QFT

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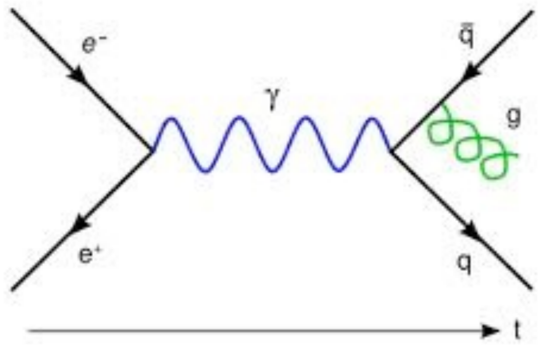
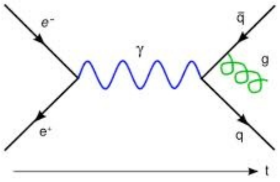
(Notes based on Dr A. Moss' lectures)



Lecture 1: Preliminaries (Classical)

What is QFT?

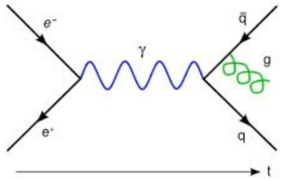


A framework for building theories that are

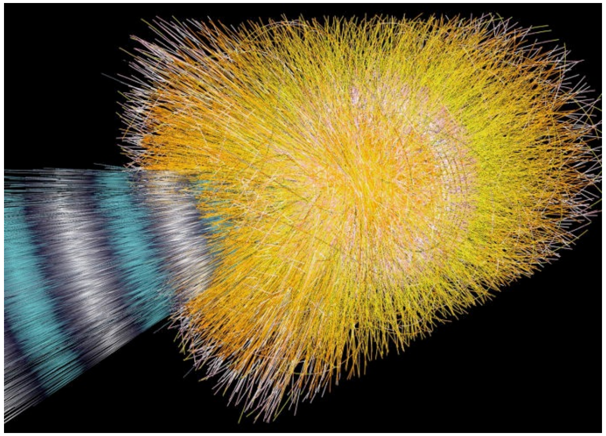
Lorentz invariant, local, causal

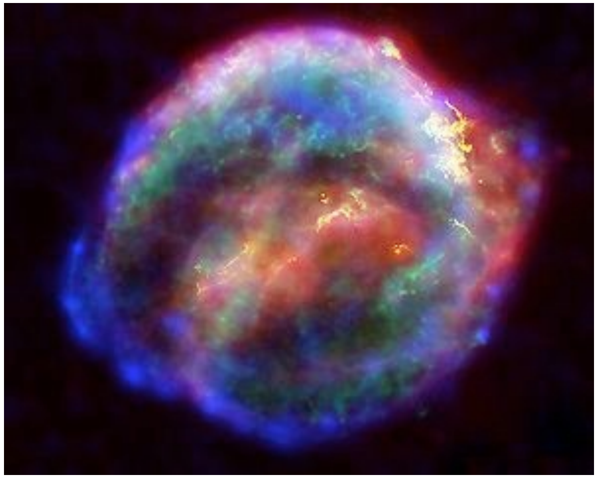
- Basic idea: Fields are fundamental, quantization of ripples in field are ‘particles’
  - Field for each fundamental particle (electrons, quarks, gluons, Higgs etc)
- Promote classical degrees of freedom (DOF) to operators
  - In quantum mechanics DOFs promoted to operators acting on Hilbert space
  - QFT is quantization of classical fields. Fields promoted to operator valued function
- Infinite number of degrees of freedom! Can cause problems
- In this course will consider canonical quantization (more transparent starting from classical picture)

Why QFT?



- Need consistent formalism to deal with multi-particle states
  - Special relativity and QM imply particle number is not conserved
  - Cannot be reconciled with wave function description
- All particles are identical - those in the lab and those on cosmological scales
- Fields generally provide local description of physics - e.g. field equations of Maxwell and Einstein





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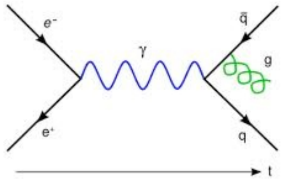
Course Outline



Lecture	Topic
1	Preliminaries - Classical mechanics, Classical Field Theory
2	Preliminaries - Canonical Quantization, Harmonic Oscillator
3-4	Free Fields - Canonical Quantization, Vacuum State, Particle States, Causality, Feynman Propagator
5-6	Interacting Fields - S-Matrix, Wick's Theorem, Feynman Diagrams
7	Spinors - Lorentz Group, Spinor representation
8	Dirac Equation
9-10	Quantization of Dirac Equation - Fermions, Feynman Rules

Units: $\hbar = c = 1$

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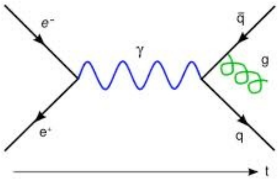
Classical Mechanics - Lagrangian



