministic objective function. In response to this, and in contrast to the classical theory, a sizable literature has developed which attempts to model stochastic preference and/or stochastic choice behavior.[1] In this literature, given a set of feasible options, say $\{a, b\}$, the agent's choice behavior is described by a probability distribution that assigns probability d to *a*, it being permissible to have $d \in (0,1)$. Intuitively, this modeling strategy tries to capture the situation where, faced repeatedly with *apparently* the same feasible set, the agent seems to choose some option some of the time, but seems to reject that option in favor of other options the rest of the time. Such behavior could be perceived either because the external observer fails to account for changes in some aspects relevant for the agent's decision-making process, thereby mis-specifying the feasible set, or because the agent's preferences themselves are subject to random shocks.[2]

The earlier literature however has paid little attention to the specific economic problem of stochastic choices made by a *competitive consumer*.[3] Recently, this problem has been explored by Bandyopadhyay, Dasgupta and Pattanaik (1999), who introduce a stochastic counterpart of the familiar weak axiom of revealed preference (WARP), called the Weak Axiom of Stochastic Revealed Preference (WASRP), and analyze its implications for demand behavior when one price changes at a time. They show that this restriction implies a stochastic counterpart of the standard non-positivity property of the own substitution effect (NPS).

A natural generalization of this analysis is to the case where more than one price is allowed to change at a time. In the deterministic case, such a general framework yields Samuelson's well-known Fundamental Substitution Theorem (see Samuelson (1947), Chapter 5), from which the deterministic NPS property follows as a special case. The obvious question to ask therefore is whether one can identify the restrictions imposed by WASRP on the stochastic substitution effect of multiple price changes, and thereby develop a generalized version of the Fundamental Substitution Theorem, from which both the classical deterministic result and the stochastic NPS property will follow as special

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[1] The contributions include Nandeibaum (1999), Barbera & Pattanaik (1986), Block & Marschak (1960), Falmagne (1978), Fishburn (1973, 1977, 1978), Georgescu-Roegen (1936, 1950, 1958), Halldin (1974), Luce (1958, 1959, 1977) and Quandt (1956).

[2] The first problem is of course familiar to econometricians and experimental decision-theorists. The second issue has received much attention from psychologists. The actual underlying reason for randomness in an agent's observed choice behavior is however not germane to our analysis.

[3] Exceptions are Halldin (1974) and Quandt (1956), whose approaches are very different from that in Bandyopadhyay, Dasgupta and Pattanaik (1999).

cases. Such a generalization is also of interest because it allows one to address the issue of recovering WASRP from intuitively plausible restrictions on stochastic demand behavior, a question left unaddressed by Bandyopadyay et al. (1999). The purpose of this paper is to investigate these problems.

We identify a restriction on stochastic demand behavior that we term "Stochastic Substitutability" (SS). We show that WASRP implies SS, a result which provides a stochastic generalization of the classical fundamental substitution theorem. We show that both the classical substitution theorem and the stochastic non-positivity of the own substitution effect follow from this result. Furthermore, in the presence of the restriction that the consumer spend all her wealth with probability one, SS is equivalent to WASRP. The standard recoverability result for the classical deterministic case follows from our generalized equivalence relationship.

Section 2 introduces the basic notation and definitions. Section 3 introduces stochastic substitutability and discusses its intuitive basis. The formal results are presented in Sections 4 and 5. We conclude in Section 6.

2. Some Basic Notation and Definitions

Let *m* denote the number of commodities, $m \geq 2$, and let *M* denote the set $\{1,\ 2,...,\ m\}$. R_+ and R_{++} will denote, respectively, the set of non-negative real numbers and the set of positive real numbers. We assume that R_+^m constitutes the consumer's consumption set. A price-wealth situation is an ordered pair *(p, W)*, where $p \in R_{++}^m$ and $W \in R_+$. Let *Z* denote the set of all possible price-wealth situations. Given a price-wealth situation, *(p, W)*, the *budget set* of the consumer is defined to be $\{x \in R_+^m \mid W \geq p.x\}$. The budget sets corresponding to price-wealth situations *(p, W)*, *(p', W')*, (p^*, W^*) etc. will be denoted, respectively, by *B*, *B'*, *B** etc.

Definition 2.1. A *stochastic demand function* (SDF) is a rule *D*, which, for every $(p, W) \in Z$, specifies exactly one probability measure[4] *q* over the class of all subsets of the budget set *B*.

Remark 2.2. Let $q = D(p, W)$, where *D* is the SDF. Then, for every subset *A* of *B*, *q(A)* is to be interpreted as the probability that, given the price-wealth situation *(p, W)*, the consumer's chosen bundle will belong to the set *A*. Throughout this paper, *q* and *q'* will denote, respectively, *D(p, W)* and *D(p', W')*, where *D* is the SDF.

