Maths

Levelling

Up

Tutorial

Contents

Introduction

A very warm welcome to our maths refresher course. It has been specifically designed to prepare you for our courses and to allay any concerns that you may have about maths. Most modules have a mathematical element for which a basic knowledge of mathematical principles is needed to understand and build upon during a module.

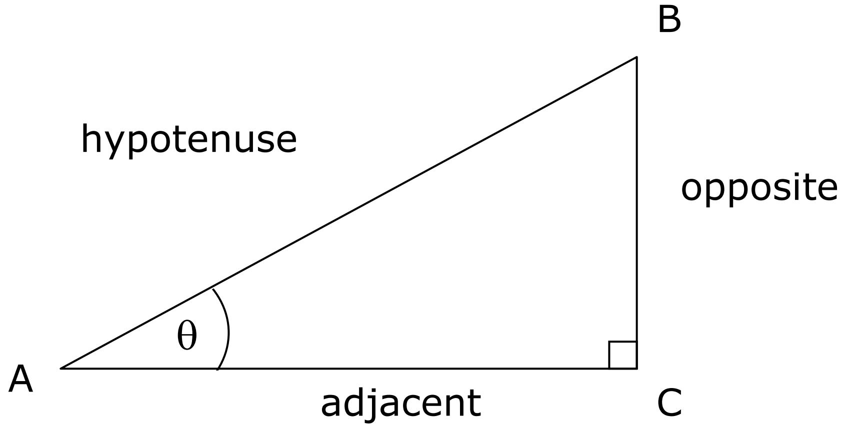
This course comprises 5 main sections, each introducing an area of mathematics which is developed through explanation, worked example and further exercises for you to try.

If you are keen on more practice then under *Recommended further reading*, there is a list of what chapters to look at in which recommended textbooks, there are also online materials available.

Trigonometry

As surveyors and navigators, our work is devoted solely to the measurement of geometry through distances and angles. Trigonometry is the mathematics that we use to relate angles to distances and is therefore the most fundamental area of mathematics for our field. We expect that you are familiar with angular units (degrees, radians), all of the trigonometric functions (sin, cos, tan) and their identities and double- and half-angle formulae.

1. Trigonometric functions and properties



Above is a right angled triangle.

BC is the side **opposite** the angle θ.

AC is the side **adjacent** to the angle θ.

AB is the **hypotenuse** of the triangle.

The trigonometric functions are given by the ratios of the lengths of these sides. If we call the length of the opposite O, the adjacent A and the hypotenuse H, then:

$$\sin\theta = \frac{O}{H}$$

$$\cos\theta = \frac{A}{H}$$

$$\tan\theta = \frac{O}{A}$$

Where $\sin\theta$ is called the sine of θ, $\cos\theta$ is the cosine of θ and $\tan\theta$ is the tangent of θ.

Evaluating sine, cosine and tangent using a calculator

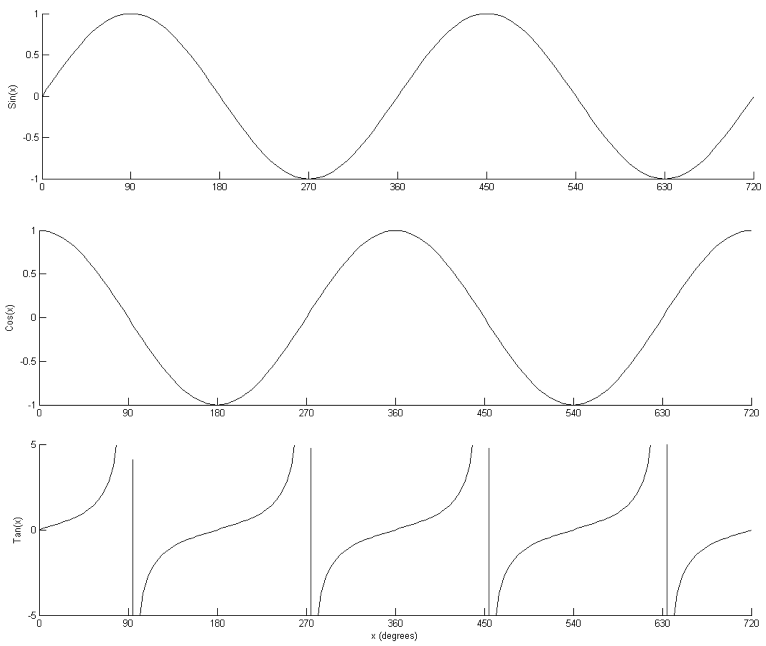
$\sin\theta$, $\cos\theta$ and $\tan\theta$ can be evaluated using a calculator. Be aware that most calculators can be set to use angles in two forms: degrees and radians. Be sure your calculator is set to the form you are using.

Sine, cosine and tangent of selected angles

θ°	**sin θ**	**cos θ**	**tan θ**
0	0	1	0
30	0·5	0·866	0·577
45	0·707	0·707	1
60	0·866	0·5	1·732
90	1	0	undefined

Properties of sine, cosine and tangent

Below are the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$.



Odd and even functions

Sine and tangent are odd functions. Then:

$$\sin(-\theta) = -\sin\theta$$
$$\tan(-\theta) = -\tan\theta$$

Cosine is an even function. Then:

$$\cos(-\theta) = \cos\theta$$

Asymptotes of tangent function

The tangent function is not defined at 90°, 270°, … These points are called asymptotes.

Period

Sine and cosine repeat every 360°; they have a period of 360°. That is:

$$\sin(\theta + 360) = \sin\theta$$
$$\cos(\theta + 360) = \cos\theta$$

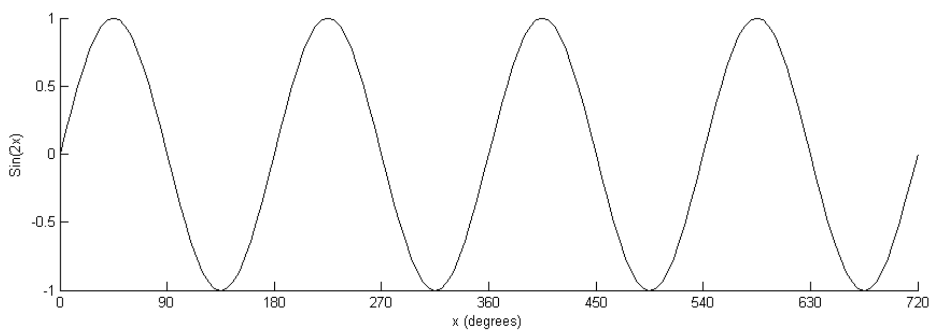
The tangent function has a period of 180°. Then:

$$\tan(\theta + 180) = \tan\theta$$

Sine and cosine are called sinusoidal functions. The graph of $\sin\theta$ passes through (0, 0).

Angular frequency

Below is the graph of $y = \sin(2x)$.

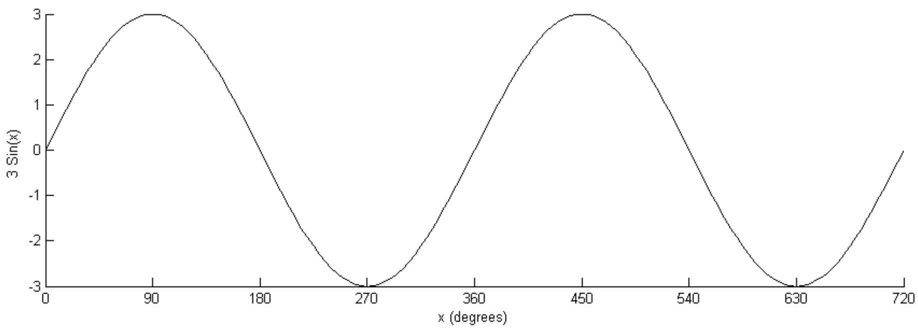


From this graph it is clear the period of sin(2x) is 180°. That is, there are two cycles in 360°. Indeed, for sin(ωx) there are ω cycles in 360°. The number ω is the angular frequency.

For an angular frequency of ω, the period is 360°/ω.

Amplitude

Below is the graph of $y = 3\sin(x)$.

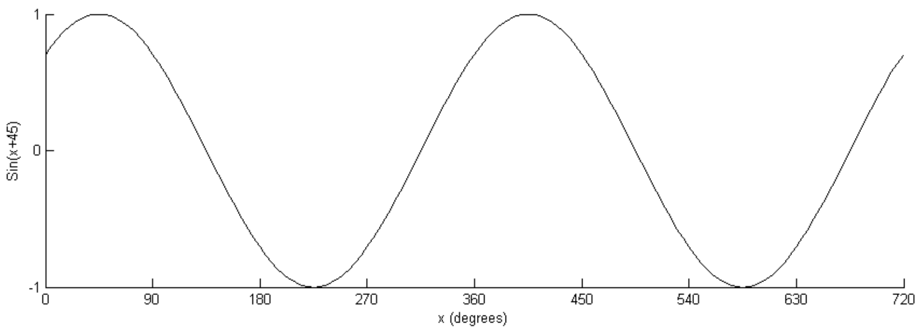


Notice that this has the same shape and period as sinx, but the maximum and minimum points of y are now 3 and -3. For $y = 4\sin(\theta)$ the maximum and minimum points of y are 4 and -4.

This 'height' of the graph is called the amplitude, so the amplitude of the graph above is 3.

Phase angles

Below is the graph of $y = \sin(x + 45°)$.



Notice that this is the same shape, period and amplitude as $y = \sin(x)$ but is out of phase by 45° (that is, shifted 45° to the right). This angle 45° is called the phase angle.

In general, $y = \sin(x + \alpha)$ has a phase angle of α.

Notice that $\sin(x+90°)=\cos(x)$.

Secant, cosecant and cotangent

In addition to sine, cosine and tangent, the following are three more trigonometric functions, the secant (sec), cosecant (cosec) and cotangent (cot) functions:

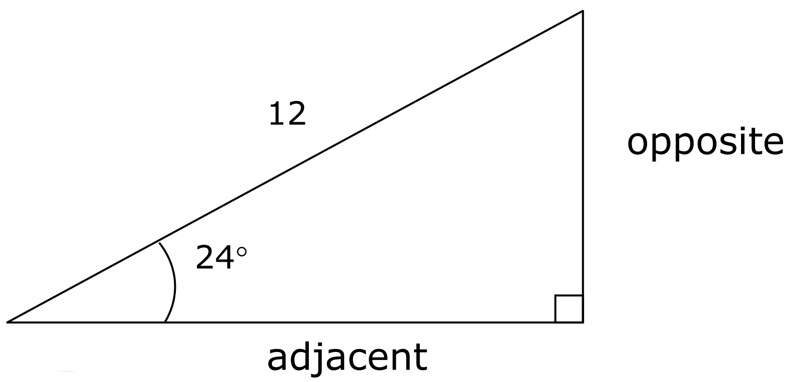
$$\sec\theta = \frac{1}{\cos\theta}$$
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$
$$\cot\theta = \frac{1}{\tan\theta}$$

These definitions can be used to calculate the secant, cosecant and cotangent functions using a calculator which only has buttons for sine, cosine and tangent. E.g.:

$$\operatorname{cosec}45° = \frac{1}{\sin 45°} = \frac{1}{0{\cdot}707} = 1{\cdot}414$$

Worked example

1. In the following triangle, one angle and the length of the hypotenuse are given. Calculate the remaining lengths.



Since we know the hypotenuse and the angle 24°, we can find the length of the side opposite the angle using the sine of 24°.

$$\sin\theta = \frac{O}{H}$$

$$\sin 24^\circ = \frac{O}{12}$$

$$\therefore O = 12\sin 24^\circ$$

The value of sin24° can be evaluated using a calculator.

$$O = 12\sin 24^\circ = 4{\cdot}880$$

So the length of the side opposite the angle is 4·880.

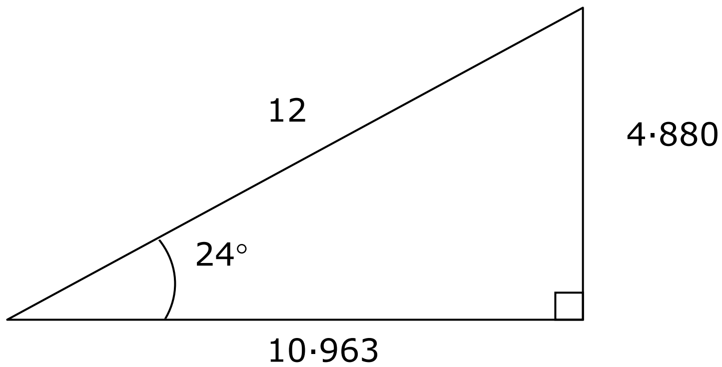
Similarly, the length of the side adjacent to the angle 24° is given by:

$$\cos\theta = \frac{A}{H}$$

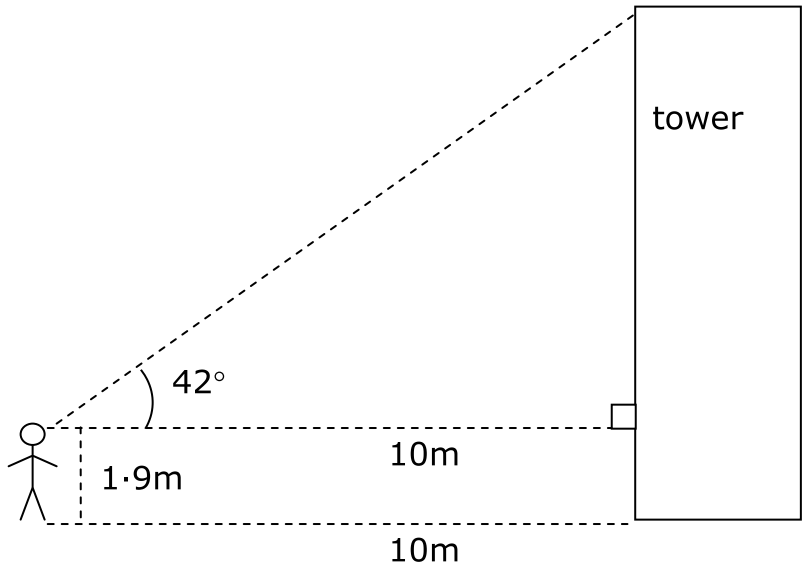
$$\cos 24^\circ = \frac{A}{12}$$

$$A = 12\cos 24^\circ = 10{\cdot}963$$

So the length of the adjacent side is 10·963.



2. Suppose we want to know the height of a tower, shown in the figure below.



In this, the person is standing 10m from the base of the tower. They have used a surveying instrument to measure the angle from their line of sight to the top of the tower at 42°. The height of the person measuring is 1·9m. How high is the tower?

The figure above shows that the horizontal line of sight of the person, line of sight to the top of the tower and the side of the tower forms a right angled triangle. The horizontal line of sight of the person is the side adjacent to the angle 42°. The side of the tower is the side opposite the angle 42°. Then we know the ratio between these two sides is the tangent of the angle:

$$\tan\theta = \frac{O}{A}$$

$$\tan 42^o = \frac{O}{10m}$$

$$\therefore O = 10m \times \tan 42^o$$

You can evaluate tan42° on a calculator. Multiplying this by 10 gives the length of O.

$$O = 10\tan 42 = 9{\cdot}004m$$

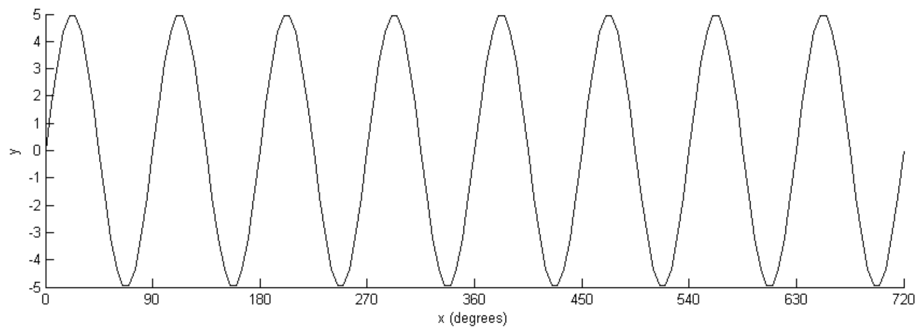
Adding the height of the surveyor gives the height of the tower

$$\begin{aligned} \text{height} &= 9{\cdot}004m + 1{\cdot}9m \\ &= 10{\cdot}904m \\ &= 10{\cdot}9m\ (1\,\text{dp}) \end{aligned}$$

So the height of the tower is 10·9m.

Exercises

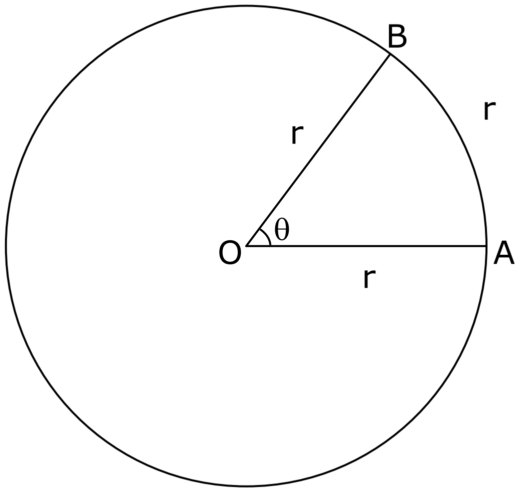
1. Calculate:
    a. sin90°
    b. cos90°
    c. tan15°
    d. sin20°
    e. cos35°
    f. sin10°
    g. tan60°
    h. sin82°
    i. cos45°
2. For the following graph, write down the period and amplitude. From the period, calculate the angular frequency and write a formula for the graph.



3. The angle to the top of a mast is measured to be 23° by surveying equipment which is 1·7m high from a distance of 5m from the base of the mast. How tall is the mast (to 1 decimal place)?
4. The angle of elevation to a 12·2m high tower is measured from a point at ground level to be 34°. How far away is the point from the tower (to 1 decimal place)?
5. From the top of an 8·9m high tower, the angle from the vertical is measured to a point at ground level to be 22°. How far away is that point from the base of the tower?

2. Circular measure and equivalence degree and radian

Using degrees, a degree is 1/360 of a full circle. An alternative measure for angles is the radian.



Above is a circle of radius r. If the arc length between the points A and B is also r, then the angle AOB, marked θ, is defined to be 1 radian. If the length of the arc is 2r then the angle is 2 radians, etc.

Since the arc length of the complete circle (circumference) is 2π, it can be seen that $2\,\pi$ radians=360°; so π radians=180°. Then:

$$1\,\text{radian} \equiv \frac{180}{\pi}\,\text{degrees} = 57{\cdot}3^{\circ}$$

1 radian is sometimes written 1 rad or 1^c, but commonly the symbol is omitted and this is simply written 1. Radians are often expressed as fractions of π.

Evaluating sine, cosine and tangent using a calculator

$\sin\theta$, $\cos\theta$ and $\tan\theta$ can be evaluated using a calculator. Be aware that most calculators can be set to use angles in both degrees and radians. Be sure your calculator is set to the form you are using.

Worked examples

1. Convert 60° to radians.

   We have that π radians=180°. Then:

$$1^\circ = \frac{\pi}{180}$$

$$60^\circ = 60 \times \frac{\pi}{180} = 1{\cdot}047$$

   So 60° is 1·047 radians. Notice also that:

$$1^\circ = \frac{\pi}{180}$$

$$60^\circ = 60 \times \frac{\pi}{180} = \frac{60\pi}{180} = \frac{\pi}{3}\,(=1.047\ 3\text{dp})$$

   So 60° can be expressed as π/3 radians. This form is more precise (no rounding) and may be more convenient.

2. Convert 3·5 radians to degrees.

$$1\,\text{radian} = \frac{180^\circ}{\pi}$$

   Then:

$$\begin{aligned} 3{\cdot}5\,\text{radians} &= 3{\cdot}5 \times \frac{180^\circ}{\pi} \\ &= \frac{630^\circ}{\pi} = 200{\cdot}535^\circ \end{aligned}$$

   So 3·5 radians is 200·535°.

3. Convert 3π/4 radians to degrees.

$$1\,\text{radian} = \frac{180^\circ}{\pi}$$

   Then:

$$\frac{3\pi}{4}\text{ radians} = \frac{3\not{\pi}}{4} \times \frac{180^\circ}{\not{\pi}}$$

$$= \frac{3}{4} \times 180^\circ = 135^\circ$$

So $3\pi/4$ radians is 135°.

Exercises

1. Convert the following angles to radians:

a. 45°	d. 240°	g. 315°
b. 30°	e. 15°	h. 90°
c. 190°	f. 85°	i. 20°

2. Convert the following from radians to degrees:

a. $\pi/3$	d. $3\pi/2$	g. 2.5
b. 2	e. $\pi/2$	h. $\pi/10$
c. $\pi/6$	f. 5	i. 1.5

3. Calculate (angles in radians):

a. $\sin \pi/3$	d. $\sin \pi/2$	g. tan2
b. $\cos \pi/3$	e. $\cos \pi/2$	h. $\sin \pi/6$
c. $\tan 3\pi/4$	f. sin2	i. $\cos \pi/6$

3. Trig identities

An identity is an equation that is true for all values of the variable. The following are commonly used trig identities. Note that $\sin^2\theta$ is a shorthand notation for $(\sin\theta)^2$.

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

Worked examples

1. Write the following in terms of $\sin\theta$ and $\cos\theta$.

$$\frac{\text{cosec}\theta}{1+\cot^2\theta}$$

We have the following identities:

$$1+\cot^2\theta = \text{cosec}^2\theta$$

So

$$\frac{\text{cosec}\theta}{1+\cot^2\theta} = \frac{\text{cosec}\theta}{\text{cosec}^2\theta} = \frac{1}{\text{cosec}\theta}$$

We know the following:

$$\text{cosec}\theta = \frac{1}{\sin\theta}$$

So then,

$$\frac{\text{cosec}\theta}{1+\cot^2\theta} = \frac{\text{cosec}\theta}{\text{cosec}^2\theta} = \frac{1}{\text{cosec}\theta} = \sin\theta$$

2. Solve the following equation:

$$2\sin^2\theta + 2\cos^2\theta + \cos\theta = 3$$

We know that, for all values of θ:

$$\sin^2\theta + \cos^2\theta = 1$$

So

$$2(\sin^2\theta + \cos^2\theta) = 2$$

Then

$$2\sin^2\theta + 2\cos^2\theta + \cos\theta = 3$$
$$2(\sin^2\theta + \cos^2\theta) + \cos\theta = 3$$
$$2 + \cos\theta = 3$$
$$\therefore \cos\theta = 1$$

We know $\cos\theta=1$ when $\theta=0$.

Exercises

1. Use the trig identities to write the following in terms of $\sin\theta$ and $\cos\theta$ and simplify.

   a. $\sec^2\theta$

   b. $\operatorname{cosec}^2\theta$

   c. $\cot\theta\left(1+\tan^2\theta\right)$

   d. $\sin^2\theta\cos^2\theta\left(\tan^2\theta+1\right)+\sin^2\theta\cos^2\theta$

   e. $\dfrac{1}{\sec^2\theta-1}+1$

4. Sum and difference formulae

The following identities are the sum and difference formulae:

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$
$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$
$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$
$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$
$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$

Worked example

Given two angles A and B, with

$$\sin A=\frac{4}{5}$$
$$\cos B=\frac{7}{25}$$

calculate $\sin(A-B)$ and $\cos(A-B)$.

