UNIVERSITY OF NOTTINGHAM

SCHOOL OF ECONOMICS

DISCUSSION PAPER NO. 99/37

Spurious rejections by Perron tests in the presence of a misplaced or second break under the null

Tae-Hwan Kim, Stephen J. Leybourne, Paul Newbold

School of Economics, University of Nottingham,
Nottingham NG7 2RD, UK

Abstract

It is known that Dickey-Fuller tests can lead to spurious rejections of the unit root null hypothesis when the true generating process is difference-stationary with a break. Suppose now that an unsuccessful attempt is made to allow for a break, either through misplaced dummy variables or through neglecting a second break. It is demonstrated that spurious rejections can now occur for a broader set of true break dates than would be the case if the possibility of a break was ignored.

1. Introduction

In a seminal paper, Perron (1989) demonstrated that Dickey-Fuller tests may have little power when the true generating process is stationary around a broken linear trend (see also Montanes and Reyes 1998). Conversely, Leybourne et al (1998) showed that when the true generating process is difference stationary, but with a break, routine application of Dickey-Fuller tests can yield spurious rejections of the unit root null hypothesis when the neglected break is relatively early. This is so for both a break in level and a break in drift, although there are important differences between the two cases. Perron (1989, 1993, 1994) and Perron and Vogelsang (1992, 1993) discuss modifications of the tests that circumvent these difficulties by incorporating dummy variables at the break date. For the procedures considered here, the Perron tests have null distributions that are invariant to the magnitude of the break when there is a single break date, exogenously (and correctly) determined.

In this paper, we concentrate on the case of an exogenously chosen break date, but entertain the possibility that an incorrect choice is made. In fact, the Perron test statistics considered are invariant to any break in the generating process at the *assumed* break date. Our results therefore apply equally to the possibly more practically important case of a generating process with two breaks, only one of which is specifically accounted for in the analysis. As in Leybourne et al (1998), we find that a neglected relatively early break can lead to spurious rejections of the unit root null hypothesis. Moreover, for all but one of the tests analysed, spurious rejections now also arise if a true break occurs relatively soon after the assumed break date. Thus, for example, if the true generating process is difference-stationary with two relatively close breaks, a rejection of the null hypothesis is likely to occur if dummy variables accounting for only the earlier of these breaks are incorporated into the analysis.

This phenomenon arises for both breaks in level and breaks in drift, though, as in Leybourne et al (1998), there are important differences between the two cases, which we analyse respectively in sections 2 and 3. We consider both additive outlier and innovational outlier variants of the Perron statistics, restricting our theoretical analysis to the former. Simulation evidence suggests only minor differences in results for corresponding variants of the two types of tests.

2. Breaks in level

Consider the simple difference-stationary generating process for the series y_t

$$y_t = d_{1t}(t) + n_t, \quad n_t = n_{t-1} + e_t, \quad t = 1, \ldots, T \tag{1}$$

where e_t are i.i.d. disturbances with mean 0 and standard deviation s. A level break of magnitude sb, occurring a fraction $t \in (0,1)$ through the series, is specified as

$$d_{1t}(t) = sb = skT^{1/2} 1[t > tT]. \tag{2}$$

In fact, if b is held constant as sample size increases, the asymptotic distributions of the test statistics are invariant to break magnitude. However, allowing break magnitude to grow at rate $T^{1/2}$ generates limiting distributional results that predict what is actually found in moderate-sized samples.

The additive outliers variant of the Perron test, with assumed break fraction $t^* \in (0,1)$, is based on two regression steps

$$y_t = \hat{a} + \hat{b}t + \hat{g}d_{1t}^*(t^*) + e_t \tag{3}$$

and

$$e_t = re_{t-1} + f\Delta d_{1t}^*(t^*) + w_t \tag{4}$$

where w_t in (4) is an error term, and $d_{1t}^*(t^*)$ is a dummy variable allowing a level break a fraction t^* through the series,

$$d_{1t}^*(t^*) = 1[t > t^*T]. \tag{5}$$

Of course, if a break is assumed at the correct place and there is no additional break, so that $t = t^*$, this is the setup analysed by Perron (1989, 1993, 1994) and Perron and Vogelsang (1993). In particular, the limiting null distribution of the test statistic, taken here to be the *t*-ratio associated with the estimate of $(r-1)$ in (4), and denoted AO_{lev}, is invariant to the break magnitude.

