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Calculus Taster Lecture

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Calculus Taster Lecture

This lecture will give you a taste of a University lecture of our first-year module

Calculus.

In particular, the topic of this lecture is "**Ordinary differential equations**".

About this session

During this lecture you will have the opportunity to participate in polls and quizzes, based on the topics presented.

You can participate in the polls today using your computer or mobile device. Please use the link provided in the Teams chat or QR code shown on the slide.

Introduction to basic ideas

Derivatives of functions

For a function $y(x)$, we denote the (first) derivative of this function with respect to variable x by

$$\frac{dy}{dx}.$$

Examples:

$$\frac{d}{dx}\left(x^a\right) = ax^{a-1}, \quad \frac{d}{dx}\left(\mathrm{e}^x\right) = \mathrm{e}^x, \quad \frac{d}{dx}\left(\sin(x)\right) = \cos(x), \quad \frac{d}{dx}\left(\cos(x)\right) = -\sin(x),$$

$$\frac{d^2}{dx^2}\left(x^a\right) = \frac{d}{dx}\left(ax^{a-1}\right) = a(a-1)x^{a-2}, \quad \frac{d^2}{dx^2}\left(\mathrm{e}^{bx}\right) = b^2\mathrm{e}^{bx}, \quad \frac{d^n}{dx^n}\left(\mathrm{e}^{bx}\right) = b^n\mathrm{e}^{bx}$$

Notation

$\frac{dy}{dx}, \frac{d}{dx}\left(f(x)\right), g'(r), \dot{x}(t)$ are different notations of the first derivative

Ordinary differential equations

Introduction

Differential equations involve unknown functions and their derivatives.

An ordinary differential equation (ODE) involves **one** unknown function of **one** independent variable, and derivatives of this function up to some finite order.

ODE: equation for $y(x),\ y'(x),\ y''(x),\ \cdots$

Task: to solve the ODEs

For example:

$y' = 2xy$ $\rightarrow$ A solution is: $y(x) = \exp(x^2)$

and

$y'' - 2y' - 3y = 4\sin(2x) - 7\cos(2x)$ $\rightarrow$ A solution is: $y(x) = \exp(-x) + \cos(2x)$

are ODEs of the unknown function y of the independent variable x.

Classification of ODEs

We need to classify ODEs so we can choose the right solution method.

ODEs are classified into different categories, according to their general characteristics.

Independent and dependent variables

$$y'' - 2y' - 3y = 4\sin(2x) - 7\cos(2x)$$

Unknown: $y(x)$

y : dependent variable
x : independent variable

$$4\ddot{x} + 3\dot{x} - x = 0$$

Unknown: $x(t)$

x : dependent variable
t : independent variable

Quiz

Go to www.menti.com and use code 6189 8241.

How many distinct ODEs do you see?

$$4y'' + 3y' - y = 0$$

$$4\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$$

$$4\frac{d^2x}{dt^2} + 3\frac{dx}{dt} - x = 0$$

$$4\ddot{x} + 3\dot{x} - x = 0$$

Classification of ODEs

Order

The **order** of an ODE is the highest derivative occurring in the equation.

For example:

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} - 10y = 0$$ is second-order

$$3\frac{dy}{dx} + 4y^3x = 0$$ is first-order

$$\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} + y = 0$$ is fourth-order

Classification of ODEs

Degree

The **degree** of an ODE is the power to which the highest-order derivative is raised, after the equation has been rationalised to contain only positive integer powers of derivatives.

For example:

$$\left(\frac{d^4y}{dx^4}\right)^2 + x\left(\frac{dy}{dx}\right)^7 + y = 0$$

fourth-order second degree

Fractional power???

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{7/4} + y = 0$$

second-order fourth degree

$$\left(\frac{dy}{dx}\right)^{7/4} = \left(-\frac{d^2y}{dx^2} - y\right) \Rightarrow \left(\frac{dy}{dx}\right)^7 = \left(-\frac{d^2y}{dx^2} - y\right)^4$$

No fractional powers

Quiz

(1) $y' + xy - (y')^3 = 0$

(2) $y''' - 5xy^2 = 0$

(3) $(y''y^3)^2 + (y')^4 = 0$

Classification of ODEs

Linearity

A **linear** ODE of order n (n = integer) has the form

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)\,.$$

This means that y and its derivatives appear linearly, that is, only raised to power 1 (or 0) and are not multiplied together.