Since

$$\sin\theta = \frac{opp}{hyp}$$

and using Pythagoras, we know

$$5^2 = 4^2 + adj^2 = 4^2 + 3^2$$

So then

$$\cos A = \frac{3}{5} = 0{\cdot}6$$

Similarly,

$$\sin B = \frac{24}{25} = 0{\cdot}96$$

Since we now have values for $\sin A$, $\cos A$, $\sin B$ and $\cos B$, we can now use the difference formulae.

$$\begin{aligned}\sin(A-B) &= \sin A\cos B - \cos A\sin B\\ &= \frac{4}{5}\cdot\frac{7}{25} - \frac{3}{5}\cdot\frac{24}{25}\\ &= \frac{28}{125} - \frac{72}{125}\\ &= -\frac{44}{125} = -0{\cdot}352\end{aligned}$$

And similarly,

$$\begin{aligned}\cos(A-B) &= \cos A\cos B + \sin A\sin B\\ &= \frac{3}{5}\cdot\frac{7}{25} - \frac{4}{5}\cdot\frac{24}{25}\\ &= \frac{21}{125} - \frac{96}{125}\\ &= -\frac{75}{125} = -0.6\end{aligned}$$

Exercises

1. Given that $\sin A = \frac{5}{13}$ and $\cos B = \frac{15}{17}$, find:

    a. $\sin(A + B)$

    b. $\sin(A - B)$

    c. $\cos(A + B)$

    d. $\cos(A - B)$

    e. $\tan(A + B)$

    f. $\tan(A - B)$

5. Double angle identities

The double angle identities are obtained from the sum and difference formulae by setting $A = B$. Then

$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Worked example

Given $\sin A = 0{\cdot}6$ and $\cos A = 0{\cdot}8$, find $\sin 2A$ and $\cos 2A$.

From the double angle identies for sine and cosine, we have:

$$\sin 2A = 2\sin A\cos A = 2\times 0{\cdot}6\times 0{\cdot}8 = 0{\cdot}96$$
$$\cos 2A = 1 - 2\sin^2 A = 1 - 2\times 0{\cdot}6^2 = 0{\cdot}28$$

Given $\tan A = 0{\cdot}6$, find $\tan 2A$.

From the double angle identity for tangent, we have:

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{2\times 0{\cdot}6}{1-0{\cdot}6^2} = \frac{1{\cdot}2}{0{\cdot}64} = 1{\cdot}875$$

Exercises

1. Given $\sin A = 0{\cdot}96$ and $\cos A = 0{\cdot}28$, find:
   a. $\sin 2A$
   b. $\cos 2A$
2. Given $\tan A = \frac{8}{15}$, find $\tan 2A$.

6. Inverse trig functions

For the function $\sin x = a$, we can find x by applying the inverse sine to a. On a calculator this is often marked $\sin^{-1}$ or achieved by pressing "inv" then "sin". Remember to be aware whether your calculator is working in degrees or radians.

$\sin^{-1} x$ is also written $\arcsin^{-1}$. Remember that $\sin^{-1} x$ **does not** mean $(\sin x)^{-1}$ or equivalently $\frac{1}{\sin x}$.

The inverses of the trig functions are not functions, since the inverse of a value a, say, will have many possible values. For example, the inverse sine of 0·5 could be $\frac{\pi}{6}, \frac{\pi}{6}+2\pi, \frac{\pi}{6}-2\pi$, etc.

To obtain the inverse trig functions, we must restrict the range of these functions. So

$$f(x) = \sin^{-1} x, \text{ with range } -\frac{\pi}{2} \le f \le \frac{\pi}{2}, \left(-90^o \le f \le 90^o\right)$$
$$f(x) = \cos^{-1} x, \text{ with range } 0 \le f \le \pi, \left(0^o \le f \le 180^o\right)$$
$$f(x) = \tan^{-1} x, \text{ with range } -\frac{\pi}{2} < f < \frac{\pi}{2}, \left(-90^o \le f \le 90^o\right)$$

The values of the inverses of the trig functions that fall within these ranges are called the principal values. The values given by calculators are the principal values.

Worked example

Find the value of the functions $f(x) = \sin^{-1} x$, $g(x) = \cos^{-1} x$ and $h(x) = \tan^{-1} x$ for $x = 0{\cdot}6$.

Using a calculator,

$$f(0{\cdot}6) = \sin^{-1} 0{\cdot}6 = 0{\cdot}644 \, (3\,\text{dp})$$
$$g(0{\cdot}6) = \cos^{-1} 0{\cdot}6 = 0{\cdot}927 \, (3\,\text{dp})$$
$$h(0{\cdot}6) = \tan^{-1} 0{\cdot}6 = 0{\cdot}540 \, (3\,\text{dp})$$

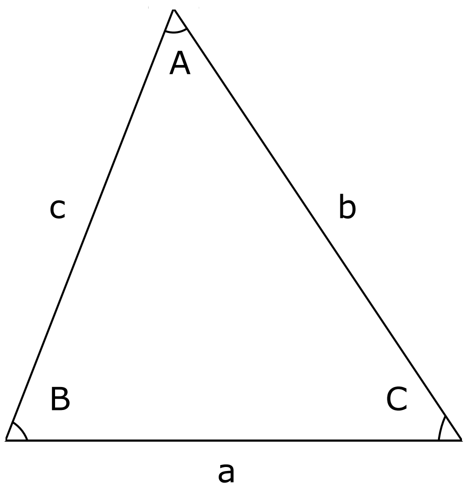
Exercises

1. Find the value of the functions $f(x) = \sin^{-1} x$ and $g(x) = \cos^{-1} x$ for the following values of x.

   a. 0·4
   b. 0·5
   c. -0·5
   d. $\frac{1}{\sqrt{2}}$
   e. 1
   f. 1·5
   g. 0
   h. -1

7. Sine and cosine rules

The sine and cosine rules are used when solving problems which involve triangles that are not right angled triangles (called oblique). The following diagram shows an oblique triangle with three angles A, B and C, and sides a, b and c.



With the sides and angles defined as in the diagram above, the sine rule is:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The cosine rule is:

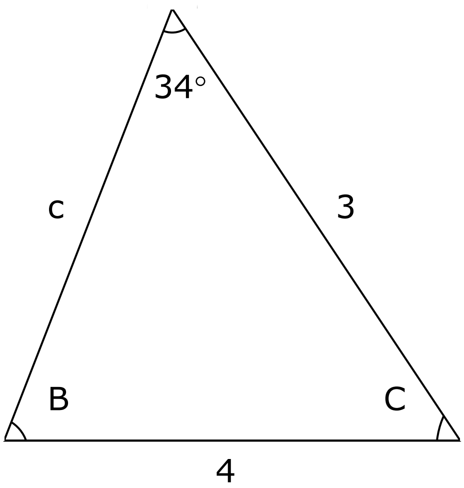
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Similarly,

$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$a^2 = b^2 + c^2 - 2bc\cos A$$

Worked examples

1. Given the angles and lengths are known as marked in the diagram below, calculate the remaining angles and lengths.



The angle B can be found using the sine rule, since

$$\frac{a}{\sin A} = \frac{4}{\sin 34°}$$
$$\frac{b}{\sin B} = \frac{3}{\sin B}$$

So

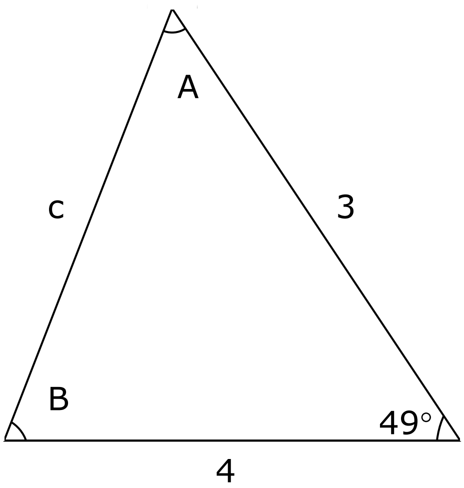
$$\frac{4}{\sin 34°} = \frac{3}{\sin B}$$
$$\therefore \sin B = \frac{3\sin 34°}{4} = 0{\cdot}419\,(3\,\text{dp})$$
$$\therefore \text{B} = \sin^{-1} 0{\cdot}419 = 24{\cdot}796\,(3\,\text{dp})$$

Since we now know 2 angles, and the angles in a triangle sum to 180°, we can calculate that C=180-(24·796+34)=121·203 (3 dp).

Finally, we can calculate the length c using the sine rule again:

$$\frac{4}{\sin 34°} = \frac{c}{\sin 121{\cdot}203°}$$
$$\therefore c = \frac{4\sin 121{\cdot}203°}{\sin 34°} = 6{\cdot}118\,(3\,\text{d.p.})$$

2. Given the angles and lengths are known as marked in the diagram below, calculate the remaining angles and lengths.



Since we do not know a pair of values for an angle and its opposite side, we cannot straight away apply the sine rule. Instead, we use the cosine rule to find the length of the side c.

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos C \\ &= 4^2 + 3^2 - 2 \times 4 \times 3 \times \cos 49^\circ \\ &= 9{\cdot}255\,(3\,\text{dp}) \end{aligned}$$

Then we take the square root to find c,

$$c = \sqrt{9{\cdot}255} = 3{\cdot}042\,(3\,\text{dp})$$

Then we find either and A or B using the sine rule.

$$\frac{3{\cdot}042}{\sin 49^\circ} = \frac{4}{\sin A}$$

$$\therefore \sin A = \frac{4\sin 49^\circ}{3{\cdot}042} = 0{\cdot}992\,(3\,\text{dp})$$

$$\therefore \text{B} = \sin^{-1} 0{\cdot}992 = 82{\cdot}905\,(3\,\text{dp})$$

And finally, A can be found since the angles in a triangle sum to 180°.

A=180-(49+82·905)=48·095 (3 dp).

We can check the value for A using the sine rule,

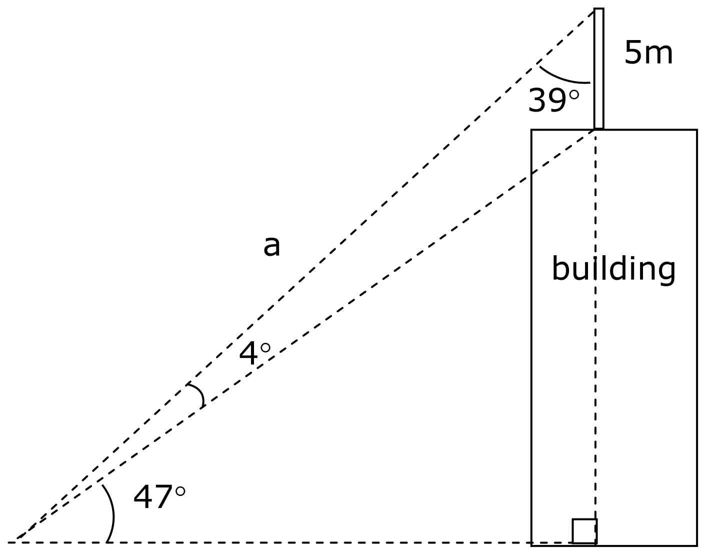
$$\frac{3{\cdot}042}{\sin 49^\circ} = \frac{3}{\sin B}$$

$$\therefore \sin B = \frac{3\sin 49^\circ}{3{\cdot}042} = 0{\cdot}744\,(3\,\text{dp})$$

$$\therefore \text{B} = \sin^{-1} 0{\cdot}744 = 48{\cdot}095(3\text{dp})$$

3. A building has a flagpole on top of height 5m. From a point on the ground the angles to the top and bottom of the flagpole are 51° and 47°, respectively. Calculate the height of the building.

The situation is detailed in the diagram below.



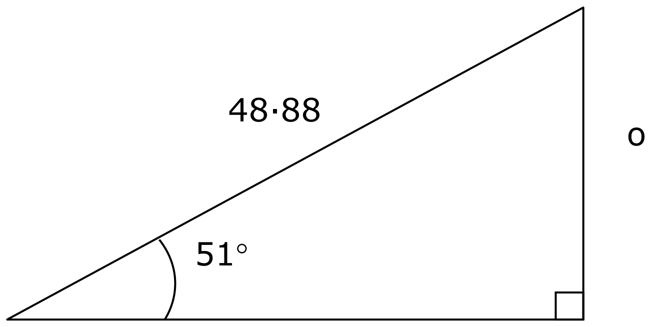
Since the angle from the point on the ground between the flagpole top and the horizontal is 51° and the larger triangle between the top of the flagpole, the point on the ground and the base of the building is a right angled triangle, we know that the remaining angle in this triangle is 39°.

For the smaller, oblique triangle we now have 2 angles and the length of a side (the flagpole). The two angles tell us the remaining angle must be 137°. We can apply the sine rule to find the length marked a on the diagram.

$$\frac{a}{\sin 137^\circ} = \frac{5m}{\sin 4^\circ}$$

$$a = \sin 137^\circ \frac{5m}{\sin 4^\circ} = 48{\cdot}88 \, (2 \text{ dp})$$

This length, a, is the hypotenuse of the larger right angled triangle. The relevant information can now be expressed as in the following diagram.

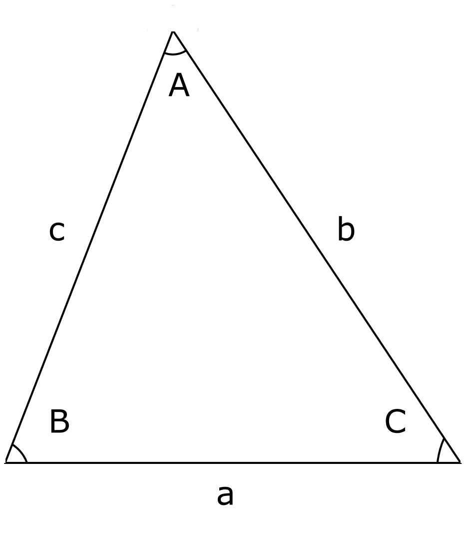


Then o, the height of the building, is given by

$$o = 48{\cdot}88 \sin 51^\circ = 37{\cdot}99 \, (2 \text{ dp})$$

Subtracting the height of the flagpole, 5m, we have the height of the building as 32·99m (2 dp).

Exercises



1. Given the angles and lengths as labelled in the above diagram, for each of the following calculate the missing values (angles or lengths):

   a. $A=30^\circ$, $a=5$, $b=3$

   b. $B=67^\circ$, $a=4$, $c=6$

   c. $B=45^\circ$, $C=56^\circ$, $c=6$

   d. $C=54^\circ$, $a=4$, $b=3$

   e. $A=56^\circ$, $C=27^\circ$, $a=3$

   f. $B=34^\circ$, $a=4$, $b=8$

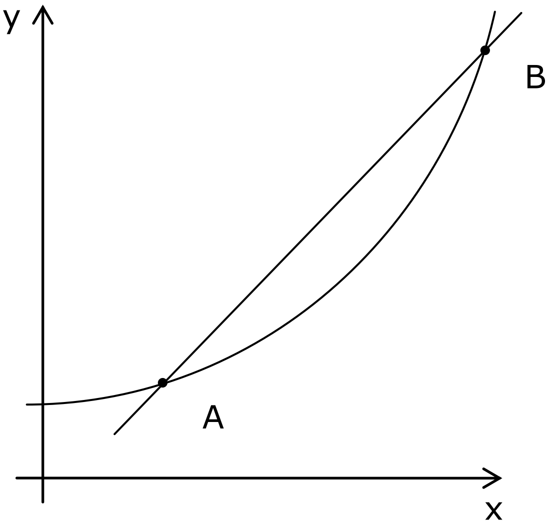
Differentiation

The best solution of a set of equations usually comes from finding the location where the errors are at a minimum. To find the minima of any set of equations we can use calculus. We expect that you will be familiar with differential calculus, ordinary and partial, and the rules regarding the derivatives of different types of functions, their multiples and ratios, and the Chain Rule. You should also understand how differentiation is used to calculate the minima and maxima of curves and surfaces. The expansion of a function using a Taylor Series is also needed.

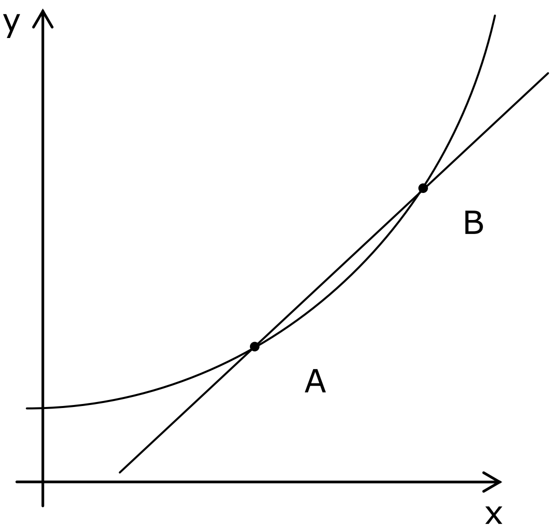
1. What is differentiation?

Gradient to a curve

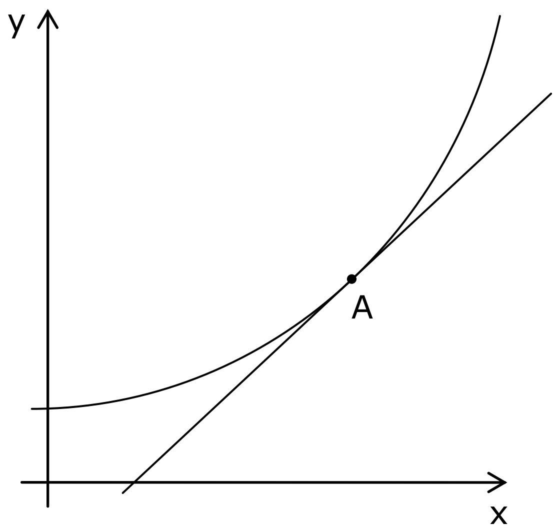
In the figure below points A and B have been joined by a straight line, known as a secant.



In the figure below points A and B have moved closer, so the secant is closer to the curve.



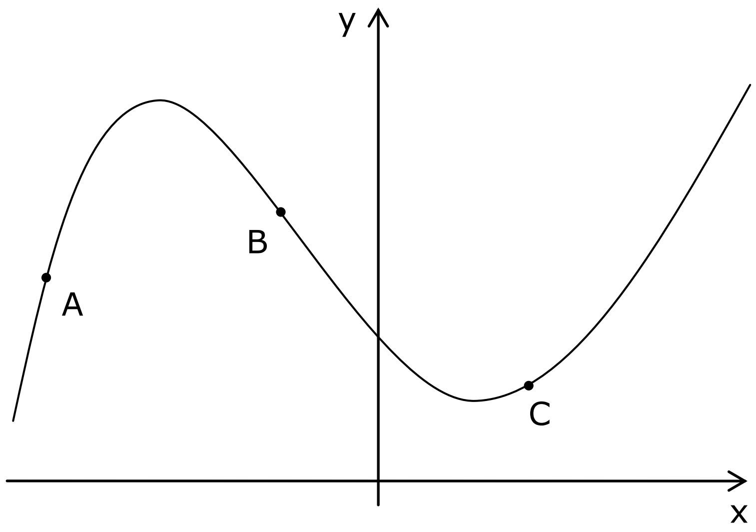
Imagine continuing to move the secant until A and B coincide. This line is called the *tangent* to the curve at A, shown below.



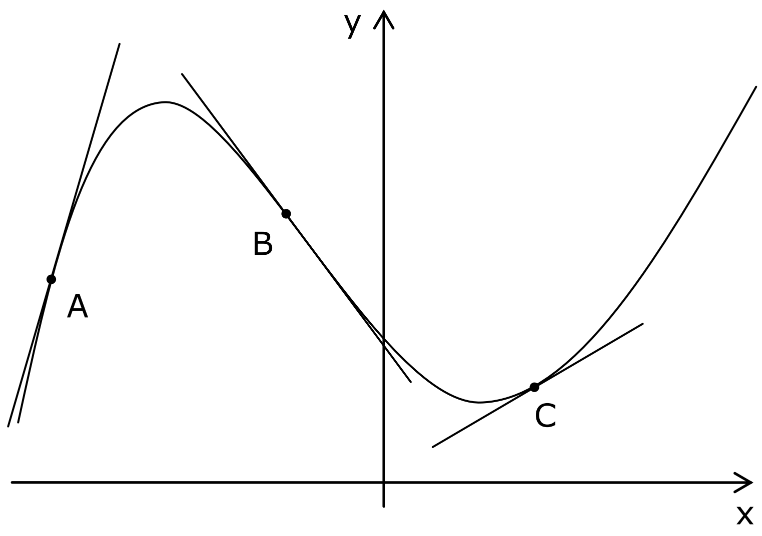
The *gradient* to the curve at A is the gradient of the tangent line at A. The steeper this gradient is at A then the greater the rate of change of the function is at that point. If the gradient is negative (i.e. sloping downwards from left to right) then the function is decreasing at A. For this reason the gradient is also called the *instantaneous rate of change* of the curve at A.

Worked example

Draw the tangents on the following curve at the points A, B and C and say whether the gradient at each point is positive or negative.



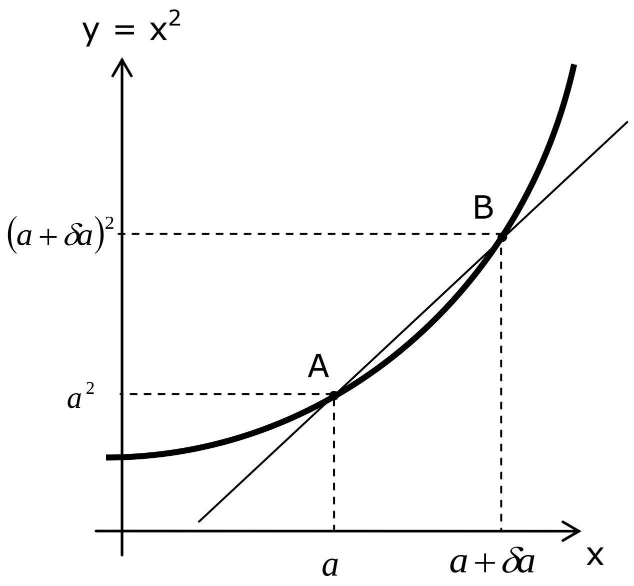
Solution: draw the tangents by eye.



From this, the gradient at A and C is positive and the gradient at B is negative. Notice the gradient at A is greater than the gradient at C and this means the curve is increasing more rapidly (i.e. steeper) at A than at C.

Finding the gradient at a point

Consider the graph of $f(x) = x^2$ and two points on that function A and B. Place A at the point $x = a$ and B at the point $x = a + \delta a$, where δa represents a small change in a.



The gradient of the secant that joins A and B is the amount which the y value increases when the x value increases by 1. This can be calculated by the difference in the y values divided by the difference in the x values.

The y value at A is a^2 and at B is $(a + \delta a)^2$. Then the gradient is given by:

$$\text{gradient} = \frac{\text{difference in y values}}{\text{difference in x value s}}$$

$$= \frac{(a+\delta a)^2 - a^2}{(a+\delta a) - a}$$

$$= \frac{a^2 + 2a\delta a + \delta a^2 - a^2}{\delta a}$$

$$= \frac{2a\delta a + \delta a^2}{\delta a}$$

$$= 2a + \delta a$$

We saw above that as A and B are placed closer, and ultimately in the same place, we approach the tangent to the curve at A. In this case A and B are separated by δa. For A and B to move closer and ultimately coincide, δa must approach zero. As δa approaches zero (written $\delta a \to 0$), the gradient$\to 2a$ at the point $x = a$.

This means that for any point a on the curve $f(x) = x^2$ the formula for finding the gradient at that point is $2a$. This gives a function $f'(x) = 2x$ from which we can find the gradient or instantaneous rate of change of $f(x) = x^2$ at any point. This function is called the derivative of $f(x)$ and the process for finding it is called differentiation.