[4] Let X be a given set and let F be an algebra, i.e., a non-empty class of subsets of X, such that, for all $A, B \in F$, $[A \cup B \in F \text{ and } A - B \in F]$, and $[X \in F]$. Then, a probability measure defined on F is a countably additive function $u : F \rightarrow [0,1]$ such that $u(X) = 1$ (see Adams and Guillemin (1996), p.42).

Definition 2.3. A stochastic demand function *D* is *degenerate* iff, for every $(p,W)\in Z$, there exists $x^* \in B$ such that $q(\{x^*\})=1$.

Definition 2.4. A *deterministic demand function* (equivalently, a *demand function*) is a rule *d* which, for every price-wealth situation *(p, W)*, specifies exactly one element *d(p, W)* of *B*.

Remark 2.5. From an intuitive point of view, the concept of a degenerate SDF is the same as the concept of a deterministic demand function used in the standard theory of consumers' behavior. More formally, given a degenerate SDF, *D*, one can "induce" a unique deterministic demand function as follows: for every $(p,W)\in Z$, *d(p, W)* is the consumption bundle $x^* \in B$ such that $q(\{x^*\})=1$. Conversely, given a deterministic demand function *d*, one can "induce" a unique degenerate SDF, *D*, such that, for every price-wealth situation (p,W), $q(\{d(p,W)\})=1$. Thus, there is a one-to-one correspondence between degenerate demand functions and deterministic demand functions.

Definition 2.6. A stochastic demand function *D* is *tight* iff, for every price-wealth situation (p,W), $q(\{x\in B \mid p.x=W\})=1$.

Remark 2.7. If the SDF is tight, then, for every price-wealth situation, the consumer spends her entire wealth with probability 1.

Definition 2.8. A stochastic demand function *D* satisfies the weak axiom of stochastic revealed preference (WASRP) iff, for all *(p, W)*, *(p', W')*$\in Z$, and for every $A\subseteq[B\cap B']$, $q(B-B')\geq q'(A)-q(A)$.

Remark 2.9. WASRP was first introduced by Bandyopadhyay, Dasgupta and Pattanaik (1999), who also discuss its intuitive justification. It is weaker than the condition of rationalizability in terms of stochastic orderings discussed by Block and Marschak (1960), Falmagne (1978), Barbera and Pattanaik (1986) and Nandeibam (1999).

Remark 2.10. WASRP constitutes the stochastic counterpart of Samuelson's WARP for deterministic demand functions. Recall that a deterministic demand function *d* satisfies WARP iff, for all price-wealth situations *(p, W)* and *(p', W')*, $[d(p,W)\neq d(p',W') \text{ and } d(p',W')\in B]$ implies $[p'.d(p,W)>W']$. It is easy to see that, if the SDF is degenerate and satisfies WASRP, then the deterministic demand function induced by *D* satisfies WARP, and, conversely, if a deterministic demand function *d* satisfies WARP, then the degenerate SDF induced by *d* must satisfy WASRP.

Definition 2.11. For every price-wealth situation *(p, W)*, let *E(p, W)* denote the set $\{e\in[0,W] \mid q(\{x\in R_+^m \mid W-e\geq p.x\})=1\}$.

E(p, W) is the set of all non-negative numbers *e* such that: (a) *e* does not exceed *W*, and (b) given *p*, if we reduce the consumer's wealth to *(W – e)*, then the set of bundles that the consumer would lose would not have a positive probability attached to it by *q*, where *q = D(p, W)*. For example, suppose, for some $x^* \in B$, we have $q(\{x^*\})=1$. Then, *E(p, W*) is simply the closed interval $[0,W-p.x^*]$

Remark 2.12. Since it contains 0, the set *E(p, W)* must be non-empty for every $(p,W)\in Z$. Furthermore, for every e in *E(p, W)*, $[q[B(p,W)-B(p,W-e)]=0]$. If the SDF is tight, then $E(p,W)=\{0\}$ for every $(p,W)\in Z$. It can be shown that the set *E(p, W)* must be compact.

3. Stochastic Substitutability

We shall now identify an intuitively plausible restriction on consumers' demand behavior, which we call Stochastic Substitutability.

Samuelson's classical substitution theorem for deterministic demand functions starts by assuming that the consumer satisfies WARP, and derives, from this assumption, empirically testable restrictions on the behavior of the consumer when changes in commodity prices are accompanied by a 'compensating change' in the consumer's wealth. The compensating wealth change is defined with reference to the initially chosen consumption bundle, say x_*, so that the consumer's wealth in the new situation is exactly that amount which is required to buy x_* under the new price vector.

In our stochastic case, however, we no longer necessarily have a uniquely chosen consumption bundle in the initial situation. Since we no longer have a unique reference bundle, we can no longer define uniquely the magnitude of compensating change in the consumer's wealth either. How then does one translate the restrictions on demand behavior imposed by the classical substitution theorem to our stochastic context?