ODEs that are not linear are said to be **nonlinear**.

Remark: $a_0(x)$, $a_1(x)$, $\cdots$ may be nonlinear in x.

$$\frac{d^2y}{dx^2} + 10\frac{dy}{dx} - 6y = 0$$ linear

$$\frac{dy}{dx} + 4x^3 y = e^{4x}$$ linear

$$\frac{dy}{dx} + e^y = 0$$ nonlinear

$$\frac{d^2y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 - 4y = e^x$$ nonlinear

$$y\frac{dy}{dx} + x^5 = 0$$ nonlinear

Classification of ODEs

(In)homogeneous linear ODEs

A linear ODE of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x)$$

is said to be homogeneous if (and only if) the right-hand side $f(x)$ vanishes identically.

Homogeneous if $f(x) = 0$ for all x

Inhomogeneous if $f(x) \neq 0$ for some x

For example:

$$\frac{d^2 y}{dx^2} + x\frac{dy}{dx} - y = 0$$ **Homogeneous**

$$\frac{d^2 y}{dx^2} + x\frac{dy}{dx} - y = \cos x$$ **Inhomogeneous**

Quiz

(1) $y' + xy - (y')^3 = 0$

(2) $y'' + 2y' + y = x^3$

(3) $y'''' + y'' + 2x = 0$

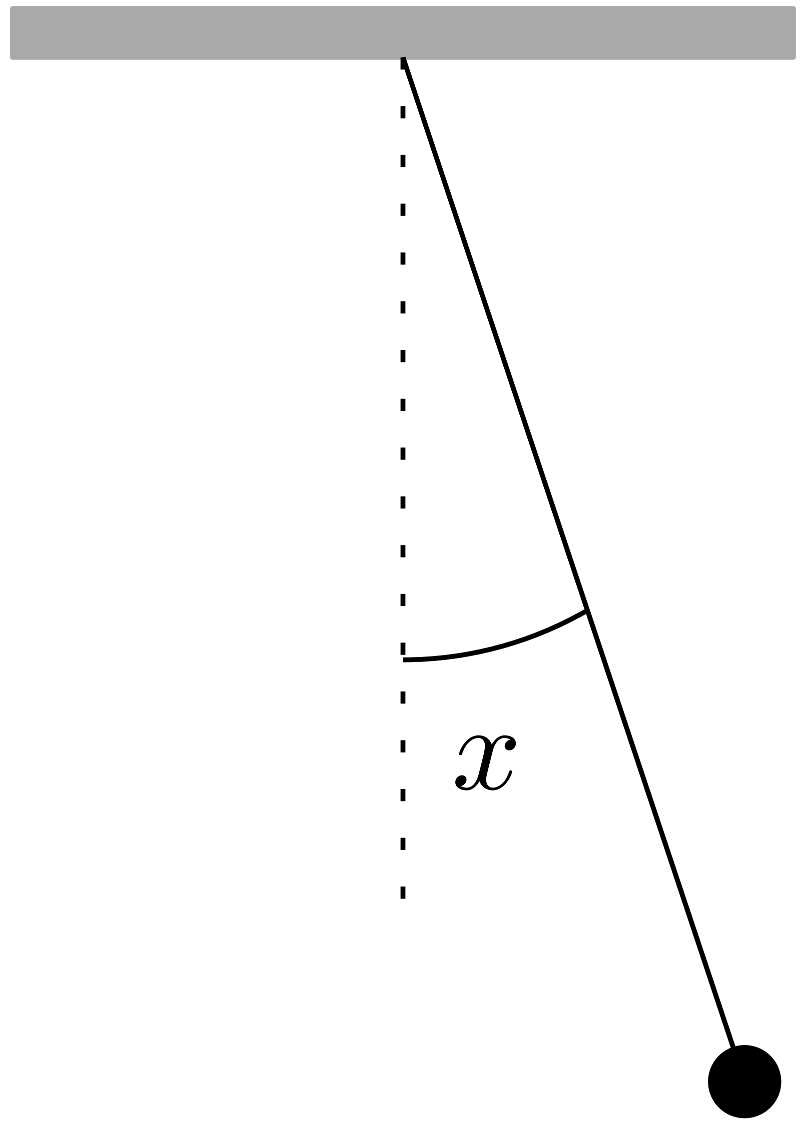
Real-life examples of ODEs

1. Simple harmonic motion

For example, a harmonic oscillator consisting of a weight attached to one end of a spring, whose other end is connected to a wall,

$$m\ddot{x}(t) = -kx(t)$$

where x is displacement, m is mass and k is a constant.



Other phenomena can be modelled by simple harmonic motion, including the motion of a simple pendulum.