Differentiation

For a function $f(x)$, differentiation is concerned with finding a function which gives the instantaneous rate of change of $f(x)$ at any point.

For a function $f(x)$, to differentiate $f(x)$ with respect to x gives the rate of change of $f(x)$ with respect to x. This is called the derivative of $f(x)$ with respect to x and is written as:

$$\frac{df(x)}{dx}$$

For example, if $f(t)$ is a function used to determine the position of an object with respect to time, t, then the derivative of $f(t)$ with respect to t is the function that gives the rate of change of position, which is the speed of the object with respect to time.

Graphically, differentiation can be regarded as the process of finding the gradient of a function at any given point.

Notation

Suppose:

$$f(x) = 3x^2$$

And let $y = f(x)$.

The derivative of $f(x)$ with respect to x can be written in the following ways:

$$\frac{df(x)}{dx} = \frac{dy}{dx} = \frac{d(3x^2)}{dx} = f'(x)$$

Worked example

If we take a function $f(t)$ which represents the distance travelled by some object as

$$f(t) = 2t^3$$

We know that the average speed over a time interval is given by:

$$\text{average speed} = \frac{\text{distance covered}}{\text{time taken}} = \frac{\text{change in distance}}{\text{change in time}}$$

We denote change using the symbol Δ, so change in distance, f is given by Δf. So if the distance changes from d_1 to d_2 as the time changes from t_1 to t_2, then the average speed is given by:

$$\text{average speed} = \frac{\delta f}{\delta t} = \frac{d_2 - d_1}{t_2 - t_1}$$

If we are interested in the speed of the object at t =0·5 we can look at the average speed for an interval which includes 0·5. Look at the average speed between 0 and 1:

$$f(0) = 2 \times 0^3 = 0$$
$$f(1) = 2 \times 1^3 = 2$$
$$\frac{\delta f}{\delta t} = \frac{2-0}{1-0} = 2$$

If we look at a smaller time interval, say from 0·25 to 0·75, we get:

$$f(0) = 2 \times 0{\cdot}25^3 = 0{\cdot}03125$$
$$f(1) = 2 \times 0{\cdot}75^3 = 0{\cdot}84375$$
$$\frac{\delta f}{\delta t} = \frac{0{\cdot}84375 - 0{\cdot}03125}{0{\cdot}75 - 0{\cdot}25} = 1{\cdot}625$$

Taking an even smaller time interval, 0·45 to 0·55, gives:

$$f(0) = 2 \times 0{\cdot}45^3 = 0{\cdot}18225$$
$$f(1) = 2 \times 0{\cdot}55^3 = 0{\cdot}33275$$
$$\frac{\delta f}{\delta t} = \frac{0{\cdot}33275 - 0{\cdot}18225}{0{\cdot}55 - 0{\cdot}45} = 1{\cdot}505$$

Continuing this process we find the average speed tends to 1·5 as the interval decreases. Try it for e.g. the interval 0·49 to 0·51.

In fact, we say the derivative is given as the limit of $\frac{\delta f}{\delta t}$ as δt tends to 0:

$$\frac{df(t)}{dt} = \lim_{\delta t \to 0}\left(\frac{\delta f}{\delta t}\right)$$

Second order differentiation

Differentiating a function with respect to a variable is called first order differentiation. Differentiating the result of a first order derivative is called second order differentiation.

Suppose:

$$y = f(x) = 3x^2$$

Then the second derivative of $y = f(x)$ is noted as follows:

$$\frac{d^2 f(x)}{dx^2} = \frac{d^2 y}{dx^2} = \frac{d^2(3x^2)}{dx^2} = f''(x)$$

In the case of distance being differentiated to give the speed, the speed can be again differentiated to find the rate of change of speed, which is the acceleration.

2. Derivatives

For functions of the form:

$$f(x) = mx^n$$

The derivative with respect to x is given by:

$$\frac{df(x)}{dx} = mnx^{n-1}$$

Taking the case $n=1$ provides an interesting result (remembering $x^0 = 1$):

$$f(x) = mx^n = mx^1$$
$$\frac{df(x)}{x} = mnx^{n-1} = m \times 1 \times x^0 = m$$

So the derivative is the constant m. This is analogous to the gradient of a straight line ($y = mx$) being constant.

Taking the derivative when $n=0$:

$$f(x) = m = mx^0$$
$$\frac{df(x)}{x} = mnx^{n-1} = m \times 0 \times x^{-1} = 0$$

So the derivative (rate of change) of a constant function with respect to x (e.g. $f(x) = m$) is zero. This is intuitive, since a constant function is not changing.

There follow some rules for finding derivatives of certain other types of function.

Trig functions:

$$f(x) = \sin x$$
$$\frac{df(x)}{dx} = \cos x$$
$$f(x) = \cos x$$
$$\frac{df(x)}{dx} = -\sin x$$
$$f(x) = \tan x$$
$$\frac{df(x)}{dx} = \sec^2 x$$

e and the natural logarithm:

$$f(x) = e^x$$
$$\frac{df(x)}{dx} = e^x$$
$$f(x) = \ln x = \log_e x$$
$$\frac{df(x)}{dx} = \frac{1}{x}$$

Worked example

1. Find the derivative of $y = f(x) = 3x^4$, with respect to x.

We know that for functions of this type:

$$\frac{dy}{dx} = mnx^{n-1}$$

Then,

$$\frac{d(3x^4)}{dx} = 3 \times 4 \times x^3 = 12x^3$$

2. Find the second derivative of $f(x)=\frac{x^6}{2}$, with respect to x.

First find the first order derivative with respect to x:

$$\frac{df(x)}{dx}=\frac{d\left(\frac{x^6}{2}\right)}{dx}=\frac{1}{2}\times 6\times x^5=3x^5$$

Then differentiate this function again with respect to x to find the second order derivative of $f(x)$:

$$\frac{d^2f(x)}{dx^2}=\frac{d^2\left(\frac{x^6}{2}\right)}{dx^2}=\frac{d(3x^5)}{dx}=3\times 5\times x^4=15x^4$$

Addition and subtraction

If a function $f(x)$ involves more than one term in x added together, then its derivative is the derivatives of each of the terms added together. So:

$$f(x)=u(x)+v(x)$$
$$\frac{df(x)}{dx}=\frac{du(x)}{dx}+\frac{dv(x)}{dx}$$
$$g(x)=u(x)-v(x)$$
$$\frac{dg(x)}{dx}=\frac{du(x)}{dx}-\frac{dv(x)}{dx}$$

E.g.:

$$f(x)=x^3+4x^2+5x+4$$
$$\frac{df(x)}{dx}=3x^2+8x+5$$

Worked example

1. If $f(x)=3x^5+x^{\frac{7}{3}}+x^{-3}+\cos x+e^x$, find $\dfrac{df(x)}{dx}$

We differentiate each of the terms separately and sum them.

$$\frac{d(3x^5)}{dx}=3\times5\times x^4=15x^4$$

$$\frac{d\left(x^{\frac{7}{3}}\right)}{dx}=\frac{7x^{\left(\frac{7}{3}-1\right)}}{3}=\frac{7x^{\frac{5}{3}}}{3}$$

$$\frac{d(x^{-3})}{dx}=-3x^{-3-1}=-3x^{-4}$$

$$\frac{d(\cos x)}{dx}=-\sin x$$

$$\frac{d(e^x)}{dx}=e^x$$

Then:

$$\frac{df(x)}{dx}=15x^4+\frac{7x^{\frac{5}{3}}}{3}-3x^{-4}-\sin x+e^x$$

Exercises

1. For the following functions, find the derivative:
   a. $f(x)=3\sin x$
   b. $g(x)=6x^4+4x$
   c. $f(t)=3e^t$
   d. $f(x)=15x^7+3x^4-89x$
   e. $g(r)=\sin r+\cos r$
   f. $f(x)=\cos x+5e^x$

2. For the following functions, find the second derivative with respect to x:
   a. $g(x)=6x^4+4x$
   b. $f(x)=3\sin x$
   c. $f(x)=15x^7+3x^4-89x$

3. Chain, product and quotient rules

Chain rule

The *chain rule*, or *function-of-a-function rule*, or *differentiation by substitution*, is used to differentiate functions that are made up of two or more simple functions, or composite functions.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Worked examples

1. Differentiate y, a function of x given by:

$$y = \frac{1}{x^2 + 3x}$$

We let $u = x^2 + 3x$ and substitute this into y to give:

$$y = \frac{1}{u} = u^{-1}$$
$$u = x^2 + 3x$$

Since y and u are now simple functions (y is a function of u and u is a function of x) we can differentiate them using the rules given in section 2 above:

$$\frac{dy}{du} = -u^{-2} = -\left(x^2 + 3x\right)^{-2}$$
$$\frac{du}{dx} = 2x + 3$$

Then we apply the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\left(x^2 + 3x\right)^{-2} \cdot (2x + 3) = -\frac{2x + 3}{\left(x^2 + 3x\right)^2}$$

2. Differentiate s, a function of t given by:

$$s = \sin\left(3t^2\right)$$

Using the substitution $u = 3t^2$ we get:

$$s = \sin u$$
$$u = 3t^2$$

Differentiating, we get:

$$\frac{ds}{du} = \cos u$$
$$\frac{du}{dt} = 6t$$

And applying the chain rule gives:

$$\frac{ds}{dt} = \frac{ds}{du} \cdot \frac{du}{dt} = 6t\cos\left(3t^2\right)$$

3. Differentiate y, a function of t given by:

$$y = e^{\frac{1}{t^2}} = e^{t^{-2}}$$

Substituting $u = t^{-2}$, gives:

$$\frac{dy}{du} = e^u = e^{t^{-2}}$$
$$\frac{du}{dt} = -2t^{-3}$$
$$\frac{dy}{dt} = -2t^{-3}e^{t^{-2}} = -\frac{2e^{\frac{1}{t^2}}}{t^3}$$

Product rule

The product rule is used when the function to be differentiated is the product of two simple functions.

If u and v are functions of x and $y = uv$, then:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Worked examples

1. Differentiate:

$$y = t^2 \sin t$$

Let u and v be functions of t so that:

$$u = t^2$$
$$v = \sin t$$

Then:

$$y = uv$$
$$\frac{du}{dt} = 2t$$
$$\frac{dv}{dt} = \cos t$$

Then apply the product rule:

$$\begin{aligned}\frac{dy}{dt} &= u\frac{dv}{dt} + v\frac{du}{dt}\\ &= t^2 \times \cos t + \sin t \times 2t\\ &= t^2 \cos t + 2t \sin t\end{aligned}$$

2. Differentiate:

$$y = 2x^5 e^x$$

Let:

$$y = uv$$
$$u = 2x^5$$
$$v = e^x$$

Then:

$$\frac{du}{dx} = 10x^4$$
$$\frac{dv}{dx} = e^x$$

Then apply the product rule:

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= 2x^5 \times e^x + e^x \times 10x^4$$
$$= 2x^5e^x + 10x^4e^x = e^x(2x^5 + 10x^4)$$

The quotient rule

The quotient rule is used when the function to be differentiated is the quotient of two simple functions.

If u and v are functions of x and $y = \frac{u}{v}$, then:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Worked examples

1. Differentiate:

$$y = \frac{\cos x}{\left(x^2 + 5\right)}$$

Let:

$$y = \frac{u}{v}$$
$$u = \cos x$$
$$v = x^2 + 5$$

Then:

$$\frac{du}{dx} = -\sin x$$
$$\frac{dv}{dx} = 2x$$

Applying the quotient rule gives:

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{\left(x^2+5\right)\times\left(-\sin x\right) - \left(\cos x\right)\times\left(2x\right)}{\left(x^2+5\right)^2} \\ &= \frac{-\sin x\left(x^2+5\right) - 2x\cos x}{\left(x^2+5\right)^2}\end{aligned}$$

2. Differentiate:

$$y = 2e^{-x}x^{15} - 34x^4e^{-x} + e^{-x}\sin x$$

Notice:

$$\begin{aligned}y &= 2e^{-x}x^{15} - 34x^4e^{-x} + e^{-x}\sin x \\ &= \frac{2x^{15} - 34x^4 + \sin x}{e^x}\end{aligned}$$

Let:

$$y = \frac{u}{v}$$
$$u = 2x^{15} - 34x^4 + \sin x$$
$$v = e^x$$

Then:

$$\frac{du}{dx} = 30x^{14} - 136x^3 + \cos x$$
$$\frac{dv}{dx} = e^x$$

Applying the quotient rule gives:

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \\ &= \frac{\left(e^x\right) \times \left(30x^{14} - 136x^3 + \cos x\right) - \left(2x^{15} - 34x^4 + \sin x\right) \times \left(e^x\right)}{\left(e^x\right)^2} \\ &= \frac{-2x^{15} + 30x^{14} + 34x^4 - 136x^3 - \sin x + \cos x}{e^x}\end{aligned}$$

Exercises

1. Differentiate:
   a. $w = e^{x^2+3}$
   b. $h = \sin\left(e^t\right)$
   c. $y = \left(x + x^2\right)^4$

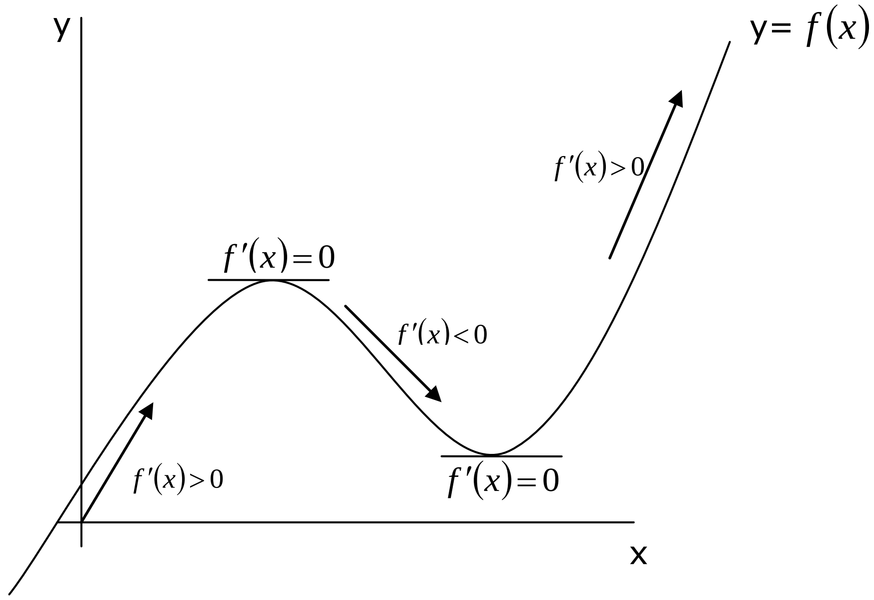
2. Differentiate:
   a. $y = e^x\left(x^5 + 2x^4 + 5x^3\right)$
   b. $g = x^3 \cos x$
   c. $k = t^2 \ln(3t)$

3. Differentiate:
   a. $y = \dfrac{x^5 + 2x^4 + 5x^3}{e^x}$
   b. $t = \dfrac{\sin x}{x^2 + 3}$
   c. $y = \dfrac{\ln x}{\sin(3x)}$

4. Max and min

Stationary points

If we have a graph of $f(x)$, some function of x, then its gradient is given by $f'(x)$. If $f'(x)>0$ then the gradient is positive and then $f(x)$ is increasing. If $f'(x)<0$ then $f(x)$ is decreasing. If $f'(x)=0$ then the function is neither increasing nor decreasing and these points are called stationary points. At stationary points the gradient is parallel to the x-axis.



A function f has a stationary point at $x=a$ if $f'(a)=0$.

Worked examples

1. Find the stationary points of

$$f(x)=4x^3+21x^2+36x$$

Differentiating we obtain:

$$f'(x)=12x^2+42x+36$$

The stationary points are where $f'(x)=0$. This gives:

$$12x^2 + 42x + 36 = 0$$

This is a quadratic equation. Solving this we get:

$$12x^2 + 42x + 36 = 6\left(2x^2 + 7x + 6\right)$$
$$2x^2 + 7x + 6 = (x+2)(2x+3)$$

So the stationary points are at $x = -2$ and $x = -\frac{3}{2}$.

We can check this answer:

$$f'(x) = 12x^2 + 42x + 36$$
$$f'(-2) = 12 \times (-2)^2 + 42 \times (-2) + 36 = 0$$
$$f'\left(-\frac{3}{2}\right) = 12 \times \left(-\frac{3}{2}\right)^2 + 42 \times \left(-\frac{3}{2}\right) + 36 = 0$$

2. Find the stationary points of g, a function of t, where:

$$g = \frac{1}{3}t^3 - t$$

Differentiating gives:

$$\frac{dg}{dt} = t^2 - 1$$

Putting this equal to zero gives the quadratic equation:

$$t^2 - 1 = 0$$
$$\Rightarrow t^2 = 1$$
$$\Rightarrow t = \sqrt{1} = \pm 1$$

So the stationary points are at $t = 1$ and $t = -1$.

Checking to confirm this:

$$\frac{dg}{dt} = t^2 - 1$$
$$\text{At t} = 1$$
$$\frac{dg}{dt} = 1^2 - 1 = 0$$
$$\text{At t} = -1$$
$$\frac{dg}{dt} = (-1)^2 - 1 = 0$$

Local maximum, local minimum and stationary points of inflection

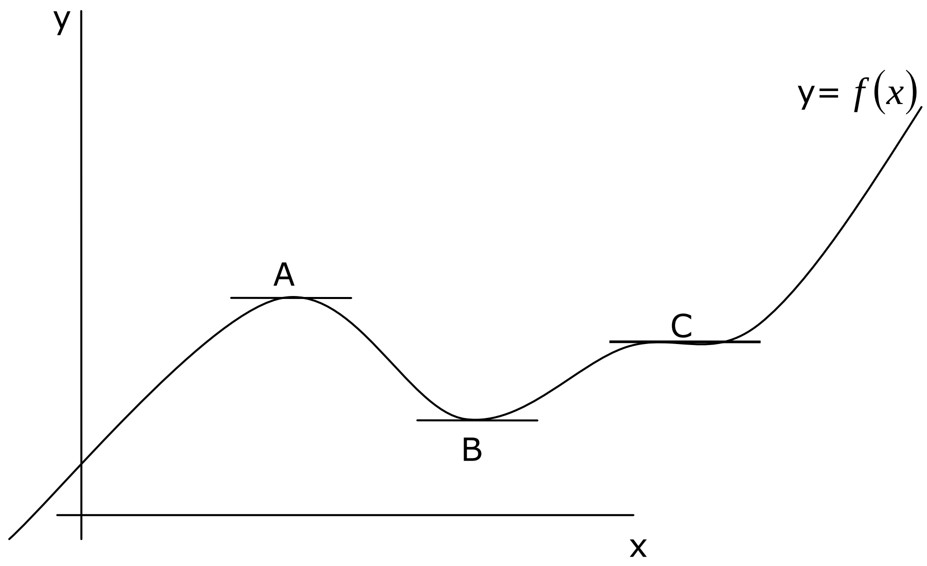
There are three kinds of stationary points:

Local maximum: Those for which all points in the vicinity of $x = a$, $f(a) > f(x)$. Then $x = a$ is at the top of a curve. Note this is referred to as a *local* maximum since it does not need to be the maximum of the whole graph. When passing through a maximum the gradient changes from positive to negative.

Local minimum: Those for which all points in the vicinity of $x = b$, $f(b) < f(x)$. Then $x = b$ is at the bottom of a curve. Again this is a *local* minimum as it is not necessarily the minimum of the whole graph. When passing through a minimum the gradient changes from positive to negative.

Stationary point of inflection: Points at which the gradient is zero but which are neither a local maximum or local minimum are called stationary points of inflection. This can be thought of as when the curve comes to a stop but does not change direction, or when the curve crosses the tangent (which is parallel to the x-axis at a stationary point). When passing through a stationary point of inflection the gradient does not change sign.

On the graph below, point A is a local maximum, point B is a local minimum and point C is a stationary point of inflection.



Worked example

Find and classify the stationary points of:

$$y = x^3 - 12x$$

First differentiate:

$$\frac{dy}{dx} = 3x^2 - 12 = 3(x^2 - 4)$$

Solving the quadratic gives us the stationary points:

$$3(x^2 - 4) = 0$$
$$\Rightarrow x^2 = 4$$
$$\Rightarrow x = \pm 2$$

So the stationary points are at $x = 2$ and $x = -2$.

To classify this, we can examine the gradient in the vicinity of $x = 2$ and $x = -2$. Start with $x = -2$.

We have confirmed above that there are no more stationary points with $x < -2$, so look at $x = -2{\cdot}5$ to discover the sign of the gradient below $x = -2$. The gradient is given by:

$$\frac{dy}{dx} = 3 \times (-2{\cdot}5)^2 - 12 = 6{\cdot}75$$

So the gradient for $x < -2$ is positive.

For the interval between the stationary points, $-2 < x < 2$, take $x = 0$. Then:

$$\frac{dy}{dx} = 3 \times (0)^2 - 12 = -12$$

So the gradient for $-2 < x < 2$ is negative.

Since the gradient has changed from positive to negative around $x = -2$, this stationary point is a local maximum.

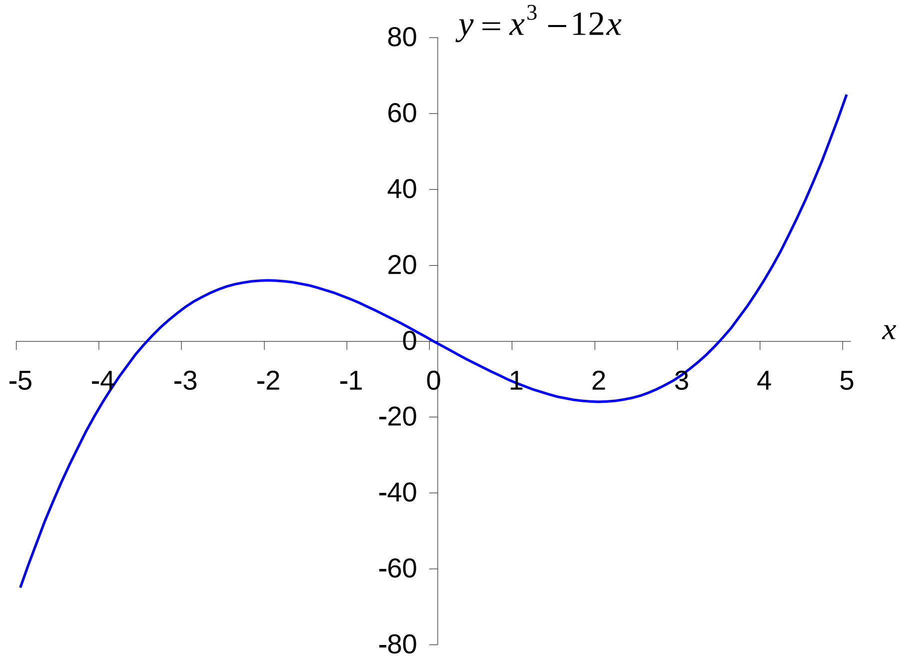
Now look at $x = 2$. For $-2 < x < 2$ we know the gradient is negative. We have confirmed there are no more stationary points with $x > 2$, so take $x = 2{\cdot}5$.

$$\frac{dy}{dx} = 3 \times (2{\cdot}5)^2 - 12 = 6{\cdot}75$$

So the gradient for $x > 2$ is positive.

Since the gradient has changed from negative to positive around $x = 2$, this stationary point is a local minimum.

The following is a graph of $y = x^3 - 12x$. The local maximum at $x = -2$ and local minimum at $x = 2$ should be clear.



