Explosive asset price bubble detection with unknown bubble length and initial condition

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Abstract

Recent research has proposed a method of detecting explosive processes that is based on forward recursions of OLS, right-tailed, Dickey-Fuller [DF] unit root tests. In this paper an alternative approach using GLS DF tests is considered. We derive limiting distributions for both mean-invariant and trend-invariant versions of OLS and GLS-based Phillips, Wu and Yu (2011, International Economic Review 52, 201–226) [PWY] test statistics under a temporary, locally explosive alternative. These limits are shown to be dependent on both the value of the initial condition and the start and end points of the temporary explosive regime. Local asymptotic power simulations show that a GLS version of the PWY statistic offers superior power when a large proportion of the data is explosive, but that the OLS approach is preferred for explosive periods of short duration as a proportion of the total sample. These power differences are magnified by the presence of an asymptotically non-negligible initial condition. We propose a union of rejections procedure that capitalises on the respective power advantages of both OLS and GLS-based approaches. This procedure achieves power close to the effective envelope provided by the two individual PWY tests across all settings of the initial condition and length of the explosive period considered in this paper. These results are shown to be robust to the point in the sample at which the temporary explosive regime occurs. An application of the union procedure to NASDAQ prices confirms the empirical value of this testing strategy.

Keywords: Explosive autoregression; Generalized least squares; Initial Condition

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1 Introduction

Detection of asset price bubbles in financial and macroeconomic time series data is an increasingly prominent issue in economics. Recent events such as the Dot-Com bubble of the late 1990s and the global financial crisis of 2007-2009 have highlighted the detrimental and wide-ranging effect that a bubble in a single asset market can have on the whole economy. A substantial body of literature has emerged which attempts both to improve our theoretical modelling of asset price bubbles and to design statistical tests which can detect these bubbles.

A key finding of this literature is that time series in which an asset price bubble is present follows an explosive process and can therefore be modelled by a simple first order autoregression [AR]. Consider the standard present value theory of finance for stock market prices. Starting with the standard no arbitrage condition, the real stock price, P_t, can be written as

$$P_t = \frac{1}{1+r} E_t \left(P_{t+1} + D_{t+1} \right)$$

where r is the risk-free discount rate ($r > 0$), D_t is the real dividend and E_t is the expectation at time t. Through recursive substitution of this condition it is shown that

$$P_t = P_t^f + B_t$$

where P_t^f denotes the market fundamentals component of prices, and B_t a bubble component. The fundamentals component is given by the discounted sum of future dividends

$$P_t^f = \sum_{t=1}^{\infty} (1+r)^{-i} E_t(D_{t+i}).$$

If the stochastic difference equation

$$B_{t+1} = (1+r)B_t + u_t$$

holds, where $E_{t-i}(u_t) = 0$ for all $i \geq 0$ then a rational bubble is said to exist. In the absence of bubbles, where $B_t = 0$, prices will be determined by expected future dividends. However, if a bubble is present, such that $B_t \neq 0$, the explosive behaviour of B_t dominates and prices will be explosive regardless of whether dividends follow a stationary or unit root process. Rational bubbles therefore appear as explosive behaviour in a price series.

In seminal work, Diba and Grossman (1988) propose a method of detecting rational bubbles that examines the first difference of a price series. Given that the first difference of an explosive series cannot be stationary, they apply unit root tests to the first differenced price series where rejection of the null hypothesis of a unit root implies that a rational bubble is not present in prices. Evans (1991) shows that methods which rely on full sample test procedures are likely to have low power, due to the typically temporary nature of explosive asset price bubbles. Price series which contain these periodically collapsing bubbles may appear much like unit root or even stationary processes across the full sample, such that full sample methods which examine the stochastic properties of price series will likely have low power in detecting temporary explosiveness. In recent work, Phillips et al. (2011) [PWY] propose a test for the presence of temporary explosive behaviour in time series that is based on the supremum of recursive, right-tailed unit root tests. Applying this test to monthly observations of the NASDAQ composite stock index with a sample period of February 1973 - June 2005, they find evidence of explosiveness in real prices but not in the real dividend series, and thus conclude that a rational bubble occurred in the NASDAQ stock market in this sample period.

The PWY test procedure is fundamentally based on sub-sample OLS Dickey-Fuller [DF] unit root tests. When considering full sample explosiveness, Harvey and Leybourne (2014) show that a DF unit root test based on Generalised Least Squares [GLS] demeaning or detrending has a power advantage over OLS DF tests. This result holds when the initial condition of the series, i.e. the deviation of the first observation of the sample away from the deterministics of the process, is both asymptotically negligible and non-negligible.

In this paper we consider a GLS version of the PWY test procedure and evaluate the properties of both OLS and GLS-based PWY approaches in detecting temporary explosiveness. Whilst PWY focus on mean-invariant tests only, we also consider a trend-invariant test. Many time series that are potentially subject to asset price bubbles, such as stock market prices or house prices, may follow a linear trend. It is well known that the power of a demeaned (but not detrended) unit root test is significantly affected by the presence of a trend. Therefore trend-invariant tests for explosive processes are likely to be required in addition to mean-invariant tests. We derive the limiting distributions of the two mean-invariant and the two trend-invariant PWY-type statistics under a temporary, locally explosive alternative, in the presence of an asymptotically non-negligible initial condition. These limits are shown to be dependent on both the value of the initial condition and the start and end points of the temporary explosive regime. We examine the asymptotic and finite sample power of OLS and GLS PWY-type statistics under varying initial conditions, lengths of the explosive period, and points in the sample at which the explosive period occurs. Our results show that the GLS version of the PWY statistic achieves higher power than its OLS counterpart when a long explosive period is considered, in line with the full sample results of Harvey and Leybourne (2014). However, when considering an explosive period of shorter length, we show that the OLS version of the PWY statistic has superior power. The presence of an asymptotically non-negligible initial condition does not affect the ranking of the two procedures, in contrast to the results seen in the left-tailed unit root testing context, but instead amplifies the power difference between OLS and GLS-based procedures.

Given that the relative power performance of OLS and GLS-based PWY tests depends on two factors which are likely to be unobserved in practice: the length of the explosive regime as a proportion of the sample and the value of the initial condition, we therefore propose a union of rejections procedure to capture the best available power offered by the two tests under these different sources of uncertainty. This union procedure displays power close to the effective envelope provided by the OLS and GLS-based tests for all settings of the initial condition and bubble length considered in this paper. An application of the union procedure to daily

NASDAQ prices demonstrates the differing results that OLS and GLS-based PWY tests can provide in practice and highlights the empirical value of our proposed union procedure.

Throughout this paper, we refer to both the *length* of the explosive period and the *magnitude* of the explosive process. The length of the explosive period describes the proportion of the sample for which the data follows an explosive process, as opposed to a unit root. The magnitude of the explosive process describes the growth rate of the explosive series, which is affected by both the explosive parameter in the AR model, as well as the value of the initial condition. We use the following notation throughout: $\lfloor . \rfloor$ to denote the integer part of the argument; $\xrightarrow{p}$ and $\xrightarrow{d}$ to denote convergence in probability and weak convergence respectively as the sample size diverges.

In the following section we discuss the role that the initial condition plays in explosive processes. Section 3 outlines the PWY test procedure and discusses a GLS variant. In Section 4 local asymptotic distributions of mean-invariant and trend-invariant OLS and GLS tests are derived and results from asymptotic simulations are reported for differing values of the initial condition and length of the explosive period. Section 5 proposes a union of rejections procedure and reports the local asymptotic power of this procedure. Section 6 evaluates the finite sample performance of all test statistics considered in this paper. An empirical application of the test procedures in this paper to NASDAQ prices is considered in Section 7. Section 8 concludes.

2 Initial conditions and explosive processes

Consider the following DGP

$$y_t = \mu + u_t \tag{1}$$
$$u_t = (1+\delta)u_{t-1} + v_t \qquad t = 2, ..., T \tag{2}$$

where v_t is assumed to follow a martingale difference sequence. Suppose that y_t is a price index, which has been normalized at the first value, such that $y_1 = 100$. Normalization of this form is standard when using macroeconomic or financial data as the common starting value allows easier comparison of growth rates across multiple series. PWY apply their testing procedure to monthly NASDAQ price data from February 1973 - June 2005, with the starting value normalized to 100. Phillips et al. (2015) consider a generalized version of the PWY bubble detection procedure and apply this to the S&P 500 price-dividend ratio from January 1871 to December 2010, again with the first observation normalized to 100. To assess the power of their proposed test procedures Phillips et al. (2015) consider Monte Carlo simulations based on a DGP with parameters calibrated from this empirical application, such that the simulated data series has to be initialised at a value of 100.

Consider using the DGP in (1) - (2) to generate an explosive process initialised at $y_1 = 100$. By back-substitution in (2), we can show that

$$u_t = (1+\delta)^{t-1}u_1 + \sum_{j=2}^{t}(1+\delta)^{j-2}v_{t-(j-2)}$$

such that

$$y_t = \mu + (1+\delta)^{t-1}u_1 + \sum_{j=2}^{t}(1+\delta)^{j-2}v_{t-(j-2)}. \tag{3}$$

We can therefore write the initial observation y_1 as

$$y_1 = \mu + u_1.$$

It is clear then that there are two observationally equivalent ways of initialising the process at $y_1 = 100$. Either we set $\mu = 100$ and $u_1 = 0$, or we could set $\mu = 0$ and $u_1 = 100$, such that the process u_t is initialised at a non-zero value. Of course, some combination of non-zero μ and u_1 is also possible, such that $\mu + u_1 = 100$. If u_t is a stationary series ($\delta < 0$), the process is mean-reverting and there should be a clear sense of the value of the underlying mean of the process. However, in an explosive context ($\delta > 0$), the distinction between the value of the underlying mean and the value of u_1 is less clear cut. In practice, it may be difficult to determine which of these set-ups is appropriate for an explosive process with non-zero initial value.

The initial condition of a process is defined as the deviation of the first observation of the series from the deterministics of that series. Setting $u_1 \neq 0$ implies that the process has a non-zero initial condition. For an explosive process with a non-zero first observation, it is therefore unclear whether the process has a non-zero initial condition, or indeed what size such an initial condition would be. The effect of the initial condition on the performance of unit root tests is well-documented in both the stationary and explosive contexts. When examining the null hypothesis of a unit root against an alternative of stationarity, Elliott (1999) considers a random initial condition drawn from its unconditional distribution and highlights the strong dependence of power functions of unit root tests on the value of this initial condition. Elliott et al. (1996) show that a GLS version of the DF test has superior power to its OLS counterpart if the initial condition is asymptotically negligible. However, as demonstrated by Muller and Elliott (2003), this power advantage does not hold in the presence of non-negligible, fixed initial conditions, such that for large values of the initial condition the OLS DF test has superior power to the GLS variant. Harvey et al. (2009) show that the relative power performance of these two tests is qualitatively similar under both random and fixed initial conditions.

Harvey and Leybourne (2014) examine whether the results found in the stationary case extend to the explosive, right-tailed context. Focusing on full-sample explosiveness, they find that GLS DF unit root tests retain superior power over OLS DF tests in the case of negligible initial conditions, although this power advantage is smaller than that exhibited in the left-tailed context. Interestingly, DF GLS tests are shown to maintain this power advantage over OLS DF tests for non-negligible, fixed initial conditions, in contrast to the results seen for left-tailed unit root tests. The difficulty in distinguishing between non-zero means and non-zero initial conditions in an explosive context motivates us to examine the effect that the presence of an asymptotically non-negligible initial condition has on the PWY test for detecting temporary explosiveness.

3 Right-tailed unit root tests

Consider a time series y_t where $t = 1, .., T$. We are interested in testing the null that y_t follows a unit root AR(1) process for the full sample, against the alternative that y_t is temporarily explosive, that is y_t follows an explosive AR(1) process for some sub-period of the full sample. In order to detect this temporary explosiveness, PWY propose a supremum test based on forward recursions of right-tailed, OLS DF unit root tests. Choosing some initial value τ_0 where $\tau_0 \in [0, 1]$, a right-tailed, OLS DF test is calculated over the first $\lfloor \tau_0 T \rfloor$ observations. The sample size is then increased by an additional observation, the DF test re-estimated, and so on for $T - \lfloor \tau_0 T \rfloor + 1$ recursions. Whilst PWY focus on the mean case only, we extend the analysis by considering a DF statistic which is both demeaned and detrended in addition to the original PWY approach.

The PWY-type statistics are therefore given by

$$\begin{aligned} PWY^{\mu}_{OLS} &= \sup_{\tau \in [\tau_0, 1]} DF^{\tau,\mu}_{OLS} \\ PWY^{\beta}_{OLS} &= \sup_{\tau \in [\tau_0, 1]} DF^{\tau,\beta}_{OLS} \end{aligned}$$

where $DF^{\tau,\mu}_{OLS}$ and $DF^{\tau,\beta}_{OLS}$ denote the right-tailed demeaned, and demeaned and detrended OLS DF tests respectively. The DF test is the standard t-test

$$DF^{\tau,i}_{OLS} = \frac{\widehat{\delta}_\tau}{s.e.\left(\widehat{\delta}_\tau\right)}$$

where $i = \{\mu, \beta\}$ and $\widehat{\delta}_\tau$ is the OLS estimate from the auxiliary regression

$$\Delta \widehat{u}_{\tau,t} = \delta_\tau \widehat{u}_{\tau,t-1} + v_{\tau,t} \tag{4}$$

where $\widehat{u}_{\tau,t} = y_t - z_t' \widehat{\theta}$ are the residuals from the OLS regression of y_t on $z_t = 1$, $\theta = \mu$ in the case of $DF^{\tau,\mu}_{OLS}$ and $z_t = (1, t)'$, $\theta = (\mu, \beta)'$ in the case of $DF^{\tau,\beta}_{OLS}$over the subsample period $t = 1, ..., \lfloor \tau T \rfloor$, with the standard error of $\widehat{\delta}_\tau$ given by $s.e.\left(\widehat{\delta}_\tau\right)$.

To explore whether a GLS variant of the PWY test can provide power gains in testing for temporary explosiveness, we modify the PWY statistic such that it is based on forward recursions of right-tailed, GLS demeaned, or demeaned and detrended DF tests. These tests are given by

$$\begin{aligned} PWY^{\mu}_{GLS} &= \sup_{\tau \in [\tau_0, 1]} DF^{\tau,\mu}_{GLS} \\ PWY^{\beta}_{GLS} &= \sup_{\tau \in [\tau_0, 1]} DF^{\tau,\beta}_{GLS} \end{aligned}$$

where $DF_{GLS}^{\tau,\mu}$ and $DF_{GLS}^{\tau,\beta}$ denote the right-tailed, GLS demeaned, and demeaned and detrended DF tests respectively. The DF test is the standard t-test

$$DF_{GLS}^{\tau,i} = \frac{\widetilde{\delta}_\tau}{s.e.\left(\widetilde{\delta}_\tau\right)}$$

where $i = \{\mu, \beta\}$ and $\widetilde{\delta}_\tau$ is the GLS estimate from the auxiliary regression

$$\Delta\tilde{u}_{\tau,t} = \tilde{\delta}_\tau \tilde{u}_{\tau,t-1} + \tilde{v}_{\tau,t}, \tag{5}$$

where on setting $\overline{\rho} = 1 + \overline{c}/T$ for some chosen constant $\overline{c}$, $\widetilde{u}_{\tau,t} = y_t - z_t'\widetilde{\theta}$ where $\widetilde{\theta}$ is obtained from the GLS regression of $y_{\overline{c}} = (y_1, y_2 - \overline{\rho}y_1, ..., y_{\tau T} - \overline{\rho}y_{\tau T-1})'$ on $z_{\overline{c}} = (z_1, z_2 - \overline{\rho}z_1, ..., z_{\tau T} - \overline{\rho}z_{\tau T-1})'$ where $z_t = 1$ in the case of $DF_{GLS}^{\tau,\mu}$ and $z_t = (1, t)'$ in the case of $DF_{GLS}^{\tau,\beta}$, with the standard error of $\widetilde{\delta}_\tau$ given by $s.e.\left(\widetilde{\delta}_\tau\right)$.

4 Asymptotic behaviour of tests

To examine the behaviour of the four tests discussed in Section 3 we consider the following DGP

$$y_t = \mu + \beta t + u_t \tag{6}$$

$$u_t = \begin{cases} T^{1/2}\sigma\alpha & t = 1 \\ u_{t-1} + v_t & t = 2, ..., \lfloor \tau_1 T \rfloor \\ (1+\delta)u_{t-1} + v_t & t = \lfloor \tau_1 T \rfloor + 1, ..., \lfloor \tau_2 T \rfloor \\ u_{t-1} + v_t & t = \lfloor \tau_2 T \rfloor + 1, ..., T \end{cases} \tag{7}$$

where v_t is assumed to follow a martingale difference sequence with conditional variance σ^2 and $\sup_t E(v_t^4) < \infty$. A unit root is imposed on y_t up to time $\lfloor \tau_1 T \rfloor$. We set $\delta \geq 0$ such that y_t follows an explosive process when $\delta > 0$ between time $\lfloor \tau_1 T \rfloor + 1$ and $\lfloor \tau_2 T \rfloor$. In the third regime, the series reverts back to a unit root process. We assume that on reversion back to a unit root there is no crash, such that the observation $u_{\lfloor \tau_2 T \rfloor + 1} = u_{\lfloor \tau_2 T \rfloor} + v_{\lfloor \tau_2 T \rfloor + 1}$. As Evans (1991) argues, this assumption may be unrealistic in that an empirically plausible rational bubble must have a significant chance of collapsing. They examine a class of rational bubble which has a probability of collapsing in each period of $1 - \pi$ where $0 < \pi \leq 1$. When the bubble collapses, the process falls to a mean value before potentially 'erupting' again. Periodically collapsing bubbles of this type therefore follow more complex AR(1) processes than the DGP in (7). One simple solution would be to model an instantaneous crash by re-initialising u_t when it reverts back to a unit root. PWY discuss doing this by setting the first observation of the new unit root regime to be equal to the last observation before the explosive regime began plus some stochastic element, such that $u_{\lfloor \tau_2 T \rfloor + 1} = u_{\lfloor \tau_1 T \rfloor} + v_{\lfloor \tau_2 T \rfloor + 1}$. Harvey et al. (2015) note that the supremum of forward recursions of DF unit root tests will tend to occur when the sub-sample of observations used contains only the pre-collapse period of the data. As a result, PWY-type tests are unlikely to be affected by the inclusion of an instantaneous crash in the DGP, and our analysis is therefore unlikely to be limited by this no-crash assumption.