Our concern now is with the case where the assumed break fraction t^* differs from the true fraction t. It can then be shown that, for fixed k in (2), the limiting distribution of the Perron test statistic is of the form

$$AO_{lev} \Rightarrow \frac{1}{s(1+I_1k^2)^{1/2}} \frac{E + I_1(E_1 + E_2)}{[F + I_1(F_1 + F_2)]^{1/2}} \tag{6}$$

where I_1 is the indicator function $I_1 = 1[t \neq t^*]$, so that in the case $t = t^*$, (6) reduces to $s^{-1}F^{-1/2}E$, which notation we use for the functionals in the limiting distribution given by Perron (1993) and Perron and Vogelsang (1993). More generally in expression (6), when $t \neq t^*$ these functionals depend on the *assumed* break fraction t^*. Also, in (6) E_2 and F_2 are random variables with means 0, while the quantities E_1 and F_1 are deterministic. In fact, as we see below, it is these deterministic quantities that dominate the behaviour of the limiting distribution for moderately large *k*, leading to predictions of spurious rejections.

Figure 1 shows E_1 and F_1 when the assumed break fraction is $t^* = 0.5$, $s = 1$, and $k = 1$. (Algebraic expressions for these quantities are given in the Appendix). Also, for comparison we show the means of $(E + E_2)$ and $(F + F_2)$. From Figure 1, it can be seen that the impact of E_1 is to induce a very large reduction in the mean of the numerator in (6) when the true break fraction t is either very small or a very little more than the assumed break fraction of 0.5. Moreover, as can be seen from the graph of F_1 in Figure 1, it is precisely for true breaks in these places that the positive quantity F_1 contributes virtually no additional compensating increment to the denominator of (6). We might therefore conjecture that spurious rejections of the unit root null are likely to occur both for very early true breaks and for break fractions a little higher than t^*.

This conjecture is confirmed by Figure 2, which graphs the results of some simulation experiments. Series of $T = 100$ observations were generated from (1), with normally distributed innovations e_t and $s = 1$. Break magnitudes *b* of (2) were set at 5 and 10. A sequence of true break fractions t was analysed, but in all experiments the assumed break fraction t^* was set at 0.5 in (5), and the test statistic AO_{lev} was calculated from the regressions (3) and (4). Nominal 5% significance levels were found by setting $t = t^* = 0.5$. Then, for other values of t, Figure 2 shows the percentages of rejections of the unit root null hypothesis at nominal 5%- levels. Here and throughout the paper, simulations were based on 20,000 replications. These results confirm our predictions. Very frequent rejections of the null hypothesis occur for both early true breaks and for true breaks a little after the assumed break date. The first of these phenomena is simply the analogue of that reported by Leybourne et al (1998) for Dickey-Fuller tests in the presence of a neglected break, while the second is an additional region of spurious rejections generated by the (misplaced) insertion of dummy variables into the estimated models. Notice that the spurious rejection phenomenon is almost equally severe in the two regions - about 30% rejections for a break of 5 standard deviations, and over 70% for a break of 10 standard deviations.

The innovational outliers variant of the test is based on the regression

$$y_t = a + bt + gd_{1t}^*(t^*) + f\Delta d_{1t}^*(t^*) + ry_{t-1} + w_t \quad (7)$$

where w_t is an error term and $d_{1t}^*(t^*)$ is defined in (5). The test statistic IO_{lev} is the t-ratio associated with the estimate of $(r-1)$ in (7). Using exactly the same generating process as for the additive outliers variant of the test, we again estimated rejection percentages for the IO_{lev} test with t^* set at 0.5. These are also shown in Figure 2, and are virtually identical to those of the AO_{lev} test in the most interesting regions, though there are slight differences outside these regions.

It is worth re-emphasising that, while our results are based on the generating process (1) and (2), they would continue to hold in the presence of an *additional* true break at time t^*T. They can thus be interpreted as the consequences of allowing for only one break when two are present as well as of misplacing a single true break.