Using second order differentiation

Stationary points can be classified using second order differentiation. Just as the first order derivative, $f'(x)$, gives the rate of change of a function $f(x)$, so the second order derivative, $f''(x)$, gives the rate of change of $f'(x)$. This can be seen as the rate of change of the gradient of $f(x)$.

If $x = a$ is a stationary point of $f(x)$, the following rules can be followed:

if $f''(a) > 0$ then $x = a$ is a maximum;

if $f''(a) < 0$ then $x = a$ is a minimum.

However, if $f''(a) = 0$ then the stationary point cannot be classified by this method. In this case looking at the sign of the gradient function either side of the stationary point will identify the point. Even though using second order differentiation only works when

$f''(a) \neq 0$ it is a much quicker method than calculating the gradient either side of the stationary point and so is a good starting point.

Worked examples

1. Find and classify the stationary points of:

$$f(x) = x^3 + 3x^2 - 45x$$

First find the stationary points using the first derivative:

$$\begin{aligned} f'(x) &= 3x^2 + 6x - 45 \\ &= 3(x^2 + 2x - 15) \\ &= 3(x+5)(x-3) \end{aligned}$$

So the stationary points are at $x = -5$ and $x = 3$.

Find the second order derivative:

$$f''(x) = 6x + 6$$

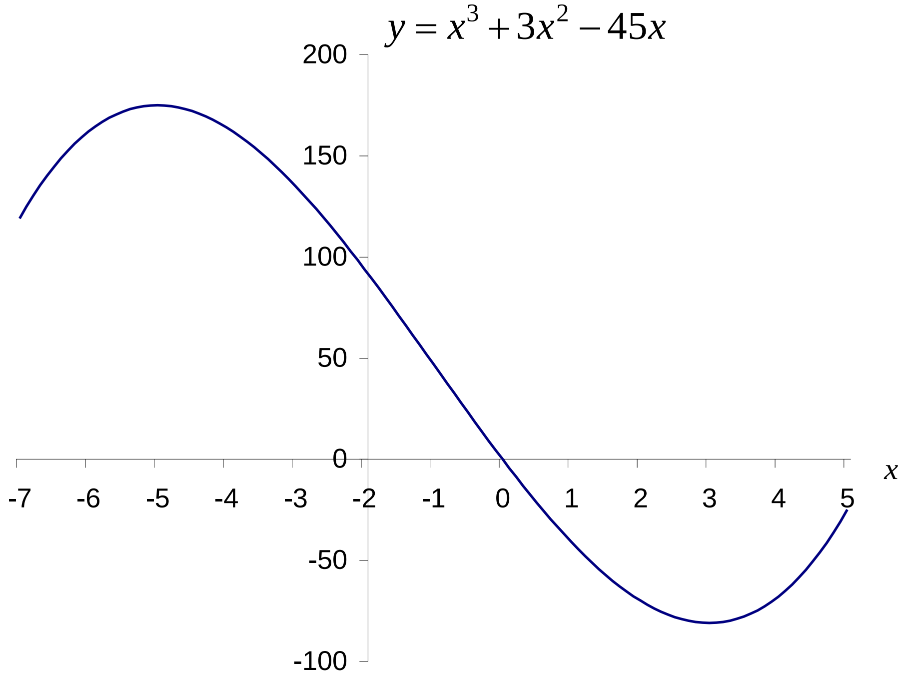
Then evaluate this for $x = -5$ and $x = 3$.

$$\begin{aligned} f''(x) &= 6x + 6 \\ f''(-5) &= 6 \times (-5) + 6 = -24 \\ f''(3) &= 6 \times 3 + 6 = 24 \end{aligned}$$

At $x = -5$ the second order derivative is negative, so $x = -5$ is a maximum.

At $x = 3$ the second order derivative is positive, so $x = 3$ is a minimum.

A sketch of the graph illustrates this:



2. Find and classify the stationary points of:

$$f(x) = 12x^3 + 5$$

First identify the stationary points from the first order derivative.

$$f'(x) = 36x^2$$

$36x^2 = 0$ when $x = 0$. So the single stationary point is at $x = 0$.

Now calculate the second order derivative:

$$f''(x) = 72x$$

At $x = 0$,

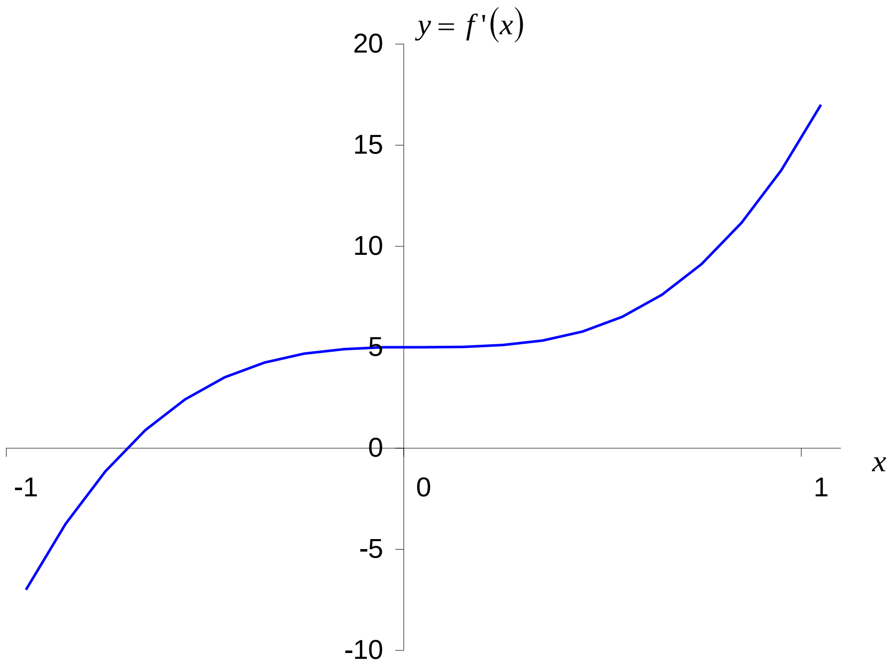
$$f''(0) = 72 \times 0 = 0$$

Since $f''(0) = 0$ we cannot classify $x = 0$ using the second order derivative.

Instead, examine the gradient $f'(x)$ around $x = 0$, say at $x = -1$ and $x = 1$.

$$f'(x) = 36x^2$$
$$f'(-1) = 36 \times (-1)^2 = 36$$
$$f'(1) = 36 \times (1)^2 = 36$$

Since the gradient is positive either side of the stationary point $x = 0$, this must be a stationary point of inflection. The graph of $f'(x)$ is given below.



Exercises

1. Find and classify the stationary points of the following functions:

   a. $f(x) = x^3 + \frac{3x^2}{2} - 6x$

   b. $g(t) = 3t^4 + 8t^3 - 6t^2 - 24t + 12$

   c. $f(x) = x^3 - x^4$

5. Taylor series, expansion, linearization

The Taylor series is used to express a function as a power series. Then we can express complex functions in terms of simple polynomials, which can be easier to deal with.

Higher order differentiation

Higher order derivatives may be defined similarly to second order derivatives, i.e. the third order derivative is found by differentiating the second order derivative, etc.

Higher order derivatives are denoted with increasing numbers of dashes (primes) after the f, or with a number as follows:

$$\frac{d^3 f(x)}{dx^3} = \frac{d^3 y}{dx^3} = \frac{d^3(3x^2)}{dx^3} = f'''(x) = f^{(3)}(x)$$
$$\frac{d^n f(x)}{dx^n} = \frac{d^n y}{dx^n} = \frac{d^n(3x^2)}{dx^n} = f^{(n)}(x)$$

Taylor series

If a function $f(x)$ is smooth (that is, it can be differentiated as often as required) at $x = a$, then $f(x)$ can be expressed as:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f^{(3)}(a) + \ldots$$

This is the Taylor series of $f(x)$ about the point $x = a$.

Worked examples

1. Find the Taylor series expansion of $f(x) = \frac{1}{1-x}$ about $x = 0{\cdot}5$.

First find the value of $f(x)$ at $x = 0{\cdot}5$:

$$f(0{\cdot}5) = \frac{1}{1-0{\cdot}5} = 2$$

Then find the derivatives of $f(x)$:

$$f'(x)=\frac{1}{(1-x)^2},\ f''(x)=\frac{2}{(1-x)^3},\ f^{(3)}(x)=\frac{3!}{(1-x)^4},\ \ldots$$

Now find the values of these at $x=0{\cdot}5$:

$$f'(0{\cdot}5)=\frac{1}{(1-0{\cdot}5)^2}=\frac{1}{(0{\cdot}5)^2}=2^2=4$$

$$f''(0{\cdot}5)=\frac{2}{(1-0{\cdot}5)^3}=\frac{2}{(0{\cdot}5)^3}=2^3\times 2!=16$$

$$f^{(3)}(0{\cdot}5)=\frac{3!}{(1-0{\cdot}5)^4}=\frac{3!}{(0{\cdot}5)^4}=2^4\times 3!=96,\ \ldots$$

So the Taylor series expansion of $f(x)=\frac{1}{1-x}$ about $x=0{\cdot}5$ is:

$$f(x)=2+(x-0{\cdot}5)\times 2^2+\frac{(x-0{\cdot}5)^2}{2!}\times 2^3\times 2!+\frac{(x-0{\cdot}5)^3}{3!}\times 2^4\times 3!+\ldots$$
$$=2+2^2(x-0{\cdot}5)+2^3(x-0{\cdot}5)^2+2^4(x-0{\cdot}5)^3+\ldots$$

2. Find the Taylor series expansion of $f(x)=e^x$ about $x=2$.

First find the value of $f(x)$ at $x=2$:

$$f(2)=e^2=7{\cdot}389\,(3\,\text{dp})$$

Then find the derivatives of $f(x)$:

$$f'(x)=e^x,\ f''(x)=e^x,\ f^{(3)}(x)=e^x,\ \ldots$$

Now find the values of these at $x=2$:

$$f'(2)=f''(2)=f^{(3)}(2)=e^2=7{\cdot}389\,(3\,\text{dp})$$

So the Taylor series expansion of $f(x)=e^x$ about $x=2$ is:

$$
\begin{aligned}
e^x &= e^2 + (x-2)e^2 + \frac{(x-2)^2}{2!}e^2 + \frac{(x-2)^3}{3!}e^2 + \ldots \\
&= e^2\left(1 + (x-2) + \frac{(x-2)^2}{2!} + \frac{(x-2)^3}{3!} + \ldots\right) \\
&= e^2\left(1 + (x-2) + \frac{(x-2)^2}{2} + \frac{(x-2)^3}{6} + \ldots\right)
\end{aligned}
$$

Exercises

1. Find the Taylor series expansions of the following:

   a. $f(x) = \dfrac{1}{2-x}$ about $x = 1$

   b. $f(x) = \cos(x)$ about $x = 2\pi$

   c. $f(x) = \ln(x)$ about $x = 2$

6. Partial differentiation

Partial differentiation is the process of differentiating functions of more than one independent variable, e.g. $f(x, y)$. (Note that differentiation involving functions of only one independent variable is called ordinary differentiation).

Many natural phenomena lead to partial differential equations since they deal with functions of more than one dimension of space or of space and time.

If we have a function $z = f(x, y)$ then z is a dependant variable and x and y are independent variables. As x and y change, z changes accordingly. Remember that with one independent variable the derivative with respect to that variable is the rate of change of the function with respect to that variable.

With a function of two variables the partial derivatives are the rates of change of the function with respect to each of the independent variables. To calculate this we first fix y and differentiate with respect to x (and so treat y as a constant in the differentiation), then vice versa.

The partial derivative of $z = f(x, y)$ with respect to x is written as:

$$\frac{\partial z}{\partial x} = f_x(x, y)$$

The partial derivative of $z = f(x, y)$ with respect to y is written as:

$$\frac{\partial z}{\partial y} = f_y(x, y)$$

Of course, a function of 3 independent variables would have 3 partial derivatives, and so on.

Second order partial differentiation

In partial differentiation, as in ordinary differentiation, it is possible to take higher derivatives. The second order partial derivative of $z = f(x, y)$ with respect to x (holding y constant) is denoted:

$$\frac{\partial^2 z}{\partial x^2} = f_{xx}(x, y)$$

Similarly, the second order partial derivative with respect to y (holding x constant) is denoted:

$$\frac{\partial^2 z}{\partial y^2} = f_{yy}(x, y)$$

It is also possible to differentiate z with respect to x and then differentiate the result with respect to y. The result is a mixed derivative and is denoted:

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = f_{xy}(x, y)$$

Notice that since the differential operator, ∂, acts on the left, the order of the $\partial y \partial x$ is read from right to left (x then y). However, the order is reversed for the alternative notation f_{xy}.

Similarly, the process of differentiating z first with respect to y then x results in a mixed derivative denoted:

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = f_{yx}(x, y)$$

Worked examples

1. Find the second order partial derivatives of $z = f(x, y) = x^2 y^2$.

First find the first order partial derivative of z with respect to x. To do this, we treat y as a constant, so the function simply becomes a constant (y^2) multiplied by x^2. Then:

$$\frac{\partial z}{\partial x} = 2xy^2$$

Similarly, to find the first order partial derivative of z with respect to y, we treat x as a constant. Then:

$$\frac{\partial z}{\partial y} = 2x^2 y$$

To find the second order derivative of z with respect to x, we differentiate the first order derivative of z with respect to x, with respect to x. Then:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial\left(2xy^2\right)}{\partial x} = 2y^2$$

Similarly, we find the second order derivative of z with respect to y:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial\left(2x^2 y\right)}{\partial x} = 2x^2$$

To find the mixed derivatives we take the first order derivative of z with respect to x and differentiate this with respect to y, and vice versa. Thus:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial\left(2xy^2\right)}{\partial y} = 4xy$$

And similarly:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial\left(2x^2 y\right)}{\partial x} = 4xy$$

2. Find the first order partial derivatives of $z = f(x, y) = x^2 + x\sin y$.

First find the first order partial derivative of z with respect to x. To do so, we take y to be a constant. Then $\sin y$ is similarly constant and the term $x\sin y$ is simply a constant multiplied by x. Then:

$$\frac{\partial z}{\partial x} = 2x + \sin y$$

Next, find the first order partial derivative of z with respect to y.

To do this we take x as constant, so x^2 is a constant term and $x\sin y$ is a constant multiplied by $\sin y$. Then:

$$\frac{\partial z}{\partial y} = x\cos y$$

3. Find the second order partial derivatives of $z = f(x, y) = x^2 y + xy^3$.

First, find the first order partial derivatives:

$$\frac{\partial z}{\partial x} = 2xy + y^3$$

$$\frac{\partial z}{\partial y} = x^2 + 3xy^2$$

Then differentiate these each with respect to x and y to find the second order partial derivatives:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial\left(2xy + y^3\right)}{\partial x} = 2y$$

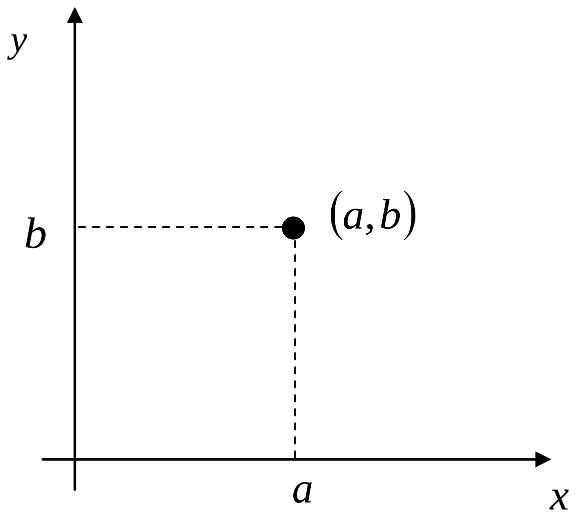
$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial\left(x^2 + 3xy^2\right)}{\partial y} = 6xy$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial\left(2xy + y^3\right)}{\partial y} = 2x + 3y^2$$

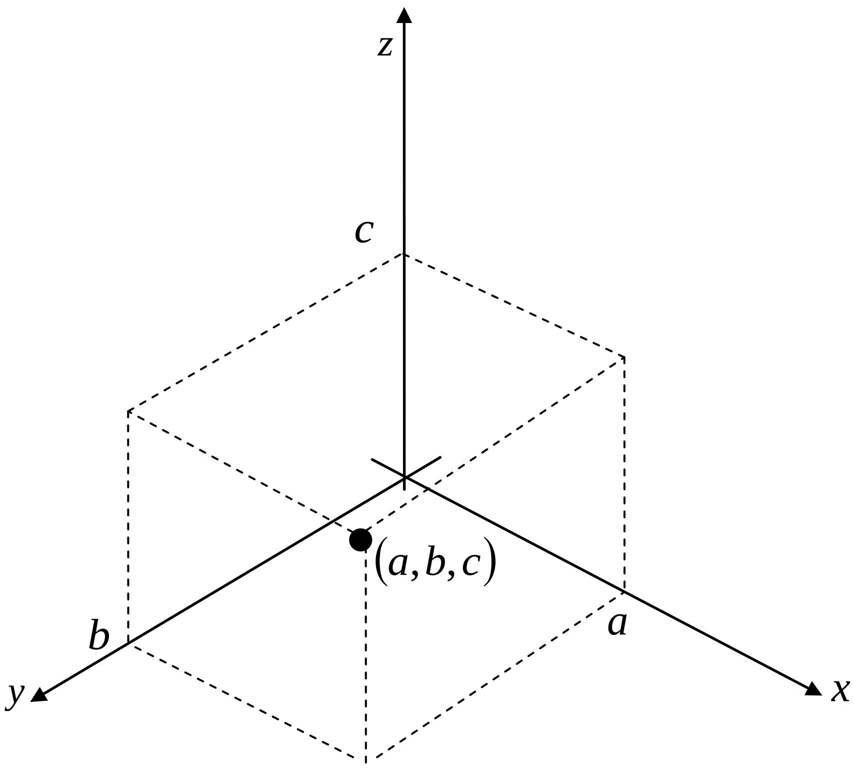
$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial\left(x^2 + 3xy^2\right)}{\partial x} = 2x + 3y^2$$

3D surfaces: max and min

We know two coordinates x and y can describe a point in 2-dimensional space:



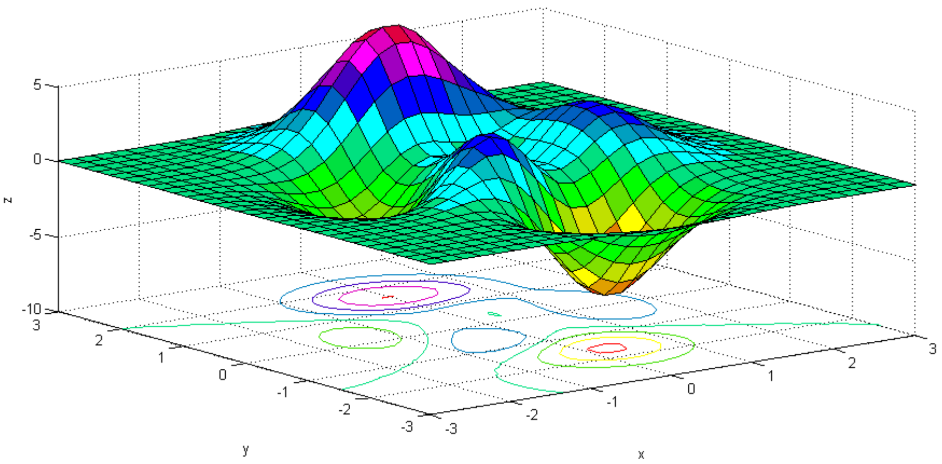
In the same way, three coordinates, x, y and z, can be used to describe a point in 3-dimensional space:



A function can have $y = f(x)$ where y is a dependant variable determined by the independent variable x (so as x varies so does y), and this function can be plotted as a line on a 2D graph.

In the same way, a function $z = f(x, y)$ can be used to describe a surface in three dimensions: x, y and z. In this case, z is the dependant variable and will vary according to the values of the independent variables x and y. This function can be plotted as a surface in 3D space.

Here is an example of a 3D surface:



Stationary points

A surface described by the function $z = f(x, y)$ has a stationary point at $(x, y) = (a, b)$ if $\frac{\partial z}{\partial x} = f_x(x, y) = 0$ and $\frac{\partial z}{\partial y} = f_y(x, y) = 0$ at this point.

Worked example

Find the stationary points of:

$$z = f(x, y) = xy(x + 2y - 2)$$

Note that:

$$f(x, y) = xy(x + 2y - 2) = x^2 y + 2xy^2 - 2xy$$

Now compute the first order partial derivatives:

$$f_x(x,y) = 2xy + 2y^2 - 2y$$
$$f_y(x,y) = x^2 + 4xy - 2x$$

The stationary points are when $f_x(x,y) = f_y(x,y) = 0$. Looking at $f_x(x,y)$:

$$f_x(x,y) = 2xy + 2y^2 - 2y = 2y(x + y - 1) = 0$$

This equation is satisfied when $y = 0$ or $x + y - 1 = 0$.

Now look at $f_y(x,y)$:

$$f_y(x,y) = x^2 + 4xy - 2x = x(x + 4y - 2) = 0$$

This equation is satisfied when $x = 0$ or $x + 4y - 2 = 0$.

We need solutions for which both $f_x(x,y)$ and $f_y(x,y)$ are zero. We have found two equations for which $f_x(x,y)$ is zero and two for which $f_y(x,y)$ is zero. Thus, we have four points where both equations are zero:

1. $\begin{array}{l} y = 0 \\ x = 0 \end{array}$

2. $\begin{array}{l} y = 0 \\ x + 4y - 2 = 0 \end{array}$

3. $\begin{array}{l} x + y - 1 = 0 \\ x = 0 \end{array}$

4. $\begin{array}{l} x + y - 1 = 0 \\ x + 4y - 2 = 0 \end{array}$

1 describes the point $(x,y) = (0,0)$

Looking at 2, we have $y=0$. Putting $y=0$ in the second equation gives:

$$x+4y-2=x+4\times 0-2=x-2=0$$

So this set of equations has its solution at the point $(x,y)=(2,0)$.

Looking at 3, similarly putting $x=0$ in the first equation we have:

$$x+y-1=0+y-1=y-1=0$$

So we have the point $(x,y)=(0,1)$.

Finally, 4 must be solved as a set of simultaneous equations:

$$\begin{aligned} x+y-1=0 \quad &(1) \\ x+4y-2=0 \quad &(2) \end{aligned}$$

Taking (2)-(1) gives:

$$0+3y-1=0 \quad (2)-(1)$$

So $3y-1=0$, which gives $y=\frac{1}{3}$. Putting this back into (1) gives:

$$x+\frac{1}{3}-1=x-\frac{2}{3}=0$$

So $x=\frac{2}{3}$. This set of equations has the solution $(x,y)=\left(\frac{2}{3},\frac{1}{3}\right)$.

Thus, $z=f(x,y)=xy(x+2y-2)$ has stationary points at:

$(0,0)$, $(2,0)$, $(0,1)$ and $\left(\frac{2}{3},\frac{1}{3}\right)$.