A natural way to approach this problem seems to be from the other end, i.e., to first define some arbitrary level of compensation instead, and then to examine the kind of restriction that stochastic demand behavior may be expected to satisfy, given this arbitrary level of change in wealth associated with a change in commodity prices. It is therefore useful to first identify the kind of restriction that WARP imposes on demand behavior in the deterministic case if one adopts such a procedure.

Let d be a deterministic demand function. Given any ordered pair of price-wealth situations $\langle (p,W),(p',W')\rangle$, let $x_*=d(p,W)$ and let $x'_*=d(p',W')$. WARP imposes restrictions on demand behavior only if $p'.x_*\le W'$. The first step is therefore to identify a restriction on the level of compensation, i.e., on $(W-W')$, which suffices to ensure this. Note that, for every $e\in E(p,W)$, it must be the case that $p.x_*\le W-e$. It follows that:

if $[(W-W')\le(p-p').x_*+e]$ for some $e\in E(p,W)$, then $[p'.x_*\le W']$. (3.1)

Now suppose the demand function *d* satisfies WARP. Then (3.1) yields:

for every $e\in E(p,W)$, if $[(W-W')\le(p-p').x_*+e]$, then either $x_*=x'_*$ or $p.x'_*>W$. (3.2)

It is easy to check that, if $p.x'_*>W$, then $(W-W')<(p-p').x'_*$. Hence, decomposing (3.2), we can get the following.

Remark 3.1. Let d be a deterministic demand function. Given any ordered pair of price-wealth situations $\langle (p,W),(p',W')\rangle$, let $x_* = d(p,W)$ and let $x'_* = d(p',W')$. If d satisfies WARP, then, for every ordered pair of price-wealth situations $\langle (p,W),(p',W')\rangle$, and for every $e \in E(p,W)$,

if $[(W-W') \leq (p-p').x_* + e]$, then $[(W-W') \leq (p-p').x'_* + e]$; (3.3)

if $[(W-W') < (p-p').x_* + e]$, then $[(W-W') < (p-p').x'_* + e]$; (3.4)

and

if $[[(W-W') = (p-p').x_* + e]$ *and* $[x_* \neq x']]$, then $[(W-W') < (p-p').x'_* + e]$. (3.5)

In Remark 3.1 we specify the restrictions that are imposed on demand behavior in the deterministic case by WARP, when the problem is reformulated in terms of arbitrary changes in wealth, i.e., in terms of arbitrary magnitudes of (W –W'), rather than in terms of exact compensation, as in the classical case. It seems natural therefore to impose restrictions on stochastic demand behavior that are stochastic analogues of the deterministic restrictions specified in (3.3), (3.4) and (3.5).

Let *D* be a stochastic demand function and let $\langle (p,W),(p',W')\rangle$ be any ordered pair of price-wealth situations. Given any $e \in E(p,W)$, obvious stochastic counterparts of the restrictions (3.3), (3.4) and (3.5) are, respectively:

$$[q'(\{x' \in B' \mid (p-p').x' \geq (W-W')-e\}) \geq q(\{x \in B \mid (p-p').x \geq (W-W')-e\})], \quad (3.6)$$

$$[q'(\{x' \in B' \mid (p-p').x' > (W-W')-e\}) \geq q(\{x \in B \mid (p-p').x > (W-W')-e\})], \quad (3.7)$$

and,

for every non-empty proper subset *A* of

$$[\{x \in B \mid (p-p').x = (W-W')-e\} \cap \{x' \in B' \mid (p-p').x' = (W-W')-e\}],$$

$$[q'(\{x' \in B' \mid (p-p').x' > (W-W')-e\}) + q'(A) \geq q(\{x \in B \mid (p-p').x > (W-W')-e\}) + q(A)]. \quad (3.8)$$

The restriction on stochastic demand behavior that we now introduce simply combines (3.6), (3.7) and (3.8), and therefore derives its intuitive plausibility from the argument developed above.

Definition 3.2. A stochastic demand function *D* satisfies *stochastic substitutability* (SS) iff, for all ordered pairs of price-wealth situations $\langle (p,W),(p',W')\rangle$, for all $e \in E(p,W)$, and for all $A \subseteq [\{x \in B \mid (p-p').x = (W-W')-e\} \cap \{x' \in B' \mid (p-p').x' = (W-W')-e\}]$, we have:

$$[q'(\{x' \in B' \mid (p-p').x' > (W-W')-e\}) + q'(A)] \geq$$
$$[q(\{x \in B \mid (p-p').x > (W-W')-e\}) + q(A)]. \quad (3.9)$$

4. The Stochastic Substitution Theorem

Using a general framework that permits simultaneous changes in several prices, in this section we explore the implications, for the stochastic demand behavior of the consumer, of the weak axiom of stochastic revealed preference. We show that WASRP implies Stochastic Substitutability. This result, which we term the Stochastic Substitution Theorem, in turn yields both Samuelson's Fundamental Substitution Theorem and the central result of Bandyopadhyay et al. (1999) as special cases.