3. Breaks in drift

We now consider the case where the true generating process is difference-stationary, but with a break in drift. The simplest generating process of this form is

$$y_t = d_{2t}(t) + n_t, \quad n_t = n_{t-1} + e_t, \quad t = 1, \ldots, T \quad (8)$$

where e_t are i.i.d. disturbances with mean 0 and standard deviation s, and

$$d_{2t}(t) = sb(t - tT)1[t > tT]. \quad (9)$$

Several variants of the Perron test might be employed when such a break is suspected. The most general possibility allows for a break in both level and slope at time t^*T. The additive outliers variant of the test is then based on the two regression steps,

$$y_t = \hat{a} + \hat{b}t + \hat{g}d_{1t}^*(t^*) + \hat{d}d_{2t}^*(t^*) + e_t \quad (10)$$

followed by the regression (4). We consider the test statistic $AO1_{tre}$ given by the t-ratio associated with the estimate of $(r-1)$. In (10), the dummy variable $d_{1t}^*(t^*)$ is defined in (5), and

$$d_{2t}^*(t^*) = (t - t^*T)1[t > t^*T]. \quad (11)$$

Insight into the behaviour of the test statistic when the assumed break fraction t^* differs from the true fraction t can be obtained via the probability limit (plim). In fact, the t-ratio diverges, but, as shown in Vogelsang and Perron (1998), the plim of

$$T^{-1/2} AO1_{tre} = \frac{T(\hat{r} - 1)}{[T^3 V\hat{a}r(\hat{r})]^{1/2}} \tag{12}$$

does exist. Our interest here is to extend their results to identify the precise region in (t, t^*) space where this plim is negative. Through an approach sketched in the Appendix, we find for the numerator of (12) that for any nonzero fixed break amount *b* of (9), when $t \neq t^*$,

$$T(\hat{r} - 1) \rightarrow_p \frac{3}{2} \frac{2t - t^* - I_2}{(I_2 - t)(t - t^*)} \tag{13}$$

where $I_2 = 1[t^* < t]$. Then, since the denominator of (13) is always positive, the plim of (12) has the same sign as the numerator of (13). It then follows that the Perron test statistic $AO1_{tre}$ diverges to $-\infty$ in the region given by:

$$R = (t < \tfrac{1}{2}t^*) \cup [(t > t^*) \cap (t < \tfrac{1}{2}t^* + \tfrac{1}{2})] \tag{14}$$

Thus, the size of the Perron test tends to 100% in this region for any fixed break amount *b*. In particular then notice, as for the breaks in level case of the previous section, there are two sets of values of the true break fraction t for which spurious rejections can be expected when the assumed break fraction $t^* \neq t$.

The expression for the plim of the denominator of (12) is considerably more involved (and does depend on b, s) and rather than provide the algebra we present in Figure 3 (a) a three-dimensional plot of the plim of $T^{-1/2} AO1_{tre}$ for $s = 1$ and $b = 3$. For ease of interpretation the function is truncated above 0, so that only values for the region R of (14) are shown. This is of most interest as it is the region where spurious rejections might be expected for sufficiently large samples. Beyond revealing the region (14), the shape of Figure 3 (a) is interesting, as the depth of the plim should give an indication of the likely relative severity of the spurious rejection phenomenon. First, notice that the function falls and then eventually rises both as t increases from 0 and as t increases from t^*. This suggests that the most severe cases of spurious rejections are likely to occur when t is a little greater than 0, and again when t is a little greater than t^*. This is in contrast to the results of the previous section, where we found, for example, a "worst case" occurs when the true break is *immediately* after the assumed break. Second, which of the two disjoint sets of t values contains the severest case of spurious rejections would seem to depend on the location of t^*, and in particular on whether the assumed break fraction is more or less than one-half.