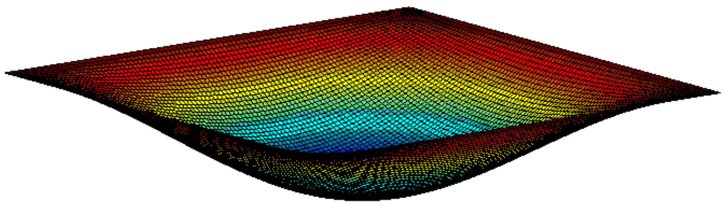
Classifying stationary points

A stationary point on the surface is either a local maximum, a local minimum or a saddle point. As with ordinary (2D) stationary points, these can be classified using second order partial differentiation. For a point $(x, y) = (a, b)$, let:

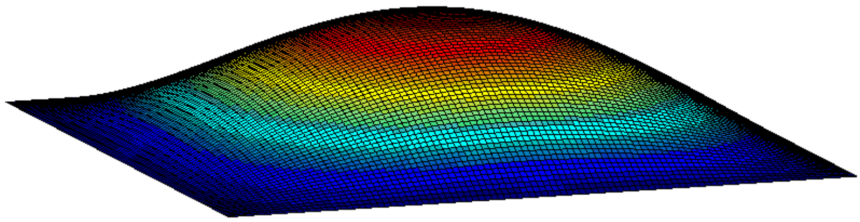
$$d = f_{xx}(a,b) f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2$$

Then:

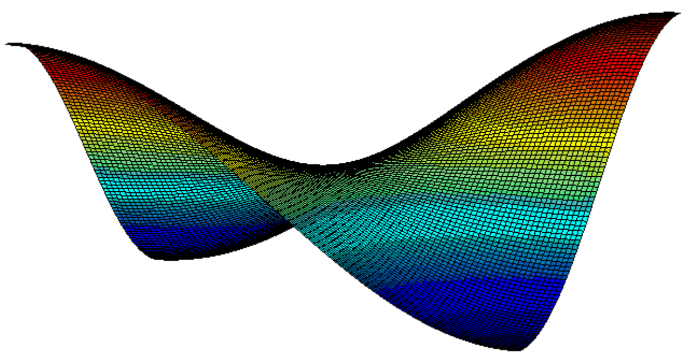
- If $d > 0$ and $f_{xx}(a,b) > 0$ then the point $(x, y) = (a, b)$ is a local minimum. An example is shown below:



- If $d > 0$ and $f_{xx}(a,b) < 0$ then the point $(x, y) = (a, b)$ is a local maximum. An example is shown below:



- If $d < 0$ then the point $(x, y) = (a, b)$ is called a saddle point. An example is shown below:



However, if $d = 0$ then the stationary point cannot be classified by this method.

Worked example

1. Find and classify the stationary points of:

$$z = f(x, y) = x^2 + y^2$$

First find the first order partial derivatives:

$$f_x(x, y) = 2x$$
$$f_y(x, y) = 2y$$

Now find points where $f_x(x, y) = f_y(x, y) = 0$. Looking at $f_x(x, y)$:

$$f_x(x, y) = 2x = 0 \text{ when } x = 0.$$

Looking at $f_y(x, y)$:

$$f_y(x, y) = 2y = 0 \text{ when } y = 0.$$

So there is a single stationary point at $(x, y) = (0,0)$.

To classify this stationary point we find the second order partial derivatives $f_{xx}(x, y)$, $f_{yy}(x, y)$ and $f_{xy}(x, y)$. The second order derivatives are found by differentiating the first order partial derivatives a second time:

$$f_{xx}(x, y) = 2$$

$$f_{yy}(x, y) = 2$$

$$f_{xy}(x, y) = 0$$

Now let:

$$d = f_{xx}(x,y)f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2$$
$$= (2)(2) - (0)^2 = 4$$

Since $d = 4 > 0$ and $f_{xx}(0,0) = 2 > 0$ then the point $(x,y) = (0,0)$ is a local minimum.

2.Classify the stationary points of:

$$z = f(x,y) = xy(x + 2y - 2) = x^2y + 2xy^2 - 2xy$$

Above, we found the stationary points of this function were:

$(0,0)$, $(2,0)$, $(0,1)$ and $\left(\frac{2}{3},\frac{1}{3}\right)$.

To classify these stationary points we find the second order partial derivatives $f_{xx}(x,y)$, $f_{yy}(x,y)$ and $f_{xy}(x,y)$. The first order partial derivatives were found above to be:

$$f_x(x,y) = 2xy + 2y^2 - 2y$$
$$f_y(x,y) = x^2 + 4xy - 2x$$

The second order derivatives are found by differentiating again:

$$f_{xx}(x,y) = 2y$$

$$f_{yy}(x,y) = 4x$$

$$f_{xy}(x,y) = 2x + 4y - 2$$

Now let:

$$d = f_{xx}(x,y)f_{yy}(x,y) - \left(f_{xy}(x,y)\right)^2$$
$$= (2y)(4x) - (2x + 4y - 2)^2$$

Evaluate d at each stationary point.

At $(x,y) = (0,0)$:

$$\begin{aligned} d &= (2y)(4x) - (2x + 4y - 2)^2 \\ &= (2\times 0)(4\times 0) - (2\times 0 - 4\times 0 + 2)^2 \\ &= 0 - 2^2 \\ &= -4 \end{aligned}$$

$d = -4 < 0$, so $(x, y) = (0,0)$ is a saddle point.

At $(x, y) = (2,0)$:

$$\begin{aligned} d &= (2y)(4x) - (2x + 4y - 2)^2 \\ &= (2\times 0)(4\times 2) - (2\times 2 - 4\times 0 + 2)^2 \\ &= 0 - (4 + 2)^2 \\ &= -36 \end{aligned}$$

$d = -36 < 0$, so $(x, y) = (2,0)$ is a saddle point.

At $(x, y) = (0,1)$:

$$\begin{aligned} d &= (2y)(4x) - (2x + 4y - 2)^2 \\ &= (2\times 1)(4\times 0) - (2\times 0 - 4\times 1 + 2)^2 \\ &= 0 - (-2)^2 \\ &= -4 \end{aligned}$$

$d = -4 < 0$, so $(x, y) = (0,1)$ is a saddle point.

At $(x, y) = \left(\frac{2}{3}, \frac{1}{3}\right)$:

$$d=(2y)(4x)-(2x+4y-2)^2$$
$$=\left(2\times\frac{1}{3}\right)\left(4\times\frac{2}{3}\right)-\left(2\times\frac{2}{3}-4\times\frac{1}{3}+2\right)^2$$
$$=\frac{16}{9}-\left(\frac{4}{3}-\frac{4}{3}+2\right)^2$$
$$=\frac{16}{9}-(2)^2$$
$$=-\frac{20}{9}$$

$d=-\frac{20}{9}<0$, so $(x,y)=\left(\frac{2}{3},\frac{1}{3}\right)$ is a saddle point.

Exercises

1. For the following functions, find the first and second order partial derivative with respect to both variables:

   a. $z=f(x,y)=y\sin x$

   b. $y=g(x,t)=6x^4t+4xt^2$

   c. $h=f(x,t)=4e^x-x^2t^4$

2. Find and classify the stationary points for the following functions:

   a. $z=f(x,y)=-2(x^2+y^2)$

   b. $z=f(x,y)=x^2-2y^2$

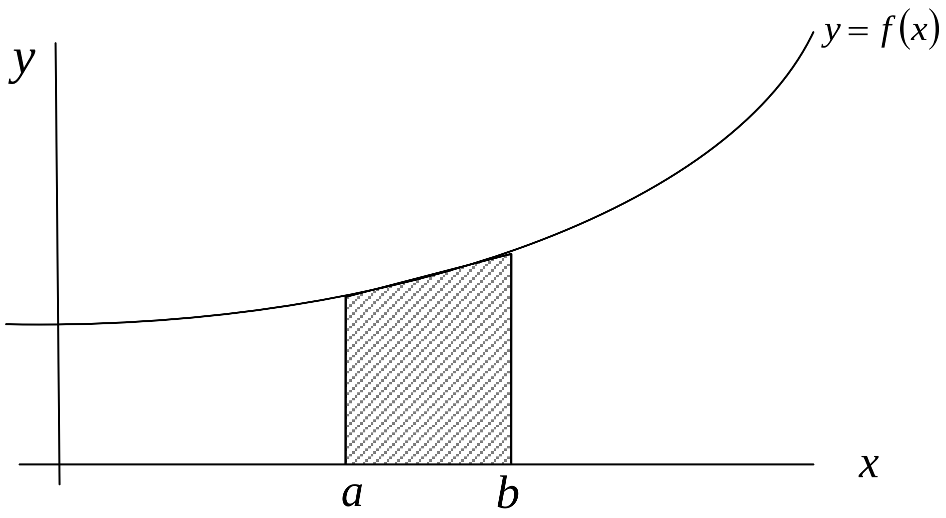
   c. $z=f(x,y)=3xy(x-y-1)$

Integration

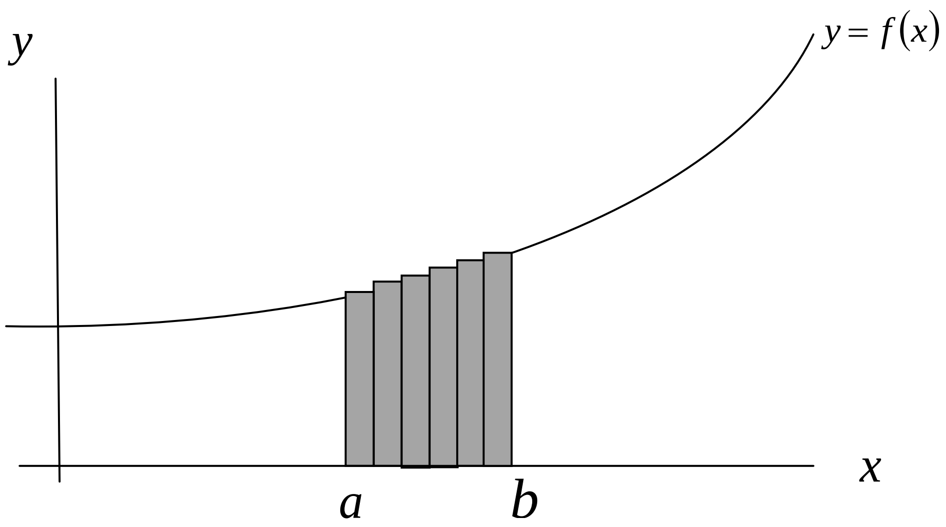
Integration is a technique in calculus used to calculate areas and volumes. We expect you will be familiar with the principles of integration, integrals of some common functions. You should also understand how integration is used to calculate definite and indefinite integrals and areas under curves.

1. What is integration?

Say we wish to find the area under a curve represented by a function $y = f(x)$. That is, we wish to discover the area of the region bounded by $y = f(x)$, the x-axis and two vertical lines $x = a$ and $x = b$. This is shown as the shaded area in the diagram below.



We can estimate this area by dividing it up into a number of thin strips (rectangles) and summing the areas of these rectangles. This is shown in the diagram below.



The sum of the areas of the grey rectangles approximates the area we are interested in. Let us say the rectangles have width δx, where we use δx to mean a small change in x.

Take the first strip. As x increases to $x + \delta x$, then for small δx we can assume $f(x) \approx f(x + \delta x)$. Then the rectangles can be taken as being of height $y = f(x)$ and width δx. Since the area of a rectangle is height multiplied by width, we say the area of each rectangle is $y\delta x$.

The area we are interested in is the sum of the areas of all the rectangles and is written as:

$$A = \sum_{x=a}^{x=b} y\delta x$$

A gives an approximation for the area under the curve. As the width of the rectangles becomes smaller (i.e. more strips are used), the approximation becomes better. If we let δx tend to zero the limit of this sum is called the definite integral and is denoted by:

$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\delta x = \int_a^b y dx = \int_a^b f(x) dx$$

a and b are called the limits of integration.

2. Integration as anti-differentiation

Earlier, we used the analogy of speed and distance to explain differentiation. If we have a function $x(t)$ which defines the position, x, with respect to time, t. Then we differentiate $x(t)$ to give the rate of change of position, x, with respect to time, t, which is the speed.

Similarly, given a function $v(t)$, which gives the speed with respect to time, t, we can attempt to perform the reverse of differentiation to get back to position. This process is called integration and the position is then the integral of speed.

For a function $f(x)$, to integrate $f(x)$ with respect to x gives the indefinite integral of $f(x)$ with respect to x, which is written as:

$$\int f(x)dx$$

Note that if $F(x)$ is the integral of some function $f(x)$ with respect to x, so that:

$$F(x)=\int f(x)dx$$

Then:

$$\frac{dF(x)}{dx}=f(x)$$

Worked example

1. Integrate $f(x)=x^2$ with respect to x.

Take integration as the opposite of differentiation. Then we are looking for some function which will differentiate to give $f(x)=x^2$.

Since in differentiation the power of x reduces by one, in integration it increases by one. Then the answer must involve a term in x^3. Try to differentiate x^3.

$$\frac{d(x^3)}{dx}=3x^2$$

The result we are looking for is x^2. Since the derivative of x^3 is 3 times x^2, it is reasonable to assume that the integral of x^2 is one third of x^3. Then

$$\int f(x)dx=\int x^2dx=\frac{x^3}{3}$$

Try to differentiate this function to check the result:

$$\frac{d\left(\frac{x^3}{3}\right)}{dx} = \frac{1}{3} \times 3x^2 = x^2$$

So we have found a function that differentiates to the function we were looking for.

However, notice that there are other functions which differentiate to give this function also. For example, differentiate $\frac{x^3}{3} + 2$:

$$\frac{d\left(\frac{x^3}{3} + 2\right)}{dx} = \frac{1}{3} \times 3x^2 + 0 = x^2$$

In fact, any function like $\frac{x^3}{3} + c$, for any constant c, gives the same answer when differentiated, since a constant differentiated with respect to a variable such as x gives zero. This arbitrary constant c is called the *constant of integration* and must be added to a function obtained through integration.

Integrals of some common functions

Taking integration as the opposite of differentiation, we can determine the integrals of some common functions.

For functions of the form

$$f(x) = mx^n$$

We have that

$$\int f(x)dx = \frac{mx^{n+1}}{n+1} + c$$

Where c is an arbitrary constant of integration. We saw above how the integral was used to determine the area under a curve between two points. This formula means we have can find the indefinite integral for any function of the form $f(x) = mx^n$. By adding constants of integration a and b we can obtain the definite integral (the area under the curve between a and b) for any such function.

Following from this are the following two results, for the integral of a constant and of zero [The symbol $\Rightarrow$ means "implies"]:

$$f(x) = m$$
$$\Rightarrow \int f(x)dx = mx + c$$
$$f(x) = 0$$
$$\Rightarrow \int f(x)dx = c$$

Where c is an arbitrary constant of integration..

For trig functions:

$$f(x) = \sin x$$
$$\Rightarrow \int f(x)dx = -\cos x + c$$
$$f(x) = \cos x$$
$$\Rightarrow \int f(x)dx = \sin x + c$$

Remembering the chain rule, we have:

$$f(x) = \sin(ax + b)$$
$$\Rightarrow \int f(x)dx = -\frac{\cos(ax + b)}{a} + c$$
$$f(x) = \cos(ax + b)$$
$$\Rightarrow \int f(x)dx = \frac{\sin(ax + b)}{a} + c$$

e and the natural logarithm:

$$f(x) = e^x$$
$$\Rightarrow \int f(x)dx = e^x + c$$
$$f(x) = \frac{1}{x}$$
$$\Rightarrow \int f(x)dx = \ln x + c = \log_e x + c$$

Remembering the chain rule, we have:

$$f(x) = e^{ax+b}$$
$$\Rightarrow \int f(x)dx = \frac{e^{ax+b}}{a} + c$$
$$f(x) = \frac{1}{ax+b}$$
$$\Rightarrow \int f(x)dx = \frac{\ln(ax+b)}{a} + c = \frac{\log_e(ax+b)}{a} + c$$

Where c is an arbitrary constant of integration.

Finding the constant of integration

We can find the constant of integration if further information is given, for instance if we know the coordinates of a point through which the integral passes.

Worked example

Find the equation of the curve of $y = f(x)$, which passes through the point $(x, y) = (3,5)$ and for which the gradient at any point is given by:

$$\frac{dy}{dx} = 4x^2$$

First find the indefinite integral:

$$f(x) = \int 4x^2 dx = \frac{4x^3}{3} + c$$

Now we have a formulae for $y = f(x)$ which gives y for any value of x. We know that at $x = 3$, $y = 5$. Thus:

$$f(x) = \frac{4x^3}{3} + c$$

$$f(3) = 5 = \frac{4 \times 2^3}{3} + c = 36 + c$$

So we know $5 = 36 + c$, therefore $c = -31$. So the equation for the curve is

$$y = f(x) = \frac{4x^3}{3} - 31$$

Exercises

1. Find the following indefinite integrals:

   a. $\int (12x + 5) dx$

   b. $\int (6x^2 + 8x^3) dx$

   c. $\int \sin(3x) dx$

   d. $\int 12\cos(6x + 3) dx$

   e. $\int e^{2x+2} dx$

   f. $\int \frac{1}{6x + 4} dx$

2. The curve $y = f(x)$ passes through $(x, y) = (2,12)$ and the gradient at any point is given by $\frac{dy}{dx} = 5x$. Find the equation for the curve.

3. The curve $y = f(x)$ passes through $(x, y) = \left(\frac{\pi}{2}, 0.5\right)$ and the gradient at any point is given by $\frac{dy}{dx} = 6\cos(3x)$. Find the equation for the curve.

3. Areas under curves

As we saw above, integration can be used to find the area under a curve of a given function. Using the techniques of integration we have learned we can calculate such an area.

Definite and indefinite integrals

Previously were some methods for finding the indefinite integral of a function $f(x)$. That is, a formula for calculating the integral over any range, the indefinite integral.

An integral which relates to a particular range of values has a definite value, since we can use the formula for the indefinite integral to calculate this value. For the definite integral from $x = a$ to b, we write:

$$\int_a^b f(x)dx$$

And this is calculated by subtracting the value of the integral at a from the value at b. If:

$$F(x) = \int f(x)dx$$

Then:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$$

Worked examples

1. Calculate the definite integral:

$$\int_2^3 (x^2 + 3x)dx$$

First integrate to find the indefinite integral.

$$F(x)=\int\left(x^2+3x\right)dx=\frac{x^3}{3}+\frac{3x^2}{2}+c$$

Notice we have remembered to include the constant of integration c.

Now take the values of $F(x)$ at $x=3$ and $x=2$ (the range required).

$$F(x)=\frac{x^3}{3}+\frac{3x^2}{2}+c$$

$$F(3)=\frac{3^3}{3}+\frac{3\times 3^2}{2}+c=22.5+c$$

$$F(2)=\frac{2^3}{3}+\frac{3\times 2^2}{2}+c=\frac{26}{3}+c=8.67+c$$

Now subtract $F(2)$ from $F(3)$ to get the integral over that range.

$$\int_2^3\left(x^2+3x\right)dx=\left[\frac{x^3}{3}+\frac{3x^2}{2}+c\right]_2^3=F(3)-F(2)$$

$$=(22.5+c)-(8.67+c)$$

$$=22.5-8.67+c-c$$

$$=13.83\,(2\text{dp})$$

Notice the constant of integration, c, is present in both $F(3)$ and $F(2)$ and so cancels out.

2. Calculate the definite integral:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}4\cos(3x+1)dx$$

First integrate to find the indefinite integral.

$$F(x)=\int 4\cos(3x+1)dx=\frac{4}{3}\sin(3x+1)+c$$

So

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 4\cos(3x+1)dx = \left[\frac{4}{3}\sin(3x+1)+c\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

Now take the values of $F(x)$ at $x=\frac{\pi}{4}$ and $x=-\frac{\pi}{4}$ (the range required).

$$F(x)=\frac{4}{3}\sin(3x+1)+c$$

$$F\left(\frac{\pi}{4}\right)=\frac{4}{3}\sin\left(3\times\frac{\pi}{4}+1\right)+c=-0.284+c$$

$$F\left(-\frac{\pi}{4}\right)=\frac{4}{3}\sin\left(3\times-\frac{\pi}{4}+1\right)+c=-1.303+c$$

So

$$\left[\frac{4}{3}\sin(3x+1)+c\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = -0.284+c-(-1.303+c)=1.019\,(3\,\text{dp})$$

Exercises

1. Find the following definite integrals:

a. $\int_0^1 8x^3 dx$

b. $\int_3^6 (2x^2+5)dx$

c. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} 12\sin(2x)dx$

d. $\int_1^2 5\cos(3x+5)dx$

e. $\int_{1.5}^2 e^{3x}dx$

f. $\int_{-1}^2 \frac{5}{5x+2}dx$

Matrices

Having made a number of distance and angle measurements, we usually wish to combine our measurements and find the position of the target we are observing. This is usually performed using Matrix Algebra which is a method by which many equations can be condensed into one equation which is, then, easily solved. We expect that you will be familiar with matrices and how they can be used to solve systems of simultaneous equations. You must also be happy with matrix and vector notation.

1. Terminology

A matrix is a rectangular array of values. The following is the notation used for:

A 2x2 matrix:

$$\underbrace{\begin{bmatrix} 5 & 0 \\ 6 & 2 \end{bmatrix}}_{2\text{ columns}} \left.\vphantom{\begin{bmatrix} 5 \\ 6 \end{bmatrix}}\right\} 2\text{ rows}$$

A 3x2 matrix:

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 4 \end{bmatrix}}_{2\text{ columns}} \left.\vphantom{\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}}\right\} 3\text{ rows}$$

An m by n matrix has m rows and n columns. The notation for this is:

$$\begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{array}{l} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \quad \vdots \\ \leftarrow \text{row } m \end{array} \\ \begin{array}{cccc} \uparrow & \uparrow & & \uparrow \\ & & \cdots & \\ \text{col } 1 & \text{col } 2 & & \text{col } n \end{array} \end{array}$$

The entries of this matrix, a_{11}, a_{12}, etc. are read "a one one", "a one two", etc. So then a_{ij} ("a eye jay") is the entry in the ith row and jth column of the matrix.

Matrices are often denoted by capital letters. So the matrix A could be:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 4 \end{bmatrix}$$

A matrix with one column is called a column vector. Vectors are often used to denote a property with multiple components, for instance the distance and direction components of position or the speed and direction components of velocity. Vectors are often denoted with a lower case bold letter. Below, $\mathbf{a}$ is a vector.

$$\mathbf{a} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

2. Simple matrix algebra – addition, subtraction, multiplication

Equality

Two matrices A and B are equal if they meet the following two conditions:

1. A and B have the same number of rows (say, m) and columns (say, n);
2. All the corresponding entries are equal (say, $a_{ij} = b_{ij}$ for all $i = 1$ to m and $j = 1$ to n).

So the following are a pair of matrices that are equal

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 4 \end{bmatrix}$$

While the following are examples of matrices that are not equal.

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 4 & 1 \\ 2 & 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 1 \\ 3 & 4 \\ 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 & 0 \\ 4 & 3 & 0 \\ 1 & 4 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 4 \\ 0 & 0 \end{bmatrix}$$

Addition and subtraction

If two matrices have the same number of rows and the same number of columns they can be added or subtracted by adding or subtracting the corresponding entries. So for general 2x2 matrices:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11} + b_{11}) & (a_{12} + b_{12}) \\ (a_{21} + b_{21}) & (a_{22} + b_{22}) \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} - \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} (a_{11} - b_{11}) & (a_{12} - b_{12}) \\ (a_{21} - b_{21}) & (a_{22} - b_{22}) \end{bmatrix}$$

Worked examples

1.

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 4 \end{bmatrix} + \begin{bmatrix} 4 & 5 \\ 3 & 2 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 7 & 5 \\ 2 & 10 \end{bmatrix}$$

2.

$$\begin{bmatrix} 23 & 34 \\ 5 & 0 \\ 17 & 60 \end{bmatrix} + \begin{bmatrix} 12 & -7 \\ 13 & 56 \\ 24 & 56 \end{bmatrix} = \begin{bmatrix} 35 & 27 \\ 18 & 56 \\ 41 & 116 \end{bmatrix}$$

3.

