QFT

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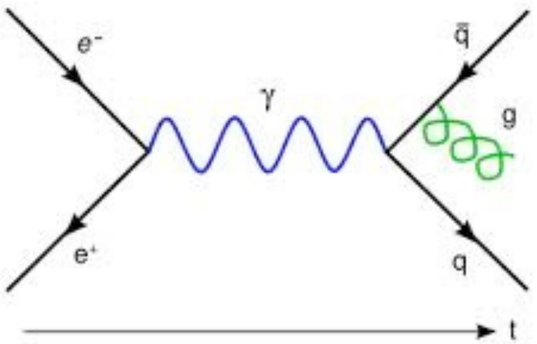
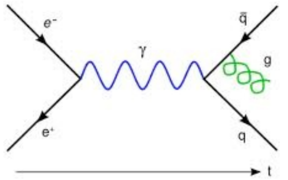
(Notes based on Dr A. Moss' lectures)



Lecture 3: Quantization of Free Fields

Canonical Quantization
(free scalar field)



- Analogous to quantum mechanics promote canonical variables to be operators acting on states

$$\phi(\mathbf{x}) \to \hat{\phi}(\mathbf{x}) \qquad \pi(\mathbf{x}) \to \hat{\pi}(\mathbf{x})$$

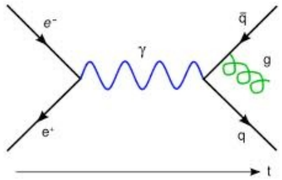
- Impose commutation relations (from Poisson brackets)

$$\left[\hat{\phi}(\mathbf{x}), \hat{\phi}(\mathbf{y})\right] = \left[\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})\right] = 0$$

$$\left[\hat{\phi}(\mathbf{x}), \hat{\pi}(\mathbf{y})\right] = i\delta^3(\mathbf{x} - \mathbf{y})$$

- 3 dimensional δ-function as we are using fields
- Note we are in the *Schrödinger picture*: operators depend only on space, all time dependence is in the states

Free Scalar Field



- Promote solution of Klein-Gordon equation to operator status

$$\hat{\phi}(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} \left[\hat{a}^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} + \hat{a}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}\right]$$

$$\hat{\pi}(\mathbf{x}) = \frac{i}{2} \int \frac{d^3k}{(2\pi)^3} \left[\hat{a}^\dagger(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} - \hat{a}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}\right]$$

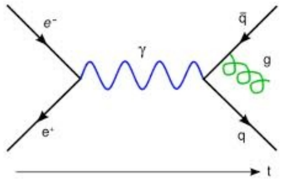
- Complex conjugate for coefficient turned into hermitian conjugate for operator
- NB: Complete analogy to QHO

$$\hat{x} = \frac{1}{\sqrt{2m\omega}}(\hat{a} + \hat{a}^\dagger)$$

$$\hat{p} = i\sqrt{\frac{m\omega}{2}}(\hat{a}^\dagger - \hat{a})$$

- Creation $\hat{a}^\dagger(\mathbf{k})$ and annihilation $\hat{a}(\mathbf{k})$ operators will be used to construct particle states

Creation/Annihilation Operators



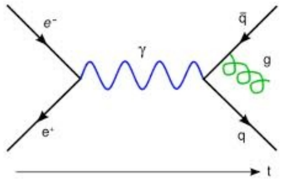
- Exercise: Take Fourier transform and solve for operator coefficients
- NB: Remember $\delta^3(\mathbf{k}) = \int \frac{d^3x}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}}$
- Find

$$\hat{a}(\mathbf{k}) = \int d^3x \left[E(\mathbf{k})\hat{\phi}(\mathbf{x}) + i\hat{\pi}(\mathbf{x})\right] e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\hat{a}^\dagger(\mathbf{k}) = \int d^3x \left[E(\mathbf{k})\hat{\phi}(\mathbf{x}) - i\hat{\pi}(\mathbf{x})\right] e^{-i\mathbf{k}\cdot\mathbf{x}}$$

- Remember we are in the *Schrödinger picture:* operators are time-independent

Digression: Schrödinger vs Heisenberg Pictures



We seem to have lost Lorentz invariance! Starting from a Hamiltonian description we separated time from space. We are now working in the Schrödinger picture, where fields are time independent operators. Of course the theory is Lorentz invariant, even if not manifest in our description so far.