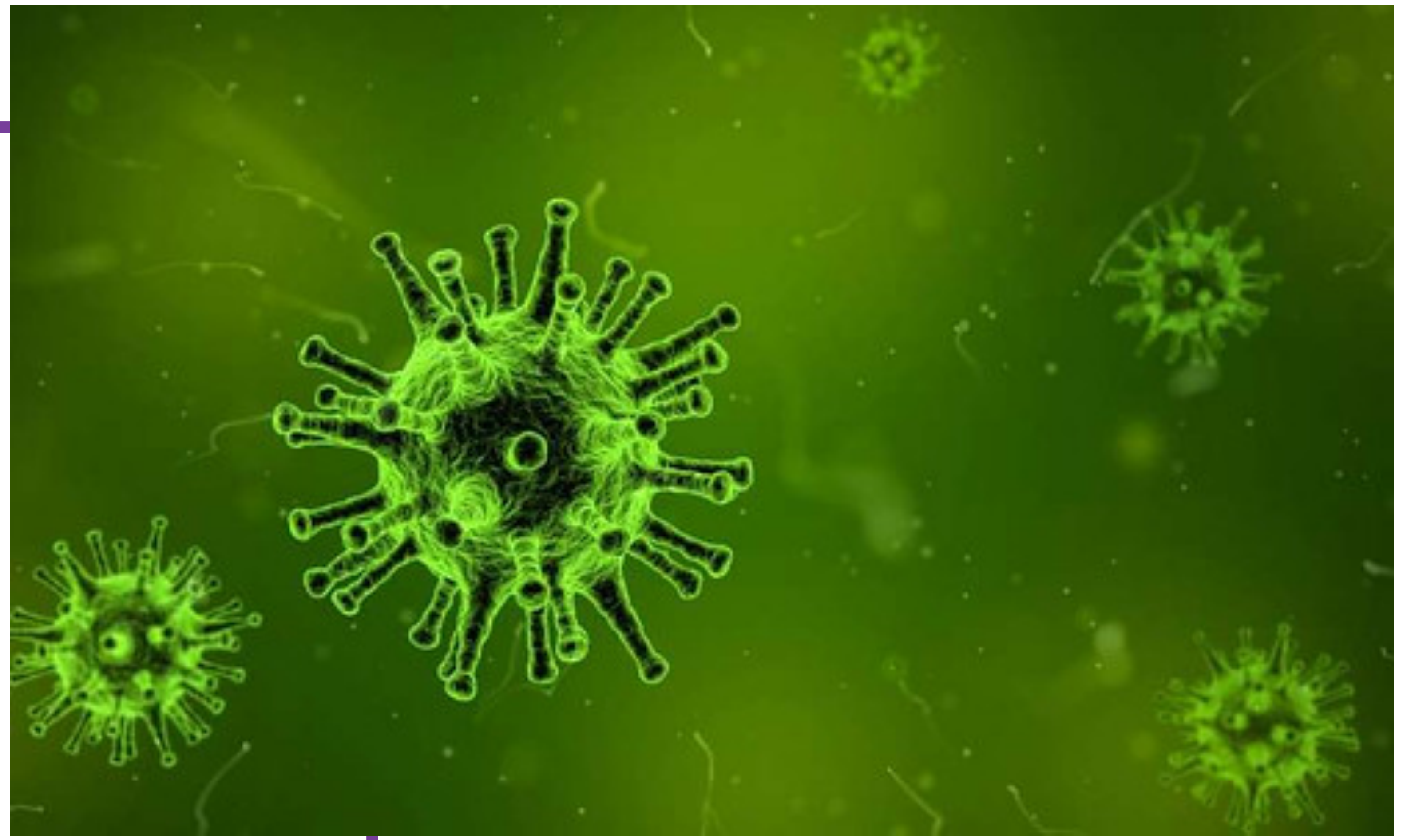
2. A ball's trajectory in football

$$m\frac{d^2\mathbf{x}}{dt^2} = \mathbf{F}(t)$$

where $\mathbf{x}(t) = (x(t), y(t), z(t))$ describes the position of a ball of mass m in three-dimensional space, and $\mathbf{F}(t) = \mathbf{F}(\dot{\mathbf{x}}(t))$ represents the forces acting on the ball (gravitational, drag and lift).

Real-life examples of ODEs

3. Modelling the dynamics of an epidemic



The SIR-model

$$\frac{dS}{dt} = -\beta IS$$

$S(t)$: susceptible population