- Consider particle in 1-D with potential V(x). Define Lagrangian in terms of kinetic and potential energy T and V by

$$L(x,\dot{x}) = T - V = \frac{1}{2}m\dot{x}^2 - V(x)$$

- Define action functional by $S = \int_{t_0}^{t_1} dt\, L(x,\dot{x})$

- Variation of action $\delta S = \int_{t_0}^{t_1} dt \left\{ \delta x \frac{\partial L}{\partial x} + \delta \dot{x} \frac{\partial L}{\partial \dot{x}} \right\}$

- Principle of least action $\delta S = 0$ leads to Euler-Lagrange equations

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \quad \xrightarrow{\text{General coordinates}} \quad \frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0$$

- For particle in 1-D potential $m\ddot{x} = -\frac{\partial V}{\partial x}$

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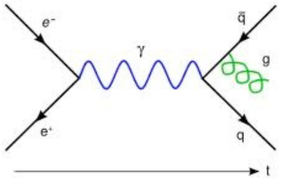
Hamiltonian Formalism



- Lagrangian formalism makes symmetries and their physical consequences explicit
- For canonical quantization need another equivalent treatment
- Define conjugate momentum $$p \equiv \frac{\partial L}{\partial \dot{x}}$$
- Define Hamiltonian $$H(x,p) \equiv p\dot{x} - L(x,\dot{x})$$
- Can derive Hamilton's equations

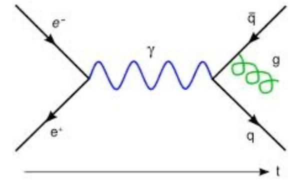
$$\frac{\partial H}{\partial x} = -\dot{p}\,, \qquad \frac{\partial H}{\partial p} = \dot{x}$$

General coordinates →

$$\frac{\partial H}{\partial x_i} = -\dot{p}^i\,, \qquad \frac{\partial H}{\partial p^i} = \dot{x}_i$$

- e.g. particle in 1-D potential $$H = \frac{1}{2}m\dot{x}^2 + V = T + V$$

Classical Field Theory



- A field is a quantity defined at every point in space and time
- In 1-D have following analogy to classical mechanics with infinite degrees of freedom:

$$x_i(t) \longrightarrow \phi(x,t)$$

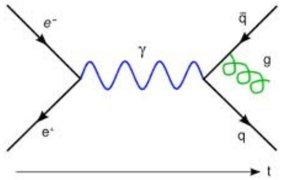
$$\dot{x}_i(t) \longrightarrow \dot{\phi}(x,t)$$

$$i \longrightarrow x \qquad\qquad \sum_i \longrightarrow \int dx$$

$$L(x_i,\dot{x}_i) \longrightarrow \mathcal{L}[\phi,\dot{\phi}]$$

- Easily generalized to 3-D $\phi(\mathbf{x},t)$, $\sum_i \longrightarrow \int d^3\mathbf{x}$
- Position has been relegated from a dynamical variable in particle mechanics to a label in field theory

Lorentz Invariance



- Would like theory to be Lorentz invariant
- Four vectors transform under

$$(x')^\mu = \Lambda^\mu{}_\nu x^\nu \qquad \Lambda^\mu{}_\sigma \eta^{\sigma\tau} \Lambda^\nu{}_\tau = \eta^{\mu\nu}$$

where $\eta^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ is the Minkowski metric

- Lorentz scalar same in all inertial frames $\phi'(x') = \phi(x)$

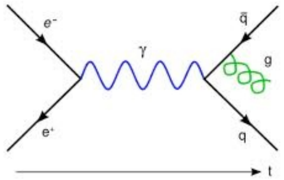
NB: $x = (\mathbf{x}, t)$ Active: $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1} x)$

- Lorentz vector transforms as $V'^\mu(x') = \Lambda^\mu{}_\nu V^\nu(x)$

$$V^\mu(x) \to V'^\mu(x) = \Lambda^\mu{}_\nu V^\nu(\Lambda^{-1} x)$$

E.g. Derivative of scalar transforms as vector $\partial_\mu \phi(x) = \frac{\partial \phi(x)}{\partial x^\mu}$

Action for Scalar Field



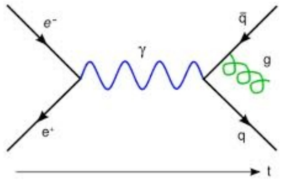
- Will consider Lagrangians depending on $\phi \, , \dot{\phi} \, , \nabla\phi$
- Define action S and Lagrangian density $\mathcal{L}$

$$S = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi) \qquad\qquad L = \int d^3x \, \mathcal{L}(\phi, \partial_\mu \phi)$$

- Invariance of integration measure d^4x ensures theory is Lorentz invariant as long as $\mathcal{L}$ is
- NB: Lagrangian density often termed Lagrangian
- Check following Lagrangian for real scalar is Lorentz invariant

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2$$

Equations of Motion



- Follow same procedure as in classical mechanics and vary action

$$\delta S = \int d^4x \left\{ \delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \delta(\partial_\mu \phi) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right\}$$