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 1 & 1 \end{bmatrix}$$

4.

$$\begin{bmatrix} 23 & 34 & 17 \\ 5 & 0 & 60 \end{bmatrix} - \begin{bmatrix} 12 & -7 & 24 \\ 13 & 56 & 56 \end{bmatrix} = \begin{bmatrix} 11 & 41 & -7 \\ -8 & -56 & 4 \end{bmatrix}$$

Scalar multiplication

To multiply a matrix by a number (scalar), c, multiply every value in the matrix by c. Then:

$$c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$$

This process can be used in reverse to simplify a matrix by removing a common factor.

Worked examples

1. Multiply the following:

$$3\begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 12 & 6 \end{bmatrix}$$

2. Extract a common factor from the matrix:

$$\begin{bmatrix} 12 & 24 & 12 \\ 6 & 18 & 6 \\ 6 & 12 & 36 \end{bmatrix} = 6\begin{bmatrix} 2 & 4 & 2 \\ 1 & 3 & 1 \\ 1 & 2 & 6 \end{bmatrix}$$

Matrix multiplication

If matrices A and B are to be multiplied then matrix A must have the same number of columns as matrix B has rows.

Then the product AB can be calculated according to the following rule:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix} = \begin{bmatrix} (a_{11}b_{11} + a_{12}b_{21}) & (a_{11}b_{12} + a_{12}b_{22}) & (a_{11}b_{13} + a_{12}b_{23}) \\ (a_{21}b_{11} + a_{22}b_{21}) & (a_{21}b_{12} + a_{22}b_{22}) & (a_{21}b_{13} + a_{22}b_{23}) \end{bmatrix}$$

Worked example

1. Calculate $C = AB$, where:

$$A = \begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix}, \ B = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix}$$

First, check the number of columns for A and the number of rows for B are equal. Both are equal to 2.

Then apply the rule:

$$C = \begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

To calculate the value of c_{11}, outline the relevant column and row as follows:

$$\begin{bmatrix} \boxed{1 \quad 5} \\ 4 \quad 2 \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 3 \\ 2 \end{matrix}} & \begin{matrix} 4 \\ 6 \end{matrix} \end{bmatrix}$$

So $c_{11} = 1 \times 3 + 5 \times 2 = 13$

Similarly:

$$C = \begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 \times 3 + 5 \times 2 & 1 \times 4 + 5 \times 6 \\ 4 \times 3 + 2 \times 2 & 4 \times 4 + 2 \times 6 \end{bmatrix} = \begin{bmatrix} 13 & 34 \\ 16 & 28 \end{bmatrix}$$

Note that for matrix multiplication, $AB \neq BA$ in general. For example notice that $C = AB \neq BA = D$ in:

$$C = \begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 13 & 34 \\ 16 & 28 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 4 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 19 & 23 \\ 26 & 22 \end{bmatrix}$$

2. Calculate $C = AB$, where:

$$A = \begin{bmatrix} 5 & 2 & 10 \\ 3 & 5 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 2 & 10 \end{bmatrix}$$

First, check the number of columns for A and the number of rows for B are equal. Both are equal to 3. Then perform the calculation row by row and column by column:

$$C = \begin{bmatrix} 5 & 2 & 10 \\ 3 & 5 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 2 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 5\times 2 + 2\times 1 + 10\times 2 & 5\times 3 + 2\times 2 + 10\times 10 \\ 3\times 2 + 5\times 1 + 0\times 2 & 3\times 3 + 5\times 2 + 0\times 10 \end{bmatrix}$$

$$= \begin{bmatrix} 32 & 119 \\ 11 & 19 \end{bmatrix}$$

Exercises

1. For the following matrices A and B, calculate A + B, A – B, AB and BA:

a. $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 4 \\ 0 & 5 \end{bmatrix}$

c. $A = \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 5 \\ 1 & 6 \end{bmatrix}$

d. $A = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix}$

e. $A = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 5 & 2 \\ 1 & 4 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 & 3 \\ 1 & 7 & 3 \\ 2 & 5 & 6 \end{bmatrix}$

f. $A = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 5 & 6 \\ 2 & 1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix}$

2. Calculate the following matrix multiplications:

a. $\begin{bmatrix} 0 & 5 & 2 \\ 3 & 4 & 0 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 1 & 4 \\ 3 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 5 & 6 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

c. $\begin{bmatrix} 5 & 3 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ 1 \end{bmatrix}$

d. $\begin{bmatrix} 7 & 2 & 6 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 6 & 2 \\ 1 & 4 & 3 \\ 7 & 3 & 1 \end{bmatrix}$

3. Simultaneous equations expressed as matrices

The following are a set of simultaneous equations:

$$ax + by = p$$
$$cx + dy = q$$

In these, x and y are the same variables in each equation, and a solution to the system of simultaneous equations is a pair of values x and y which satisfy both equations.

This relationship can be expressed in matrix form. Let:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} p \\ q \end{bmatrix}$$

Then:

$$M\mathbf{x} = \mathbf{c}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

Using matrix multiplication we know that p and q are given by:

$$\begin{bmatrix} \boxed{a \quad b} \\ c \quad d \end{bmatrix}\begin{bmatrix} \boxed{\begin{matrix} x \\ y \end{matrix}} \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

A set of 3 simultaneous equations can be expressed similarly:

$$ax + by + cz = p$$
$$dx + ey + fz = q$$
$$gx + hy + iz = r$$

This set yields the following matrix equation:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Worked example

The following are a set of simultaneous equations.

$$4x + 3y = 45$$
$$x + 2y = 23$$

These can be expressed as matrices. Let:

$$M = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$$
$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\mathbf{c} = \begin{bmatrix} 45 \\ 23 \end{bmatrix}$$

Then:

$$M\mathbf{x} = \mathbf{c}$$
$$\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 45 \\ 23 \end{bmatrix}$$

The solution of this set of simultaneous equations is now a problem of finding the column vector $\mathbf{x}$ which satisfies the matrix equation. To do this, we must learn about matrix inversion.

Exercises

1. Express the following sets of simultaneous equations as matrices (*but do not solve them*):

a. $$4x + 3y = 29$$ $$9x - y = 42$$

b. $$\frac{1}{2}x + 8y = 42$$ $$3x + 2y = 22$$

c. $$2x + y + z = 6$$ $$y - 3z = -3$$ $$4x + 2y - 2z = 4$$

d. $$3x + y = 11$$ $$2y - z = 2$$ $$8x + 2z = 32$$

4. Determinants and matrix inversion

Transpose of a matrix

Given a matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The transpose of A, written A^T, is formed by writing the rows as columns, which can be thought of as reflecting the diagonal from top left to bottom right. Thus, $(m_{ij})^T = m_{ji}$, and A^T is given by:

$$A^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$$

For a 3x3 matrix the process is equivalent:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Worked examples

1. Find the transpose of A:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

Reflect along the top left to bottom right diagonal:

$$A^T = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}^T = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$$

2. Find the transpose of B:

$$B = \begin{bmatrix} 3 & 6 & 4 \\ 2 & 7 & 2 \\ 1 & 0 & 5 \end{bmatrix}$$

Again, reflecting along the main diagonal:

$$B^T = \begin{bmatrix} 3 & 6 & 4 \\ 2 & 7 & 2 \\ 1 & 0 & 5 \end{bmatrix}^T = \begin{bmatrix} 3 & 2 & 1 \\ 6 & 7 & 0 \\ 4 & 2 & 5 \end{bmatrix}$$

Determinants of 2x2 matrices

Given a matrix A:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant of A, written $\det A$ or using straight bars instead of square brackets, is given by:

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Worked example

Find the determinant of the matrix A:

$$A = \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} = 3 \times 5 - 4 \times 2 = 7$$

The determinant of A is 7.

Minors and cofactors

Each element of a matrix has associated with it a number, called its *minor*. The minor of m_{ij} is the determinant of the matrix that remains if row i and column j are removed from $\det M$, and is written M_{ij}.

If:

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

For m_{11} we reduce $\det M$ to M_{11}:

$$\begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} = \begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix}$$

To find M_{23} we reduce $\det M$ to:

$$\begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} = \begin{vmatrix} m_{11} & m_{12} \\ m_{31} & m_{32} \end{vmatrix}$$

To calculate the cofactors of M we must create for M a sign pattern as follows:

$$M = \begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

So the sign associated with m_{ij} is $(-1)^{i+j}$ (effectively, if $i+j$ is even then the sign is positive, if $i+j$ is odd then the sign is negative).

The *cofactor* of m_{ij} is given by $(-1)^{i+j} M_{ij}$. Thus, the matrix of cofactors of M is:

$$\begin{bmatrix} (-1)^2 M_{11} & (-1)^3 M_{12} & (-1)^4 M_{13} \\ (-1)^3 M_{21} & (-1)^4 M_{22} & (-1)^5 M_{23} \\ (-1)^4 M_{31} & (-1)^5 M_{32} & (-1)^6 M_{33} \end{bmatrix} = \begin{bmatrix} +\begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix} & -\begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix} & +\begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix} \\ -\begin{vmatrix} m_{12} & m_{13} \\ m_{32} & m_{33} \end{vmatrix} & +\begin{vmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{vmatrix} & -\begin{vmatrix} m_{11} & m_{12} \\ m_{31} & m_{32} \end{vmatrix} \\ +\begin{vmatrix} m_{12} & m_{13} \\ m_{22} & m_{23} \end{vmatrix} & -\begin{vmatrix} m_{11} & m_{13} \\ m_{21} & m_{23} \end{vmatrix} & +\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix} \end{bmatrix}$$

Worked example

Find the matrix of cofactors of A:

$$A = \begin{bmatrix} 3 & 6 & 4 \\ 2 & 7 & 2 \\ 1 & 0 & 5 \end{bmatrix}$$

First find the minors. For the top left element, 3, we reduce $\det A$ to:

$$\begin{vmatrix} 3 & 6 & 4 \\ 2 & 7 & 2 \\ 1 & 0 & 5 \end{vmatrix} = \begin{vmatrix} 7 & 2 \\ 0 & 5 \end{vmatrix} = 7 \times 5 - 2 \times 0 = 35$$

To obtain the matrix of cofactors, repeat this process for each element, and apply the sign pattern:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

So the matrix of cofactors of A is:

$$\begin{bmatrix} +\begin{vmatrix} 7 & 2 \\ 0 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 2 & 7 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 6 & 4 \\ 0 & 5 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 3 & 6 \\ 1 & 0 \end{vmatrix} \\ +\begin{vmatrix} 6 & 4 \\ 7 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} & +\begin{vmatrix} 3 & 6 \\ 2 & 7 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (7\times 5-2\times 0) & -(2\times 5-2\times 1) & (2\times 0-7\times 1) \\ -(6\times 5-4\times 0) & (3\times 5-4\times 1) & -(3\times 0-6\times 1) \\ (6\times 2-4\times 7) & -(3\times 2-4\times 2) & (3\times 7-6\times 2) \end{bmatrix}$$

$$= \begin{bmatrix} 35 & -8 & -7 \\ -30 & 11 & 6 \\ -16 & 2 & 9 \end{bmatrix}$$

The adjoint matrix

The adjoint matrix of M is the transposed matrix of cofactors of M, written $\text{Adj}\,M$.

If:

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

Then $\text{Adj}\,M$ is given by:

$$\text{Adj}\,M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}^T = \begin{bmatrix} M_{11} & M_{21} & M_{31} \\ M_{12} & M_{22} & M_{32} \\ M_{13} & M_{23} & M_{33} \end{bmatrix}$$

Worked example

For A:

$$A = \begin{bmatrix} 3 & 6 & 4 \\ 2 & 7 & 2 \\ 1 & 0 & 5 \end{bmatrix}$$

Find $\text{Adj}\,A$.

From the previous example we know that the matrix of cofactors of A is:

$$\begin{bmatrix} 35 & -8 & -7 \\ -30 & 11 & 6 \\ -16 & 2 & 9 \end{bmatrix}$$

Thus:

$$\text{Adj}\,A = \begin{bmatrix} 35 & -8 & -7 \\ -30 & 11 & 6 \\ -16 & 2 & 9 \end{bmatrix}^T = \begin{bmatrix} 35 & -30 & -16 \\ -8 & 11 & 2 \\ -7 & 6 & 9 \end{bmatrix}$$

Determinants of 3x3 matrices

3x3 determinants can be evaluated in terms of 2x2 determinants. If M is a 3x3 matrix:

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

Then the determinant is

$$\det M = \begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix}$$

Using the matrix of cofactors the following formula is used to calculate the determinant:

$$\det M = +m_{11}\begin{vmatrix} m_{22} & m_{23} \\ m_{32} & m_{33} \end{vmatrix} - m_{12}\begin{vmatrix} m_{21} & m_{23} \\ m_{31} & m_{33} \end{vmatrix} + m_{13}\begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix}$$

We have moved along row 1 of M and multiplied m_{1j} by the corresponding cofactor, $(-1)^{1+j} M_{1j}$.

It is possible to calculate the determinant by expanding along any row or column, not just the first row. The process is the same, remembering to pay attention to the signs. For example, expanding along the second row:

$$\det M = -m_{21}\begin{vmatrix} m_{12} & m_{13} \\ m_{32} & m_{33} \end{vmatrix} + m_{22}\begin{vmatrix} m_{11} & m_{13} \\ m_{31} & m_{33} \end{vmatrix} - m_{23}\begin{vmatrix} m_{11} & m_{12} \\ m_{31} & m_{32} \end{vmatrix}$$

Equally, expanding down the third *column*:

$$\det M = +m_{13}\begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix} - m_{23}\begin{vmatrix} m_{11} & m_{12} \\ m_{31} & m_{32} \end{vmatrix} + m_{33}\begin{vmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{vmatrix}$$

Expanding along different rows or columns can make the calculation easier. For instance, if one or more of the matrix elements are zero (e.g. $0\begin{vmatrix} m_{21} & m_{22} \\ m_{31} & m_{32} \end{vmatrix} = 0$).

Worked example

Find the determinant of A:

$$A = \begin{bmatrix} 7 & 1 & 4 \\ 2 & 6 & 5 \\ 3 & 0 & 8 \end{bmatrix}$$

There is a zero in the third row on the second column. The calculation will be simplified if we expand along the third row or the second column. Choose the third row.

Start with the first element on the third row, 3. Removing the corresponding row and column leaves:

$$\begin{vmatrix} \cancel{7} & 1 & 4 \\ \cancel{2} & 6 & 5 \\ \cancel{3} & \cancel{0} & \cancel{8} \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 6 & 5 \end{vmatrix}$$

Now move to the second element on the third row. Again, deleting the corresponding rows and columns leaves:

$$\begin{vmatrix} 7 & 1 & 4 \\ 2 & 6 & 5 \\ 3 & 0 & 8 \end{vmatrix} = \begin{vmatrix} 7 & 4 \\ 2 & 5 \end{vmatrix}$$

Finally, do the same with the third element on the third row:

$$\begin{vmatrix} 7 & 1 & 4 \\ 2 & 6 & 5 \\ 3 & 0 & 8 \end{vmatrix} = \begin{vmatrix} 7 & 1 \\ 2 & 6 \end{vmatrix}$$

Then:

$$\begin{aligned} \det A &= \begin{vmatrix} 7 & 1 & 4 \\ 2 & 6 & 5 \\ 3 & 0 & 8 \end{vmatrix} \\ &= 3\begin{vmatrix} 1 & 4 \\ 6 & 5 \end{vmatrix} - 0\begin{vmatrix} 7 & 4 \\ 2 & 5 \end{vmatrix} + 8\begin{vmatrix} 7 & 1 \\ 2 & 6 \end{vmatrix} \\ &= 3(1 \times 5 - 4 \times 6) - 0 + 8(7 \times 6 - 1 \times 2) \\ &= 263 \end{aligned}$$

The determinant of A is 263.

The identity matrix

The identity matrix, *I* has the property that *IM*=*MI*=*M* for any matrix *M*. The 2x2 identity matrix is:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The 3x3 identity matrix is:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The identity matrix I is similar to the number 1 in numeric algebra, where $1x = x1 = x$.

Worked example

$$\begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1\times1+5\times0 & 1\times0+5\times1 \\ 4\times1+2\times0 & 4\times0+2\times1 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 4 & 2 \end{bmatrix}$$

Inversion

For a matrix M, provided $\det M \neq 0$, M has an inverse, M^{-1}, with the property:

$$MM^{-1} = M^{-1}M = I$$

Inverse of 2x2 matrices

For a matrix M with:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The inverse of M can be found by calculating the determinant and, if this is non-zero, applying the following formula:

$$M^{-1} = \frac{1}{\det M}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The following proof of this might be a helpful way to remember this arrangement. Take:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & -ab+ba \\ cd-dc & -bc+da \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

Remember that $ad - bc$ is the formula for calculating the determinant, so divide by the $\det M$ to get the identity matrix I.

$$\frac{1}{\det M}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{\det M}\begin{bmatrix} \det M & 0 \\ 0 & \det M \end{bmatrix} = I$$

Worked example

Find the inverse of the matrix A:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

First calculate the determinant:

$$\det A = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 3 \times 4 - 2 \times 1 = 10$$

So an inverse exists, and is given by:

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

Check this:

$$AA^{-1} = \frac{1}{10}\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So AA^{-1} gives the identity matrix, as expected.

Inverse of 3x3 matrices

The inverse of a 3x3 matrix M can be found by calculating the determinant and, if this is non-zero, applying the following formula:

$$M^{-1} = \frac{1}{\det M}\,\mathrm{Adj}\,M$$

Worked example

Find the inverse of A:

$$A = \begin{bmatrix} 3 & 6 & 4 \\ 2 & 7 & 2 \\ 1 & 0 & 5 \end{bmatrix}$$

From a previous example, we know the matrix of cofactors of A is

$$\begin{bmatrix} 35 & -8 & -7 \\ -30 & 11 & 6 \\ -16 & 2 & 9 \end{bmatrix}$$

Therefore we can apply the formula to find $\det A$ (using the third row, since this has a zero):

$$\det A = 1 \times (-16) + 0 \times 2 + 5 \times 9 = 29$$

Since the determinant is not zero, we know A has an inverse.

From a previous example (or transposing the matrix of cofactors above), we know $\operatorname{Adj} A$:

$$\operatorname{Adj} A = \begin{bmatrix} 35 & -30 & -16 \\ -8 & 11 & 2 \\ -7 & 6 & 9 \end{bmatrix}$$

Thus the inverse is given by:

$$A^{-1} = \frac{1}{\det A} \operatorname{Adj} A = \frac{1}{29} \begin{bmatrix} 35 & -30 & -16 \\ -8 & 11 & 2 \\ -7 & 6 & 9 \end{bmatrix}$$

To be sure, check that:

$$\frac{1}{29} \begin{bmatrix} 3 & 6 & 4 \\ 2 & 7 & 2 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 35 & -30 & -16 \\ -8 & 11 & 2 \\ -7 & 6 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Exercises

1. For the following matrices A and B, calculate the following:

$$A^T \quad \det A \quad A^{-1}$$
$$B^T \quad \det B \quad B^{-1}$$

a. $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 4 \\ 0 & 5 \end{bmatrix}$

c. $A = \begin{bmatrix} 2 & 5 \\ 1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 5 \\ 1 & 6 \end{bmatrix}$

d. $A = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix}$

e. $A = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 5 & 2 \\ 1 & 4 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 4 & 3 \\ 1 & 7 & 3 \\ 2 & 5 & 6 \end{bmatrix}$

f. $A = \begin{bmatrix} 3 & 1 & 0 \\ 2 & 5 & 6 \\ 2 & 1 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix}$

5. Solution of simultaneous equations using matrices

The following is a set of simultaneous equations:

$$ax + by = p$$
$$cx + dy = q$$

In these, x and y are the same variables in each equation, and a solution to the system of simultaneous equations is a pair of values x and y which satisfy both equations.

This relationship can be expressed in matrix form. Let:

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$
$$\mathbf{c} = \begin{bmatrix} p \\ q \end{bmatrix}$$

Then:

$$M\mathbf{x} = \mathbf{c}$$
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}$$

If M has an inverse, then multiplying both sides by M^{-1} gives:

$$M^{-1}M\mathbf{x} = M^{-1}\mathbf{c}$$
$$\Rightarrow I\mathbf{x} = M^{-1}\mathbf{c}$$
$$\Rightarrow \mathbf{x} = M^{-1}\mathbf{c}$$

So the solution for $\mathbf{x}$ can be found using matrices.

Worked examples

1. Solve the following simultaneous equations by matrix methods:

$$3x + 6y = 21$$
$$2x + 5y = 16$$

This system can be written in matrix form as:

$$\begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 21 \\ 16 \end{bmatrix}$$

Thus:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix}^{-1}\begin{bmatrix} 21 \\ 16 \end{bmatrix}$$

First, find the determinant of the matrix:

$$\begin{vmatrix} 3 & 6 \\ 2 & 5 \end{vmatrix} = 3 \times 5 - 6 \times 2 = 3$$

Then invert the matrix:

$$\begin{bmatrix} 3 & 6 \\ 2 & 5 \end{bmatrix}^{-1} = \frac{1}{3}\begin{bmatrix} 5 & -6 \\ -2 & 3 \end{bmatrix}$$

Use this to find the solution:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5 & -6 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 21 \\ 16 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

So the solution is $x = 3, y = 2$. Check this to confirm:

$$3x + 6y = 3 \times 3 + 6 \times 2 = 21$$
$$2x + 5y = 2 \times 3 + 5 \times 2 = 16$$

2. Solve the following simultaneous equations by matrix methods:

$$6x + 2y + 3z = 32$$
$$4x + 4y - z = 26$$
$$2x - 3y = -6$$

This system can be written in matrix form as:

$$\begin{bmatrix} 6 & 2 & 3 \\ 4 & 4 & -1 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 32 \\ 26 \\ -6 \end{bmatrix}$$

Thus:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 & 2 & 3 \\ 4 & 4 & -1 \\ 2 & -3 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 32 \\ 26 \\ -6 \end{bmatrix}$$

First, find the determinant of the matrix:

$$\begin{vmatrix} 6 & 2 & 3 \\ 4 & 4 & -1 \\ 2 & -3 & 0 \end{vmatrix} = 2\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} - (-3)\begin{vmatrix} 6 & 3 \\ 4 & -1 \end{vmatrix} + 0\begin{vmatrix} 6 & 2 \\ 4 & 4 \end{vmatrix}$$
$$= 2(2 \times (-1) - 3 \times 4) + 3(6 \times (-1) - 3 \times 4) + 0$$
$$= -28 - 54 = -82$$

Calculate the matrix of cofactors:

$$\begin{bmatrix} +\begin{vmatrix} 4 & -1 \\ -3 & 0 \end{vmatrix} & -\begin{vmatrix} 4 & -1 \\ 2 & 0 \end{vmatrix} & +\begin{vmatrix} 4 & 4 \\ 2 & -3 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ -3 & 0 \end{vmatrix} & +\begin{vmatrix} 6 & 3 \\ 2 & 0 \end{vmatrix} & -\begin{vmatrix} 6 & 2 \\ 2 & -3 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} & -\begin{vmatrix} 6 & 3 \\ 4 & -1 \end{vmatrix} & +\begin{vmatrix} 6 & 2 \\ 4 & 4 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -3 & -2 & -20 \\ -9 & -6 & 22 \\ -14 & 18 & 16 \end{bmatrix}$$

Then invert the matrix:

$$\begin{bmatrix} 6 & 2 & 3 \\ 4 & 4 & -1 \\ 2 & -3 & 0 \end{bmatrix}^{-1} = \frac{1}{-82}\begin{bmatrix} -3 & -9 & -14 \\ -2 & -6 & 18 \\ -20 & 22 & 16 \end{bmatrix} = \frac{1}{82}\begin{bmatrix} 3 & 9 & 14 \\ 2 & 6 & -18 \\ 20 & -22 & -16 \end{bmatrix}$$

Then:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{82}\begin{bmatrix} 3 & 9 & 14 \\ 2 & 6 & -18 \\ 20 & -22 & -16 \end{bmatrix}\begin{bmatrix} 32 \\ 26 \\ -6 \end{bmatrix} = \frac{1}{82}\begin{bmatrix} 246 \\ 328 \\ 164 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$$

So $x = 3, y = 4, z = 2$.