These predictions are closely verified by the simulation results presented in Figure 4, based on series of $T = 100$ observations generated from (8), (9) with e_t normally distributed with $s = 1$, and values of 1.5 and 3.0 for *b*. Percentages of rejections of the unit root null hypothesis for nominal 5%-level versions for the $AO1_{tre}$ test are shown. In part (a) of the figure, the assumed break fraction is $t^* = 0.5$. Two regions of serious spurious rejections are revealed - one where the true break occurs quite early in the series, and the second where the true break is soon after the assumed break. The position is virtually identical in these two regions, the most severe cases occurring a proportion 0.1 from the beginning of the series and the same amount from the assumed break. In part (b) of Figure 4, the assumed break fraction is $t^* = 0.85$. In that case, the initial region of spurious rejections is broader and more severe, while the second region is narrower and less severe than in the $t^* = 0.5$ case.

The innovational outliers variant of the test is based on the regression

$$y_t = a + bt + gd^*_{1t}(t^*) + f\Delta d^*_{1t}(t^*) + dd^*_{2t}(t^*) + ry_{t-1} + w_t \tag{15}$$

where w_t is an error term. Figure 4 also shows simulation evidence on the test statistic $IO1_{tre}$, the t-ratio associated with the estimate of $(r-1)$ from (15). The picture is broadly similar to that for the additive outliers variant of the test, though it appears that the spurious rejection phenomenon is a little more severe and extends to a somewhat broader range of values of the true break fraction for the innovation outliers test.

The tests discussed so far in this section permit breaks in both level and slope. An additive outliers test incorporating just the latter is based on the two regression steps

$$y_t = \hat{a} + \hat{b}t + \hat{d}d^*_{2t}(t^*) + e_t \tag{16}$$

and

$$e_t = re_{t-1} + w_t \tag{17}$$

We then consider the test statistic $AO2_{tre}$, the t-ratio associated with the estimate of $(r-1)$ from (17). As noted for example in footnote 4 of Perron (1994), the corresponding innovational outlier variant of this test is usually not recommended as the asymptotic null distribution of the test statistic is not invariant to the magnitude of any (correctly specified) break under the null hypothesis.

We investigate the plim of $T^{-1/2}AO2_{tre}$, again for the generating process (8), (9). Writing the quantity of interest in the same form as the right-hand side of (12), it can be shown that, for $t \neq t^*$,

$$T(\hat{r}-1) \to_p \frac{3}{2}\frac{(t-2t^*+I_2)[2tt^*+I_2(1-3t)+(t-2t^*)(1-I_2)]}{(t-I_2)(t^*-I_2)[3tt^*+I_2(t^*-4t)+(t-4t^*)(1-I_2)]}. \qquad (18)$$

The function (18), and therefore the plim of $T^{-1/2}AO2_{tre}$, is negative in the region

$$R = \{(t < t^*) \cap [t < 2t^*(2t^*+1)^{-1}]\} \cup \{(t > t^*) \cap [t < (3-2t^*)^{-1}]\}. \qquad (19)$$

It follows that the test statistic $AO2_{tre}$ diverges to $-\infty$ in the region (19), so that for fixed break magnitudes *b*, spurious rejections can be expected in this region for sufficiently large sample sizes.

The algebraic expression for the plim of $T^{-1/2}AO2_{tre}$ is extremely lengthy. The function, truncated above 0, is graphed for $s = 1$ and $b = 3$ in Figure 3 (b). The contrast with Figure 3 (a) is quite striking, suggesting that the spurious rejection phenomenon will seem quite different in the two cases. In particular, notice that the region associated with $t > t^*$ - that is, the second element in (19) - is both quite small and shallow, extending only over relatively low values of t. That suggests that, unless the true and misplaced breaks are quite early, a second region of spurious rejections will not be found. Indeed, even in these cases, spurious rejections will occur on either side of the assumed break date and be restricted to quite low values of t. Figure 3 (b) also suggests that, the greater is t^*, the broader will be the interval in which spurious rejections are found, and the more severe will be these rejections. Notice that, as t^* approaches 0, *R* of (19) tends to $(t < 1/3)$, while as t^* approaches 1, *R* tends to $(t < 2/3)$. (The corresponding region for spurious rejections by Dickey-Fuller tests in the presence of a break under the null is shown by Leybourne et al, 1998, to be $t < 1/2$.). Thus, while the insertion and location of a misplaced break in the $AO2_{tre}$ test can be expected to have some impact on the spurious rejection phenomenon, that impact should be far less dramatic than for the $AO1_{tre}$ test.