Proposition 4.1 (Stochastic Substitution Theorem). *WASRP implies SS.*

We shall prove Proposition 4.1 via the two following lemmas.

Lemma 4.2. *Let the SDF satisfy WASRP. Let* $(p,W) \in Z$, $e \in E(p,W)$ *and let* $W'' = W - e$. *Then, for every subset A of B,* $q(A) = q''(A \cap B'')$, *where* $B'' = B(p,W'')$ *and* $q'' = D(p,W'')$.

Proof of Lemma 4.2: By construction, $B'' \subseteq B$ and $[q[B - B''] = 0]$ (see Remark 2.12). Hence, by WASRP, for every subset a of B'', we have $[q(\text{a}) \le q''(\text{a})]$ and $[q''(\text{a}) \le q(\text{a})]$. Hence,

for every subset a of B'', $q(\text{a}) = q''(\text{a})$. (4.1)

Since $q(B - B'') = 0$, it is obvious that:

for every subset A of B, $q(A) = q(A \cap B'')$. (4.2)

Lemma 3.7. follows from (4.1) and (4.2). ◊

Lemma 4.3. *If an SDF satisfies WASRP, then, for all ordered pairs of price-wealth situations* $\langle (p,W),(p',W') \rangle$, *and for all*

$A \subseteq [\{x \in B \mid (p - p').x = (W - W')\} \cap \{x' \in B' \mid (p - p').x' = (W - W')\}]$, *we must have:*

$$[q'(\{x' \in B' \mid (p - p').x' > (W - W')\}) + q'(A)] \ge [q(\{x \in B \mid (p - p').x > (W - W')\}) + q(A)]. \tag{4.3}$$

Proof of Lemma 4.3: Let the SDF satisfy WASRP. Consider any ordered pair of price-wealth situations $\langle (p,W),(p',W') \rangle$ and any

$$A \subseteq [\{x \in B \mid (p - p').x = (W - W')\} \cap \{x' \in B' \mid (p - p').x' = (W - W')\}].$$

We need to show that the SDF satisfies (4.3) for this case.

Let $a = W - W'$. We first show that:

$$[B - B'] \subseteq \{x \in B \mid (p - p').x < a\}; \tag{4.4}$$

$$\{x' \in B' \mid (p - p').x' < a\} \subseteq \{x \in B \mid (p - p').x < a\}. \tag{4.5}$$

and

$$\{x \in B \mid (p - p').x = a\} = \{x' \in B' \mid (p - p').x' = a\}. \tag{4.6}$$

First consider any $x \in [B - B']$. Clearly, we must have $W \geq p.x$ and $W' < p'.x$. Hence $[a = W - W' > (p - p').x]$. (4.4) follows immediately. Now consider any $x' \in B'$ such that $(p - p').x' < a$. Since $[W = W' + a]$, for any such x', it must be the case that $[W \geq p'.x' + a > p.x']$. Hence $x' \in B$; (4.5) follows. Claim (4.6) can be established in an analogous fashion.

By (4.4) and (4.5), $\{x \in B \mid (p - p').x < a\}$ can be partitioned into $[B - B']$ and $\{x' \in B' \mid (p - p').x' < a\}$. Hence,

$$q(\{x \in B \mid (p - p').x < a\}) = q(\{x' \in B' \mid (p - p').x' < a\}) + q(B - B'). \tag{4.7}$$

Let $\overline{B} = [\{x \in B \mid (p - p').x = a\} \cap \{x' \in B' \mid (p - p').x' = a\}]$.

Since, by (4.5), $\{x' \in B' \mid (p - p').x' < a\} \subseteq [B \cap B']$, by WASRP,

$$q(B - B') \geq [q'(\{x' \in B' \mid (p - p').x' < a\}) - q(\{x' \in B' \mid (p - p').x' < a\})] + [q'(\overline{B} - A) - q(\overline{B} - A)]$$

which yields:

$$q'(\{x' \in B' \mid (p - p').x' < a\}) + q'(\overline{B} - A) \leq [q(B - B') + q(\{x' \in B' \mid (p - p').x' < a\})] + q(\overline{B} - A). \tag{4.8}$$

By (4.7) and (4.8), we have:

$$[q'(\{x' \in B' \mid (p - p').x' < a\}) + q'(\overline{B} - A)] \leq [q(\{x \in B \mid (p - p').x < a\}) + q(\overline{B} - A)]. \tag{4.9}$$

Noting (4.6), (4.3) follows immediately from (4.9). ◊

Proof of Proposition 4.1: Let $e \in E(p, W)$, $a = W - W'$, and let $W'' = W - e$. Thus, $W' = W'' - (a - e)$. Let the budget set corresponding to *(p, W")* be denoted by *B"*, and let *q"* denote *D(p, W")*. Let $\overline{B} = [\{x \in B \mid (p - p').x = a - e\} \cap \{x' \in B' \mid (p - p').x' = a - e\}]$.