The null hypothesis that y_t follows a unit root throughout is given by $H_0 : \delta = 0$, and the alternative that y_t exhibits temporary explosiveness is given by $H_1 : \delta > 0$. We focus on local alternative hypotheses of the form $\delta = c/T$ where $c \geq 0$. Following Harvey and Leybourne (2014), we consider two possibilities for the initial condition, u_1: it is either asymptotically negligible, $u_1 = o_p\left(T^{1/2}\right)$, or asymptotically non-negligible, where $u_1 = T^{1/2}\sigma\alpha$ and $\alpha \neq 0$ is a finite constant. Under the null hypothesis, α acts as a mean shift, such that increasing α or increasing μ both increase the mean of y_t. Under the alternative, when $\delta > 0$, we can note from (3) that the presence of a non-zero initial condition adds a deterministic explosive component to the data. In this sense, both α and the explosive parameter c contribute to the magnitude of the explosive process. Note that to illustrate the results, our analysis assumes that v_t is not serially correlated. Following PWY, serial correlation is permitted to enter the model provided that the usual ADF lag augmentation is applied to the DF tests by including lags of $\Delta\widehat{u}_{\tau,t}$ and $\Delta\tilde{u}_{\tau,t}$ in the auxiliary regressions (4) and (5) respectively. The asymptotic distributions of PWY^i_{OLS} and PWY^i_{GLS} where $i = \{\mu, \beta\}$ are given in the following theorem.

Theorem 1 *Let y_t be generated according to (6)-(7) where we assume $\beta = 0$ in the case of the two demeaned tests. For $c \geq 0$,*

$$
\begin{aligned}
PWY^{\mu}_{OLS} &\xrightarrow{d} \sup_{\tau\in[\tau_0,1]} \frac{K^{\mu}_{c,\alpha}(\tau,\tau,\tau_1,\tau_2)^2 - K^{\mu}_{c,\alpha}(0,\tau,\tau_1,\tau_2)^2 - \tau}{2\sqrt{\int_0^{\tau} K^{\mu}_{c,\alpha}(r,\tau,\tau_1,\tau_2)^2 dr}} \equiv L^{OLS^{\mu}}_{c,\tau_1,\tau_2} \\
PWY^{\mu}_{GLS} &\xrightarrow{d} \sup_{\tau\in[\tau_0,1]} \frac{K^{\mu,G}_{c,\alpha}(\tau,\tau_1,\tau_2)^2 - \tau}{2\sqrt{\int_0^{\tau} K^{\mu,G}_{c,\alpha}(r,\tau_1,\tau_2)^2 dr}} \equiv L^{GLS^{\mu}}_{c,\tau_1,\tau_2} \\
PWY^{\beta}_{OLS} &\xrightarrow{d} \sup_{\tau\in[\tau_0,1]} \frac{K^{\beta}_{c,\alpha}(\tau,\tau,\tau_1,\tau_2)^2 - K^{\beta}_{c,\alpha}(0,\tau,\tau_1,\tau_2)^2 - \tau}{2\sqrt{\int_0^{\tau} K^{\beta}_{c,\alpha}(r,\tau,\tau_1,\tau_2)^2 dr}} \equiv L^{OLS^{\beta}}_{c,\tau_1,\tau_2} \\
PWY^{\beta}_{GLS} &\xrightarrow{d} \sup_{\tau\in[\tau_0,1]} \frac{K^{\beta,G}_{c,\bar{c},\alpha}(\tau,\tau,\tau_1,\tau_2)^2 - \tau}{2\sqrt{\int_0^{\tau} K^{\beta,G}_{c,\bar{c},\alpha}(r,\tau,\tau_1,\tau_2)^2 dr}} \equiv L^{GLS^{\beta}}_{c,\tau_1,\tau_2}
\end{aligned}
$$

where

$$
\begin{aligned}
K^{\mu}_{c,\alpha}(r,\tau,\tau_1,\tau_2) &= K_{c,\alpha}(r,\tau_1,\tau_2) - \tau^{-1}\int_0^{\tau} K_{c,\alpha}(s,\tau_1,\tau_2)ds \\
K^{\mu,G}_{c,\alpha}(r,\tau_1,\tau_2) &= K_{c,\alpha}(r,\tau_1,\tau_2) - \alpha \\
K^{\beta}_{c,\alpha}(r,\tau,\tau_1,\tau_2) &= K_{c,\alpha}(r,\tau_1,\tau_2) - \frac{2}{\tau}\left(2-\frac{3}{\tau}r\right)\int_0^{\tau} K_{c,\alpha}(s,\tau_1,\tau_2)ds \\
&\quad +\frac{6}{\tau^2}\left(1-\frac{2}{\tau}r\right)\int_0^{\tau} sK_{c,\alpha}(s,\tau_1,\tau_2)ds \\
K^{\beta,G}_{c,\bar{c},\alpha}(r,\tau,\tau_1,\tau_2) &= K_{c,\alpha}(r,\tau_1,\tau_2) - \alpha - \\
&\quad \left[\begin{array}{c} (\tau-\bar{c}\tau^2+\bar{c}^2\tau^3/3)^{-1}\{(1-\bar{c}\tau)K_{c,\alpha}(\tau,\tau_1,\tau_2) \\ +\bar{c}^2\int_0^{\tau} sK_{c,\alpha}(s,\tau_1,\tau_2)ds\} - \frac{1-\bar{c}\tau+\bar{c}^2\tau^2/2}{\tau-\bar{c}\tau^2+\bar{c}^2\tau^3/3}\alpha \end{array}\right] r
\end{aligned}
$$

with

$$
K_{c,\alpha}(r,\tau_1,\tau_2) = \begin{cases} \alpha & r=0 \\ \alpha+W(r) & r<\tau_1 \\ e^{c(r-\tau_1)}\alpha + e^{c(r-\tau_1)}W(\tau_1) + \int_{\tau_1}^{r} e^{c(r-s)}dW(s) & \tau_1<r<\tau_2 \\ W(r)-W(\tau_2)+e^{c(\tau_2-\tau_1)}\alpha & \\ +e^{c(\tau_2-\tau_1)}W(\tau_1) + \int_{\tau_1}^{\tau_2} e^{c(\tau_2-s)}dW(s) & r>\tau_2 \end{cases}
$$

and $W(r)$ a standard Brownian motion process.

Proof: See Appendix.

The limit distributions of PWY^{μ}_{OLS}, PWY^{β}_{OLS}, PWY^{μ}_{GLS} and PWY^{β}_{GLS} under the null hypothesis are given by $L^{OLS^{\mu}}_{0,\tau_1,\tau_2}$, $L^{OLS^{\beta}}_{0,\tau_1,\tau_2}$, $L^{GLS^{\mu}}_{0,\tau_1,\tau_2}$ and $L^{GLS^{\beta}}_{0,\tau_1,\tau_2}$ respectively. These are the limit distributions obtained from Theorem 1, with $c=0$.

Elliott et al. (1996) choose $\bar{c}$ such that when testing the null of a unit root against the alternative of stationarity, the Gaussian point optimal invariant test of $c=0$ against $c=\bar{c}$, which forms the asymptotic Gaussian local power envelope, has a power of 0.50. Using a nominal 0.05 level test, Harvey and Leybourne (2014) repeat the exercise in the context of testing the unit root null against an explosive alternative, yielding approximate values of $\bar{c}_{\mu}=1.6$ and $\bar{c}_{\tau}=2.4$ for the demeaned and detrended DF tests respectively. In the context of temporary explosiveness, the optimal choice of $\bar{c}_{\mu}$ and $\bar{c}_{\tau}$ will change depending on the bubble start and end points, τ_1 and τ_2, as well as the length of the bubble, $\tau_2-\tau_1$. Of course, these bubbles start and end points are likely to be unknown in practice, making it impossible to know the ideal setting of $\bar{c}_{\mu}$ and $\bar{c}_{\tau}$. For simplicity, we therefore choose to adopt the full sample values employed by Harvey and Leybourne (2014) in what follows.

The value of τ_0, the length of the first sub-sample, should be chosen such that it gives a sufficiently large number of observations to ensure that the initial estimation of the sub-sample DF statistic is satisfactory, but not too large that a temporary explosive period early in the sample is missed. In practice, the choice of τ_0 is likely to be driven by the sample size T, with smaller values of τ_0 being possible for larger sample sizes. Throughout this paper, we set $\tau_0 = 0.1$ so that our work is in line with that of PWY. Asymptotic null critical values, given in Table 1, are generated by direct simulation of the limit distributions using IID $N(0, 1)$ random variates, with the integrals approximated by normalised sums of 1,000 steps. Simulations are conducted using 10,000 Monte Carlo replications throughout the paper.

To evaluate the performance of the tests for different lengths of the explosive interval, we consider two $[\tau_1, \tau_2]$ pairs: $[0.45, 0.55]$ and $[0.2, 0.8]$. We therefore have a short bubble where 10% of the data is explosive, and a longer bubble where the explosive interval covers 60% of the data. Both of these non-collapsing bubbles are centred within the sample, although we extend our analysis to consider non-centered bubbles in Section 6. To consider the effect of the initial condition on the performance of the tests, we set $\alpha = \{0, 2, 10\}$. Figures 1 and 2 plot local asymptotic power curves of nominal 0.05-level PWY^{μ}_{OLS}, PWY^{β}_{OLS}, PWY^{μ}_{GLS} and PWY^{β}_{GLS} tests for different values of c, obtained via direct simulation of the limiting distributions above.

Figure 1 reports asymptotic power for the short explosive regime, $[\tau_1, \tau_2] = [0.45, 0.55]$. Figures 1(a) and 1(b) consider power where $\alpha = 0$, with 1(a) considering the two mean-invariant tests and 1(b) considering the two trend-invariant tests. Figures 1(c) and 1(d) report powers in the mean and trend case respectively, where $\alpha = 2$, and Figures 1(e) and 1(f) examine $\alpha = 10$. Consider first Figure 1(a) where the initial condition is asymptotically negligible. Asymptotic power of both PWY^{μ}_{OLS} and PWY^{μ}_{GLS} increases as the magnitude of the explosive bubble, c, increases. For small values of c, PWY^{μ}_{GLS} has slightly higher power than PWY^{μ}_{OLS}, but this power ranking is reversed at approximately $c = 14$, beyond which PWY^{μ}_{OLS} retains a small power advantage. The difference in power between the two tests is always very small, with a maximum disparity of 0.033 observed at $c = 16.8$. In Figure 1(b), PWY^{β}_{OLS} has a power advantage over PWY^{β}_{GLS} across all values of c. These power differences are larger than those observed in the mean case. For $c = 11.2$, PWY^{β}_{OLS} has power of 0.423 and PWY^{β}_{GLS} of 0.292 yielding a power advantage of 0.131.

Consider now Figures 1(c) and 1(d), where $\alpha = 2$ such that the initial condition is asymptotically non-negligible. In both the mean and trend cases, with $i = \{\mu, \beta\}$, PWY^{i}_{OLS} has superior power to PWY^{i}_{GLS} across almost all values of c. The power difference between the two tests is greater than that exhibited when $\alpha = 0$, with PWY^{μ}_{OLS} and PWY^{β}_{OLS} having a maximum power advantage of 0.105 and 0.231 respectively. Figures 1(e) and 1(f) examine power for $\alpha = 10$. The relative power advantage of PWY^{i}_{OLS} over PWY^{i}_{GLS} has increased from the $\alpha = 2$ case, with observed maximum power differences of 0.167 and 0.318 respectively. These results demonstrate that when a short explosive period is considered, PWY^{i}_{OLS} generally has superior power performance to PWY^{i}_{GLS} for both asymptotically negligible and non-negligible initial conditions. This is contrary to the full-sample results of Harvey and Leybourne (2014), where PWY^{i}_{GLS} was shown to outperform PWY^{i}_{OLS} at all values of the initial condition. The relative power advantage of PWY^{i}_{OLS} increases as the value of the initial condition increases, with the difference in powers always greater in the two trend-invariant tests than in the mean-invariant case.

Figure 2 reports power results where the longer explosive regime, $[\tau_1, \tau_2] = [0.2, 0.8]$, is considered. Figures 2(a) and 2(b) report powers for the mean-invariant and trend-invariant tests respectively, where $\alpha = 0$. Figures 2(c) and 2(d) consider $\alpha = 2$ and 2(e) and 2(f) consider $\alpha = 10$. Consider first Figures 2(a) and 2(b). As before, the asymptotic local power of all tests is increasing in c. However, for this longer explosive regime, PWY^{μ}_{GLS} has a small power advantage compared to PWY^{μ}_{OLS} for c values up to approximately 4.5, beyond which the powers of the two tests are near identical. The maximum power advantage of PWY^{μ}_{GLS} over PWY^{μ}_{OLS} is approximately 0.115 observed at $c = 2$. In the trend case, PWY^{β}_{GLS} outperforms PWY^{β}_{OLS} at almost all settings of c, with near identical power observed for the few remaining c settings. As with the short explosive period considered in Figure 1, the power difference between the two tests is greater when considering the trend-invariant case, with PWY^{β}_{GLS} having a maximum power advantage of 0.182 at $c = 3.5$.

In Figures 2(c) and 2(d), where $\alpha = 2$, the relative power performance of the mean-invariant tests is shown to closely correspond to that exhibited in Figure 2(a). In the trend case, the power advantage of PWY^{β}_{GLS} over PWY^{β}_{OLS} is much larger than that obtained for $\alpha = 0$, with a maximum power difference of 0.528 observed at $c = 2$. Figure 2(e) and 2(f) considers $\alpha = 10$, and again PWY^{μ}_{GLS} has a power advantage over PWY^{μ}_{OLS} for the vast majority of c settings. The magnitude of the power difference between tests is similar to that observed for $\alpha = 2$, with powers of 0.739 and 0.630 respectively for PWY^{μ}_{GLS} and PWY^{μ}_{OLS} observed at $c = 0.4$, yielding a maximum power advantage of 0.109. In the trend case PWY^{β}_{GLS} now exhibits a maximum power advantage over PWY^{β}_{OLS} of 0.489 at $c = 0.9$, slightly lower than that for $\alpha = 2$. We therefore find that, for a longer explosive period, PWY^{i}_{GLS} generally outperforms PWY^{i}_{OLS} irrespective of the value of the initial condition. These results coincide with the full sample analysis of Harvey and Leybourne (2014). We note that the initial condition plays a role in the relative superiority of PWY^{β}_{GLS}, with a much larger power advantage observed in the case of $\alpha = 2$ and $\alpha = 10$ than for an asymptotically negligible initial condition. However, unlike in the case of a short explosive period, the power differences of the two mean-invariant tests appear to be somewhat less affected by the size of the initial condition.

To further investigate the impact that the initial condition has on asymptotic power, we examine the performance of PWY^{i}_{OLS} and PWY^{i}_{GLS} for a given c value across different values of α. Figures 3 and 4 display power for the two $[\tau_1, \tau_2]$ pairs: $[0.45, 0.55]$ and $[0.2, 0.8]$ respectively. We set $\alpha = \{1, ..., 10\}$ and we select a value of c that yields power approaching one for the largest value of α. Consider first Figure 3, where $c = 2$. In both the mean and trend case, the powers of PWY^{i}_{OLS} and PWY^{i}_{GLS} are increasing as α increases, due to the role the initial condition plays in the magnitude of the explosive bubble process. However, beyond approximately $\alpha = 2.5$ in the mean case and $\alpha = 1$ in the trend case, the powers of the two tests begin to grow at different rates, such that the relative power advantage of PWY^{i}_{OLS} to PWY^{i}_{GLS} is increasing with α. Now considering Figure 4, we set $c = 0.8$ in Figure 4(a) and $c = 1$ in Figure 4(b). Again, in both the mean and trend case, the power of both tests is increasing as α increases. The relative power ranking of the mean-invariant tests is generally unaffected by the size of the initial condition.

However, in the trend case, we observe that the power advantage of PWY^{β}_{GLS} over PWY^{β}_{OLS} increases substantially with α. This is in contrast to the full sample, left-tailed results of Muller and Elliott (2003) where a GLS-based unit root test is shown to have decreasing power as the size of the initial condition increases.

Overall, the results in this section highlight the important role that the length of the explosive period plays in the performance of recursive, right-tailed unit root tests. Whilst for longer explosive periods PWY^{i}_{GLS} has superior power performance to PWY^{i}_{OLS}, this power ranking is reversed for short explosive periods. With the exception of mean-invariant tests under the longer explosive period, the power difference between tests is shown to be small when the initial condition is asymptotically negligible, but much larger under the presence of an asymptotically non-negligible initial condition. As both the length of the explosive period and the value of the initial condition are unlikely to be known in practice, it will often be unclear which test procedure should be employed. This uncertainty suggests that a composite procedure which capitalises on the respective power advantages of both tests is required.

5 A union of rejections strategy

We consider a union of rejections testing approach, in line with that of Harvey et al. (2009, 2012) to combine inference from the two mean-invariant tests, or the two trend-invariant tests. The union of rejections procedure is a simple decision rule where the null hypothesis of a unit root is rejected if either of the individual tests reject. We can write our proposed union of rejections strategies as

$$\begin{aligned} U^{\mu} &: \text{ Reject } H_0 \text{ if } PWY^{\mu}_{OLS} > \lambda_{\zeta} cv^{\zeta,\mu}_{OLS} \text{ or } PWY^{\mu}_{GLS} > \lambda_{\zeta} cv^{\zeta,\mu}_{GLS} \\ U^{\beta} &: \text{ Reject } H_0 \text{ if } PWY^{\beta}_{OLS} > \lambda_{\zeta} cv^{\zeta,\beta}_{OLS} \text{ or } PWY^{\beta}_{GLS} > \lambda_{\zeta} cv^{\zeta,\beta}_{GLS} \end{aligned}$$

where $cv^{\zeta,\mu}_{OLS}, cv^{\zeta,\mu}_{GLS}, cv^{\zeta,\beta}_{OLS}$ and $cv^{\zeta,\beta}_{GLS}$ denote the asymptotic null critical values of $PWY^{\mu}_{OLS}, PWY^{\mu}_{GLS}, PWY^{\beta}_{OLS}$ and PWY^{β}_{GLS} respectively for a significance level ζ. If we were to use the decision rule

$$\text{Reject } H_0 \text{ if } PWY^{i}_{OLS} > cv^{\zeta,i}_{OLS} \text{or } PWY^{i}_{GLS} > cv^{\zeta,i}_{GLS}$$

such that both test statistics are compared to the critical values given in Table 1, U^{i} would be oversized. We therefore incorporate a scaling constant, λ_{ζ}, which is calculated such that the asymptotic size of U^{μ} and U^{β} is equivalent to the nominal size ζ. The decision rules can also be written as:

$$\begin{aligned} U^{\mu} &: \text{ Reject } H_0 \text{ if } \max\left(PWY^{\mu}_{OLS}, \frac{cv^{OLS^{\mu}}_{\zeta}}{cv^{GLS^{\mu}}_{\zeta}} PWY^{\mu}_{GLS}\right) > \lambda_{\zeta} cv^{\zeta,\mu}_{OLS} \\ U^{\beta} &: \text{ Reject } H_0 \text{ if } \max\left(PWY^{\beta}_{OLS}, \frac{cv^{\zeta,\beta}_{OLS}}{cv^{\zeta,\beta}_{GLS}} PWY^{\beta}_{GLS}\right) > \lambda_{\zeta} cv^{\zeta,\beta}_{OLS} \end{aligned}$$

Under the null hypothesis, where $\delta = 0$,

$$\max\left(PWY^{\mu}_{OLS}, \frac{cv^{\zeta,\mu}_{OLS}}{cv^{\zeta,\mu}_{GLS}} PWY^{\mu}_{GLS}\right) \xrightarrow{d} \max\left(L^{OLS^{\mu}}_{c,\tau_1,\tau_2}, \frac{cv^{\zeta,\mu}_{OLS}}{cv^{\zeta,\mu}_{GLS}} L^{GLS^{\mu}}_{c,\tau_1,\tau_2}\right) \tag{8}$$

$$\max\left(PWY^{\beta}_{OLS}, \frac{cv^{\zeta,\beta}_{OLS}}{cv^{\zeta,\beta}_{GLS}} PWY^{\beta}_{GLS}\right) \xrightarrow{d} \max\left(L^{OLS^{\beta}}_{c,\tau_1,\tau_2}, \frac{cv^{\zeta,\beta}_{OLS}}{cv^{\zeta,\beta}_{GLS}} L^{GLS^{\beta}}_{c,\tau_1,\tau_2}\right) \tag{9}$$

To obtain the appropriate value for the scaling constant λ_ζ, we can simulate the limit distribution of U^i (i.e. the RHS of equations 8 and 9) and calculate the ζ level critical value $cv^{\zeta,\mu}_U$ or $cv^{\zeta,\beta}_U$. Computing $\lambda_\zeta = cv^{\zeta,i}_U / cv^{\zeta,i}_{OLS}$ will then give the value for the scaling constant that ensures U^i is asymptotically correctly sized. Asymptotic scaling constants calculated in this way are given in Table 1.