Figure 5 shows simulation results for the application of the $AO2_{tre}$ test, using the same generating models as for the $AO1_{tre}$ test in Figure 4. Again, as in Figure 4, assumed break fractions of $t^* = 0.5$ and 0.85 were used. As predicted, only one region of spurious rejections is found in these cases - that region increasing in width, and deepening in severity as t^* increases. Notice also that the spurious rejection phenomenon is a little more severe for the $AO2_{tre}$ test than for the $AO1_{tre}$ test of Figure 4, in the sense that the rejection frequency is somewhat higher in the worst cases for the former than for the latter.

4. Summary

We have shown theoretically and by simulation that Perron tests based on a misplaced structural break can generate two regions of spurious rejections of the unit root null hypothesis. The first of these is where the true break in the series is relatively early, mirroring the findings of Leybourne et al (1998) of spurious rejections from Dickey-Fuller tests in the presence of a break under the null. However, spurious rejections now also occur if the true break in the series is relatively soon after the assumed break. This phenomenon can be equally severe in the second region as in the first.

If a time series has a structural break, then it is entirely possible, if the assumed break is a little earlier than the true break, for spurious rejections of the unit root null hypothesis to occur through application of the Perron test in cases where this phenomenon would not be found through routine application of the Dickey-Fuller test, ignoring the possibility of a break. The results can also be interpreted as a demonstration of the potential for spurious rejections of the unit root null when specific allowance in the analysis is made for only the first of two breaks close in time. Of course, these results should not be interpreted as suggesting that unit root tests, such as the Perron tests, that allow for the possibility of structural breaks are of limited value. On the contrary, our results point in the other direction, indicating the sensitivity of the test outcomes to specification of both the number and location of any breaks. It is quite clear that any determination of the issue of unit autoregressive roots is intimately linked with successful resolution of these issues concerning breaks.

Finally, we also note that our findings also extend to tests such as that of Zivot and Andrews (1992) which endogenise the break point, and are based on the most negative value of the Perron-type unit root test across all possible values for t^*. The critical values of these tests are calculated assuming no break under the null. However, when a break exists, our results suggest that the actual critical values will be much more negative than when no break exists. Thus, spurious rejections of the unit root null will arise if the conventional critical values for such tests are employed.

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Appendix

Breaks in level

A theoretical prediction of spurious rejections found for the additive outliers variant of the Perron test is based on the following results, which we state without proof: Consider the generating process (1), (2) with k fixed. Then the test statistic AO_{lev}, the *t*-ratio associated with the least squares estimate of $(r-1)$ in (4), has the limiting null distribution given by (6). In that expression, E and F, here taken to be functions of t^*, are the random variables given in the corresponding result of Perron (1993) and Perron and Vogelsang (1993) for the case $t = t^*$. Also, E_2 and F_2 are random variables (functions of the Wiener process in $C[0,1]$) with means 0. The fixed numbers E_1 and F_1 are given by

$$E_1 \equiv -s^2k^2(I_2 - t + t^*) - s\,k(t - 1/2)H_1/D + s^2k^2(1 - t^* - I_2)H_2/D \qquad \text{(A.1)}$$

and

$$\begin{aligned} F_1 \equiv\ & s^2k^2(t - t^*)(2I_2 - 1 - t + t^*) + H_1^2/12D^2 \qquad \text{(A.2)} \\ & + s^2k^2(1 - t^*)t^*H_2^2/D^2 - s\,k(t - t^*)(1 - t - t^*)H_1/D \\ & + 2s^2k^2(t - t^*)(t^* - 1 + I_2)H_2/D + s\,k(1 - t^*)t^*H_1H_2/D^2 \end{aligned}$$

where

$$D \equiv (1/12)s^2k^2t^*(1 - t^*)[1 - 3t^*(1 - t^*)]$$

$$H_1 \equiv (1/2)s^3k^3t^*(1 - t^*)(t - t^*)(I_2 - t)$$

$$H_2 \equiv (-1/12)s^2k^2(t^* - 1 + I_2)(t - t^*)[1 + 3(I_2 - t^*)(1 - t - t^*)].$$

The functions E_1 and F_1 given by (A.1) and (A.2) are graphed in Figure 1 for k = 1, s = 1 and assumed break fraction t^* = 0.5. It is the behaviour of these functions for moderate k that suggests the possibility of spurious rejections when the Perron test is applied with a misplaced break.