First consider any $A \subseteq \overline{B}$. Clearly, $[q'(A) \geq q'(A \cap B'')]$. Now, since $W'' = W - e$ and $e \in E(p, W)$, by Lemma 4.2, we must have: (i) $[q(A) = q''(A \cap B'')]$ and (ii) $[q(\{x \in B \mid (p - p').x > a - e\}) = q''(\{x'' \in B'' \mid (p - p').x'' > a - e\})]$. Hence, to establish Proposition 4.1, it is sufficient to show that:

for all $\hat{A} \subseteq [\{x \in B'' \mid (p - p').x = (a - e)\} \cap \{x' \in B' \mid (p - p').x' = (a - e)\}]$,

$$[q'(\{x' \in B' \mid (p - p').x' > a - e\}) + q'(\hat{A})] \geq [q''(\{x'' \in B'' \mid (p - p').x'' > a - e\}) + q''(\hat{A})]. \tag{4.10}$$

Noting that, given the specification of *W"*, $(a - e) = W'' - W'$, (4.10) follows from Lemma 4.3. ◊

Remark 4.4. We now show that the basic result regarding the implications of WARP for consumers' demand behavior, namely, the Fundamental Substitution Theorem due to Samuelson[5] can be derived from our Stochastic Substitution Theorem (Proposition 4.1).

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[5] See Samuelson (1947) and Mas-Colell, Whinston and Green (1995, pp.28-32).

Corollary 4.5 (Samuelson's Fundamental Substitution Theorem). *Let d be a deterministic demand function. Given any ordered pair of price-wealth situations* $\langle (p,W),(p',W')\rangle$, *let* $x_* = d(p,W)$ *and let* $x'_* = d(p',W')$. *If d satisfies WARP, then, for every ordered pair of price-wealth situations* $\langle (p,W),(p',W')\rangle$ *such that* $[W' = p'.x_*]$, *it must also satisfy:*

$$[0 \geq (p-p')(x_* - x'_*)]; \tag{4.11}$$

and

$$[0 > (p-p')(x_* - x'_*)] \text{ if } x_* \neq x'_*. \tag{4.12}$$

Proof: Let *D* be the degenerate stochastic demand function induced by *d* (see Remark 2.5); then,

$$q(\{x_*\}) = q'(\{x'_*\}) = 1.$$

Let $e = W - p.x_*$ and $a = W - W'$. Clearly, given $q(\{x_*\}) = 1$, $e \in E(p,W)$. Also,

$$(p-p').x_* = a - e. \tag{4.13}$$

Hence,

$$q(\{x \in B \mid (p-p').x \geq a-e\}) = 1. \tag{4.14}$$

Suppose that *d* satisfies WARP. Then *D* satisfies WASRP (see Remark 2.10). Then, by Proposition 4.1, the SDF must satisfy SS. Hence, it must satisfy (3.9) for

$$A = \{x \in B \mid (p-p').x = a-e\} \cap \{x' \in B' \mid (p-p').x' = a-e\}.$$

Then, using (4.6), we have:

$$q'(\{x' \in B' \mid (p-p').x' \geq a-e\}) \geq q(\{x \in B \mid (p-p').x \geq a-e\}). \tag{4.15}$$

Given (4.14), (4.15) implies

$$q'(\{x' \in B' \mid (p-p').x' \geq a-e\}) = 1;$$

which yields:

$$(p-p').x'_* \geq a-e. \tag{4.16}$$

Combining (4.13) and (4.16), we get (4.11).

Now suppose $x'_* \neq x_*$, but $(p-p').x'_* = a-e$. Then, clearly,
$[q(\{x \in B \mid (p-p').x > a-e\}) + q(\{x_*\}) = 1]$, but $[q'(\{x' \in B \mid (p-p').x > a-e\}) + q'(\{x_*\}) = 0]$; which violates SS. Hence, using (4.16), we get:

$$\text{if } x'_* \neq x_*, \text{ then } (p-p').x'_* > a-e. \tag{4.17}$$

Combining (4.13) and (4.17), we get (4.12). ◊

Remark 4.6. Since Samuelson's Substitution Theorem implies non-positivity of the own-price substitution effect for deterministic demand functions, it follows from Corollary 4.5 that this fundamental result in standard (deterministic) demand theory is also generated by our Stochastic Substitution Theorem. The central theorem presented in Bandyopadhyay et al. (1999) is a stochastic extension of this result. Their theorem can also be shown to follow from the Stochastic Substitution Theorem as a special case.