Figures 1 and 2 display the powers of U^μ and U^β at the nominal 0.05 level for the explosive intervals $[\tau_1, \tau_2] = [0.45, 0.55]$ and $[0.2, 0.8]$ respectively for initial conditions $\alpha = \{0, 2, 10\}$. Consider first Figures 1(a) and 1(b). In both the mean and trend case, the power of U^i is near identical to that exhibited by PWY^i_{OLS}, the better performing of the two individual test procedures. This pattern extends to Figures 1(c)-1(f), where asymptotically non-negligible initial conditions are considered. Therefore, when a short explosive period is present in the data, applying the union procedure is costless in the sense that there is no power loss associated with using U^i over PWY^i_{OLS}, but a power gain is made by using U^i instead of PWY^i_{GLS}. Figure 2 considers the longer explosive period where we previously showed that PWY^i_{GLS} generally outperforms PWY^i_{OLS}. Figures 2(a) and 2(b) display the power of U^i when the initial condition is asymptotically negligible. In both the mean and trend cases, the power of U^i is either very similar, or tracking slightly below that of PWY^i_{GLS}. U^i outperforms PWY^i_{OLS} (or has near identical power) at all values of c. When considering a non-negligible initial condition, as in Figures 2(c)-2(f), U^i retains its power advantage over PWY^i_{OLS}, exhibiting power either equivalent to or slightly lower than that obtained by PWY^i_{GLS}.

Figures 3 and 4 display the power of U^i for explosive intervals of $[0.45, 0.55]$ and $[0.2, 0.8]$ respectively across different values of α. In all cases, we see that the power of U^i is either equivalent to, or slightly below that of the best-performing individual test, whilst always greater than that of the worst-performing test.

The results from this section demonstrate the obvious advantage of employing a union of rejections strategy when testing for temporary explosive behaviour. When the powers of PWY^i_{OLS} and PWY^i_{GLS} are similar, U^i will have near identical power to the better-performing of the two tests, regardless of the length of the explosive period. As the size of the initial condition increases, such that the power difference between the two individual tests also increases, the power of U^i always closely tracks that of the better-performing test, providing a substantial power advantage over either PWY^i_{GLS} when a short explosive interval is considered, or PWY^i_{OLS} when a longer explosive interval is considered. A practitioner employing a union of rejections procedure would overcome any potential power loss arising from uncertainty over both the length of the explosive period and the size of the initial condition.

Throughout this paper we abstract from the issue of whether or not a trend is present in the time series and assume that the practitioner is able to make an informed judgement about this. Of course, all four individual tests could be combined in a union procedure in line with Harvey et al. (2012), who demonstrate the need for a higher scale value when combining an increasing number of tests, and the subsequent impact on power that this has.

6 Finite sample power comparison

To assess the extent to which the local asymptotic power results are an accurate representation of finite sample behaviour, we consider a number of finite sample simulations. Finite sample critical values for conventional levels of significance and a sample size of $T = 150$ are given in Table 2. Note that whilst we use finite sample critical values in line with PWY, we continue to use asymptotic union scale values as is standard in a union of rejections context. As in the asymptotic simulations, we set $\alpha = \{0, 2, 10\}$. Finite sample results for $T = 150$ are given in Figures 5 and 6 for the $[\tau_1, \tau_2]$ pairs: $[0.45, 0.55]$ and $[0.2, 0.8]$ respectively. For both the short and long explosive period, the finite sample powers of PWY^i_{OLS} and PWY^i_{GLS} are closely aligned with their local asymptotic counterparts. As before, the power of U^i is always greater than that of the worst-performing individual test (PWY^i_{GLS} in the $[0.45, 0.55]$ case and PWY^i_{OLS} in the $[0.2, 0.8]$ case), whilst exhibiting power that closely tracks that obtained by the better-performing test.

To assess whether the location of the explosive period affects the power performance of the test statistics, we also consider two additional $[\tau_1, \tau_2]$ pairs: $[0.15, 0.25]$ and $[0.75, 0.85]$. We therefore have two bubbles of short length, located near the beginning and end of the sample respectively. Figures 7 and 8 display the power of PWY^i_{OLS}, PWY^i_{GLS} and U^i for both asymptotically negligible and non-negligible initial conditions, setting $\alpha = \{0, 2, 10\}$ as before.

For these short explosive intervals we again find that PWY^i_{OLS} is generally superior to PWY^i_{GLS}. We note an exception to this in Figures 7(d) and 7(f) where the two trend-invariant tests are considered for the interval $[0.15, 0.25]$ with $\alpha = 2$ and $\alpha = 10$ respectively. Here, despite the small interval length, there is a power ranking reversal as c increases such that PWY^i_{GLS} has a small power advantage over PWY^i_{OLS} across most values of c. This result does not occur when the same explosive interval length is placed later in the sample at $[0.75, 0.85]$ in Figures 8(d) and 8(f), suggesting that the relative power of PWY^i_{OLS} to PWY^i_{GLS} is adversely affected if the explosive period is early in the sample. The power performance of U^i is unaffected by the location of the explosive interval within the sample. As such, U^i offers a method for detecting explosive processes that is robust to the length of the explosive period, the point in the sample at which the explosive period occurs, and the value of the initial condition.

It is well documented that financial returns exhibit time-varying volatility, thus an applied researcher may be interested in the performance of bubble detection procedures under conditionally heteroskedastic errors. In Section 4 we assume that the innovation process v_t follows a martingale difference sequence, such that ARCH/GARCH errors are permitted in our asymptotic framework. As a result, the asymptotic size and power of the test procedures discussed in this paper should not be affected by conditional heteroskedasticity of this nature. To evaluate the finite sample performance of tests under conditional heteroskedasticity, we consider the temporarily explosive DGP in (6) - (7), but now allowing for GARCH errors such that $v_t = \eta_t \sqrt{h_t}$ with $\eta_t \sim NIID(0,1)$ and

$$h_t = \omega + \gamma v_{t-1}^2 + \phi h_{t-1}. \tag{10}$$

Phillips et al. (2015) fit a GARCH error process of this form to the S&P 500 price-dividend ratio over the sample period January 2004 to December 2007 using maximum likelihood estimation. Assuming no trend or initial condition is present ($\beta = 0$ and $\alpha = 0$), and setting $\mu = dT^{-\psi}$, they obtain the following estimates: $y_0 = 376.8$, $d = 1$, $\psi = 1$, $\omega = 30.69$, $\gamma = 0$ and $\phi = 0.61$. Simulating empirical sizes of PWY_{OLS}^{μ} with these estimates used as parameter values in the GARCH equation, and using sample sizes from $T = 100$ to $T = 1600$, they find that conditional heteroskedasticity of this degree has little impact on the size of the procedure.

To examine the power of all the test procedures considered in this paper under an empirically relevant conditional heteroskedasticity set up, we consider the DGP in (6) - (7) with the GARCH error process in (10), setting $\omega = 30$, $\gamma = 0$ and $\phi = 0.6$, approximately the S&P 500 estimated coefficients taken from Phillips et al. (2015). We consider the same $[\tau_1, \tau_2]$ and c settings as considered in Figures 5 - 6 and display results in Figure 9. The power profiles of the six test procedures are very similar to those observed in the IID error case, demonstrating the robustness of these procedures to empirically realistic degrees of conditional heteroskedasticity.[1]

7 Empirical application

To illustrate the differing performances of the test statistics considered in this paper, we apply the four individual tests PWY_{OLS}^{μ}, PWY_{GLS}^{μ}, PWY_{OLS}^{β} and PWY_{GLS}^{β}, as well as the two union procedures U^{μ} and U^{β} to logarithms of real daily NASDAQ closing prices. Daily NASDAQ closing prices are obtained from Yahoo Finance. Monthly US Consumer Price Index data is obtained from the Federal Reserve Bank of St. Louis FRED database and linearly interpolated to a daily frequency to convert nominal prices into real prices. PWY employ PWY_{OLS}^{μ} on the NASDAQ stock index using monthly data from February 1973 - June 2005 and conclude that the series exhibits explosive behaviour. The sample period used here is 2 January 1996 - 31 December 2015, yielding 5036 observations (note that there are approximately 252 trading days in each year). Our choice of sample period is motivated by the date estimation of Harvey et al. (2017) who propose a dating procedure based on minimum sum of squared residual estimators combined with Bayesian Information Criterion (BIC) model selection. This dating algorithm provides consistent estimates of the start and end dates of explosive regimes and is shown to outperform recursive unit root test methods for dating in finite samples. Using this procedure,

[1]We report results for $\alpha = 0$ only here. Unreported simulations confirm that the comparative power performance of the six test procedures when considering non-zero initial conditions and GARCH errors are qualitatively similar to that observed in Figures 5 - 6.

Harvey et al. (2017) find evidence of explosive behaviour in NASDAQ prices starting in November 1998 and ending in September 2000[2]. The sample period used here covers this period of explosiveness.

PWY^i_{OLS} and PWY^i_{GLS} tests are computed using augmented DF tests where the optimal number of lags is selected using BIC up to a maximum lag of 14. Table 3 reports the four individual test statistics and the rejections obtained from the two union procedures. PWY^{μ}_{OLS} fails to reject the unit root null at conventional levels of significance, whilst PWY^{μ}_{GLS} rejects the null at a 0.01 level. The mean-invariant union, U^{μ}, is able to pick up this rejection at a 0.05 level. Considering the trend-invariant tests, it is the GLS-based procedure, PWY^{β}_{GLS}, that fails to reject the null hypothesis of a unit root. In contrast, PWY^{β}_{OLS} rejects the null at a 0.01 level of significance. The trend-invariant union procedure, U^{β}, picks up this rejection, also at a 0.01 level. This demonstrates that inference from PWY tests can depend on whether the procedure employs OLS or GLS DF tests. In both the mean and trend case (where a rejection comes respectively from the GLS test only and the OLS test only), the union of rejections procedure is able to reject the null hypothesis.

8 Conclusion

In this paper we examine the power performance of the PWY test for detecting explosive behaviour in comparison to a GLS-based procedure. Limit distributions for both mean-invariant and trend-invariant versions of the two tests are derived, and these limits are shown to be dependent on the value of the initial condition and the start and end date of the temporary explosive regime. Asymptotic and finite sample simulations show that the GLS-based test offers superior power when an explosive period of long length is considered, whilst the original OLS PWY test has a power advantage for explosive periods of short length. The power rankings of the two tests are unaffected by the presence of an asymptotically non-negligible initial condition, but, in general, an increase in the size of the initial condition increases the magnitude of the power differences between OLS and GLS-based approaches. A union of rejections procedure is shown to capitalise on the relative power advantages provided by these competing tests across all values of initial condition and lengths of the explosive regime considered in this paper. Further simulations show that the union procedure is robust to the point in the sample at which the explosive period occurs. An application of our proposed union procedure to daily NASDAQ price data demonstrates the empirical value of this testing strategy.

Phillips et al. (2015) consider detecting multiple bubbles using a generalised version of the PWY procedure in which the starting point of each recursive DF test is no longer fixed at the first observation of the sample, but instead allowed to vary across the sample. When considering a long time series that contains multiple periods of explosiveness, and where the explosive intervals are likely to be of different lengths, it may be the case that a GLS-based multiple bubble detection procedure

[2]Harvey et al. (2017) also apply the Phillips et al. (2015) procedure for detecting multiple bubbles to NASDAQ prices, which yields an identical start date, but an end date of December 2000.

can outperform the original OLS procedure in certain circumstances. Additionally, the focus of this paper is on detection rather than dating of temporary explosive behaviour. Both PWY and Phillips et al. (2015) propose dating procedures, based on recursions of OLS DF tests, which can be used to estimate the start and end points of a detected explosive regime. In light of the results of this paper, it may be interesting to examine the performance of a GLS-based dating procedure. We leave these possibilities to future work.

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Appendix: Proof of Theorem 1

By backward substitution in (7) we obtain

$$u_t = \begin{cases} T^{1/2}\sigma\alpha & t=1 \\ T^{1/2}\sigma\alpha + \sum_{i=2}^{t} v_i & t=2,...,\lfloor \tau_1 T \rfloor \\ \begin{array}{l} (1+\delta)^{t-\lfloor \tau_1 T \rfloor}\left(T^{1/2}\sigma\alpha + \sum_{i=2}^{\lfloor \tau_1 T \rfloor} v_i\right) \\ +\sum_{i=\lfloor \tau_1 T \rfloor+1}^{t}(1+\delta)^{i-\lfloor \tau_1 T \rfloor-1} v_{t-(i-\lfloor \tau_1 T \rfloor-1)} \end{array} & t=\lfloor \tau_1 T \rfloor+1,...,\lfloor \tau_2 T \rfloor \\ u_{\lfloor \tau_2 T \rfloor+1} + \sum_{i=\lfloor \tau_2 T \rfloor+2}^{t} v_i & t=\lfloor \tau_2 T \rfloor+2,...,T \end{cases}$$

The third of these parts comes from the backward recursion:

$$\begin{aligned} u_{\lfloor \tau_1 T+1 \rfloor} &= (1+\delta)u_{\lfloor \tau_1 T \rfloor} + v_{\lfloor \tau_1 T \rfloor+1} \\ u_{\lfloor \tau_1 T+2 \rfloor} &= (1+\delta)u_{\lfloor \tau_1 T \rfloor+1} + v_{\lfloor \tau_1 T \rfloor+2} \\ &= (1+\delta)^2 u_{\lfloor \tau_1 T \rfloor} + (1+\delta)v_{\lfloor \tau_1 T \rfloor+1} + v_{\lfloor \tau_1 T \rfloor+2} \\ u_{\lfloor \tau_1 T+3 \rfloor} &= (1+\delta)u_{\lfloor \tau_1 T \rfloor+2} + v_{\lfloor \tau_1 T \rfloor+3} \\ &= (1+\delta)^3 u_{\lfloor \tau_1 T \rfloor} + (1+\delta)^2 v_{\lfloor \tau_1 T \rfloor+1} + (1+\delta)v_{\lfloor \tau_1 T \rfloor+2} + v_{\lfloor \tau_1 T \rfloor+3} \\ &\vdots \\ u_t &= (1+\delta)^{t-\lfloor \tau_1 T \rfloor} u_{\lfloor \tau_1 T \rfloor} + \sum_{i=\lfloor \tau_1 T \rfloor+1}^{t}(1+\delta)^{t-i} v_i \quad t=\lfloor \tau_1 T \rfloor+1,...,\lfloor \tau_2 T \rfloor \\ &= (1+\delta)^{t-\lfloor \tau_1 T \rfloor}\left(T^{1/2}\sigma\alpha + \sum_{i=1}^{\lfloor \tau_1 T \rfloor} v_i\right) \\ &\quad +\sum_{i=\lfloor \tau_1 T \rfloor+1}^{t}(1+\delta)^{t-i} v_i \qquad t=\lfloor \tau_1 T \rfloor+1,...,\lfloor \tau_2 T \rfloor \end{aligned}$$

and subsequently, using $\delta = c/T$

$$T^{-1/2}u_{\lfloor rT \rfloor} = \begin{cases} \sigma\alpha & \lfloor rT \rfloor = 1 \\ \sigma\alpha + T^{-1/2}\sum_{i=2}^{\lfloor rT \rfloor} v_i & \lfloor rT \rfloor = 2,...,\lfloor \tau_1 T \rfloor \\ \begin{array}{l} (1+c/T)^{\lfloor rT \rfloor-\lfloor \tau_1 T \rfloor}\left(\sigma\alpha + T^{-1/2}\sum_{i=2}^{\lfloor \tau_1 T \rfloor} v_i\right) \\ +T^{-1/2}\sum_{i=\lfloor \tau_1 T \rfloor+1}^{\lfloor rT \rfloor}(1+c/T)^{\lfloor rT \rfloor-i} v_i \end{array} & \lfloor rT \rfloor = \lfloor \tau_1 T \rfloor+1,...,\lfloor \tau_2 T \rfloor \\ T^{-1/2}u_{\lfloor \tau_2 T \rfloor+1} + T^{-1/2}\sum_{i=\lfloor \tau_2 T \rfloor+2}^{\lfloor rT \rfloor} v_i & \lfloor rT \rfloor = \lfloor \tau_2 T \rfloor+1,...,T \end{cases}$$

Weak convergence of standardised partial sums of v_t to the standard Brownian motion process $W(r)$ is a standard result, so we can write

$$\begin{aligned} T^{-1/2}\sum_{i=2}^{\lfloor rT \rfloor} v_i &\xrightarrow{d} \sigma W(r) \\ T^{-1/2}\sum_{i=2}^{\lfloor \tau_1 T \rfloor} v_i &\xrightarrow{d} \sigma W(\tau_1) \\ T^{-1/2}\sum_{i=\lfloor \tau_2 T \rfloor+2}^{\lfloor rT \rfloor} v_i &\xrightarrow{d} \sigma W(r) - \sigma W(\tau_2) \end{aligned}$$