Breaks in drift

We sketch how to obtain the plim given by (13). Consider the first regression step (10). The Frisch-Waugh-Lovell Theorem gives us

$$\hat{q} = A^{-1}(b_1 + b_2)$$

where $\hat{q} \equiv \begin{bmatrix} \hat{b} \\ \hat{g} \\ \hat{d} \end{bmatrix}$, $A \equiv \begin{bmatrix} \sum \bar{t}^2 & \sum \bar{t}\bar{d}_{1t}^* & \sum \bar{t}\bar{d}_{2t}^* \\ \sum \bar{t}\bar{d}_{1t}^* & \sum \bar{d}_{1t}^{*2} & \sum \bar{d}_{1t}^*\bar{d}_{2t}^* \\ \sum \bar{t}\bar{d}_{2t}^* & \sum \bar{d}_{1t}^*\bar{d}_{2t}^* & \sum \bar{d}_{2t}^{*2} \end{bmatrix}$, $b_1 \equiv \begin{bmatrix} \sum \bar{t}\bar{d}_{2t} \\ \sum \bar{d}_{1t}^*\bar{d}_{2t} \\ \sum \bar{d}_{2t}^*\bar{d}_{2t} \end{bmatrix}$, $b_1 \equiv \begin{bmatrix} \sum \bar{t}\bar{v}_t \\ \sum \bar{d}_{1t}^*\bar{v}_t \\ \sum \bar{d}_{2t}^*\bar{v}_t \end{bmatrix}$,

$\bar{d}_{it} \equiv \bar{d}_{it}(t)$, $\bar{d}_{it}^* \equiv \bar{d}_{it}(t^*)$, and all variables are expressed as deviations from their sample means. It can be shown that $D_2^{-1} A D_1^{-1} \to_p A^*$, $D_2^{-1} b_1 \to_p b_1^*$, $D_2^{-1} b_2 \to_p 0$ where $D_1 \equiv \begin{bmatrix} 1 & 0 & 0 \\ 0 & T^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $D_2 \equiv \begin{bmatrix} T^3 & 0 & 0 \\ 0 & T^2 & 0 \\ 0 & 0 & T^3 \end{bmatrix}$, A^* is a non-singular 3×3 matrix, b_1^* is a 3×1 vector. Therefore, $D_1 \hat{q} \to_p A^{*-1} b_1^*$. In order to obtain the plim for each component in $\hat{q}$, first derive all plims in A^* and b_1^*, and then use the Symbolic Math Toolbox in MATLAB to compute $A^{*-1} b_1^*$ symbolically. Then we have the following results:

$$\hat{b} \to_p (1 - I_2)(t^* + 2t)(t - t^*)^2 / (t^* - I_2)^3 \quad \text{(A.3)}$$

$$T^{-1}\hat{g} \to_p (t - I_2)^2 (t - t^*)^2 / (t^* - I_2)^2$$

$$\hat{d} \to_p (t - I_2)^2 (3t^* - 2t - I_2)^3 / (t^* - I_2)^3 .$$

From the second step, one can compute that $\hat{r} - 1 = E_d / F_d$ where $E_d \equiv \sum e_{t-1} \Delta e_t - e_{[t^*T]} \Delta e_{[t^*T]+1}$ and $F_d \equiv \sum e_{t-1}{}^2 - e_{[t^*T]}{}^2$. With straightforward but tedious algebra, one can show using the results in (A.3) that

$$T^{-2} E_d \to_p (t - I_2)^2 (2t - t^* - I_2)(t - t^*)^2 / 2(t^* + I_2 - 2t^* I_2)^3$$

$$T^{-3} F_d \to_p -(t - I_2)^3 (t - t^*)^3 / 3(t^* + I_2 - 2t^* I_2)^3$$

which delivers the desired result in (13).