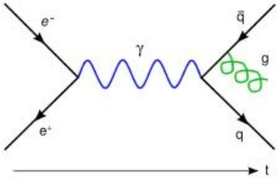
$$\delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right\} \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right)$$

- Principle of least action leads to Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0 \quad \Rightarrow \quad \frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) = 0$$

General number of fields

Hamiltonian



- Similarly define momentum conjugate $\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$
- Hamiltonian density $\mathcal{H}$

$$\mathcal{H}(\phi, \pi) = \pi(x)\dot{\phi}(x) - \mathcal{L}(x) \qquad H = \int d^3x \, \mathcal{H}(\phi, \pi)$$

- Hamilton's equations

$$\dot{\phi}(x) = \frac{\partial H}{\partial \pi(x)}\,, \qquad \dot{\pi}(x) = -\frac{\partial H}{\partial \phi(x)}$$

- Straightforward generalization to multiple fields

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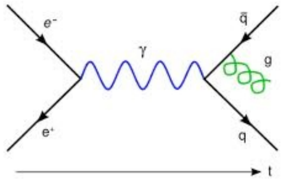
Klein-Gordon Equation



- Consider Lagrangian $\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2$
- Derivatives of Lagrangian $\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi\,, \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi$
- Euler-Lagrange equation then gives Klein-Gordon equation

$$\partial_\mu\partial^\mu\phi + m^2\phi = \left(\Box + m^2\right)\phi = 0$$

- NB Minkowski: $\Box\phi = \ddot{\phi} - \nabla^2\phi$
- Hamiltonian $H = \frac{1}{2}\int d^3x\left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right]$
- Interpret Hamiltonian as total energy
- Easy to generalize to other potential not $V(\phi) = \frac{1}{2}m^2\phi^2$

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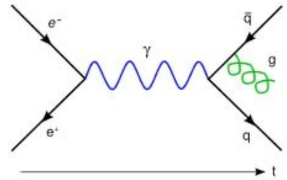
Plane Wave Solution



- Consider real solutions to KG equation. Plane wave ansatz:

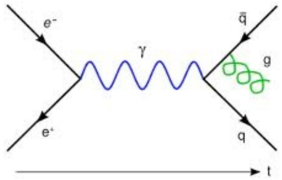
$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} \left[f(k)e^{-ik\cdot x} + f^\star(k)e^{+ik\cdot x}\right] \qquad k\cdot x = k^0x^0 - \mathbf{k}\cdot\mathbf{x}$$

- Substitute into KG equation. Find $(k^0)^2 - \mathbf{k}^2 = m^2$
- Identify energy as positive branch $E(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$
- Existence of negative energy states - interpretation of $\phi(x)$ as quantum field gives rise to anti-particles
- Integrate out k^0 dependence

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} \left[a^\star(\mathbf{k})e^{ik\cdot x} + a(\mathbf{k})e^{-ik\cdot x}\right]$$

Lorentz invariant measure

Symmetries



- Symmetries play an important role in particle physics and field theory
- Noether's theorem: Invariance of the action under continuous symmetry transformation gives rise to a conserved current $j^\mu(x)$ such that

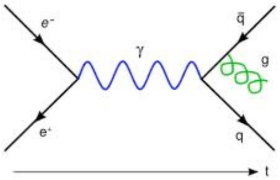
$$\partial_\mu j^\mu = 0$$

- Conserved current implies conserved charge associated with this symmetry

$$Q_V = \int_V d^3x\, j^0 \qquad \frac{dQ_V}{dt} = -\int_V d^3x\, \nabla \cdot \mathbf{j} = -\int_A \mathbf{j} \cdot d\mathbf{s}$$

- Charge is conserved *locally*

Noether's Theorem
for translational invariance



- Consider infinitesimal translation $x^\mu \to x^\mu + \epsilon^\mu$
- Change in Lagrangian is $\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)$
- Euler-Lagrange equations give $\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right)$
- Under translation $\phi(x) \to \phi(x) - \epsilon^\mu\partial_\mu\phi(x)$

$$\mathcal{L}(x) \to \mathcal{L}(x) - \epsilon^\mu\partial_\mu\mathcal{L}(x)$$

NB Lagrangian has no explicit coordinate dependence

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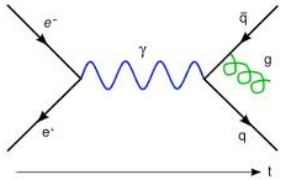
Noether's Theorem

for translational invariance



- For invariance of action for general ϵ^μ find 4 conserved currents

$$(j^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \delta^\mu_\nu \mathcal{L} \equiv T^\mu{}_\nu$$

- $T^\mu{}_\nu$ is the energy-momentum tensor which satisfies

$$\partial_\mu T^\mu{}_\nu = 0$$

- Translation symmetry gives rise to conservation of energy-momentum
- Other symmetries give other conserved currents - e.g. Lorentz transformation and angular momentum