Following Phillips (1987), from the third part of the above

$$T^{-1/2}\sum_{i=\lfloor \tau_1 T \rfloor+1}^{\lfloor rT \rfloor}(1+c/T)^{\lfloor rT \rfloor-i} v_i$$

converges in distribution to the Ornstein-Uhlenbeck process

$$W_c(r) = \int_{\tau_1}^{r} e^{c(r-s)} dW(s)$$

We can therefore show that

$$\begin{aligned} T^{-1/2} u_{\lfloor rT \rfloor} &\overset{d}{\to} \sigma \begin{cases} \alpha & r = 0 \\ \alpha + W(r) & r < \tau_1 \\ e^{c(r-\tau_1)}\alpha + e^{c(r-\tau_1)} W(\tau_1) + \int_{\tau_1}^{r} e^{c(r-s)} dW(s) & \tau_1 < r < \tau_2 \\ \begin{array}{l} W(r) - W(\tau_2) + e^{c(\tau_2-\tau_1)}\alpha + e^{c(\tau_2-\tau_1)} W(\tau_1) \\ + \int_{\tau_1}^{\tau_2} e^{c(\tau_2-s)} dW(s) \end{array} & r > \tau_2 \end{cases} \\ &\equiv \sigma K_{c,\alpha}(r, \tau_1, \tau_2) \end{aligned}$$

Asymptotic distribution of PWY^{μ}_{OLS}

Here we have $\beta = 0$ and the recursive test statistic calculated for the sub-sample $t = 1, ..., \lfloor \tau T \rfloor$ is based on the t-ratio for $\hat{\delta}_\tau$ from the estimated regression

$$\Delta \hat{u}_{\tau,t} = \hat{\delta}_\tau \hat{u}_{\tau,t-1} + \hat{v}_{\tau,t}, \qquad t = 1, ..., \lfloor \tau T \rfloor$$

where

$$\hat{u}_{\tau,t} = y_t - \lfloor \tau T \rfloor^{-1} \sum_{s=1}^{\lfloor \tau T \rfloor} y_s$$

i.e.

$$t_\tau = \frac{\hat{\delta}_\tau}{s.e.(\hat{\delta}_\tau)}$$

where

$$\begin{aligned} \hat{\delta}_\tau &= \frac{\sum_{t=2}^{\lfloor \tau T \rfloor} \Delta \hat{u}_{\tau,t} \hat{u}_{\tau,t-1}}{\sum_{t=2}^{\lfloor \tau T \rfloor} \hat{u}^2_{\tau,t-1}} \\ s.e.(\hat{\delta}_\tau)^2 &= \frac{\hat{\sigma}^2_\tau}{\sum_{t=2}^{\lfloor \tau T \rfloor} \hat{u}^2_{\tau,t-1}} \\ \hat{\sigma}^2_\tau &= \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} \hat{v}^2_{\tau,t} \end{aligned}$$

Note that in what follows we can set $\mu = 0$ without loss of generality, so that $y_t = u_t$. Consider the recursively demeaned y_t, i.e. $\hat{u}_{\tau,t}$. Since $y_t = u_t$ we have

$$\begin{aligned} \hat{u}_{\tau,t} &= u_t - \lfloor \tau T \rfloor^{-1} \sum_{s=1}^{\lfloor \tau T \rfloor} u_s \\ T^{-1/2} \hat{u}_{\tau,\lfloor rT \rfloor} &= T^{-1/2} u_{\lfloor rT \rfloor} - \lfloor \tau T \rfloor^{-1} \sum_{\lfloor sT \rfloor = 1}^{\lfloor \tau T \rfloor} T^{-1/2} u_{\lfloor sT \rfloor} \\ &\xrightarrow{d} \sigma K_{c,\alpha}(r, \tau_1, \tau_2) - \sigma \tau^{-1} \int_0^\tau K_{c,\alpha}(s, \tau_1, \tau_2) ds \\ &\equiv \sigma K^{\mu}_{c,\alpha}(r, \tau, \tau_1, \tau_2) \end{aligned}$$

where

$$T^{-3/2} \sum_{\lfloor sT \rfloor = 1}^{\lfloor \tau T \rfloor} u_{\lfloor sT \rfloor} \xrightarrow{d} \sigma \int_0^\tau K_{c,\alpha}(s, \tau_1, \tau_2) ds$$

by the Functional Central Limit Theorem.

The recursive parameter estimate $\hat{\delta}_\tau$:

$$\begin{aligned}
\hat{\delta}_\tau &= \frac{\sum_{t=2}^{\lfloor \tau T \rfloor} \Delta \hat{u}_{\tau,t} \hat{u}_{\tau,t-1}}{\sum_{t=2}^{\lfloor \tau T \rfloor} \hat{u}_{\tau,t-1}^2} \\
T\hat{\delta}_\tau &= \frac{T^{-1}\sum_{t=2}^{\lfloor \tau T \rfloor} \Delta \hat{u}_{\tau,t} \hat{u}_{\tau,t-1}}{T^{-2}\sum_{t=2}^{\lfloor \tau T \rfloor} \hat{u}_{\tau,t-1}^2}
\end{aligned}$$

For the denominator we have

$$\begin{aligned}
T^{-2}\sum_{t=2}^{\lfloor \tau T \rfloor} \hat{u}_{\tau,t-1}^2 &= T^{-1}\sum_{\lfloor rT \rfloor=2}^{\lfloor \tau T \rfloor} (T^{-1/2}\hat{u}_{\tau,\lfloor rT \rfloor-1})^2 \\
&\xrightarrow{d} \sigma^2 \int_0^\tau K_{c,\alpha}^\mu(r,\tau,\tau_1,\tau_2)^2 dr
\end{aligned}$$

Now consider the numerator

$$T^{-1}\sum_{t=2}^{\lfloor \tau T \rfloor} \Delta \hat{u}_{\tau,t} \hat{u}_{\tau,t-1} = \frac{1}{2}\left\{ (T^{-1/2}\hat{u}_{\lfloor \tau T \rfloor})^2 - (T^{-1/2}\hat{u}_1)^2 - T^{-1}\sum_{t=2}^{\lfloor \tau T \rfloor} (\Delta \hat{u}_{\tau,t})^2 \right\}$$

using

$$\begin{aligned}
\hat{u}_{\tau,t} &= \Delta \hat{u}_{\tau,t} + \hat{u}_{\tau,t-1} \\
\sum_{t=2}^{\lfloor \tau T \rfloor} \hat{u}_{\tau,t}^2 &= \sum_{t=2}^{\lfloor \tau T \rfloor} (\Delta \hat{u}_{\tau,t})^2 + \sum_{t=2}^{\lfloor \tau T \rfloor} \hat{u}_{\tau,t-1}^2 + 2\sum_{t=2}^{\lfloor \tau T \rfloor} \Delta \hat{u}_{\tau,t} \hat{u}_{\tau,t-1} \\
\sum_{t=2}^{\lfloor \tau T \rfloor} \Delta \hat{u}_{\tau,t} \hat{u}_{\tau,t-1} &= \frac{1}{2}\left\{ \sum_{t=2}^{\lfloor \tau T \rfloor} \hat{u}_{\tau,t}^2 - \sum_{t=2}^{\lfloor \tau T \rfloor} \hat{u}_{\tau,t-1}^2 - \sum_{t=2}^{\lfloor \tau T \rfloor} (\Delta \hat{u}_{\tau,t})^2 \right\} \\
&= \frac{1}{2}\left\{ \hat{u}_{\lfloor \tau T \rfloor}^2 - \hat{u}_1^2 - \sum_{t=2}^{\lfloor \tau T \rfloor} (\Delta \hat{u}_{\tau,t})^2 \right\}
\end{aligned}$$

Then

$$\begin{aligned}
T^{-1}\sum_{t=2}^{\lfloor \tau T \rfloor} \Delta \hat{u}_{\tau,t} \hat{u}_{\tau,t-1} &= \frac{1}{2}\left\{ (T^{-1/2}\hat{u}_{\lfloor \tau T \rfloor})^2 - (T^{-1/2}\hat{u}_1)^2 - T^{-1}\sum_{t=2}^{\lfloor \tau T \rfloor} (\Delta \hat{u}_{\tau,t})^2 \right\} \\
&\xrightarrow{d} \frac{1}{2}\left\{ \sigma^2 K_{c,\alpha}^\mu(\tau,\tau,\tau_1,\tau_2)^2 - \sigma^2 K_{c,\alpha}^\mu(0,\tau,\tau_1,\tau_2)^2 - \tau\sigma^2 \right\}
\end{aligned}$$

The limit of the last term comes from

$$
\begin{aligned}
T^{-1}\sum_{t=2}^{\lfloor \tau T\rfloor}(\Delta\hat{u}_{\tau,t})^2 &= T^{-1}\sum_{t=2}^{\lfloor \tau T\rfloor}(\hat{u}_{\tau,t}-\hat{u}_{\tau,t-1})^2 \\
&= T^{-1}\sum_{t=2}^{\lfloor \tau T\rfloor}\left\{\left(u_t-\lfloor \tau T\rfloor^{-1}\sum_{s=1}^{\lfloor \tau T\rfloor}u_s\right)-\left(u_{t-1}-\lfloor \tau T\rfloor^{-1}\sum_{s=1}^{\lfloor \tau T\rfloor}u_s\right)\right\}^2 \\
&= T^{-1}\sum_{t=2}^{\lfloor \tau T\rfloor}(\Delta u_t)^2 \\
&= \tau(\tau T)^{-1}\sum_{t=2}^{\lfloor \tau T\rfloor}(\Delta u_t)^2 \\
&= \tau\left\{\begin{array}{ll}(\tau T)^{-1}\sum_{t=2}^{\lfloor \tau T\rfloor}v_t^2 & \lfloor \tau T\rfloor\le\lfloor \tau_1 T\rfloor \\ (\tau T)^{-1}\sum_{t=2}^{\lfloor \tau T\rfloor}v_t^2+o_p(1) & \lfloor \tau_1 T\rfloor<\lfloor \tau T\rfloor\le\lfloor \tau_2 T\rfloor \\ (\tau T)^{-1}\sum_{t=2}^{\lfloor \tau T\rfloor}v_t^2+o_p(1) & \lfloor \tau T\rfloor>\lfloor \tau_2 T\rfloor\end{array}\right. \\
&\xrightarrow{p}\tau\sigma^2
\end{aligned}
$$

since

$$
\begin{aligned}
\Delta u_t &= \left\{\begin{array}{ll} v_t & t=2,...,\lfloor \tau_1 T\rfloor \\ cT^{-1}u_{t-1}+v_t & t=\lfloor \tau_1 T\rfloor+1,...,\lfloor \tau_2 T\rfloor \\ v_t & t=\lfloor \tau_2 T\rfloor+1,...,T\end{array}\right. \\
&= \left\{\begin{array}{ll} v_t & t=2,...,\lfloor \tau_1 T\rfloor \\ v_t+o_p(1) & t=\lfloor \tau_1 T\rfloor+1,...,\lfloor \tau_2 T\rfloor \\ v_t & t=\lfloor \tau_2 T\rfloor+1,...,T\end{array}\right.
\end{aligned}
$$

So

$$
\begin{aligned}
T\hat{\delta}_\tau &\xrightarrow{d} \frac{\frac{1}{2}\left\{\sigma^2K_{c,\alpha}^{\mu}(\tau,\tau,\tau_1,\tau_2)^2-\sigma^2K_{c,\alpha}^{\mu}(0,\tau,\tau_1,\tau_2)^2-\tau\sigma^2\right\}}{\sigma^2\int_0^\tau K_{c,\alpha}^{\mu}(r,\tau,\tau_1,\tau_2)^2dr} \\
&= \frac{K_{c,\alpha}^{\mu}(\tau,\tau,\tau_1,\tau_2)^2-K_{c,\alpha}^{\mu}(0,\tau,\tau_1,\tau_2)^2-\tau}{2\int_0^\tau K_{c,\alpha}^{\mu}(r,\tau,\tau_1,\tau_2)^2dr}
\end{aligned}
$$

Next consider $\hat{\sigma}_\tau^2$ and then $s.e.(\hat{\delta}_\tau)$. We obtain

$$\begin{aligned}
\hat{\sigma}_\tau^2 &= \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} \hat{v}_{\tau,t}^2 \\
&= \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} (\Delta \hat{u}_{\tau,t} - \hat{\delta}_\tau \hat{u}_{\tau,t-1})^2 \\
&= \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} (\Delta \hat{u}_{\tau,t})^2 + \hat{\delta}_\tau^2 \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} \hat{u}_{\tau,t-1}^2 - 2\hat{\delta}_\tau \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} \Delta \hat{u}_{\tau,t} \hat{u}_{\tau,t-1} \\
&= \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} (\Delta u_t)^2 + O_p(T^{-2}) O_p(T) - 2 O_p(T^{-1}) O_p(1) \\
&= \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} (\Delta u_t)^2 + o_p(1) \\
&\xrightarrow{p} \sigma^2
\end{aligned}$$

and

$$\begin{aligned}
\{T s.e.(\hat{\delta}_\tau)\}^2 &= \frac{\hat{\sigma}_\tau^2}{T^{-2} \sum_{t=2}^{\lfloor \tau T \rfloor} \hat{u}_{\tau,t-1}^2} \\
&\xrightarrow{d} \frac{\sigma^2}{\sigma^2 \int_0^\tau K_{c,\alpha}^\mu(r,\tau,\tau_1,\tau_2)^2 dr} \\
&= \frac{1}{\int_0^\tau K_{c,\alpha}^\mu(r,\tau,\tau_1,\tau_2)^2 dr}
\end{aligned}$$

Putting it all together we find

$$\begin{aligned}
t_\tau &= \frac{T\hat{\delta}_\tau}{T s.e.(\hat{\delta}_\tau)} \\
&\xrightarrow{d} \frac{\frac{K_{c,\alpha}^\mu(\tau,\tau,\tau_1,\tau_2)^2 - K_{c,\alpha}^\mu(0,\tau,\tau_1,\tau_2)^2 - \tau}{2\int_0^\tau K_{c,\alpha}^\mu(r,\tau,\tau_1,\tau_2)^2 dr}}{\sqrt{\frac{1}{\int_0^\tau K_{c,\alpha}^\mu(r,\tau,\tau_1,\tau_2)^2 dr}}} \\
&= \frac{K_{c,\alpha}^\mu(\tau,\tau,\tau_1,\tau_2)^2 - K_{c,\alpha}^\mu(0,\tau,\tau_1,\tau_2)^2 - \tau}{2\sqrt{\int_0^\tau K_{c,\alpha}^\mu(r,\tau,\tau_1,\tau_2)^2 dr}}
\end{aligned}$$

Asymptotic distribution of PWY_{GLS}^{μ}

Here we have $\beta = 0$ and the recursive test statistic calculated for the sub-sample $t = 1, ..., \lfloor \tau T \rfloor$ is based on the t-ratio for $\tilde{\delta}_{\tau}$ from the estimated regression

$$\Delta \tilde{u}_{\tau,t} = \tilde{\delta}_{\tau} \tilde{u}_{\tau,t-1} + \tilde{v}_{\tau,t}, \qquad t = 1, ..., \lfloor \tau T \rfloor$$

where

$$\tilde{u}_{\tau,t} = y_t - \tilde{\mu}_{\tau}$$

with $\tilde{\mu}_{\tau}$ the recursive GLS estimate of the mean with quasi-differencing parameter $\bar{\rho} = 1 + \bar{c}T^{-1}$, i.e.

$$\begin{aligned}
\tilde{\mu}_{\tau} &= \frac{y_1 + (1-\bar{\rho}) \sum_{t=2}^{\lfloor \tau T \rfloor} (y_t - \bar{\rho} y_{t-1})}{1 + (\lfloor \tau T \rfloor - 1)(1-\bar{\rho})^2} \\
&= \frac{y_1 - \bar{c}T^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} (\Delta y_t - \bar{c}T^{-1} y_{t-1})}{1 + (\lfloor \tau T \rfloor - 1)\bar{c}^2 T^{-2}} \\
&= y_1 + o_p(1)
\end{aligned}$$

so the recursive t-ratio here is

$$t_{\tau} = \frac{\tilde{\delta}_{\tau}}{s.e.(\tilde{\delta}_{\tau})}$$

where

$$\begin{aligned}
\tilde{\delta}_{\tau} &= \frac{\sum_{t=2}^{\lfloor \tau T \rfloor} \Delta \tilde{u}_{\tau,t} \tilde{u}_{\tau,t-1}}{\sum_{t=2}^{\lfloor \tau T \rfloor} \tilde{u}_{\tau,t-1}^2} \\
s.e.(\tilde{\delta}_{\tau})^2 &= \frac{\tilde{\sigma}_{\tau}^2}{\sum_{t=2}^{\lfloor \tau T \rfloor} \tilde{u}_{\tau,t-1}^2} \\
\tilde{\sigma}_{\tau}^2 &= \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} \tilde{v}_{\tau,t}^2
\end{aligned}$$

Note that in what follows we can set $\mu = 0$ without loss of generality, so that $y_t = u_t$. First consider the recursively demeaned y_t, i.e. $\tilde{u}_{\tau,t}$. Since $y_t = u_t$ we have

$$\begin{aligned}
\tilde{u}_{\tau,t} &= u_t - u_1 + o_p(1) \\
T^{-1/2} \tilde{u}_{\tau,\lfloor rT \rfloor} &= T^{-1/2}(u_{\lfloor rT \rfloor} - u_1) + o_p(1) \\
&\xrightarrow{d} \sigma K_{c,\alpha}(r, \tau_1, \tau_2) - \sigma \alpha \\
&= \sigma K_{c,\alpha}^{\mu,G}(r, \tau_1, \tau_2)
\end{aligned}$$

Now consider the recursive parameter estimate $\tilde{\delta}_{\tau}$:

$$\begin{aligned}
\tilde{\delta}_{\tau} &= \frac{\sum_{t=2}^{\lfloor \tau T \rfloor} \Delta \tilde{u}_{\tau,t} \tilde{u}_{\tau,t-1}}{\sum_{t=2}^{\lfloor \tau T \rfloor} \tilde{u}_{\tau,t-1}^2} \\
T\tilde{\delta}_{\tau} &= \frac{T^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} \Delta \tilde{u}_{\tau,t} \tilde{u}_{\tau,t-1}}{T^{-2} \sum_{t=2}^{\lfloor \tau T \rfloor} \tilde{u}_{\tau,t-1}^2}
\end{aligned}$$