$$\frac{dI}{dt} = \beta IS - \gamma I$$

$I(t)$: infected

$$\frac{dR}{dt} = \gamma I$$

$R(t)$: removed (by death or recovery)

Basic reproduction number:

$$\mathscr{R}_0 = \frac{\beta}{\gamma} > 1 \quad \Rightarrow$$

number of infected individuals increases

If you would like to hear more, please attend the *Popular maths talk* on "**Using maths in the fight against Covid-19**" (5pm, May 6th).

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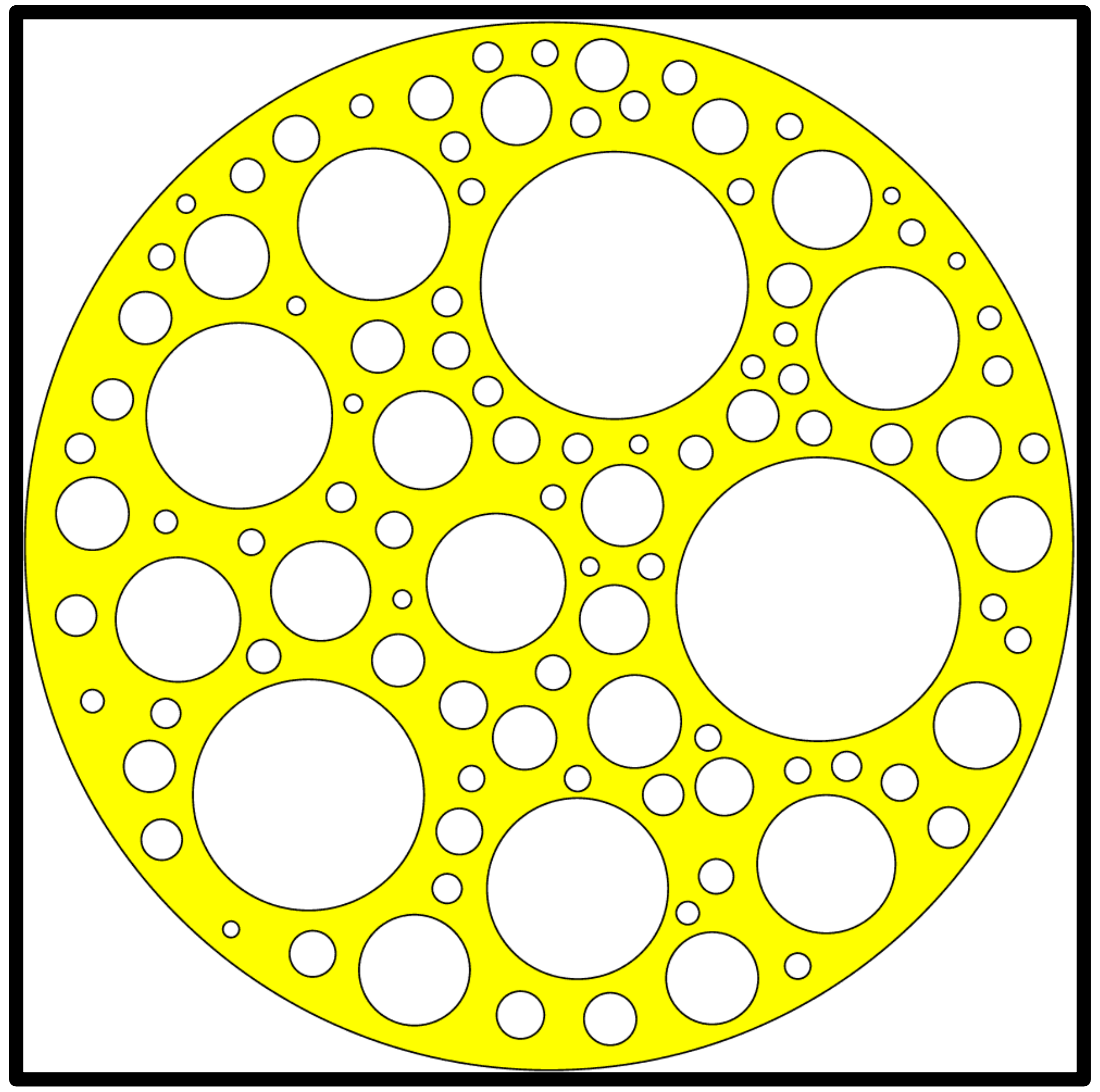
Maths Courses at the University of Nottingham