Exercises

1. Solve the following simultaneous equations by matrix methods:

a. $$\begin{aligned} 4x + 3y &= 29 \\ 9x - y &= 42 \end{aligned}$$

b. $$\begin{aligned} \frac{1}{2}x + 8y &= 42 \\ 3x + 2y &= 22 \end{aligned}$$

c. $$\begin{aligned} 2x + y + z &= 6 \\ y - 3z &= -3 \\ 4x + 2y - 2z &= 4 \end{aligned}$$

d. $$\begin{aligned} 3x + y &= 11 \\ 2y - z &= 2 \\ 8x + 2z &= 32 \end{aligned}$$

Statistics

All measurements are subject to error and when we combine them together to find the position of our target we want to make sure that our solution is the best one. Therefore, we need to characterise that error and make sure we interpret it accordingly. This is the basis of Statistics. You will need to be familiar with very basic probability and simple probability distributions, how to calculate the mean, median, mode and standard deviation of a set of measurements and what exactly they tell us about the data.

1. Basic probability

If a coin is thrown, there are two possible outcomes – heads (H) or tails (T).

For an event with a chance outcome we define the probability, P, so that for a coin the probability the outcomes being heads is $P(H)$ and the probability of tails is $P(T)$.

Probability is measured on a scale from 0 to 1, so that for some outcome A to an event, $P(A)=0$ indicates that A is impossible whereas $P(A)=1$ indicates A is certain.

For a fair coin the chances of either outcome are equal. Then, $P(H)=0{\cdot}5$ and $P(T)=0{\cdot}5$.

If we toss a fair coin a large number of times, we would expect that roughly half the time heads would be the outcome. Intuitively then, $P(A)$ is the proportion of the time we would expect A to happen if the event were repeated many times.

For an event with several possible mutually exclusive outcomes, $E_1, E_2, E_3, ..., E_n$, the probabilities of all the possible outcomes add to 1 (that is, it is certain that one of all possible outcomes will happen). Then:

$$P(E_1)+P(E_2)+P(E_3)+...+P(E_n)=1$$

An experiment or measurement is operated or taken in a manner which is consistent and repeatable. Then if that experiment or measurement has some chance outcome (error) then probability applies. If we take a measurement a number of times, we can expect different outcomes to apply with difference frequencies, and this can be used to suggest the likelihood that a particular outcome is correct.

2. Mean, median, mode, variance and standard deviation

From a set of data we can calculate several useful values which help describe the set of data.

Mean

Let the data be the values:

$$x_1, x_2, x_3, \ldots x_n$$

The mean, denoted $\bar{x}$, is the value commonly thought of as the average and is given by:

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Worked example

Calculate the mean of the following set of data:

$$2, 3, 6, 4, 9, 6, 5, 8, 2, 7, 5, 9, 2, 10, 3, 12$$

There are 16 values, so $n = 16$. Sum the values and divide by 16:

$$\bar{x} = \frac{2+3+6+4+9+6+5+8+2+7+5+9+2+10+3+12}{16} = \frac{93}{16} = 5{\cdot}8125$$

So the mean is 5·8125.

Median

The median value is the middle score. To find it, put the values in numerical order then choose the value in the middle. If there are two values in the middle take the average of them.

Worked example

Find the median of the following set of data:

$$2, 3, 6, 4, 9, 6, 5, 8, 2, 7, 5, 9, 2, 10, 3, 12$$

First, sort the data into numerical order:

$$2, 2, 2, 3, 3, 4, 5, 5, 6, 6, 7, 8, 9, 9, 10, 12$$

Now choose the middle value. Since there are an even number of values, there are two middle values, 5 and 6, so the median is:

$$\text{median} = \frac{5 \times 6}{2} = 5\cdot 5$$

Mode

The mode of a set of data is the value with the highest frequency. To find it, make a tally of how often each number appears and the mode is the value that appears most often. If more than one value is equally common then all the joint most common values are said to be the mode.

Worked example

Find the mode of the following set of data:

$$2, 3, 6, 4, 9, 6, 5, 8, 2, 7, 5, 9, 2, 10, 3, 12$$

Draw a frequency table:

number	freq
2	3
3	2
4	1
5	2
6	1
7	1
8	1
9	1
10	1
12	1

Then the mode is the most common value. For this set of data, this is 2.

Variance

The variance is the average of the squared differences of the data from the mean. It is a measure of how spread out around the mean the data are. The variance, denoted σ^2, is given by:

$$\sigma^2 = \frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n-1}$$

Worked example

Find the variance of the following set of data:

$$2, 3, 6, 4, 9, 6, 5, 8, 2, 7, 5, 9, 2, 10, 3, 12$$

From the example above we know the mean of this data set is $\bar{x} = 5{\cdot}8125$. Then the variance is:

$$\sigma^2 = \frac{\begin{pmatrix} (2-5{\cdot}8125)^2 + (3-5{\cdot}8125)^2 + (6-5{\cdot}8125)^2 + (4-5{\cdot}8125)^2 + (9-5{\cdot}8125)^2 \\ +(6-5{\cdot}8125)^2 + (5-5{\cdot}8125)^2 + (8-5{\cdot}8125)^2 + (2-5{\cdot}8125)^2 + (7-5{\cdot}8125)^2 \\ +(5-5{\cdot}8125)^2 + (9-5{\cdot}8125)^2 + (2-5{\cdot}8125)^2 + (10-5{\cdot}8125)^2 + (3-5{\cdot}8125)^2 \\ +(12-5{\cdot}8125)^2 \end{pmatrix}}{15}$$

$$= \frac{\begin{pmatrix} 14{\cdot}5352 + 7{\cdot}9102 + 0{\cdot}0352 + 3{\cdot}2852 + 10{\cdot}1602 + 0{\cdot}0352 + 0{\cdot}6602 \\ +4{\cdot}7852 + 14{\cdot}5352 + 1{\cdot}4102 + 0{\cdot}6602 + 10{\cdot}1602 + 14{\cdot}5352 \\ +17{\cdot}5352 + 7{\cdot}9102 + 38{\cdot}2852 \end{pmatrix}}{15}$$

$$= \frac{146{\cdot}4375}{15} = 9{\cdot}7625$$

So the variance is 9·7625.

Standard deviation

The standard deviation the most commonly used measure of how spread out around the mean the data are. The standard deviation, denoted σ, is the square root of the variance. This means it gives the same measure as the variance in units which relate to the original data. The standard deviation is given by:

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n-1}}$$

Worked example

Find the standard deviation of the following set of data:

$$2, 3, 6, 4, 9, 6, 5, 8, 2, 7, 5, 9, 2, 10, 3, 12$$

From the example above we know the mean of this data set is $\bar{x} = 5{\cdot}8125$ and the variance is $\sigma^2 = 9{\cdot}7625$. Then the standard deviation is:

$$\sigma = \sqrt{\sigma^2} = \sqrt{9{\cdot}7625} = 3{\cdot}1245$$

So the standard deviation is 3·1245.

Exercises

1. Calculate the mean, median, mode, variance and standard deviation of the following sets of data:

    a. $7, 4, 7, 9, 2, 6, 2, 5, 6, 10, 8, 6, 10, 9$

    b. $2, 10, 2, 9, 5, 10, 6, 4, 5, 4, 3, 3, 9, 10, 10, 7$

    c. $10, 5, 8, 5, 10, 10, 4, 1, 1, 3, 9, 10, 10, 4, 3, 10, 5, 3$

    d. $1, 3, 5, 7, 5, 1, 5, 8, 9, 3, 5, 5, 7$

3. Basic distributions – normal, binomial

Normal distribution

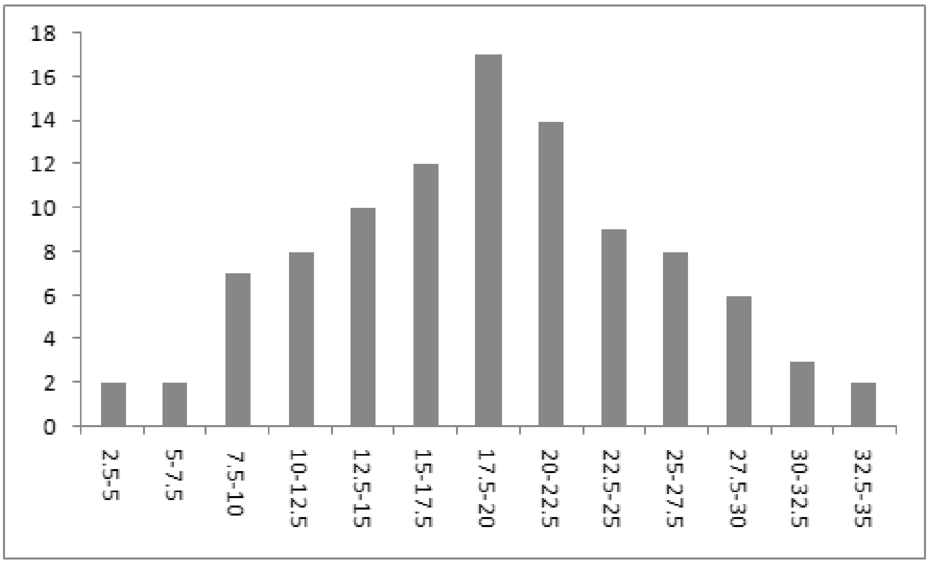
Many sets of data, particularly those involving measurements, can be approximated by the normal distribution. Take the following set of data:

0·5, 0·3, 5·9, 6·7, 9·7, 8, 7·9, 8·2, 8·9, 9·4, 8·5, 11·4, 10·5, 11·9, 10·9, 10·5, 11·4, 10·9, 11, 13·5, 13, 14·8, 13·6, 14·4, 13·5, 14·7, 14·9, 14·3, 14·9, 15·3, 17·1, 15·4, 16·5, 17·2, 16·7, 16·1, 15·5, 16·7, 16·3, 15·2, 15·3, 19·3, 18, 18·7, 17·9, 18·6, 18·3, 19·9, 17·6, 17·8, 19·2, 19·9, 19·7, 19·3, 17·8, 18·6, 18·1, 18·9, 22·5, 20·2, 20·4, 21·1, 22, 21·3, 20·7, 22, 22·3, 20·5, 20·8, 22·2, 21·9, 24·1, 24·6, 24·7, 23·6, 23·2, 22·5, 22·6, 23·9, 23, 23, 26·9, 25·4, 26·4, 26·2, 27·4, 26·4, 27, 27·4, 28·3, 28·7, 28·3, 28·2, 28·2, 28·5, 32·1, 31·5, 31·7, 34, 33·6

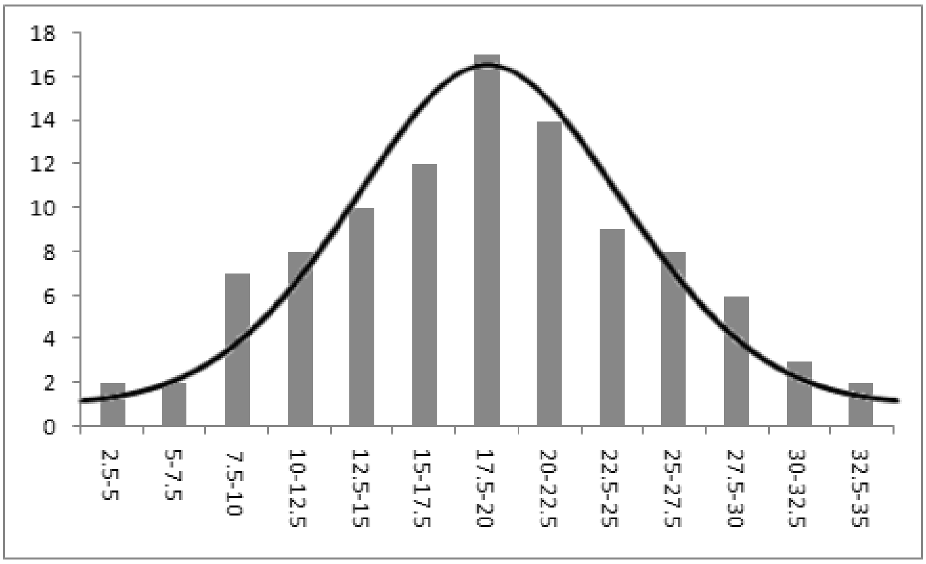
First, group these values to draw a frequency table:

range	frequency
2·5-5	2
5-7·5	2
7·5-10	7
10-12·5	8
12·5-15	10
15-17·5	12
17·5-20	17
20-22·5	14
22·5-25	9
25-27·5	8
27·5-30	6
30-32·5	3
32·5-35	2

Now we plot these values on a bar chart:



We can see from looking at this chart that the data tends to cluster around the 17·5-20 range and that as we get further from this point the frequency decreases. We can plot a curve of best fit onto this chart:

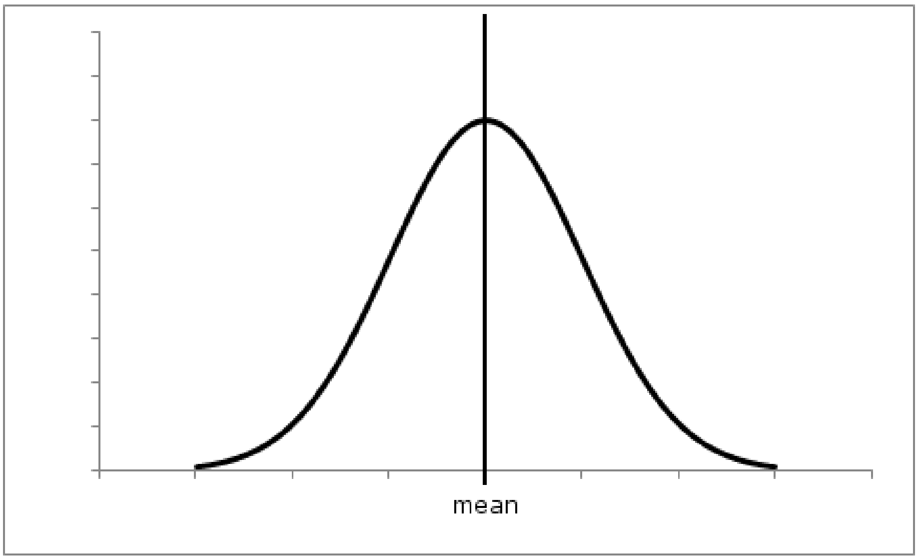


This "bell shaped" curve is called the normal curve.

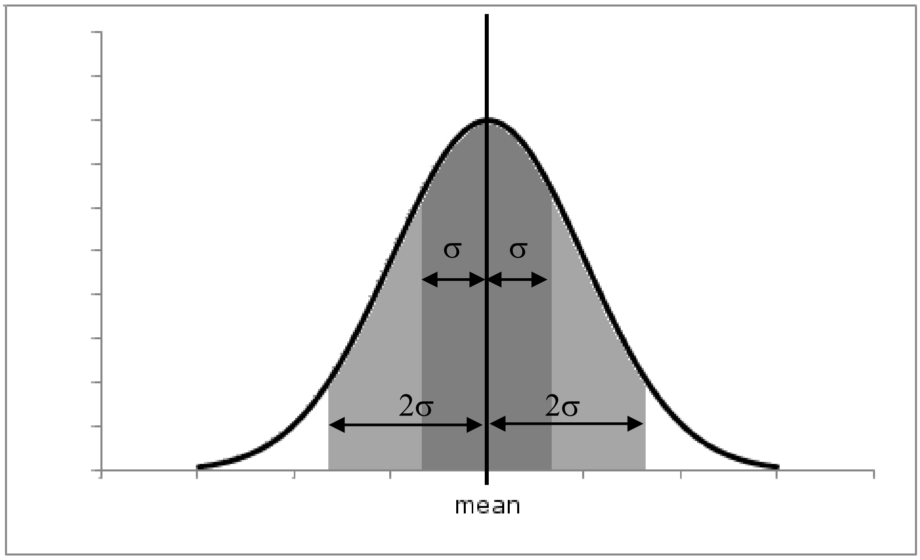
For the data set given, the mean is 18·743. We see from the chart that the normal curve is approximately centred on the mean value.

Properties of the normal distribution

Many data sets from real world data follow the normal distribution. As more data are included in a sample the fit to the normal curve becomes smoother. The diagram below shows a normal distribution, symmetrical about the mean.



For the standard normal distribution, about 68% of the values lie within one standard deviation of the mean (the area shaded dark grey in the diagram below) and about 95% of the values lie within 2 standard deviations of the mean (the area shaded lighter grey in the diagram below). Further, 99·7% of data lie within 3 standard deviations of the mean.



This means that if we know a set of data follows the normal distribution and we know the mean and standard deviation, we can make some statements concerning the likelihood that any data point is within one or two standard deviations of the mean.

Binomial distribution

If a random event has two outcomes that are independent and mutually exclusive then the binomial distribution can be used to determine the probability of getting a certain number of outcomes.

If such an event occurs n times and the probability of one outcome ("success") is p, then the binomial distribution gives the probability for obtaining r successes:

$$P(r) = \binom{n}{r} p^r (1-p)^{n-r}$$

Where $\binom{n}{r}$ is a binomial coefficient which gives the number of ways of picking r unordered outcomes from n possibilities and is given by:

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Worked example

1. A fair coin is tossed 10 times. What is the probability of throwing exactly 2 heads?

The probability of getting a head each time the coin is thrown is $\mathrm{p} = 0{\cdot}5$. The probability of getting exactly 2 heads is:

$$P(2\text{ heads}) = \binom{10}{2} \times 0{\cdot}5^2 \times (1 - 0{\cdot}5)^8$$
$$= \frac{10!}{2! \times 8!} \times 0{\cdot}5^2 \times (1 - 0{\cdot}5)^8$$
$$= 45 \times 0{\cdot}25 \times 0{\cdot}00391$$
$$= 0{\cdot}0439$$

2. A fair die is rolled 5 times. What is the probability of rolling exactly 3 sixes?

The probability of rolling a six is $\mathrm{p} = \frac{1}{6} \approx 0{\cdot}167$. The probability of rolling exactly 3 sixes from 5 rolls is:

$$P(3\text{ sixes}) = \binom{5}{3} \times \left(\frac{1}{6}\right)^3 \times \left(1 - \frac{1}{6}\right)^2$$
$$= \frac{5!}{3! \times 2!} \times \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right)^2$$
$$= 10 \times 0{\cdot}00463 \times 0{\cdot}694$$
$$= 0{\cdot}0322$$

Exercises

1. A fair coin is tossed 12 times. What is the probability of throwing exactly 5 heads?

2. A fair die is rolled 9 times. What is the probability of rolling exactly 2 sixes?

4. Precision, accuracy

A measurement is taken for some length. It is important to remember the limitations of the process of measuring that length. The measurement that has been taken should be considered only an approximation to the correct value. An awareness of error can allow us to ensure that measurements contain sufficiently minimal error that they can be relied on. (This will depend on the context – a 0.5mm error in a distance measured in km is not likely to be significant; a 0.5mm error in a measurement of mm is likely to be significant).

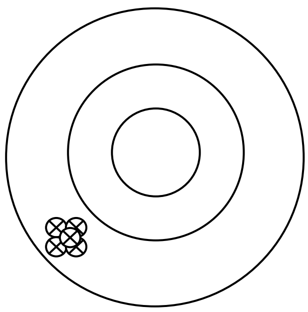
The *accuracy* of a measurement is a measure of how close the measurement is to the true value. Accuracy can be improved by the elimination of systematic errors.

The *precision* is a measure of how consistent a result is when a measurement is repeated. If a measurement is taken several times, the precision is indicated by the standard deviation or variance of the measurements. A precise measurement can be reproduced at will.

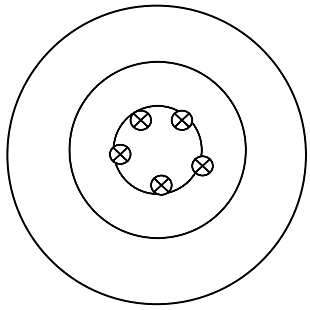
Note that a precise measurement is not necessarily an accurate one and an accurate measurement may be obtained from a set of measurements which are not precise. So a measurement may be: accurate and precise; precise but not accurate; accurate but not precise; or, neither accurate or precise.

In the diagrams below an analogy is used whereby arrows are fired at the centre of the target and the dots (⊗) indicate the positions of the arrows.

In the first diagram, the results are precise but not accurate, in that there is a high degree of repeatability but the results are not close to the centre:



In the next diagram the results are accurate but not precise, in that all the arrows are near the centre but they are not very close to each other:



If a measurement is both precise and accurate it can be called *valid*.

Calibration

A measuring device can be calibrated by using it to take some known measurement and judging precision and accuracy.

The valid measurement is called the reference value. A set of measurements are taken with the device to be calibrated. The difference between the mean of those measurements and the reference value is called the bias.

Each measurement will include an element of random error. Such errors are believed to follow the normal distribution. We have seen above that for a normal distribution, 68% of measurements lie within one standard deviation of the mean, 95% lie with 2 standard deviations and 99·7% lie within 3 standard deviations. So then, greater precision is indicated by a small value for the standard deviation, since almost all measurements lie within 3 standard deviations of the mean of the results.

Worked example

Consider the following two sets of measurements.

Set 1	**Set 2**
105·1mm	102·5mm
105·8mm	102·9mm
105·2mm	102·4mm

105·9mm	102·7mm
105·2mm	102·5mm

a. Calculate the mean and standard deviation for the 2 data sets.

b. Which of the two data sets are more precise?

c. Given that the two sets of measurements are taken by two different measuring instruments both measuring a standard length known to be 105mm, which device is more accurate?

d. Measurements are needed to 1 decimal place. Can either instrument be said to be producing valid measurements?

a.

For set 1: The mean of set 1 is 105·44mm. The standard deviation is 0·378mm.

For set 2: The mean of set 2 is 102·60mm. The standard deviation is 0·2mm.

b.

Set 2 is more precise, since it has a smaller standard deviation.

With a standard deviation of 0·2 and mean 102·60 the normal distribution says 99·7% of the results taken in set 2 will be within the range 102·60±0·6mm. For set 1, with standard deviation 0·378 and mean 105·44, the normal distribution says 99·7% of the results will be in the range 105·44±1·13mm.

c.

Calculate the bias from each instrument.

For set 1: The mean of set 1 is 105·44mm. The bias is the difference between the mean and the reference value:

105·44 - 105= 0·44mm

For set 2: The mean of set 2 is 102·60mm. The bias for set 2 is therefore:

102·60 - 105= -2·40mm

Set 1 is the more accurate, since it has a lower bias.

d.

No. The instrument used to produce set 1 is producing accurate but not precise measurements and that for set 2 is producing precise but not accurate measurements.