For the denominator we have

$$
\begin{aligned}
T^{-2}\sum_{t=2}^{\lfloor \tau T\rfloor}\tilde{u}_{\tau,t-1}^2 &= T^{-1}\sum_{\lfloor rT\rfloor=2}^{\lfloor \tau T\rfloor}(T^{-1/2}\tilde{u}_{\tau,t-1})^2 \\
&\xrightarrow{d} \sigma^2\int_0^{\tau}K_{c,\alpha}^{\mu,G}(r,\tau_1,\tau_2)^2dr
\end{aligned}
$$

by the Continuous Mapping Theorem. Now consider the numerator

$$
\begin{aligned}
T^{-1}\sum_{t=2}^{\lfloor \tau T\rfloor}\Delta\tilde{u}_{\tau,t}\tilde{u}_{\tau,t-1} &= \frac{1}{2}\left\{(T^{-1/2}\tilde{u}_{\lfloor \tau T\rfloor})^2-(T^{-1/2}\tilde{u}_1)^2-T^{-1}\sum_{t=2}^{\lfloor \tau T\rfloor}(\Delta\tilde{u}_{\tau,t})^2\right\} \\
&\xrightarrow{d} \frac{1}{2}\left\{\sigma^2K_{c,\alpha}^{\mu,G}(\tau,\tau_1,\tau_2)^2-\sigma^2K_{c,\alpha}^{\mu,G}(0,\tau_1,\tau_2)^2-\tau\sigma^2\right\}
\end{aligned}
$$

with the limit of the last term coming from

$$
\begin{aligned}
T^{-1}\sum_{t=2}^{\lfloor \tau T\rfloor}(\Delta\tilde{u}_{\tau,t})^2 &= T^{-1}\sum_{t=2}^{\lfloor \tau T\rfloor}(\tilde{u}_{\tau,t}-\tilde{u}_{\tau,t-1})^2 \\
&= T^{-1}\sum_{t=2}^{\lfloor \tau T\rfloor}\{(u_t-u_1)-(u_{t-1}-u_1)\}^2+o_p(1) \\
&= T^{-1}\sum_{t=2}^{\lfloor \tau T\rfloor}(\Delta u_t)^2+o_p(1) \\
&= \tau(\tau T)^{-1}\sum_{t=2}^{\lfloor \tau T\rfloor}(\Delta u_t)^2+o_p(1) \\
&\xrightarrow{p} \tau\sigma^2
\end{aligned}
$$

Note that $K_{c,\alpha}^{\mu,G}(0,\tau_1,\tau_2)=W(0)=0$ so we can simplify to

$$
T^{-1}\sum_{t=2}^{\lfloor \tau T\rfloor}\Delta\tilde{u}_{\tau,t}\tilde{u}_{\tau,t-1}\xrightarrow{d}\frac{1}{2}\left\{\sigma^2K_{c,\alpha}^{\mu,G}(\tau,\tau_1,\tau_2)^2-\tau\sigma^2\right\}
$$

So

$$
\begin{aligned}
T\tilde{\delta}_{\tau} &\xrightarrow{d} \frac{\frac{1}{2}\left\{\sigma^2K_{c,\alpha}^{\mu,G}(\tau,\tau_1,\tau_2)^2-\tau\sigma^2\right\}}{\sigma^2\int_0^{\tau}K_{c,\alpha}^{\mu,G}(r,\tau_1,\tau_2)^2dr} \\
&= \frac{K_{c,\alpha}^{\mu,G}(\tau,\tau_1,\tau_2)^2-\tau}{2\int_0^{\tau}K_{c,\alpha}^{\mu,G}(r,\tau_1,\tau_2)^2dr}
\end{aligned}
$$

Next consider $\tilde{\sigma}_\tau^2$ and then $s.e.(\tilde{\delta}_\tau)$. We obtain

$$\begin{aligned}
\tilde{\sigma}_\tau^2 &= \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} \tilde{v}_{\tau,t}^2 \\
&= \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} (\Delta \tilde{u}_{\tau,t} - \tilde{\delta}_\tau \tilde{u}_{\tau,t-1})^2 \\
&= \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} (\Delta \tilde{u}_{\tau,t})^2 + \tilde{\delta}_\tau^2 \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} \tilde{u}_{\tau,t-1}^2 - 2\tilde{\delta}_\tau \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} \Delta \tilde{u}_{\tau,t} \tilde{u}_{\tau,t-1} \\
&= \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} (\Delta u_t)^2 + O_p(T^{-2}) O_p(T) - 2 O_p(T^{-1}) O_p(1) \\
&= \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} (\Delta u_t)^2 + o_p(1) \\
&\xrightarrow{p} \sigma^2
\end{aligned}$$

and

$$\begin{aligned}
\{T s.e.(\tilde{\delta}_\tau)\}^2 &= \frac{\tilde{\sigma}_\tau^2}{T^{-2} \sum_{t=2}^{\lfloor \tau T \rfloor} \tilde{u}_{\tau,t-1}^2} \\
&\xrightarrow{d} \frac{\sigma^2}{\sigma^2 \int_0^\tau K_{c,\alpha}^{\mu,G}(r, \tau_1, \tau_2)^2 dr} \\
&= \frac{1}{\int_0^\tau K_{c,\alpha}^{\mu,G}(r, \tau_1, \tau_2)^2 dr}
\end{aligned}$$

Putting it all together we find

$$\begin{aligned}
t_\tau &= \frac{T \tilde{\delta}_\tau}{T s.e.(\tilde{\delta}_\tau)} \\
&\xrightarrow{d} \frac{K_{c,\alpha}^{\mu,G}(\tau, \tau_1, \tau_2)^2 - \tau}{2\sqrt{\int_0^\tau K_{c,\alpha}^{\mu,G}(r, \tau_1, \tau_2)^2 dr}}
\end{aligned}$$

Asymptotic distribution of PWY_{OLS}^{β}

In this case the recursive test statistic calculated for the sub-sample $t = 1, ..., \lfloor \tau T \rfloor$ is based on the t-ratio for $\hat{\delta}_\tau$ from the estimated regression

$$\Delta \hat{u}_{\tau,t} = \hat{\delta}_\tau \hat{u}_{\tau,t-1} + \hat{v}_{\tau,t}, \qquad t = 1, ..., \lfloor \tau T \rfloor$$

where

$$\hat{u}_{\tau,t} = y_t - \hat{\mu}_\tau - \hat{\beta}_\tau t$$

with $\hat{\mu}_\tau$ and $\hat{\beta}_\tau$ the recursively detrending estimates. The t-ratio is then

$$t_\tau = \frac{\hat{\delta}_\tau}{s.e.(\hat{\delta}_\tau)}$$

with

$$\begin{aligned}
\hat{\delta}_\tau &= \frac{\sum_{t=2}^{\lfloor \tau T \rfloor} \Delta \hat{u}_{\tau,t} \hat{u}_{\tau,t-1}}{\sum_{t=2}^{\lfloor \tau T \rfloor} \hat{u}_{\tau,t-1}^2} \\
s.e.(\hat{\delta}_\tau)^2 &= \frac{\hat{\sigma}_\tau^2}{\sum_{t=2}^{\lfloor \tau T \rfloor} \hat{u}_{\tau,t-1}^2} \\
\hat{\sigma}_\tau^2 &= \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} \hat{v}_{\tau,t}^2
\end{aligned}$$

In what follows we can set $\mu = \beta = 0$, so that $y_t = u_t$ throughout. First consider the properties of the recursively detrending estimates:

$$\begin{aligned}
\begin{bmatrix} \hat{\mu}_\tau \\ \hat{\beta}_\tau \end{bmatrix} &= \begin{bmatrix} \lfloor \tau T \rfloor & \sum_{t=1}^{\lfloor \tau T \rfloor} t \\ \sum_{t=1}^{\lfloor \tau T \rfloor} t & \sum_{t=1}^{\lfloor \tau T \rfloor} t^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{t=1}^{\lfloor \tau T \rfloor} u_t \\ \sum_{t=1}^{\lfloor \tau T \rfloor} t u_t \end{bmatrix} \\
\begin{bmatrix} T^{-1/2} \hat{\mu}_\tau \\ T^{1/2} \hat{\beta}_\tau \end{bmatrix} &= \begin{bmatrix} T^{-1} \lfloor \tau T \rfloor & T^{-2} \sum_{t=1}^{\lfloor \tau T \rfloor} t \\ T^{-2} \sum_{t=1}^{\lfloor \tau T \rfloor} t & T^{-3} \sum_{t=1}^{\lfloor \tau T \rfloor} t^2 \end{bmatrix}^{-1} \begin{bmatrix} T^{-3/2} \sum_{t=1}^{\lfloor \tau T \rfloor} u_t \\ T^{-5/2} \sum_{t=1}^{\lfloor \tau T \rfloor} t u_t \end{bmatrix} \\
&\xrightarrow{d} \begin{bmatrix} \tau & \tau^2/2 \\ \tau^2/2 & \tau^3/3 \end{bmatrix}^{-1} \begin{bmatrix} \sigma \int_0^\tau K_{c,\alpha}(r, \tau_1, \tau_2) dr \\ \sigma \int_0^\tau r K_{c,\alpha}(r, \tau_1, \tau_2) dr \end{bmatrix} \\
&= \begin{bmatrix} \frac{4}{\tau} & -\frac{6}{\tau^2} \\ -\frac{6}{\tau^2} & \frac{12}{\tau^3} \end{bmatrix} \begin{bmatrix} \sigma \int_0^\tau K_{c,\alpha}(r, \tau_1, \tau_2) dr \\ \sigma \int_0^\tau r K_{c,\alpha}(r, \tau_1, \tau_2) dr \end{bmatrix} \\
&= \begin{bmatrix} \frac{4\sigma}{\tau} \int_0^\tau K_{c,\alpha}(r, \tau_1, \tau_2) dr - \frac{6\sigma}{\tau^2} \int_0^\tau r K_{c,\alpha}(r, \tau_1, \tau_2) dr \\ \frac{12\sigma}{\tau^3} \int_0^\tau r K_{c,\alpha}(r, \tau_1, \tau_2) dr - \frac{6\sigma}{\tau^2} \int_0^\tau K_{c,\alpha}(r, \tau_1, \tau_2) dr \end{bmatrix}
\end{aligned}$$

where

$$T^{-5/2} \sum_{t=1}^{rT} t u_t \xrightarrow{d} \int_0^\tau r K_{c,\alpha}(r, \tau_1, \tau_2) dr$$

is a standard result given in Hamilton (1994).

Next consider the recursively detrended y_t, i.e. $\hat{u}_{\tau,t}$. Since $y_t = u_t$ we have

$$
\begin{aligned}
\hat{u}_{\tau,t} &= u_t - \hat{\mu}_\tau - \hat{\beta}_\tau t \\
T^{-1/2}\hat{u}_{\tau,\lfloor rT \rfloor} &= T^{-1/2}u_{\lfloor rT \rfloor} - T^{-1/2}\hat{\mu}_\tau - T^{-1/2}\hat{\beta}_\tau t \\
&= T^{-1/2}u_{\lfloor rT \rfloor} - T^{-1/2}\hat{\mu}_\tau - T^{1/2}\hat{\beta}_\tau r \\
&\xrightarrow{d} \sigma K_{c,\alpha}(r,\tau_1,\tau_2) - \frac{4\sigma}{\tau}\int_0^\tau K_{c,\alpha}(s,\tau_1,\tau_2)ds + \frac{6\sigma}{\tau^2}\int_0^\tau sK_{c,\alpha}(s,\tau_1,\tau_2)ds \\
&\quad - \left\{\frac{12\sigma}{\tau^3}\int_0^\tau sK_{c,\alpha}(s,\tau_1,\tau_2)ds - \frac{6\sigma}{\tau^2}\int_0^\tau K_{c,\alpha}(s,\tau_1,\tau_2)ds\right\} r \\
&= \sigma K_{c,\alpha}(r,\tau_1,\tau_2) - \frac{2\sigma}{\tau}\left(2 - \frac{3}{\tau}r\right)\int_0^\tau K_{c,\alpha}(s,\tau_1,\tau_2)ds \\
&\quad + \frac{6\sigma}{\tau^2}\left(1 - \frac{2}{\tau}r\right)\int_0^\tau sK_{c,\alpha}(s,\tau_1,\tau_2)ds \\
&\equiv \sigma K^\beta_{c,\alpha}(r,\tau,\tau_1,\tau_2)
\end{aligned}
$$

Now consider the recursive parameter estimate $\hat{\delta}_\tau$:

$$
\begin{aligned}
\hat{\delta}_\tau &= \frac{\sum_{t=2}^{\lfloor \tau T \rfloor} \Delta\hat{u}_{\tau,t}\hat{u}_{\tau,t-1}}{\sum_{t=2}^{\lfloor \tau T \rfloor} \hat{u}^2_{\tau,t-1}} \\
T\hat{\delta}_\tau &= \frac{T^{-1}\sum_{t=2}^{\lfloor \tau T \rfloor} \Delta\hat{u}_{\tau,t}\hat{u}_{\tau,t-1}}{T^{-2}\sum_{t=2}^{\lfloor \tau T \rfloor} \hat{u}^2_{\tau,t-1}}
\end{aligned}
$$

For the denominator we have

$$
\begin{aligned}
T^{-2}\sum_{t=2}^{\lfloor \tau T \rfloor} \hat{u}^2_{\tau,t-1} &= T^{-1}\sum_{\lfloor rT \rfloor=2}^{\lfloor \tau T \rfloor} (T^{-1/2}\hat{u}_{\tau,\lfloor rT \rfloor -1})^2 \\
&\xrightarrow{d} \sigma^2 \int_0^\tau K^\beta_{c,\alpha}(r,\tau,\tau_1,\tau_2)^2 dr
\end{aligned}
$$

Now consider the numerator

$$
\begin{aligned}
T^{-1}\sum_{t=2}^{\lfloor \tau T \rfloor} \Delta\hat{u}_{\tau,t}\hat{u}_{\tau,t-1} &= \frac{1}{2}\left\{(T^{-1/2}\hat{u}_{\lfloor \tau T \rfloor})^2 - (T^{-1/2}\hat{u}_1)^2 - T^{-1}\sum_{t=2}^{\lfloor \tau T \rfloor}(\Delta\hat{u}_{\tau,t})^2\right\} \\
&\xrightarrow{d} \frac{1}{2}\left\{\sigma^2 K^\beta_{c,\alpha}(\tau,\tau,\tau_1,\tau_2)^2 - \sigma^2 K^\beta_{c,\alpha}(0,\tau,\tau_1,\tau_2)^2 - \tau\sigma^2\right\}
\end{aligned}
$$

The limit of the last term comes from

$$
\begin{aligned}
T^{-1}\sum_{t=2}^{\lfloor\tau T\rfloor}(\Delta\hat{u}_{\tau,t})^2 &= T^{-1}\sum_{t=2}^{\lfloor\tau T\rfloor}(\hat{u}_{\tau,t}-\hat{u}_{\tau,t-1})^2 \\
&= T^{-1}\sum_{t=2}^{\lfloor\tau T\rfloor}\left\{\left(u_t-\hat{\mu}_\tau-\hat{\beta}_\tau t\right)-\left(u_{t-1}-\hat{\mu}_\tau-\hat{\beta}_\tau(t-1)\right)\right\}^2 \\
&= T^{-1}\sum_{t=2}^{\lfloor\tau T\rfloor}(\Delta u_t-\hat{\beta}_\tau)^2 \\
&= \tau(\tau T)^{-1}\sum_{t=2}^{\lfloor\tau T\rfloor}(\Delta u_t)^2+T^{-1}\lfloor\tau T\rfloor\hat{\beta}_\tau^2-2\hat{\beta}_\tau T^{-1}\sum_{t=2}^{\lfloor\tau T\rfloor}\Delta u_t \\
&= \tau(\tau T)^{-1}\sum_{t=2}^{\lfloor\tau T\rfloor}(\Delta u_t)^2+O_p(T^{-1})-2O_p(T^{-1/2})O_p(T^{-1/2}) \\
&= \tau(\tau T)^{-1}\sum_{t=2}^{\lfloor\tau T\rfloor}(\Delta u_t)^2+o_p(1) \\
&\xrightarrow{p} \tau\sigma^2
\end{aligned}
$$

So

$$
\begin{aligned}
T\hat{\delta}_\tau &\xrightarrow{d} \frac{\frac{1}{2}\left\{\sigma^2K_{c,\alpha}^{\beta}(\tau,\tau,\tau_1,\tau_2)^2-\sigma^2K_{c,\alpha}^{\beta}(0,\tau,\tau_1,\tau_2)^2-\tau\sigma^2\right\}}{\sigma^2\int_0^\tau K_{c,\alpha}^{\beta}(r,\tau,\tau_1,\tau_2)^2dr} \\
&= \frac{K_{c,\alpha}^{\beta}(\tau,\tau,\tau_1,\tau_2)^2-K_{c,\alpha}^{\beta}(0,\tau,\tau_1,\tau_2)^2-\tau}{2\int_0^\tau K_{c,\alpha}^{\beta}(r,\tau,\tau_1,\tau_2)^2dr}
\end{aligned}
$$

Next consider $\hat{\sigma}_\tau^2$ and then $s.e.(\hat{\delta}_\tau)$. We again obtain $\hat{\sigma}_\tau^2\xrightarrow{p}\sigma^2$ and

$$
\begin{aligned}
\{Ts.e.(\hat{\delta}_\tau)\}^2 &= \frac{\hat{\sigma}_\tau^2}{T^{-2}\sum_{t=2}^{\lfloor\tau T\rfloor}\hat{u}_{\tau,t-1}^2} \\
&\xrightarrow{d} \frac{\sigma^2}{\sigma^2\int_0^\tau K_{c,\alpha}^{\beta}(r,\tau,\tau_1,\tau_2)^2dr} \\
&= \frac{1}{\int_0^\tau K_{c,\alpha}^{\beta}(r,\tau,\tau_1,\tau_2)^2dr}
\end{aligned}
$$

Putting it all together we find

$$
\begin{aligned}
t_\tau &= \frac{T\hat{\delta}_\tau}{Ts.e.(\hat{\delta}_\tau)} \\
&\xrightarrow{d} \frac{\frac{K^\beta_{c,\alpha}(\tau,\tau,\tau_1,\tau_2)^2 - K^\beta_{c,\alpha}(0,\tau,\tau_1,\tau_2)^2 - \tau}{2\int_0^\tau K^\beta_{c,\alpha}(r,\tau,\tau_1,\tau_2)^2 dr}}{\sqrt{\frac{1}{\int_0^\tau K^\beta_{c,\alpha}(r,\tau,\tau_1,\tau_2)^2 dr}}} \\
&= \frac{K^\beta_{c,\alpha}(\tau,\tau,\tau_1,\tau_2)^2 - K^\beta_{c,\alpha}(0,\tau,\tau_1,\tau_2)^2 - \tau}{2\sqrt{\int_0^\tau K^\beta_{c,\alpha}(r,\tau,\tau_1,\tau_2)^2 dr}}
\end{aligned}
$$