- Things already look better in the Heisenberg picture. Take $e^{i\hat{H}t}\hat{a}(\mathbf{k})e^{-i\hat{H}t}$ and use $\hat{H}\hat{a}(\mathbf{k}) = \hat{a}(\mathbf{k})(\hat{H} - E(\mathbf{k}))$ to show that: $e^{i\hat{H}t}\hat{a}(\mathbf{k})e^{-i\hat{H}t} = \hat{a}(\mathbf{k})e^{-iE(\mathbf{k})t}$

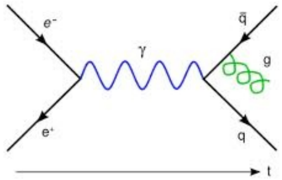
- Thus time dep. can be absorbed in exponential and the Heisenberg picture field is:

$$\hat{\phi}(x) = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})}\left[\hat{a}^\dagger(\mathbf{k})e^{ik\cdot x} + \hat{a}(\mathbf{k})e^{-ik\cdot x}\right]$$

(wave-particle duality apparent, resolves -ve energy issues of RQM)

- In this picture we have equal-time commutation relations like $\left[\hat{\phi}(x), \hat{\pi}(y)\right] \equiv \left[\hat{\phi}(\mathbf{x},t), \hat{\pi}(\mathbf{y},t)\right] = i\delta^3(\mathbf{x}-\mathbf{y})$

Commutation Relations



- Exercise: Use (equal-time) commutation relations for canonical field variables to show

$$\left[\hat{a}^\dagger(\mathbf{k_1}), \hat{a}^\dagger(\mathbf{k_2})\right] = \left[\hat{a}(\mathbf{k_1}), \hat{a}(\mathbf{k_2})\right] = 0$$

$$\left[\hat{a}(\mathbf{k_1}), \hat{a}^\dagger(\mathbf{k_2})\right] = (2\pi)^3 2E(\mathbf{k_1})\delta^3(\mathbf{k_1} - \mathbf{k_2})$$

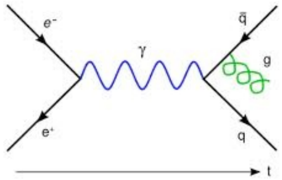
Cf. $\left[\hat{a}, \hat{a}^\dagger\right] = 1$ in QHO

- Exercise: Use equal-time commutation relations to show that in the Heisenberg picture:

$$\frac{d}{dt}\hat{\phi}(x) \equiv i\left[\hat{H}, \hat{\phi}(x)\right] = \hat{\pi}(x) \qquad \frac{d}{dt}\hat{\pi}(x) = \nabla^2\hat{\phi}(x) - m^2\hat{\phi}(x)$$

- Nearly ready to construct Hilbert space upon which these act. First will consider the Hamiltonian and vacuum state

Hamiltonian



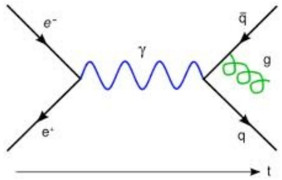
- Recall Hamiltonian of the Klein-Gordon field:

$$\hat{H} = \frac{1}{2}\int d^3x \left[\hat{\pi}^2 + (\nabla\hat{\phi})^2 + m^2\hat{\phi}^2\right]$$

- Substitute in expressions:

$$\begin{aligned}\hat{H} = \frac{1}{8}\int \frac{d^3x\, d^3k\, d^3q}{(2\pi)^6 E(\mathbf{k})E(\mathbf{q})} \times \\ E(\mathbf{k})E(\mathbf{q})\left[-(\hat{a}^\dagger(\mathbf{k})e^{ik\cdot x} - \hat{a}(\mathbf{k})e^{-ik\cdot x})(\hat{a}^\dagger(\mathbf{q})e^{iq\cdot x} - \hat{a}(\mathbf{q})e^{-iq\cdot x})\right] \\ + \left[(-i\mathbf{k}\,\hat{a}^\dagger(\mathbf{k})e^{ik\cdot x} + i\mathbf{k}\,\hat{a}(\mathbf{k})e^{-ik\cdot x})(-i\mathbf{q}\,\hat{a}^\dagger(\mathbf{q})e^{iq\cdot x} + i\mathbf{q}\,\hat{a}(\mathbf{q})e^{-iq\cdot x})\right] \\ + m^2\left[(\hat{a}^\dagger(\mathbf{k})e^{ik\cdot x} + \hat{a}(\mathbf{k})e^{-ik\cdot x})(\hat{a}^\dagger(\mathbf{q})e^{iq\cdot x} + \hat{a}(\mathbf{q})e^{-iq\cdot x})\right]\end{aligned}$$