Joel Feinstein

School of Mathematical Sciences

About me – Joel Feinstein

- Associate Professor, Pure Mathematics
- Outreach Officer
- Teaching Support Officer
- I teach the first-year module **Foundations of Pure Mathematics**
- My research includes work on Swiss cheeses!



School of Mathematical Sciences

- Department of Mathematics formed in 1919
- School of Mathematical Sciences formed in 1998
- Moved to current, purpose-built, home in 2011
- Situated in a lovely campus with great facilities
- Over 70 academic staff

Maths Courses at Nottingham

Single-Subject Degrees

- **Mathematics BSc** (3 years)
- **Mathematics MMath** (4 years)
- **Mathematics (International Study) BSc** (4 years)
- **Mathematics with a Year in Industry BSc** (4 years)
- **Mathematics with a Year in Industry MMath** (5 years)
- **Statistics BSc** (3 years)

Maths Courses at Nottingham

Joint Degrees

- **Financial Mathematics BSc** (3 years)
  - with Nottingham University Business School
- **Mathematics and Economics BSc** (3 years)
  - with School of Economics
- **Mathematical Physics BSc/MSci** (3/4 years)
  - coordinated by School of Physics & Astronomy
- **Natural Sciences BSc/MSci** (3/4 years)
  - coordinated across schools involved
  - available with a year abroad

Careers with Mathematics

The most popular employment sectors nationally for maths graduates are*:

- Business, HR and finance professionals (42%)

  e.g., Consultant, Actuarial Graduate, Analyst, Strategic Consultant, Accountant

- IT professionals (12%)

  e.g., Software Engineer, Data Analyst, Cyber Security Associate, Technology Analyst

- Education professionals (9%)

  e.g., Teacher of Mathematics, Teaching Assistant

*Source: *What do graduates do?* (HECSU 2018)

Top four employers for our graduates:

- Deloitte
- PwC
- Ernst & Young
- KPMG

Some useful links

University of Nottingham, School of Mathematical Sciences and our maths courses:

https://tinyurl.com/mathsuon

https://tinyurl.com/mathscourseuon

Complete sets of videos for the first-year module **Foundations of Pure Mathematics**:

https://tinyurl.com/uonfpm

Any questions?

Please give us feedback on this session using the link in the Q&A chat

Future Maths Taster Sessions: https://tinyurl.com/uonmathstaster