Asymptotic distribution of PWY_{GLS}^{β}

In this case the recursive test statistic calculated for the sub-sample $t = 1, ..., \lfloor \tau T \rfloor$ is based on the t-ratio for $\tilde{\delta}_\tau$ from the estimated regression

$$\Delta \tilde{u}_{\tau,t} = \tilde{\delta}_\tau \tilde{u}_{\tau,t-1} + \tilde{v}_{\tau,t}, \qquad t = 1, ..., \lfloor \tau T \rfloor$$

where

$$\tilde{u}_{\tau,t} = y_t - \tilde{\mu}_\tau - \tilde{\beta}_\tau t$$

with $\tilde{\mu}_\tau$ and $\tilde{\beta}_\tau$ the recursively GLS detrending estimates. The t-ratio is then

$$t_\tau = \frac{\tilde{\delta}_\tau}{s.e.(\tilde{\delta}_\tau)}$$

with

$$\begin{aligned}
\tilde{\delta}_\tau &= \frac{\sum_{t=2}^{\lfloor \tau T \rfloor} \Delta \tilde{u}_{\tau,t} \tilde{u}_{\tau,t-1}}{\sum_{t=2}^{\lfloor \tau T \rfloor} \tilde{u}_{\tau,t-1}^2} \\
s.e.(\tilde{\delta}_\tau)^2 &= \frac{\tilde{\sigma}_\tau^2}{\sum_{t=2}^{\lfloor \tau T \rfloor} \tilde{u}_{\tau,t-1}^2} \\
\tilde{\sigma}_\tau^2 &= \lfloor \tau T \rfloor^{-1} \sum_{t=2}^{\lfloor \tau T \rfloor} \tilde{v}_{\tau,t}^2
\end{aligned}$$

In what follows we can set $\mu = \beta = 0$, so that $y_t = u_t$ throughout. First consider the properties of the recursively detrending estimates:

$$\begin{aligned}
\left[\begin{array}{c} \tilde{\mu}_\tau \\ \tilde{\beta}_\tau \end{array} \right] &= \left[\begin{array}{cc} 1 + (\lfloor \tau T \rfloor - 1)(1 - \bar{\rho})^2 & 1 + (1 - \bar{\rho}) \sum_{t=2}^{\lfloor \tau T \rfloor} \{t - \bar{\rho}(t-1)\} \\ 1 + (1 - \bar{\rho}) \sum_{t=2}^{\lfloor \tau T \rfloor} \{t - \bar{\rho}(t-1)\} & 1 + \sum_{t=2}^{\lfloor \tau T \rfloor} \{t - \bar{\rho}(t-1)\}^2 \end{array} \right]^{-1} \\
&\quad \left[\begin{array}{c} u_1 + (1 - \bar{\rho}) \sum_{t=2}^{\lfloor \tau T \rfloor} (u_t - \bar{\rho} u_{t-1}) \\ u_1 + \sum_{t=2}^{\lfloor \tau T \rfloor} \{t - \bar{\rho}(t-1)\}\{u_t - \bar{\rho} u_{t-1}\} \end{array} \right]
\end{aligned}$$

$$\begin{aligned}
&T^{-1/2} \left[\begin{array}{cc} 1 & 0 \\ 0 & T^{-1} \end{array} \right]^{-1} \left[\begin{array}{c} \tilde{\mu}_\tau \\ \tilde{\beta}_\tau \end{array} \right] \\
&= \left(\begin{array}{c} \left[\begin{array}{cc} 1 + (\lfloor \tau T \rfloor - 1)(1 - \bar{\rho})^2 & 1 + (1 - \bar{\rho}) \sum_{t=2}^{\lfloor \tau T \rfloor} \{t - \bar{\rho}(t-1)\} \\ 1 + (1 - \bar{\rho}) \sum_{t=2}^{\lfloor \tau T \rfloor} \{t - \bar{\rho}(t-1)\} & 1 + \sum_{t=2}^{\lfloor \tau T \rfloor} \{t - \bar{\rho}(t-1)\}^2 \end{array} \right] \\ \left[\begin{array}{cc} 1 & 0 \\ 0 & T^{-1} \end{array} \right] \end{array} \right)^{-1} \\
&\quad T^{-1/2} \left[\begin{array}{c} u_1 + (1 - \bar{\rho}) \sum_{t=2}^{\lfloor \tau T \rfloor} (u_t - \bar{\rho} u_{t-1}) \\ u_1 + \sum_{t=2}^{\lfloor \tau T \rfloor} \{t - \bar{\rho}(t-1)\}\{u_t - \bar{\rho} u_{t-1}\} \end{array} \right]
\end{aligned}$$

$$
\begin{aligned}
&\begin{bmatrix} T^{-1/2}\tilde{\mu}_\tau \\ T^{1/2}\tilde{\beta}_\tau \end{bmatrix} \\
= &\left(\begin{bmatrix} 1+(\lfloor\tau T\rfloor-1)(1-\bar{\rho})^2 & T^{-1}[1+(1-\bar{\rho})\sum_{t=2}^{\lfloor\tau T\rfloor}\{t-\bar{\rho}(t-1)\}] \\ 1+(1-\bar{\rho})\sum_{t=2}^{\lfloor\tau T\rfloor}\{t-\bar{\rho}(t-1)\} & T^{-1}[1+\sum_{t=2}^{\lfloor\tau T\rfloor}\{t-\bar{\rho}(t-1)\}^2] \end{bmatrix}\right)^{-1} \\
&\begin{bmatrix} T^{-1/2}u_1+T^{-1/2}(1-\bar{\rho})\sum_{t=2}^{\lfloor\tau T\rfloor}(u_t-\bar{\rho}u_{t-1}) \\ T^{-1/2}u_1+T^{-1/2}\sum_{t=2}^{\lfloor\tau T\rfloor}\{t-\bar{\rho}(t-1)\}\{u_t-\bar{\rho}u_{t-1}\} \end{bmatrix} \\
\equiv &\begin{bmatrix} A_1 & T^{-1}A_2 \\ A_2 & A_3 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}
\end{aligned}
$$

Taking each term separately:

$$
\begin{aligned}
A_1 &= 1+(\lfloor\tau T\rfloor-1)(1-\bar{\rho})^2 \\
&= 1-\bar{c}^2T^{-2}(\lfloor\tau T\rfloor-1) \\
&\xrightarrow{d} 1
\end{aligned}
$$

$$
\begin{aligned}
A_2 &= 1+(1-\bar{\rho})\sum_{t=2}^{\lfloor\tau T\rfloor}\{t-\bar{\rho}(t-1)\} \\
&= 1-\bar{c}T^{-1}\sum_{t=2}^{\lfloor\tau T\rfloor}\{1-\bar{c}T^{-1}(t-1)\} \\
&\xrightarrow{d} 1-\bar{c}\tau+\bar{c}^2\tau^2/2
\end{aligned}
$$

$$
\begin{aligned}
A_3 &= T^{-1}[1+\sum_{t=2}^{\lfloor\tau T\rfloor}\{t-\bar{\rho}(t-1)\}^2] \\
&= T^{-1}[1+\sum_{t=2}^{\lfloor\tau T\rfloor}\{1-\bar{c}T^{-1}(t-1)\}^2] \\
&\xrightarrow{d} \tau-\bar{c}\tau^2+\bar{c}^2\tau^3/3
\end{aligned}
$$

$$
\begin{aligned}
B_1 &= T^{-1/2}u_1+T^{-1/2}(1-\bar{\rho})\sum_{t=2}^{\lfloor\tau T\rfloor}(u_t-\bar{\rho}u_{t-1}) \\
&= T^{-1/2}u_1-\bar{c}T^{-3/2}\sum_{t=2}^{\lfloor\tau T\rfloor}(u_t-\bar{\rho}u_{t-1}) \\
&\xrightarrow{p} \sigma\alpha
\end{aligned}
$$

$$
\begin{aligned}
B_2 &= T^{-1/2}u_1 + T^{-1/2}\sum_{t=2}^{\lfloor \tau T\rfloor}\{t-\bar{\rho}(t-1)\}\{u_t-\bar{\rho}u_{t-1}\} \\
&= T^{-1/2}u_1 + T^{-1/2}\sum_{t=2}^{\lfloor \tau T\rfloor}\{1-\bar{c}T^{-1}(t-1)\}(\Delta u_t-\bar{c}T^{-1}u_{t-1}) \\
&= T^{-1/2}u_1 + T^{-1/2}\sum_{t=2}^{\lfloor \tau T\rfloor}\{\Delta u_t-\bar{c}T^{-1}u_{t-1}\} - \bar{c}T^{-3/2}\sum_{t=2}^{\lfloor \tau T\rfloor}t(\Delta u_t-\bar{c}T^{-1}u_{t-1}) + o_p(1) \\
&= T^{-1/2}u_1 + T^{-1/2}(u_{\lfloor \tau T\rfloor}-u_1) - \bar{c}T^{-3/2}\sum_{t=2}^{\lfloor \tau T\rfloor}u_{t-1} - \bar{c}T^{-3/2}\sum_{t=2}^{\lfloor \tau T\rfloor}t\Delta u_t \\
&\quad +\bar{c}^2T^{-5/2}\sum_{t=2}^{\lfloor \tau T\rfloor}tu_{t-1} + o_p(1) \\
&= T^{-1/2}u_{\lfloor \tau T\rfloor} - \bar{c}T^{-3/2}\sum_{t=2}^{\lfloor \tau T\rfloor}u_{t-1} - \bar{c}T^{-3/2}\sum_{t=2}^{\lfloor \tau T\rfloor}t\Delta u_t + \bar{c}^2T^{-5/2}\sum_{t=2}^{\lfloor \tau T\rfloor}tu_{t-1} + o_p(1) \\
&\xrightarrow{d} \sigma K_{c,\alpha}(\tau,\tau_1,\tau_2) - \bar{c}\sigma\int_0^\tau K_{c,\alpha}(s,\tau_1,\tau_2)ds - \bar{c}\sigma\int_0^\tau s dK_{c,\alpha}(s,\tau_1,\tau_2) \\
&\quad +\bar{c}^2\sigma\int_0^\tau sK_{c,\alpha}(s,\tau_1,\tau_2)ds \\
&= \sigma K_{c,\alpha}(\tau,\tau_1,\tau_2) - \bar{c}\sigma\int_0^\tau K_{c,\alpha}(s,\tau_1,\tau_2)ds \\
&\quad -\bar{c}\sigma\left\{\tau K_{c,\alpha}(\tau,\tau_1,\tau_2) - \int_0^\tau K_{c,\alpha}(s,\tau_1,\tau_2)ds\right\} + \bar{c}^2\sigma\int_0^\tau sK_{c,\alpha}(s,\tau_1,\tau_2)ds \\
&= \sigma(1-\bar{c}\tau)K_{c,\alpha}(\tau,\tau_1,\tau_2) + \bar{c}^2\sigma\int_0^\tau sK_{c,\alpha}(s,\tau_1,\tau_2)ds
\end{aligned}
$$

Substituting back in gives

$$\begin{aligned}
\begin{bmatrix} T^{-1/2}\tilde{\mu}_\tau \\ T^{1/2}\tilde{\beta}_\tau \end{bmatrix} &\xrightarrow{d} \begin{bmatrix} 1 & 0 \\ 1-\bar{c}\tau+\bar{c}^2\tau^2/2 & \tau-\bar{c}\tau^2+\bar{c}^2\tau^3/3 \end{bmatrix}^{-1} \\
&\quad \begin{bmatrix} \sigma\alpha \\ \sigma(1-\bar{c}\tau)K_{c,\alpha}(\tau,\tau_1,\tau_2)+\bar{c}^2\sigma\int_0^\tau sK_{c,\alpha}(s,\tau_1,\tau_2)ds \end{bmatrix} \\
&= \sigma \begin{bmatrix} 1 & 0 \\ -\frac{1-\bar{c}\tau+\bar{c}^2\tau^2/2}{\tau-\bar{c}\tau^2+\bar{c}^2\tau^3/3} & (\tau-\bar{c}\tau^2+\bar{c}^2\tau^3/3)^{-1} \end{bmatrix} \\
&\quad \begin{bmatrix} \alpha \\ (1-\bar{c}\tau)K_{c,\alpha}(\tau,\tau_1,\tau_2)+\bar{c}^2\int_0^\tau sK_{c,\alpha}(s,\tau_1,\tau_2)ds \end{bmatrix} \\
&= \sigma \begin{bmatrix} \alpha \\ (\tau-\bar{c}\tau^2+\bar{c}^2\tau^3/3)^{-1}\{(1-\bar{c}\tau)K_{c,\alpha}(\tau,\tau_1,\tau_2) \\ +\bar{c}^2\int_0^\tau sK_{c,\alpha}(s,\tau_1,\tau_2)ds\} - \frac{1-\bar{c}\tau+\bar{c}^2\tau^2/2}{\tau-\bar{c}\tau^2+\bar{c}^2\tau^3/3}\alpha \end{bmatrix}
\end{aligned}$$

Next consider the recursively detrended y_t, i.e. $\tilde{u}_{\tau,t}$. Since $y_t = u_t$ we have

$$\begin{aligned}
\tilde{u}_{\tau,t} &= u_t - \tilde{\mu}_\tau - \tilde{\beta}_\tau t \\
T^{-1/2}\tilde{u}_{\tau,\lfloor rT\rfloor} &= T^{-1/2}u_{\lfloor rT\rfloor} - T^{-1/2}\tilde{\mu}_\tau - T^{-1/2}\tilde{\beta}_\tau t \\
&= T^{-1/2}u_{\lfloor rT\rfloor} - T^{-1/2}\tilde{\mu}_\tau - T^{1/2}\tilde{\beta}_\tau r \\
&\xrightarrow{d} \sigma K_{c,\alpha}(r,\tau_1,\tau_2) - \sigma\alpha - \\
&\quad \sigma \begin{bmatrix} (\tau-\bar{c}\tau^2+\bar{c}^2\tau^3/3)^{-1}\{(1-\bar{c}\tau)K_{c,\alpha}(\tau,\tau_1,\tau_2) \\ +\bar{c}^2\int_0^\tau sK_{c,\alpha}(s,\tau_1,\tau_2)ds\} - \frac{1-\bar{c}\tau+\bar{c}^2\tau^2/2}{\tau-\bar{c}\tau^2+\bar{c}^2\tau^3/3}\alpha \end{bmatrix} r \\
&\equiv \sigma K^{\beta,G}_{c,\bar{c},\alpha}(r,\tau,\tau_1,\tau_2)
\end{aligned}$$

Now consider the recursive parameter estimate $\tilde{\delta}_\tau$:

$$\begin{aligned}
\tilde{\delta}_\tau &= \frac{\sum_{t=2}^{\lfloor\tau T\rfloor}\Delta\tilde{u}_{\tau,t}\tilde{u}_{\tau,t-1}}{\sum_{t=2}^{\lfloor\tau T\rfloor}\tilde{u}^2_{\tau,t-1}} \\
T\tilde{\delta}_\tau &= \frac{T^{-1}\sum_{t=2}^{\lfloor\tau T\rfloor}\Delta\tilde{u}_{\tau,t}\tilde{u}_{\tau,t-1}}{T^{-2}\sum_{t=2}^{\lfloor\tau T\rfloor}\tilde{u}^2_{\tau,t-1}}
\end{aligned}$$

For the denominator we have

$$\begin{aligned}
T^{-2}\sum_{t=2}^{\lfloor\tau T\rfloor}\tilde{u}^2_{\tau,t-1} &= T^{-1}\sum_{\lfloor rT\rfloor=2}^{\lfloor\tau T\rfloor}(T^{-1/2}\tilde{u}_{\tau,\lfloor rT\rfloor-1})^2 \\
&\xrightarrow{d} \sigma^2\int_0^\tau K^{\beta,G}_{c,\bar{c},\alpha}(r,\tau,\tau_1,\tau_2)^2dr
\end{aligned}$$

Now consider the numerator

$$\begin{aligned}
T^{-1}\sum_{t=2}^{\lfloor\tau T\rfloor}\Delta\tilde{u}_{\tau,t}\tilde{u}_{\tau,t-1} &= \frac{1}{2}\left\{(T^{-1/2}\tilde{u}_{\lfloor\tau T\rfloor})^2 - (T^{-1/2}\tilde{u}_1)^2 - T^{-1}\sum_{t=2}^{\lfloor\tau T\rfloor}(\Delta\tilde{u}_{\tau,t})^2\right\} \\
&\xrightarrow{d} \frac{1}{2}\left\{\sigma^2K^{\beta,G}_{c,\bar{c},\alpha}(\tau,\tau,\tau_1,\tau_2)^2 - \sigma^2K^{\beta,G}_{c,\bar{c},\alpha}(0,\tau,\tau_1,\tau_2)^2 - \tau\sigma^2\right\} \\
&= \frac{1}{2}\left\{\sigma^2K^{\beta,G}_{c,\bar{c},\alpha}(\tau,\tau,\tau_1,\tau_2)^2 - \tau\sigma^2\right\}
\end{aligned}$$

using $T^{-1}\sum_{t=2}^{\lfloor \tau T \rfloor}(\Delta\tilde{u}_{\tau,t})^2 \xrightarrow{p} \tau\sigma^2$ as before, and noting that $K^{\beta,G}_{c,\bar{c},\alpha}(0,\tau,\tau_1,\tau_2) = 0$.