Hamiltonian



- Integrate over d^3x and d^3q :

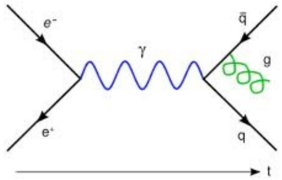
$$\hat{H} = \frac{1}{8}\int \frac{d^3k}{(2\pi)^3 E(\mathbf{k})^2} \times$$
$$\Big[(-E(\mathbf{k})^2 + \mathbf{k}^2 + m^2)(\hat{a}^\dagger(\mathbf{k})\hat{a}^\dagger(-\mathbf{k})e^{2iE(\mathbf{k})t} + \hat{a}(\mathbf{k})\hat{a}(-\mathbf{k})e^{-2iE(\mathbf{k})t})\Big]$$
$$+\big[(E(\mathbf{k})^2 + \mathbf{k}^2 + m^2)(\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k}))\big]$$

- Simplify using $E(\mathbf{k})^2 = \mathbf{k}^2 + m^2$:

$$\hat{H} = \frac{1}{4}\int \frac{d^3k}{(2\pi)^3}(\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) + \hat{a}(\mathbf{k})\hat{a}^\dagger(\mathbf{k}))$$

- Cf. quantum harmonic oscillator: $\hat{H} = \frac{1}{2}\omega\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right)$
- Hamiltonian is continuous sum of excitation energies of harmonic oscillators

Vacuum State



- In analogy with harmonic oscillator define vacuum by insisting it is annihilated by *all* $\hat{a}(\mathbf{k})$

$$\hat{a}(\mathbf{k})|0\rangle = 0$$

- Energy of ground state given by

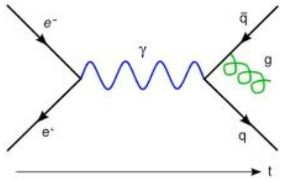
$$\hat{H}|0\rangle = E_0|0\rangle = \left[\int d^3k \frac{1}{2} E(\mathbf{k})\delta^3(0)\right]|0\rangle = \infty|0\rangle$$

- Two sources of infinity
  - Infrared divergence (space is infinitely large)

$$V = (2\pi)^3\delta^3(0) \qquad \epsilon_0 = \frac{E_0}{V} = \int \frac{d^3k}{(2\pi)^3}\frac{E(\mathbf{k})}{2}$$

  - Ultra-violet divergence (integrating to infinitely high momenta)

Renormalization



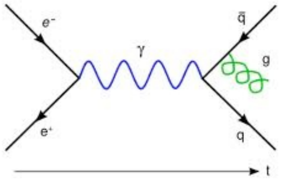
- In physics we are *mainly* interested in energy *differences* (i.e. we do not measure energy of the ground state)
- Redefine Hamiltonian $\hat{H}^R = \hat{H} - E_0$
- Find renormalized Hamiltonian

$$\hat{H}^R = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k}) \qquad \hat{H}^R|0\rangle = 0$$

- Simple way to automate removal of vacuum contribution is normal ordering

QFT

Normal Ordering



- All annihilation operators placed to the right of any creation operator, e.g.

$$:\frac{1}{2}\left(\hat{a}^{\dagger}(\mathbf{k})\hat{a}(\mathbf{k})+\hat{a}(\mathbf{k})\hat{a}^{\dagger}(\mathbf{k})\right):=\hat{a}^{\dagger}(\mathbf{k})\hat{a}(\mathbf{k})$$

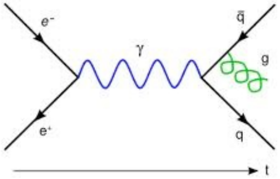
- Renormalized Hamiltonian is now

$$\hat{H}^{R}=:\hat{H}:$$

- We will normal order operators for the remainder of this lecture in this manner

(Issue arises from ordering ambiguity. E.g. if write HO Hamiltonian as $H=\frac{1}{2}(\omega q-ip)(\omega q+ip)$ then get $\hat{H}=\omega a^{\dagger}a$ upon quantisation)

Cosmological Constant



- One caveat to the statement that "we are *mainly* interested in energy *differences"* is the cosmological constant
- Gravity sees everything in the energy-momentum tensor
- Einstein equations:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$$

- $\Lambda = E_0 / V$ is the cosmological constant
- Observations suggests that $\Lambda \sim (10^{-3} eV)^4$
- Many orders of magnitude lower than zero point energy of fields in standard model