Exercises

1. Consider the following three sets of measurements.

Set 1	**Set 2**	**Set 3**
100.126	100.026	100.004
100.107	99.937	100.004
100.086	100.014	99.998
100.119	100.014	99.997
100.079	100.046	100.002
100.114	100.059	100.002
100.075	100.069	99.997
100.13	100.078	100.003
100.115	100.074	99.996
100.092	100.051	99.997

a. Calculate the mean and standard deviation for the 3 data sets.

b. Which of the three data sets are most precise?

c. Given that the three sets of measurements are taken by three different measuring instruments measuring a standard length known to be 100mm, which device is most accurate?

d. Measurements are needed to 1 decimal place. Can any instrument be said to be producing valid measurements?

Recommended further reading (optional)

There are many good textbooks that will guide you in the above. Specific book chapters for parts of this text are included below.

For example, a good introduction may be found in: J. Berry and P. Wainwright, "Foundation Mathematics for Engineers", Macmillan Press 1993 (ISBN 0-333-52717-8), specifically Chapters 4 and 5 on Trigonometry, Chapter 7 on Differentiation and Chapter 12 on Matrices.

For Statistics, we recommend that you browse through some of the more easy reading introductions to the subject. An example is: D. Rowntree, "Statistics Without Tears: A Primer for Non-mathematicians", Pelican Books, 1987 (Chapters 1-5). This book is a short, entertaining discussion of the subject and will familiarise you with the basics of the topic such that you are ready for anything that we may throw at you!

The MathTutor online video resource may be useful. This is available at: www.mathtutor.ac.uk. Topics contain video tutorial, summary text and exercises.

Section specific optional further reading

Trigonometry

More on Trigonometry is available in:

- Berry and Wainwright chapters 4 and 5, “Trigonometric Functions” and “Trigonometry”
- HELM workbook 4 “Trigonometry”
- MathTutor “Trigonometry” course.

1. Trigonometric functions and properties

For more on the derivation of sine, cosine and tangent, see Berry and Wainwright, chapters 4.1-4.5.

For more on the properties of sine, cosine and tangent, see Berry and Wainwright, chapter 4.6.

For more on asymptotes of tangent function, see Berry and Wainwright, exercise 4.5.1.

For more on the period of trig functions, see Berry and Wainwright, chapter 4.9.1.

For more on angular frequency, see Berry and Wainwright, chapter 4.9.2.

For more on amplitude, see Berry and Wainwright, chapter 4.9.3.

For more on phase angles, see Berry and Wainwright, chapter 4.9.4.

For more on secant, cosecant and cotangent, see Berry and Wainwright, chapter 4.11.

2. Circular measure and equivalence degree and radian

For more on circular measure (radians), see Berry and Wainwright, chapter 4.12.

3. Trig identities

For a derivation of the trig identities, see Berry and Wainwright, chapter 5.2.1.

4. Sum and difference formulae

For more on sum and difference formulae, see Berry and Wainwright, chapter 5.2.2.

5. Double angle identities

For a derivation of the double angle identities, see Berry and Wainwright, chapter 5.2.3.

6. Inverse trig functions

For more on inverse trig functions, see Berry and Wainwright, chapter 5.3.

7. Sine and cosine rules

For more on the sine and cosine rules, see Berry and Wainwright, chapters 5.5-5.6.

Differentiation

More on Differentiation is available in:

- Berry and Wainwright chapter 7, "Differentiation"
- HELM workbooks 11 and 12, "Differentiation" and "Applications of Differentiation"
- MathTutor "Differentiation" course.

More on Taylor series is available in HELM workbook 16, "Sequences and Series" section 5, "Maclaurin and Taylor series".

1. What is differentiation?

For more on the concept of differentiation, see Berry and Wainwright, chapters 7.1-7.5.

For more on second derivatives, see Berry and Wainwright, chapter 7.7.2.

2. Derivatives

For more on differentiation rules, see Berry and Wainwright, chapter 7.6.

For more on differentiation of functions that are sums or differences of other functions, see Berry and Wainwright, chapter 7.6.1.

For more on the differentiation of trig functions, see Berry and Wainwright, chapter 7.13.

For more on the differentiation of exponential functions, see Berry and Wainwright, chapter 7.14.

3. Chain, product and quotient rules

For more on the chain rule, see Berry and Wainwright, chapter 7.9.

For more on the product and quotient rules, see Berry and Wainwright, chapter 7.11.

4. Max and min

For more on maxima and minima, see Berry and Wainwright, chapters 7.10 and 7.12.

5. Taylor series, expansion, linearization

For more on Taylor series, see HELM workbook 16.5.

6. Partial differentiation

More on partial differentiation is available in HELM workbook 18 "Functions of several variables" section 2 "Partial Derivatives".

Integration

More on Integration is available in:

- Berry and Wainwright chapter 8, "Integration"
- HELM workbooks 13, 14 and 15, "Integration," "Applications of Integration I" and "Applications of Integration II"
- MathTutor "Integration" course.

1. What is differentiation?

For a more detailed description of integration, see Berry and Wainwright, chapter 8.1.

2. Integration a anti-differentiation

For more on indefinite integration and integration as anti-differentiation, see Berry and Wainwright, chaptes 8.4-8.5.

3. Areas under curves

For more on areas under curves and definite integration, see Berry and Wainwright, chapters 8.3-8.6.

Matrices

More on Matrices and Vectors is available in:

- Berry and Wainwright chapter 12, “Matrices” and chapter 11, “Vectors”.
- HELM workbook 7, “Matrices”

1. Terminology

For more on matrix terminology and notation, see Berry and Wainwright, chapters 12.1 and 12.2.

2. Simple matrix algebra – addition, subtraction, multiplication

For more on matrix addition, see Berry and Wainwright, chapter 12.2.1.

For more on matrix subtraction, see Berry and Wainwright, chapter 12.2.2.

For more on the scalar product of a matrix, see Berry and Wainwright, chapter 12.3.1.

For more on matrix multiplication, see Berry and Wainwright, chapter 12.3.2.

3. Simultaneous equations expressed as matrices

For more on expressing simultaneous equations in matrix form, see Berry and Wainwright, chapters 12.1, 12.4

4. Determinants and matrix inversion

For more on the transpose of a matrix, see Berry and Wainwright, chapter 12.6.3.

For more on 2x2 determinants, see Berry and Wainwright, chapter 12.5.1.

For more on minors and cofactors, see Berry and Wainwright, chapter 12.6.4.

For more on the adjoint matrix, see Berry and Wainwright, chapter 12.6.5.

For more on 3x3 determinants, see Berry and Wainwright, chapter 12.5.2-12.5.5

For more on the identity matrix, see Berry and Wainwright, chapter 12.3.3.

For more on inversion of a 2x2 matrix, see Berry and Wainwright, chapter 12.6.1.

For more on inversion of a 3x3 matrix, see Berry and Wainwright, chapter 12.6.6.

5. Solution of simultaneous equations using matrices

For more on solving simultaneous equations using matrix methods, see Berry and Wainwright, chapters 12.6.7, 12.8.

Statistics

More on Statistics is available in Rowntree.

2. Mean, median, mode, variance and standard deviation

For more on mean, median, mode, variance and standard deviation, see Rowntree, chapter 3.

3. Basic distributions – normal, binomial

For more on the normal distribution, see Rowntree, chapter 4.

4. Precision, accuracy

For more on accuracy and error, see Rowntree, pages 35-37.

Solutions to exercises

Trigonometry

1. Trigonometric functions and properties

1.

a. $\sin 90^\circ = 1$
b. $\cos 90^\circ = 0$
c. $\tan 15^\circ = 0{\cdot}2679$
d. $\sin 20^\circ = 0{\cdot}3420$
e. $\cos 35^\circ = 0{\cdot}8192$
f. $\sin 10^\circ = 0{\cdot}1736$
g. $\tan 60^\circ = 1{\cdot}7321$
h. $\sin 82^\circ = 0{\cdot}9903$
i. $\cos 45^\circ = 0{\cdot}7071$

2. The period of the graph is 90° and the amplitude is 5. The angular frequency is $\omega=4$ and the graph is given by $y=5\sin(4\theta)$

3. The height of the mast is 3·8m.

4. The distance to the tower is 18·1m.

5. The point is 3·6m from the base of the tower.

2. Circular measure and equivalence degree and radian

1.

e. $\pi/4=0{\cdot}785$
f. $\pi/6=0{\cdot}524$
g. $19\pi/18=3{\cdot}316$
h. $24\pi/18=4{\cdot}189$
i. $\pi/12=0{\cdot}262$
j. $85\pi/180=1{\cdot}484$
k. $7\pi/4=5{\cdot}498$
l. $\pi/2=1{\cdot}571$
m. $\pi/9=0{\cdot}349$

2.

a. 60°
b. $114{\cdot}592^\circ$
c. 30°
d. 270°
e. 90°
f. $286{\cdot}479^\circ$
g. $143{\cdot}240^\circ$
h. 18°
i. $85{\cdot}944^\circ$

3.

a. 0·866
b. 0·5
c. -1
d. 1
e. 0
f. 0·909

g. -2·185 h. 0·5 i. 0·866

3. Trig identities

1.

a. $\frac{1}{\cos^2\theta}$

b. $\frac{1}{\sin^2\theta}$

c. $\frac{1}{\sin\theta\cos\theta}$

d. $\sin^2\theta$

e. $\frac{1}{\sin^2\theta}$

4. Sum and difference formulae

1.

a. $\frac{171}{221}$ =0·774 (3 dp)

b. $-\frac{21}{221}$ = -0·095 (3 dp)

c. $\frac{140}{221}$ = 0·633 (3 dp)

d. $\frac{220}{221}$ = 0·995 (3 dp)

e. $\frac{171}{140}$ = 1·221 (3 dp)

f. $-\frac{21}{220}$ = -0·0955 (3 dp)

5. Double angle identities

1.

   a. 0·5376

   b. -0·8432

2. $\frac{240}{161}$ = 1·491 (3 dp)

6. Inverse trig functions

1. To 3 dp:

   a. f(x)=0·412; g(x)=1·59

   b. f(x)=0·524; g(x)=1·047

   c. f(x)=-0·524; g(x)=2·094

   d. f(x)=0·785; g(x)=0·785

   e. f(x)=1·571; g(x)=0

   f. f(x)=0·524; g(x)=1·047

   g. f(x)=0; g(x)=1·571

   h. f(x)=-1·571; g(x)=3·142

7. Sine and cosine rules

1. To 3 dp:

   a. B=17·458°, C=132·542°, c=7·368

   b. b=5·766, A=39·687°, C=73·313°

c. A=79°, b=5·118, a=7·104

d. c=3·300, A=78·662°, B=47·338°

e. B=97°, c=1·643, b=3·592

f. A=16·236, C=129·764, c=10·997

Differentiation

2. Derivatives

1.

a. $\frac{df(x)}{dx} = 3\cos x$

b. $\frac{dg(x)}{dx} = 24x^3 + 4$

c. $\frac{df(t)}{dx} = 3e^t$

d. $\frac{df(x)}{dx} = 105x^6 + 12x^3 - 89$

e. $\frac{dg(r)}{dx} = \cos r - \sin r$

f. $\frac{df(x)}{dx} = -\sin x + 5e^x$

2.

a. $\frac{d^2 g(x)}{dx^2} = 72x^2$

b. $\frac{d^2 f(x)}{dx^2} = -3\sin x$

c. $\frac{d^2 f(x)}{dx^2} = 630x^5 + 9x^2 - 89$

3. Chain, product and quotient rules

1.

a. $\frac{dw}{dx} = (2x+3)e^{x^2+3x}$ (chain rule)

b. $\frac{dh}{dt} = e^t \cos(e^t)$ (chain rule)

c. $\frac{dy}{dx} = 4(x+x^2)^3(1+2x)$ (chain rule)

2.

a. $\frac{dy}{dx} = e^x(x^5 + 7x^4 + 13x^3 + 15x^2)$ (product rule)

b. $\frac{dg}{dx} = -x^3 \sin x + 3x^2 \cos x$ (product rule)

c. $\frac{dk}{dt} = t + 2t\ln(3t)$ (product rule and chain rule)

3.

a. $\frac{dy}{dx} = \frac{-x^5 + 3x^4 + 3x^3 + 15x^2}{e^x}$ (quotient rule)

b. $\frac{dt}{dx} = \frac{(x^2+3)\cos x - 2x\sin x}{(x^2+3)^2}$ (quotient rule)

c. $\frac{dy}{dx} = \frac{\frac{\sin(3x)}{x} - 3\cos(3x)\ln x}{\sin^2 3x}$ (quotient rule and chain rule)

4. Max and min

1.

a.
$$f'(x) = 3x^2 + 3x - 6$$
$$= 3(x-1)(x+2)$$
$$f''(x) = 6x + 3$$

x=-2 is a maximum; x=1 is a minimum.

b.
$$g'(t) = 12t^3 + 24t^2 - 12t - 24$$
$$= 12(t+1)(t-1)(t+2)$$
$$g''(t) = 36t^2 + 48t - 12$$

t=-2 is a minimum; t=-1 is a maximum; t=1 is a minimum.

c.
$$f'(x) = 3x^2 - 4x^3 = x^2(3-4x)$$
$$f''(x) = 6x - 12x^2$$

$x = \frac{3}{4}$ is a maximum; x=0 is a point of inflection (examining the gradient since $f''(0) = 0$).

5. Taylor Series, expansion, linearization

1.

a. $\frac{1}{2-x} = 1 + (x-1) + (x-1)^2 + (x-1)^3 + \dots$

b. $\cos(x) = 1 - \frac{(x-2\pi)^2}{2!} + \frac{(x-2\pi)^4}{4!} - \dots$

c. $\ln(x) = \ln 2 + \frac{(x-2)}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{16} - \ldots$

6. Partial differentiation

1.

a.

$$\frac{\partial z}{\partial x} = y\cos x \qquad \frac{\partial z}{\partial y} = \sin x$$

$$\frac{\partial^2 z}{\partial x^2} = -y\sin x \qquad \frac{\partial^2 z}{\partial y^2} = 0$$

$$\frac{\partial^2 z}{\partial y \partial x} = \cos x \qquad \frac{\partial^2 z}{\partial x \partial y} = \cos x$$

b.

$$\frac{\partial y}{\partial x} = 24x^3 t + 4t^2 \qquad \frac{\partial y}{\partial t} = 6x^4 + 8xt$$

$$\frac{\partial^2 y}{\partial x^2} = 72x^2 t \qquad \frac{\partial^2 y}{\partial t^2} = 8x$$

$$\frac{\partial^2 y}{\partial t \partial x} = 24x^3 + 8t \qquad \frac{\partial^2 y}{\partial x \partial t} = 24x^3 + 8t$$

c.

$$\frac{\partial h}{\partial x} = 4e^x - 2xt^4 \qquad \frac{\partial h}{\partial t} = -4x^2 t^3$$

$$\frac{\partial^2 h}{\partial x^2} = 4e^x - 2t^4 \qquad \frac{\partial^2 h}{\partial t^2} = -12x^2 t^2$$

$$\frac{\partial^2 h}{\partial t \partial x} = -8xt^3 \qquad \frac{\partial^2 h}{\partial x \partial t} = -8xt^3$$

2.

a. A local maximum at $(x, y) = (0,0)$

b. A saddle point at $(x, y) = (0,0)$

c. $(x, y) = (0,0)$: a saddle point
$(x, y) = (1,0)$: a saddle point
$(x, y) = (0,-1)$: a saddle point
$(x, y) = \left(\frac{1}{3}, -\frac{1}{3}\right)$: a local maximum

Integration

2. Integration as anti-differentiation

1.

a. $6x^2 + 5x + c$

b. $2x^3 + 2x^4 + c$

c. $-\frac{1}{3}\cos(3x) + c$

d. $2\sin(6x+3) + c$

e. $\frac{1}{2}e^{2x+2} + c$

f. $\frac{1}{6}\ln(6x+4) + c$

2. $y = f(x) = \frac{5}{2}x^2 + 2$

3. $y = f(x) = 2\sin(3x) + 2.5$

2. Areas under curves

1.

a. 2

b. 141

c. 6

d. -3.32 (2 dp)

e. 104.47 (2 dp)

f. 2.27 (2 dp)

Matrices

2. Simultaneous equations expressed as matrices

1.

a. $\begin{bmatrix} 4 & 3 \\ 9 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 29 \\ 42 \end{bmatrix}$

b. $\begin{bmatrix} \frac{1}{2} & 8 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 42 \\ 22 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -3 \\ 4 & 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 4 \end{bmatrix}$

d. $\begin{bmatrix} 3 & 1 & 0 \\ 0 & 2 & -1 \\ 8 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \\ 32 \end{bmatrix}$

3. Simple matrix algebra –addition, subtraction, multiplication

1.

a.

$$A + B = \begin{bmatrix} 4 & 2 \\ 3 & 6 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & 4 \\ 9 & 8 \end{bmatrix}$$

$$BA = \begin{bmatrix} 3 & 2 \\ 8 & 12 \end{bmatrix}$$

b.

$$A + B = \begin{bmatrix} 11 & 7 \\ 1 & 7 \end{bmatrix}$$

$$A - B = \begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 28 & 31 \\ 7 & 14 \end{bmatrix}$$

$$BA = \begin{bmatrix} 32 & 29 \\ 5 & 10 \end{bmatrix}$$

c.

$$A + B = \begin{bmatrix} 3 & 10 \\ 2 & 13 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} (= I)$$

$$AB = \begin{bmatrix} 7 & 40 \\ 8 & 47 \end{bmatrix}$$

$$BA = \begin{bmatrix} 7 & 40 \\ 8 & 47 \end{bmatrix}$$

(N.B. $AB = BA$
by coincidence)

d.

$$A + B = \begin{bmatrix} 9 & 6 \\ 2 & 4 \end{bmatrix}$$

$$A - B = \begin{bmatrix} -1 & -2 \\ -4 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 26 & 18 \\ 4 & -1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 16 & 22 \\ 11 & 9 \end{bmatrix}$$

e.

$$A+B=\begin{bmatrix} 7 & 6 & 4 \\ 4 & 12 & 5 \\ 3 & 9 & 15 \end{bmatrix}$$

$$A-B=\begin{bmatrix} 1 & -2 & -2 \\ 2 & -2 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$AB=\begin{bmatrix} 16 & 35 & 24 \\ 18 & 57 & 36 \\ 25 & 77 & 69 \end{bmatrix}$$

$$BA=\begin{bmatrix} 27 & 38 & 38 \\ 28 & 49 & 42 \\ 29 & 53 & 66 \end{bmatrix}$$

f.

$$A+B=\begin{bmatrix} 5 & 1 & 1 \\ 6 & 8 & 10 \\ 3 & 1 & 12 \end{bmatrix}$$

$$A-B=\begin{bmatrix} 1 & 1 & -1 \\ -2 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}$$

$$AB=\begin{bmatrix} 10 & 3 & 7 \\ 30 & 15 & 52 \\ 15 & 3 & 41 \end{bmatrix}$$

$$BA=\begin{bmatrix} 8 & 3 & 7 \\ 26 & 23 & 46 \\ 13 & 6 & 35 \end{bmatrix}$$

2.

a. $\begin{bmatrix} 11 & 24 \\ 19 & 22 \end{bmatrix}$

b. $\begin{bmatrix} 35 \\ 13 \end{bmatrix}$

c. $\begin{bmatrix} 53 \\ 10 \end{bmatrix}$

d. $\begin{bmatrix} 58 & 68 & 26 \\ 35 & 34 & 13 \end{bmatrix}$

4. Determinants and matrix inversion

1.

a.

$$A^T = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \quad A^{-1} = \frac{1}{10}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.2 \\ -0.1 & 0.3 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \quad B^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 0.5 \end{bmatrix}$$

$$\det A = 10$$

$$\det B = 2$$

b.

$$A^T = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \quad A^{-1} = \frac{1}{5}\begin{bmatrix} 2 & -3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.6 \\ -0.2 & 0.8 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 7 & 0 \\ 4 & 5 \end{bmatrix} \quad B^{-1} = \frac{1}{35}\begin{bmatrix} 5 & -4 \\ 0 & 7 \end{bmatrix}$$

$$\det A = 5$$

$$\det B = 35$$

c.

$$A^T = \begin{bmatrix} 2 & 1 \\ 5 & 7 \end{bmatrix} \quad A^{-1} = \frac{1}{9}\begin{bmatrix} 7 & -5 \\ -1 & 2 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 1 \\ 5 & 6 \end{bmatrix} \quad B^{-1} = \frac{1}{1}\begin{bmatrix} 6 & -5 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ -1 & 1 \end{bmatrix}$$

$$\det A = 9$$

$$\det B = 1$$

d.

$$A^T = \begin{bmatrix} 4 & -1 \\ 2 & 3 \end{bmatrix} \quad A^{-1} = \frac{1}{14}\begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 5 & 3 \\ 4 & 1 \end{bmatrix} \quad B^{-1} = \frac{1}{-7}\begin{bmatrix} 1 & -4 \\ -3 & 5 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -1 & 4 \\ 3 & -5 \end{bmatrix}$$

$$\det A = 14$$

$$\det B = -7$$

e.

$$A^T = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 5 & 4 \\ 1 & 2 & 9 \end{bmatrix} \quad A^{-1} = \frac{1}{105}\begin{bmatrix} 37 & -14 & -1 \\ -25 & 35 & -5 \\ 7 & -14 & 14 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 7 & 5 \\ 3 & 3 & 6 \end{bmatrix} \quad B^{-1} = \frac{1}{54}\begin{bmatrix} 27 & -9 & -9 \\ 0 & 12 & -6 \\ -9 & -7 & 17 \end{bmatrix}$$

$$\det A = 105$$

$$\det B = 54$$

f.

$$A^T = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 5 & 1 \\ 0 & 6 & 7 \end{bmatrix} \quad A^{-1} = \frac{1}{85}\begin{bmatrix} 29 & -7 & 6 \\ -2 & 21 & 18 \\ -8 & -1 & 13 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix} \quad B^{-1} = \frac{1}{27}\begin{bmatrix} 15 & 0 & -3 \\ -16 & 9 & -4 \\ -3 & 0 & 6 \end{bmatrix}$$

$$\det A = 85$$

$$\det B = 27$$

5. Solution of simultaneous equations using matrices

1.

a. $x = 5, y = 3$

b. $x = 4, y = 5$

c. $x = 0{\cdot}5,\ y = 3,\ z = 2$

d. $x = 2,\ y = 5,\ z = 8$

Statistics

2. Mean, median, mode, variance and standard deviation

1.

   a. $\bar{x} = 6.5$
   $\text{median} = 6.5$
   $\text{mode} = 6$
   $\sigma^2 = 6.8846$
   $\sigma = 2.6239$

   b. $\bar{x} = 6.1875$
   $\text{median} = 5.5$
   $\text{mode} = 10$
   $\sigma^2 = 9.4958$
   $\sigma = 3.0815$

   c. $\bar{x} = 6.1667$
   $\text{median} = 5$
   $\text{mode} = 10$
   $\sigma^2 = 11.5588$
   $\sigma = 3.3998$

   d. $\bar{x} = 4.9231$
   $\text{median} = 5$
   $\text{mode} = 5$
   $\sigma^2 = 6.0769$
   $\sigma = 2.4651$

3. Basic distributions – normal, binomial

1. 0.193

2. 0.279

4. Precision, accuracy

a. Set 1: mean 100·1043; standard deviation 0·0199.
   Set 2: mean 100·0368; standard deviation 0·0422.
   Set 3: mean 100; standard deviation 0·0033.

b. Set 3 is most precise.

c. Set 3 is most accurate.

d. The bias of Set 3 is 0. According the normal distribution, 99·7% of results from set 3 are within 0·0098mm of the mean, which is less than would affect a measurement of 1 decimal place, so the results are sufficiently precise. Thus, the instrument used to produce Set 3 is producing valid measurements.