So

$$
\begin{aligned}
T\hat{\delta}_\tau &\xrightarrow{d} \frac{\frac{1}{2}\left\{\sigma^2 K^{\beta,G}_{c,\bar{c},\alpha}(\tau,\tau,\tau_1,\tau_2)^2 - \sigma^2 K^{\beta,G}_{c,\bar{c},\alpha}(0,\tau,\tau_1,\tau_2)^2 - \tau\sigma^2\right\}}{\sigma^2\int_0^\tau K^{\beta,G}_{c,\bar{c},\alpha}(r,\tau,\tau_1,\tau_2)^2 dr} \\
&= \frac{K^{\beta,G}_{c,\bar{c},\alpha}(\tau,\tau,\tau_1,\tau_2)^2 - \tau}{2\int_0^\tau K^{\beta,G}_{c,\bar{c},\alpha}(r,\tau,\tau_1,\tau_2)^2 dr}
\end{aligned}
$$

Next consider $\tilde{\sigma}^2_\tau$ and then $s.e.(\tilde{\delta}_\tau)$. We again obtain $\tilde{\sigma}^2_\tau \xrightarrow{p} \sigma^2$ and

$$
\begin{aligned}
\{Ts.e.(\tilde{\delta}_\tau)\}^2 &= \frac{\tilde{\sigma}^2_\tau}{T^{-2}\sum_{t=2}^{\lfloor \tau T \rfloor}\tilde{u}^2_{\tau,t-1}} \\
&\xrightarrow{d} \frac{\sigma^2}{\sigma^2\int_0^\tau K^{\beta,G}_{c,\bar{c},\alpha}(r,\tau,\tau_1,\tau_2)^2 dr} \\
&= \frac{1}{\int_0^\tau K^{\beta,G}_{c,\bar{c},\alpha}(r,\tau,\tau_1,\tau_2)^2 dr}
\end{aligned}
$$

Putting it all together we find

$$
\begin{aligned}
t_\tau &= \frac{T\tilde{\delta}_\tau}{Ts.e.(\tilde{\delta}_\tau)} \\
&\xrightarrow{d} \frac{K^{\beta,G}_{c,\bar{c},\alpha}(\tau,\tau,\tau_1,\tau_2)^2 - \tau}{2\sqrt{\int_0^\tau K^{\beta,G}_{c,\bar{c},\alpha}(r,\tau,\tau_1,\tau_2)^2 dr}}
\end{aligned}
$$

Table 1: Asymptotic critical values of PWY^{μ}_{OLS}, PWY^{μ}_{GLS}, PWY^{β}_{OLS}, PWY^{β}_{GLS}, and λ^{μ}_{ζ} and λ^{β}_{ζ} values for significance level ζ

ζ	PWY^{μ}_{OLS}	PWY^{μ}_{GLS}	PWY^{β}_{OLS}	PWY^{β}_{GLS}	λ^{μ}_{ζ}	λ^{β}_{ζ}
0.10	1.166	2.319	0.298	4.592	1.111	1.238
0.05	1.433	2.626	0.529	5.215	1.096	1.168
0.01	1.923	3.223	1.007	6.415	1.081	1.089

Table 2: Finite sample critical values of PWY^{μ}_{OLS}, PWY^{μ}_{GLS}, PWY^{β}_{OLS}, PWY^{β}_{GLS} for significance level ζ and $T = 150$

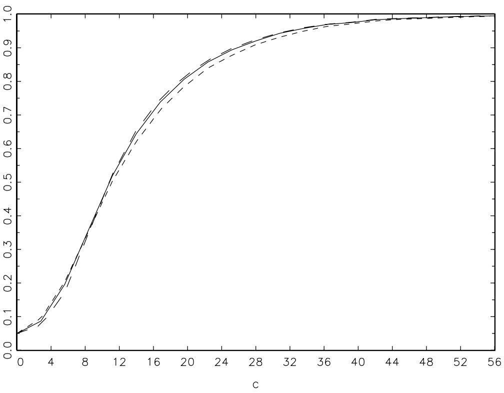
ζ	PWY^{μ}_{OLS}	PWY^{μ}_{GLS}	PWY^{β}_{OLS}	PWY^{β}_{GLS}
0.10	1.174	2.498	0.308	5.950
0.05	1.467	2.906	0.572	6.633
0.01	2.137	3.634	1.137	7.980

Table 3: Application of PWY^{μ}_{OLS}, PWY^{μ}_{GLS}, PWY^{β}_{OLS}, PWY^{β}_{GLS}, U^{μ} and U^{β} to NASDAQ prices

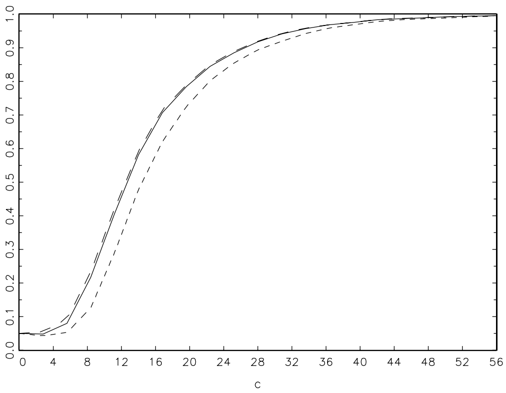
PWY^{μ}_{OLS}	PWY^{μ}_{GLS}	PWY^{β}_{OLS}	PWY^{β}_{GLS}	U^{μ}	U^{β}
1.098	3.227***	1.455***	2.896	**	***

** and *** indicate rejections at a 0.05 and 0.01 level respectively

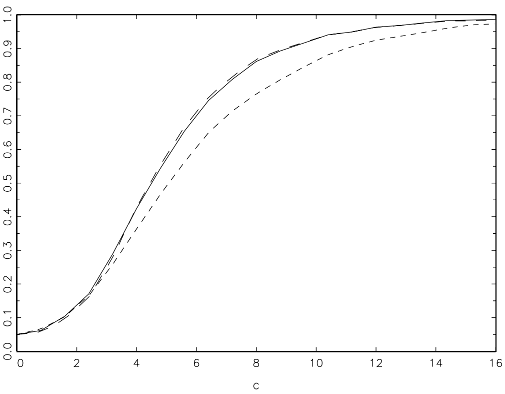
Figure 1: Asymptotic power of PWY^i_{OLS}, PWY^i_{GLS} and U^i for $[\tau_1, \tau_2] = [0.45, 0.55]$, where $i = \{\mu, \beta\}$



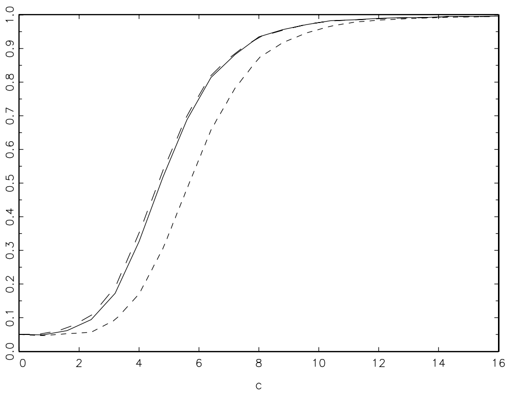
(a) $i = \mu$, $\alpha = 0$



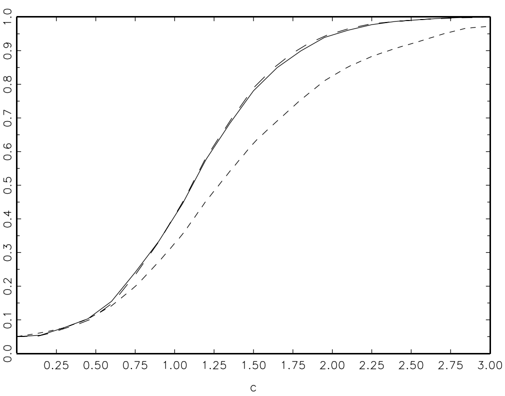
(b) $i = \beta$, $\alpha = 0$



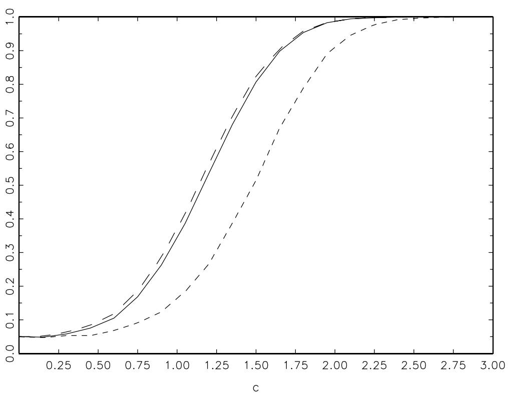
(c) $i = \mu$, $\alpha = 2$



(d) $i = \beta$, $\alpha = 2$



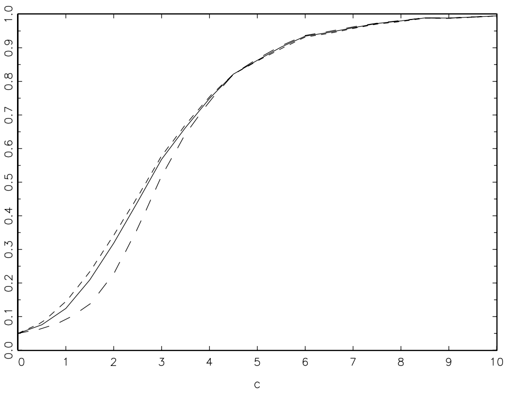
(e) $i = \mu$, $\alpha = 10$



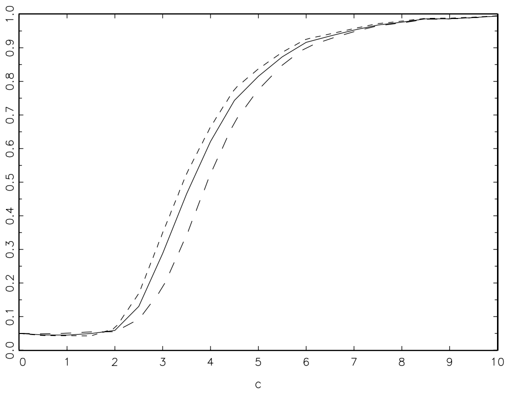
(f) $i = \beta$, $\alpha = 10$

$-\,-$ PWY^i_{OLS}, - - - PWY^i_{GLS}, — U^i

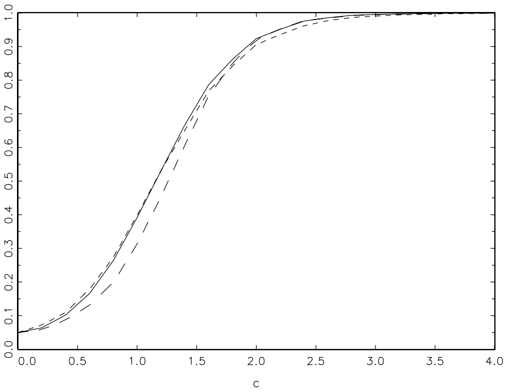
Figure 2: Asymptotic power of PWY^i_{OLS}, PWY^i_{GLS} and U^i for $[\tau_1, \tau_2] = [0.2, 0.8]$, where $i = \{\mu, \beta\}$



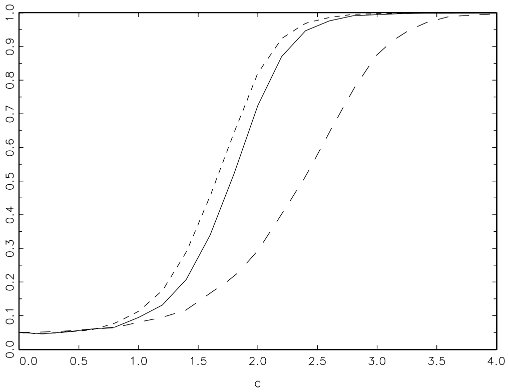
(a) $i = \mu$, $\alpha = 0$



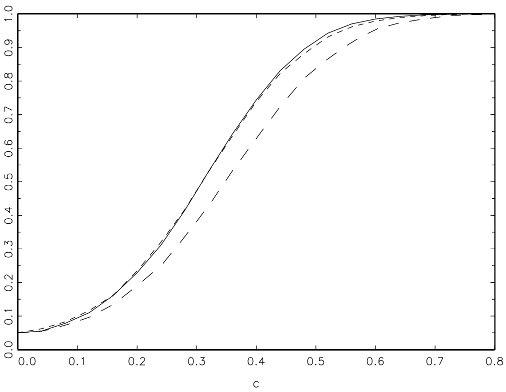
(b) $i = \beta$, $\alpha = 0$



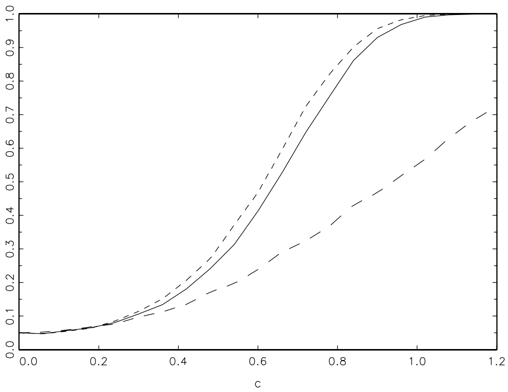
(c) $i = \mu$, $\alpha = 2$



(d) $i = \beta$, $\alpha = 2$



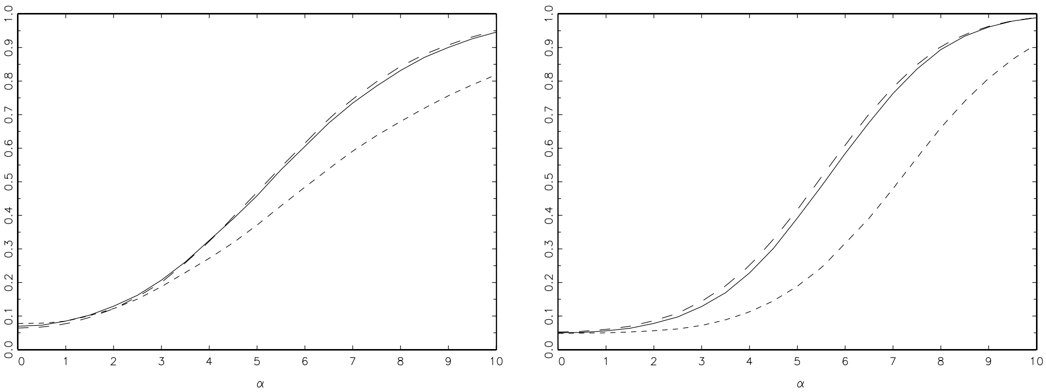
(e) $i = \mu$, $\alpha = 10$



(f) $i = \beta$, $\alpha = 10$

$--\ PWY^i_{OLS}$, - - - PWY^i_{GLS}, — U^i

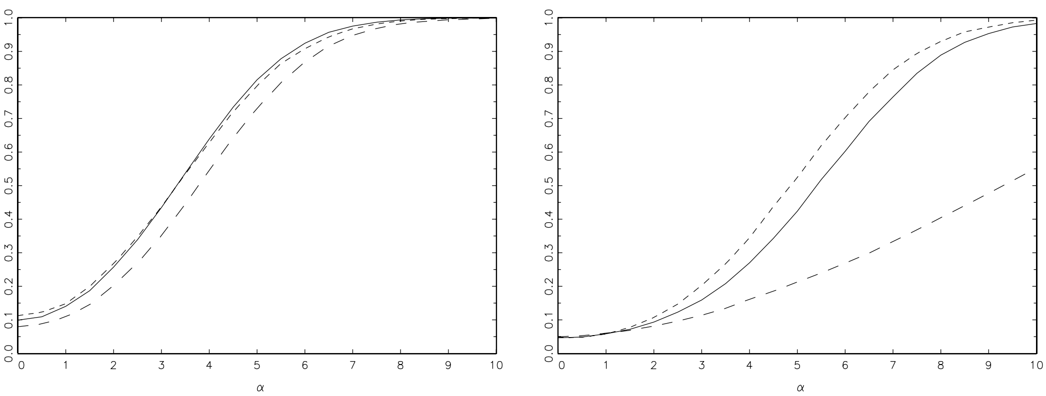
Figure 3: Asymptotic power of PWY^i_{OLS}, PWY^i_{GLS} and U^i for $[\tau_1, \tau_2] = [0.45, 0.55]$, where $i = \{\mu, \beta\}$



(a) $i = \mu$, $c = 2$ (b) $i = \beta$, $c = 2$

$-\,-\ PWY^i_{OLS}$, - - - PWY^i_{GLS}, — U^i

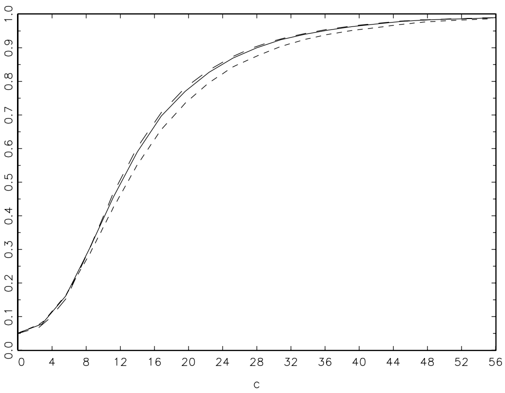
Figure 4: Asymptotic power of PWY^i_{OLS}, PWY^i_{GLS} and U^i for $[\tau_1, \tau_2] = [0.2, 0.8]$, where $i = \{\mu, \beta\}$



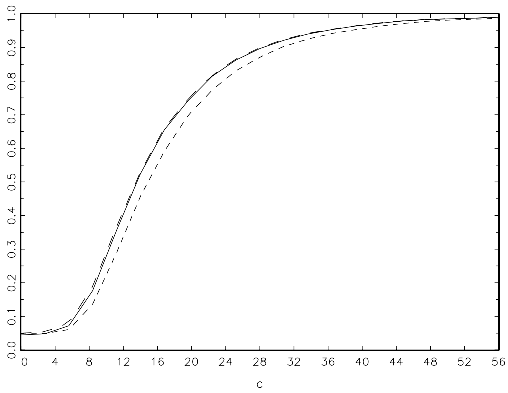
(a) $i = \mu$, $c = 0.8$ (b) $i = \beta$, $c = 1$

$-\,-\ PWY^i_{OLS}$, - - - PWY^i_{GLS}, — U^i

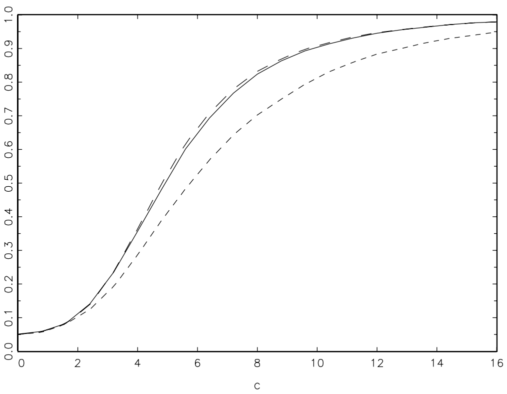
Figure 5: Finite sample power of PWY^i_{OLS}, PWY^i_{GLS} and U^i for $[\tau_1, \tau_2] = [0.45, 0.55]$, where $i = \{\mu, \beta\}$