(Note fermions contribute with opposite sign, relevance of SUSY)

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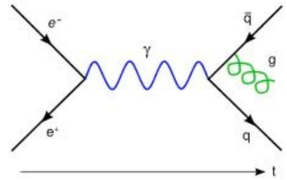
Single Particles



- Will now consider excitations of the field. For normal ordered Hamiltonian easy to check

$$[\hat{H},\hat{a}^\dagger(\mathbf{k})] = E(\mathbf{k})\hat{a}^\dagger(\mathbf{k}) \quad [\hat{H},\hat{a}(\mathbf{k})] = -E(\mathbf{k})\hat{a}(\mathbf{k})$$

(Cf. harmonic oscillator: $\left[\hat{H},\hat{a}^\dagger\right] = \omega\hat{a}^\dagger \quad \left[\hat{H},\hat{a}\right] = -\omega\hat{a}$)

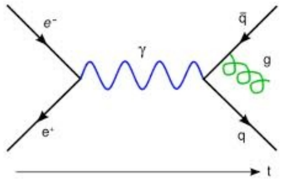
- Define state $|\mathbf{k}\rangle$ to be one obtained by $|\mathbf{k}\rangle = \hat{a}^\dagger(\mathbf{k})|0\rangle$

  Energy of this state $\hat{H}|\mathbf{k}\rangle = E(\mathbf{k})|\mathbf{k}\rangle$

- Can study other quantum numbers. Recall classical total momentum. After quantizing and normal ordering (exercise)

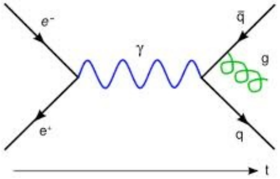
$$P^i = \int d^3x\,\dot{\phi}\,\partial^i\phi \quad \longrightarrow \quad \hat{\mathbf{P}} = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})}\mathbf{k}\,\hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})$$

Single Particles



- Acting on state with momentum operator $\hat{\mathbf{P}}|\mathbf{k}\rangle = \mathbf{k}|\mathbf{k}\rangle$
- Can interpret state $|\mathbf{k}\rangle$ as a single particle state for relativistic particle with mass m and momentum $\mathbf{k}$
- Particle of momentum $\mathbf{k}$ is excited Fourier mode of the field
- Can perform similar exercise on classical total angular momentum, turn into operator and show particle carries no internal angular momentum - it is a spin-0 particle.

Multiple Particles



- Can compute commutation relation $[\hat{\mathbf{P}}, \hat{a}^\dagger(\mathbf{k})] = \mathbf{k}\,\hat{a}^\dagger(\mathbf{k})$
- Extend to multiple $\hat{a}^\dagger(\mathbf{k})$ acting on vacuum

$$\hat{\mathbf{P}}\,\hat{a}^\dagger(\mathbf{k}_1)\ldots\hat{a}^\dagger(\mathbf{k}_N)|0\rangle = (\mathbf{k}_1 + \ldots \mathbf{k}_N)\hat{a}^\dagger(\mathbf{k}_1)\ldots\hat{a}^\dagger(\mathbf{k}_N)|0\rangle$$

- Interpret as n-particle state

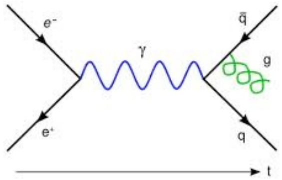
$$|\mathbf{k}_1, \ldots, \mathbf{k}_N\rangle = \hat{a}^\dagger(\mathbf{k}_1)\ldots\hat{a}^\dagger(\mathbf{k}_N)|0\rangle$$

- Since $\hat{a}^\dagger(\mathbf{k})$ commute state is symmetric under exchange of any two particles, e.g.

$$|\mathbf{k}_1, \mathbf{k}_2\rangle = |\mathbf{k}_2, \mathbf{k}_1\rangle$$

- Particles are *bosons*

Multiple Particles



- Have a Hilbert space for each n-particle state. Sum of these Hilbert spaces for all n is *Fock space*

- Define number operator $\hat{N} = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})$

- Gives number of bosons in particular state

$$\hat{N}|\mathbf{k}_1, \ldots, \mathbf{k}_N\rangle = n|\mathbf{k}_1, \ldots, \mathbf{k}_N\rangle$$

- Commutes with Hamiltonian $[\hat{N}, \hat{H}] = 0$

- Particle number is *conserved* in free scalar field theory - will not be the case in interacting theories