Corollary 4.7 (Bandyopadhyay, Dasgupta and Pattanaik (1999)). *Let D be an SDF satisfying WASRP. Let* $i \in M$, $(p,W),(p',W') \in Z$ *and* $b \in [0, W/p_i]$ *be such that* $\lfloor p_i > p_i';\ p_k = p_k'$ *for all* $k \in [M - \{i\}]$; *and* $W' = W - (p_i - p_i').b \rfloor$. *Then:*

$$q'(\{x' \in B' \mid x_i' \geq b\}) \geq q(\{x \in B \mid x_i \geq b\}); \tag{4.18}$$

and

$$q'(\{x' \in B' \mid x_i' > b\}) \geq q(\{x \in B \mid x_i > b\}). \tag{4.19}$$

Proof: Let $a = W - W'$ and let $e = 0$. By Proposition 4.1, if the SDF satisfies WASRP, it must satisfy (3.9) for $A = \{x \in B \mid (p-p').x = a\} \cap \{x' \in B' \mid (p-p').x' = a\}$. Then, using (4.6), we get:

$$q'(\{x' \in B' \mid (p-p').x' \geq a\}) \geq q(\{x \in B \mid (p-p').x \geq a\}). \tag{4.20}$$

Furthermore, if the SDF satisfies SS, it must satisfy (3.9) for $A = \phi$. Hence, Proposition 4.1 implies

$$q'(\{x' \in B' \mid (p-p').x' > a\}) \geq q(\{x \in B \mid (p-p').x > a\}). \tag{4.21}$$

As only the price of the *i-th* commodity differs between *p* and *p'*, $a = (p_i - p_i')b = (p-p').b$. Then (4.18) and (4.19) follow immediately from (4.20) and (4.21), respectively. ◊

5. Recovering the Weak Axiom of Stochastic Revealed Preference

Since, by the Stochastic Substitution Theorem, WASRP implies SS, a natural question to ask is whether the empirical content of WASRP is completely exhausted by SS, i.e., whether WASRP can be recovered as a consequence of SS alone. In this section, we first show that WASRP is in fact stronger than SS. However, given the additional restriction that the consumer spends her entire wealth with probability one, SS is equivalent to WASRP.

Proposition 5.1. *SS does not imply WASRP.*

Proof. The proof consists of an example. Let λ be a positive real number. Define an SDF, *D*, such that, for every price-wealth situation *(p, W)*, *D* assigns probability 1 to the set $\{x\}$, where:

$$\lfloor\ x_1 = \lambda \quad if \quad W/p_1 > \lambda \quad ; x_1 = 0 \ \ otherwise\ \rfloor, \text{ and } \lfloor\ \forall k \in M - \{1\},\ [x_k = 0]\ \rfloor.$$

We shall first show that D satisfies SS.

Consider any arbitrary ordered pair of price-wealth situations $\langle (p,W),(p',W') \rangle$, and let $q(\{x_*\}) = 1$, $q'(\{x_*'\}) = 1$. If D violates SS for this pair of price-wealth situations, then it must be the case that, for some $e \in E(p,W)$ and for some $A \subseteq [\{x \in B \mid (p-p').x = (W-W') - e\} \cap \{x' \in B' \mid (p-p').x' = (W-W') - e\}]$,

$$[q'(\{x' \in B' \mid (p-p').x' > (W-W') - e\}) + q'(A)]$$
$$< [q(\{x \in B \mid (p-p').x > (W-W') - e\}) + q(A)].$$

Since the SDF is degenerate, this is possible only if $[[(p-p').x_* \geq (p-p').x'] \textit{ and } x_* \neq x'_*]$. However, it is easy to check that, if $x_* \neq x'_*$, it must be the case that $(p-p').x_* < (p-p').x'_*$. Thus, *D* must satisfy SS.

To see that *D* violates WASRP, first consider a price-wealth situation (p,W) such that $W/p_1 = \mathsf{l}$. Let 0^{n-1} be the (n-1)-dimensional null vector. By the specification of *D*, $q(\{(0,0^{n-1})\}) = 1$. Now consider another price vector, p', such that $[\forall k \in N - \{1\}, [p_k = p'_k]]$ and $p_1 > p'_1$, so that $W/p'_1 > \mathsf{l}$. By the specification of *D*, $q'(\{(\mathsf{l}, 0^{n-1})\}) = 1$. Clearly, $(\mathsf{l}, 0^{n-1}), (0, 0^{n-1}) \in B(p,W) \cap B(p',W)$, $q'(\{(\mathsf{l}, 0^{n-1})\}) - q(\{(\mathsf{l}, 0^{n-1})\}) = 1$, but $[q(B(p,W) - B(p',W)) = 0]$. This however violates WASRP. ◊

We now show that, for tight SDFs, WASRP is implied by SS.