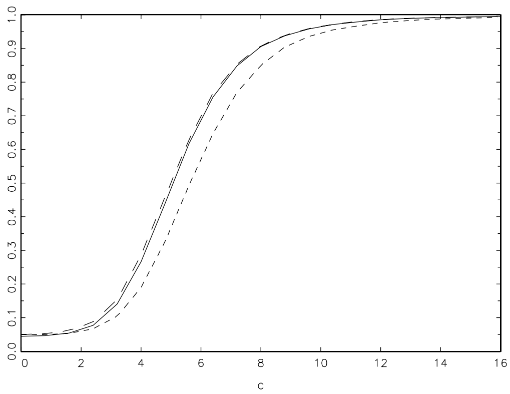
(a) $i = \mu$, $\alpha = 0$



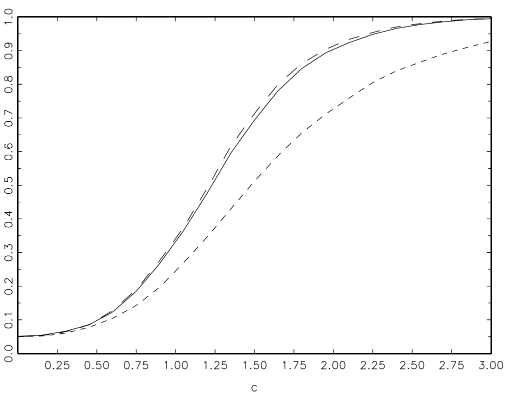
(b) $i = \beta$, $\alpha = 0$



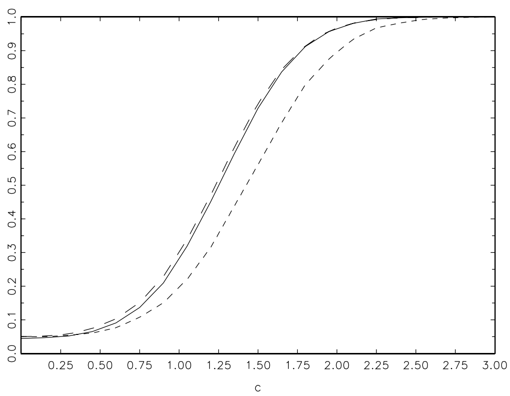
(c) $i = \mu$, $\alpha = 2$



(d) $i = \beta$, $\alpha = 2$



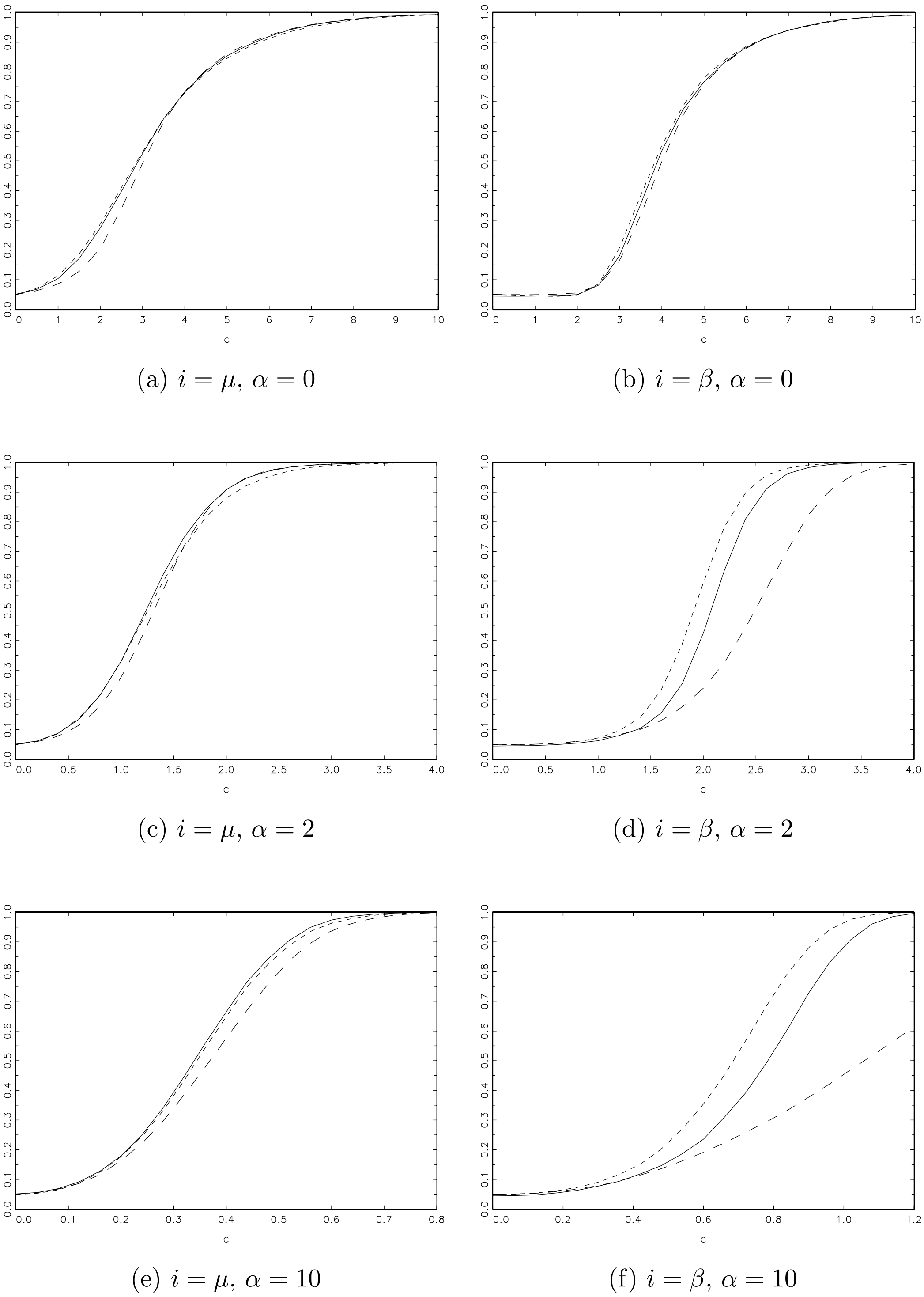
(e) $i = \mu$, $\alpha = 10$



(f) $i = \beta$, $\alpha = 10$

$- -\ PWY^i_{OLS}$, - - - PWY^i_{GLS}, — U^i

Figure 6: Finite sample power of PWY^i_{OLS}, PWY^i_{GLS} and U^i for $[\tau_1, \tau_2] = [0.2, 0.8]$, where $i = \{\mu, \beta\}$



(a) $i = \mu$, $\alpha = 0$

(b) $i = \beta$, $\alpha = 0$

(c) $i = \mu$, $\alpha = 2$

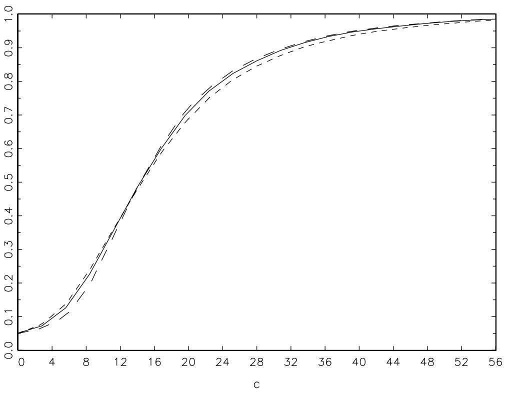
(d) $i = \beta$, $\alpha = 2$

(e) $i = \mu$, $\alpha = 10$

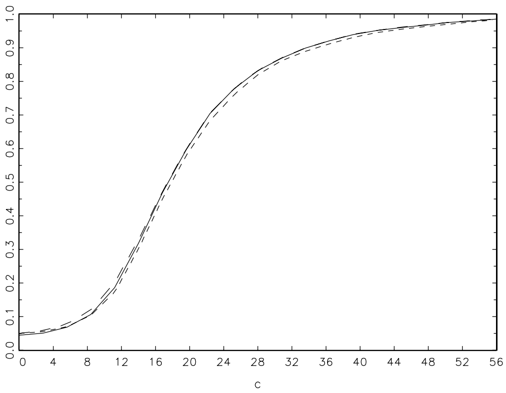
(f) $i = \beta$, $\alpha = 10$

$- -$ PWY^i_{OLS}, - - - PWY^i_{GLS}, — U^i

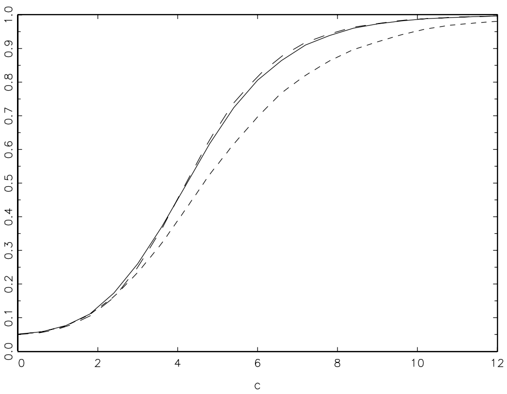
Figure 7: Finite sample power of PWY^i_{OLS}, PWY^i_{GLS} and U^i for $[\tau_1, \tau_2] = [0.15, 0.25]$, where $i = \{\mu, \beta\}$



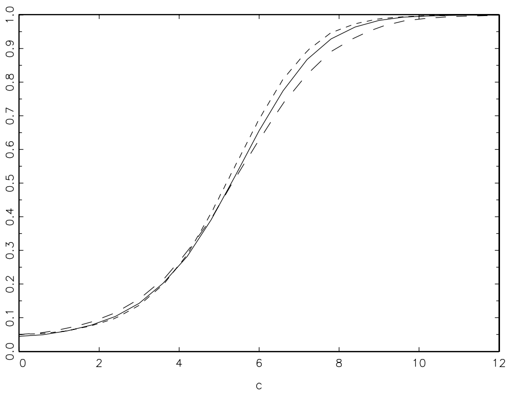
(a) $i = \mu$, $\alpha = 0$



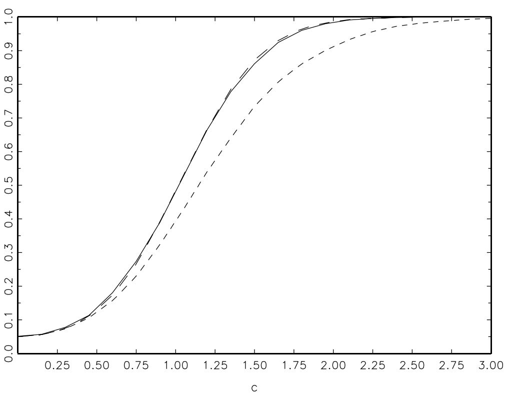
(b) $i = \beta$, $\alpha = 0$



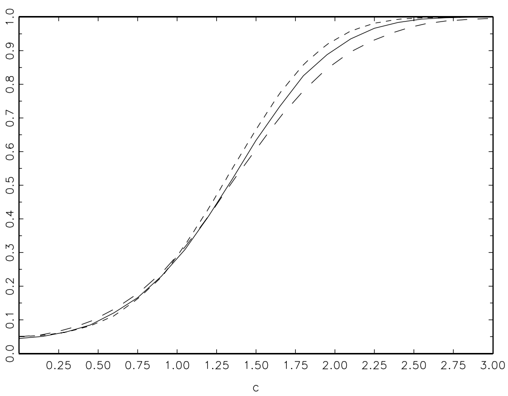
(c) $i = \mu$, $\alpha = 2$



(d) $i = \beta$, $\alpha = 2$



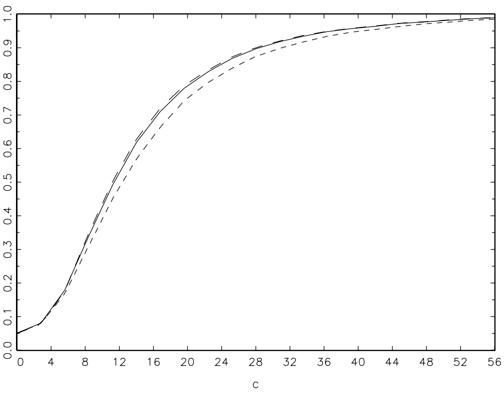
(e) $i = \mu$, $\alpha = 10$



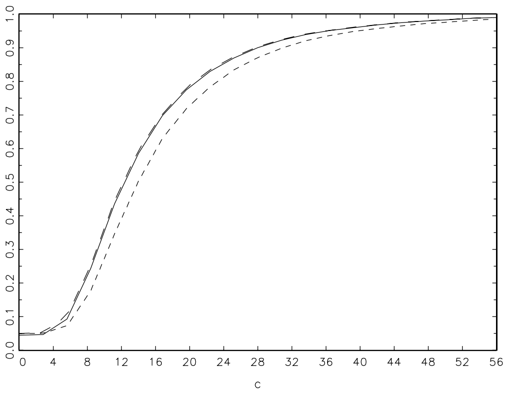
(f) $i = \beta$, $\alpha = 10$

$- -$ PWY^i_{OLS}, - - - PWY^i_{GLS}, — U^i

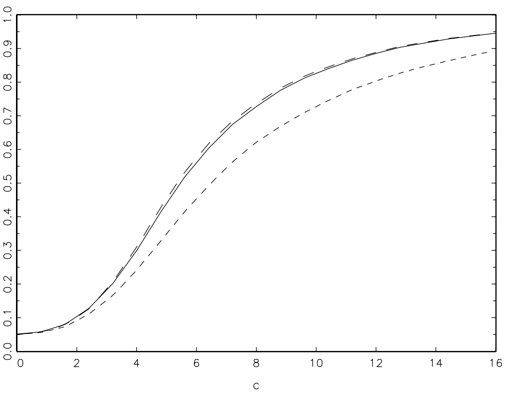
Figure 8: Finite sample power of PWY^i_{OLS}, PWY^i_{GLS} and U^i for $[\tau_1, \tau_2] = [0.75, 0.85]$, where $i = \{\mu, \beta\}$



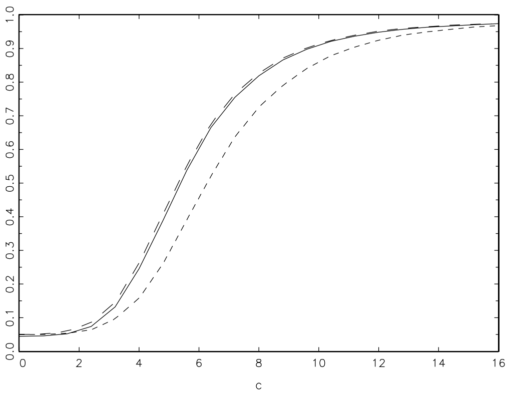
(a) $i = \mu$, $\alpha = 0$



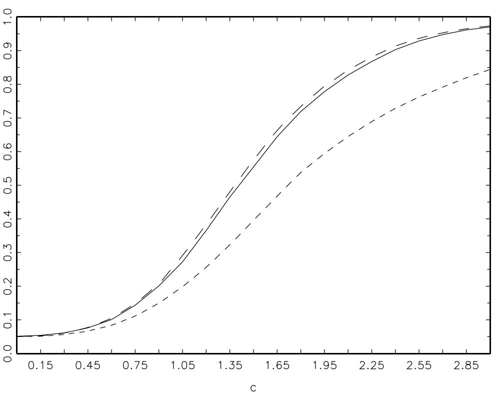
(b) $i = \beta$, $\alpha = 0$



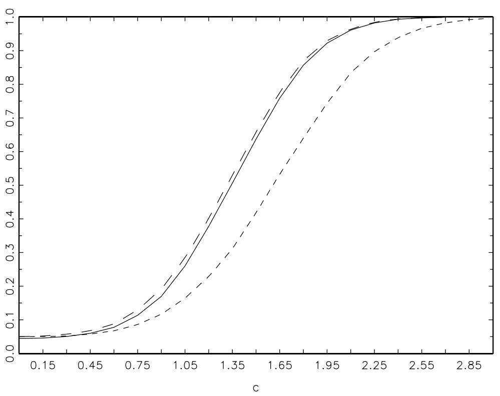
(c) $i = \mu$, $\alpha = 2$



(d) $i = \beta$, $\alpha = 2$



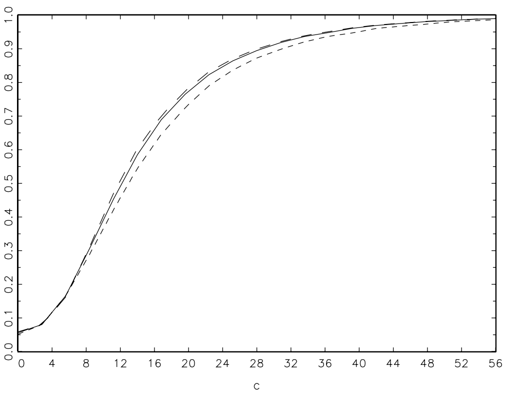
(e) $i = \mu$, $\alpha = 10$



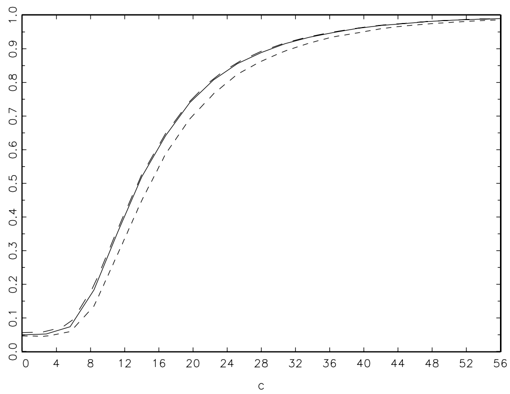
(f) $i = \beta$, $\alpha = 10$

$- -$ PWY^i_{OLS}, - - - PWY^i_{GLS}, — U^i

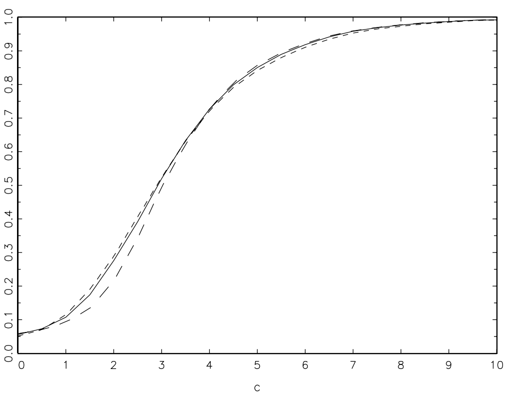
Figure 9: Finite sample power of PWY^i_{OLS}, PWY^i_{GLS} and U^i under GARCH errors, with $\omega = 30$, $\gamma = 0$ and $\phi = 0.6$, for $\alpha = 0$, where $i = \{\mu, \beta\}$



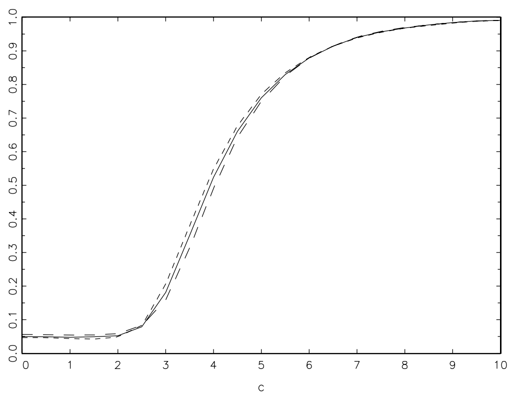
(a) $[\tau_1, \tau_2] = [0.45, 0.55]$, $i = \mu$



(b) $[\tau_1, \tau_2] = [0.45, 0.55]$, $i = \beta$



(c) $[\tau_1, \tau_2] = [0.2, 0.8]$, $i = \mu$



(d) $[\tau_1, \tau_2] = [0.2, 0.8]$, $i = \beta$

$- -$ PWY^i_{OLS}, - - - PWY^i_{GLS}, — U^i