Proposition 5.2. *Given tightness, WASRP is violated for a pair of price-wealth situations only if SS is violated for that pair of price-wealth situations.*

Proof: Let *D* be a tight SDF, let $\langle (p,W),(p',W') \rangle$ be any ordered pair of price-wealth situations and let A be any subset of $B \cap B'$. Let *D* satisfy SS for this pair of price-wealth situations. We need to show that:

$$q(B - B') \geq q'(A) - q(A). \tag{5.1}$$

Let $a = W - W'$. We shall first show that:

$$\{x \in B \mid p.x = W \textit{ and } (p-p').x < a\} \subseteq [B - B']; \tag{5.2}$$

and

$$\{x' \in B' \mid p'.x' = W' \textit{and } (p-p').x' > a\} \subseteq [B' - B]. \tag{5.3}$$

Consider, for any $r \in R^m_+$, the term $t = (p-p')r - a = (W' - p'.r) - (W - p.r)$. Obviously,

$$(p-p').r \begin{array}{c} < \\ = \\ > \end{array} a \text{ iff } t \begin{array}{c} < \\ = \\ > \end{array} 0. \tag{5.4}$$

Claims (5.2) and (5.3) follow immediately from (5.4).

Since the SDF is tight, from (5.2) and (5.3) we get, respectively,

$$q(\{x \in B \mid (p-p').x < a\}) \leq q(B - B'); \tag{5.5}$$

and

$$q'(\{x' \in A \mid (p-p').x' > a\}) = 0. \tag{5.6}$$

Now consider the set $\widetilde{A}=[[\{x\in B\,|\,(p-p').x=a\}\cap\{x'\in B'\,|\,(p-p').x'=a\}]-A]$. Since $0\in E(p,W)$, and the SDF satisfies the restrictions imposed by SS for $\langle(p,W),(p',W')\rangle$, we have:

$$q'(\{x'\in B'\,|\,(p-p').x'>a\})+q'(\widetilde{A})\geq q(\{x\in B\,|\,(p-p').x>a\})+q(\widetilde{A}). \tag{5.7}$$

Noting that (5.4) also implies $\{x\in B\,|\,(p-p').x=a\}=\{x'\in B'\,|\,(p-p').x'=a\}$, we then have:

$$q'(\{x'\in B'\,|\,(p-p').x'<a\})+q'(\{x'\in A\,|\,(p-p').x'=a\})$$
$$\leq q(\{x\in B\,|\,(p-p').x<a\})+q(\{x\in A\,|\,(p-p').x=a\}). \tag{5.8}$$

Inequalities (5.5) and (5.8) together yield

$$q'(\{x'\in B'\,|\,(p-p').x'<a\})+q'(\{x'\in A\,|\,(p-p').x'=a\})-q(\{x\in A\,|\,(p-p').x=a\})$$
$$\leq q(B-B'). \tag{5.9}$$

Clearly, (5.9) implies

$$q'(\{x'\in A\,|\,(p-p').x'\leq a\})-q(A)\leq q(B-B'). \tag{5.10}$$

(5.6) and (5.10) together yield (5.1). ◊

Remark 5.3. Proposition 4.1 and Proposition 5.2 together imply that, given tightness, WASRP is equivalent to SS. That SS does not imply WASRP in the absence of tightness is shown by the example used to establish Proposition 5.1, which violates tightness. This example also shows that SS does not imply tightness.

Remark 5.4. The recoverability condition for WARP (see Mas-Colell et al. (1995), pp. 31-32) follows from Proposition 5.2.

Corollary 5.5. *If, for every $(p,W)\in Z$, $[p.d(p,W)=W]$, and if, for every ordered pair of price-wealth situations $\langle(p,W),(p',W')\rangle$ such that $[W'=p'.d(p,W)]$, d satisfies (4.11) and (4.12), then it must also satisfy WARP.*

Proof: If WARP is violated, there must be a compensated price change for which it is violated.[6] Therefore, to establish Corollary 5.5, it suffices to establish that, given the restrictions on *d*, WARP cannot be violated for any compensated price change.

Consider any ordered pair of price-wealth situations $\langle(p,W),(p',W')\rangle$ such that $[W'=p'.d(p,W)]$. Let $x_*=d(p,W)$, $x'_*=d(p',W')$ and let $a=W-W'$. Then

$$(p-p').x_*=a\,. \tag{5.11}$$

Suppose that *d* satisfies (4.11) and (4.12). Then (5.11) yields:

$$(p-p').x'_*>a \text{ if } x_*\neq x'_*. \tag{5.12}$$

Consider the degenerate SDF, *D*, induced by *d*. Let *A* be any arbitrary subset of $\{x\in B\,|\,(p-p').x=a\}\cap\{x'\in B'\,|\,(p-p').x'=a\}$. First suppose $x_*\in A$. Then, clearly, (5.12)

[6] See Mas-Colell et al. (1995) pp. 31-32.

implies that $q'(\{x' \in B' \mid (p-p').x' > a\}) + q'(A) = 1$. Now suppose $x_* \notin A$. Then, by (5.11), $q(\{x \in B \mid (p-p').x > a\}) + q(A) = 0$. Thus (3.9) must be satisfied in both cases. By the specification of *d*, *D* is tight. Hence, $E(p,W) = \{0\}$. Thus, *D* cannot violate SS for this compensated price change. Hence, by Proposition 5.2, *D* cannot violate WASRP for this compensated price change either. It follows that *d* cannot violate WARP for any compensated price change. ◊

Remark 5.6. That tightness is a restriction independent of both WASRP and SS is intuitively apparent, and can be formally established by the following examples. First consider an SDF, *D*, such that, for every price-wealth situation, *D* assigns probability 1 to the singleton set containing the null vector. This SDF satisfies WASRP (and hence, by Proposition 4.1, SS as well) but violates tightness. Now consider a degenerate SDF, *D*, such that, for every price-wealth situation (p,W), *D* assigns probability 1 to the set $\{x\}$, where: $\left[x_1 = \frac{W}{p_1} \text{ if } \frac{W}{p_1} \leq \frac{W}{p_2};\ x_1 = 0 \text{ otherwise} \right]$, $\left[x_2 = 0 \text{ if } \frac{W}{p_1} \leq \frac{W}{p_2};\ x_2 = \frac{W}{p_2} \text{ otherwise} \right]$, and $\lfloor\ \forall k \in N - \{1,2\},\ [x_k = 0]\ \rfloor$. Clearly, *D* satisfies tightness. However, the deterministic demand function induced by D violates WARP. Hence, D violates WASRP (and hence SS as well).

6. Conclusion

In this paper, we have generalized the stochastic revealed preference approach to consumers' behavior introduced by Bandyopadhyay, Dasgupta and Pattanaik (1999). We have identified a restriction on stochastic demand behavior that we have termed stochastic substitutability. We have shown that the Weak Axiom of Stochastic Revealed Preference introduced by Bandyopadyay, Dasgupta and Pattanaik (1999) implies stochastic substitutability, and in turn is implied by it when the consumer spends her entire wealth with probability one. The existing results in the literature such as the classical substitution theorem, the deterministic recoverability result and the central result in Bandyopadyay, Dasgupta and Pattanaik (1999) have all been shown to follow as special cases from the results presented in this paper.

References

Adams, M. R. and V. Guillemin (1996): *Measure Theory and Probability* (Boston: Birkhauser).

Bandyopadhyay, T., I. Dasgupta and P. K. Pattanaik (1999): “Stochastic Revealed Preference and the Theory of Demand”, *Journal of Economic The*ory (84): 95-110.

Barbera, S. and P. K. Pattanaik (1986): “Falmagne and the Rationalizability of Stochastic Choices in terms of Random Orderings”, *Econometrica* (54): 707-715.

Block, H. D. and J. Marschak (1960): “Random Orderings and Stochastic Theories of Response” in I. Olkin, S. Ghurye, W. Hoeffding, W. Madow and H. Mann (eds.) *Contributions to Probability and Statistics* (Stanford: Stanford University Press).

Falmagne, J. C. (1978): “A Representation Theorem for Finite Random Scale Systems”, *Journal of Mathematical Psychology* (18): 52-72.

Fishburn, P. (1978): “Choice Probabilities and Choice Functions”, *Journal of Mathematical Psychology* (18): 205-219.

--- (1977): “Models of Individual Preference and Choice”, *Synthese* (36): 287-314.

--- (1973): “Binary Choice Probabilities: On the Varieties of Stochastic Transitivity”, *Journal of Mathematical Psycho*logy (10): 327-352.

Georgescu-Roegen, N. (1958): “Threshold in Choice and the Theory of Demand”, *Econometrica* (26): 157-168.

--- (1950): “The Theory of Choice and the Constancy of Economic Laws”, *Quarterly Journal of Economics* (64): 125-138.

--- (1936): “The Pure Theory of Consumer’s Behaviour”, *Quarterly Journal of Economics* (50): 545-593.

Halldin, C. (1974): “The Choice Axiom, Revealed Preference and the Theory of Demand”, *Theory and Decision* (5): 139-160.

Luce, R. D. (1977): “The Choice Axiom after Twenty Years”, *Journal of Mathematical Psychology* (15): 215-233.

--- (1959): *Individual Choice Behaviour* (New York: Wiley).

--- (1958): “A Probabilistic Theory of Utility”, *Econometrica* (26): 193-224.

Mas-Colell, A., M. D. Whinston and J. Green (1995): *Microeconomic Theory* (New York: Oxford University Press).

Nandeibam, S. (1999):

Quandt, R. (1956): “A Probabilistic Theory of Consumer Behaviour”, *Quarterly Journal of Economics* (70): 507-536.

Samuelson, P. A. (1947): *Foundations of Economic Analysis* (Cambridge, Mass.: Harvard University Press).