QFT

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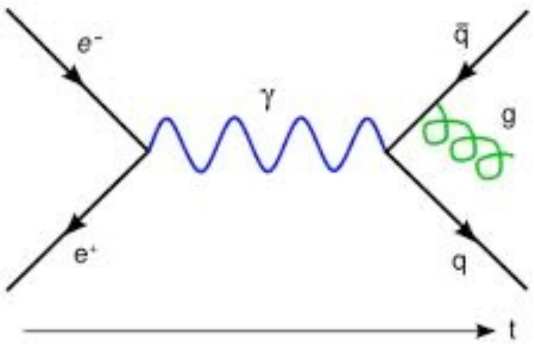
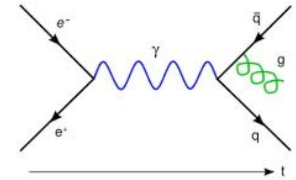
(Notes based on Dr A. Moss' lectures)



Lecture 4:

Free Fields - Causality, Feynman Propagator, Complex Scalar

Recap: n-particle states



- Commutation relation for momentum $[\hat{\mathbf{P}}, \hat{a}^\dagger(\mathbf{k})] = \mathbf{k}\, \hat{a}^\dagger(\mathbf{k})$
- Consider multiple $\hat{a}^\dagger(\mathbf{k})$ acting on vacuum

$$\hat{\mathbf{P}}\, \hat{a}^\dagger(\mathbf{k}_1) \ldots \hat{a}^\dagger(\mathbf{k}_N)|0\rangle = (\mathbf{k}_1 + \ldots \mathbf{k}_N)\hat{a}^\dagger(\mathbf{k}_1) \ldots \hat{a}^\dagger(\mathbf{k}_N)|0\rangle$$

- Interpret as n-particle state

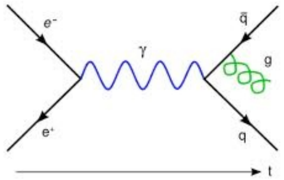
$$|\mathbf{k}_1, \ldots, \mathbf{k}_N\rangle = \hat{a}^\dagger(\mathbf{k}_1) \ldots \hat{a}^\dagger(\mathbf{k}_N)|0\rangle$$

- Since $\hat{a}^\dagger(\mathbf{k})$ commute state is symmetric under exchange of any two particles, e.g.

$$|\mathbf{k}_1, \mathbf{k}_2\rangle = |\mathbf{k}_2, \mathbf{k}_1\rangle$$

- Particles are *bosons*

Recap: n-particle states



- Have a Hilbert space for each n-particle state. Sum of these Hilbert spaces for all n is *Fock space*

- Define number operator $\hat{N} = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} \hat{a}^\dagger(\mathbf{k})\hat{a}(\mathbf{k})$

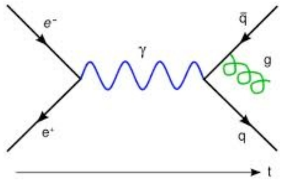
- Gives number of bosons in particular state

$$\hat{N}|\mathbf{k}_1, \ldots, \mathbf{k}_N\rangle = n|\mathbf{k}_1, \ldots, \mathbf{k}_N\rangle$$

- Commutes with Hamiltonian $[\hat{N}, \hat{H}] = 0$

- Particle number is *conserved* in free scalar field theory - will not be the case in interacting theories

Causality



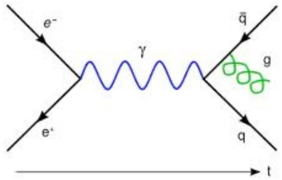
- So far we have imposed equal-time commutation relations

$$\left[\hat{\phi}(\mathbf{x},t),\hat{\phi}(\mathbf{y},t)\right]=\left[\hat{\pi}(\mathbf{x},t),\hat{\pi}(\mathbf{y},t)\right]=0$$

$$\left[\hat{\phi}(\mathbf{x},t),\hat{\pi}(\mathbf{y},t)\right]=i\delta^3(\mathbf{x}-\mathbf{y})$$

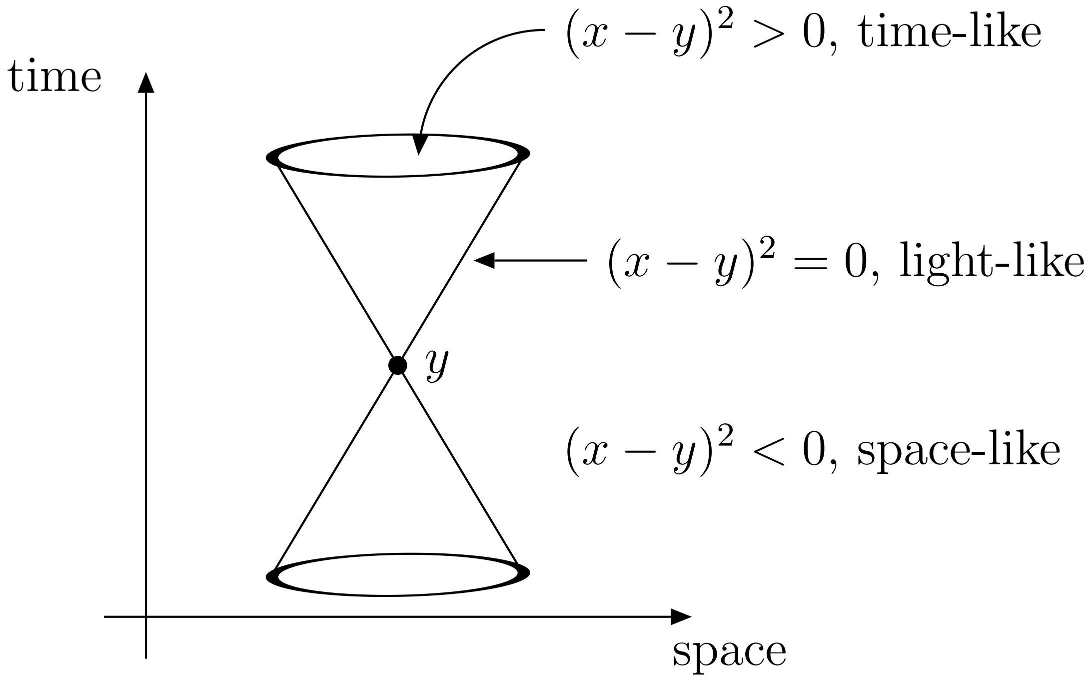
- What about arbitrary space-time separations?

Causality

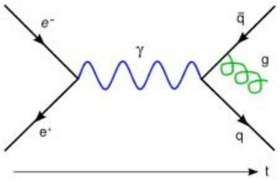


- In order for our theory to be causal we require local space-like separated operators to commute, i.e.

$$[\mathcal{O}_1(x), \mathcal{O}_2(y)] = 0 \qquad \text{for} \qquad (x-y)^2 < 0$$



Causality



- This condition ensures measurement at *x* cannot effect that at *y* if they are not causally connected

- Can show (leave $[\hat{\pi}(x), \hat{\pi}(y)]$ as exercise)

$$[\hat{\phi}(x), \hat{\pi}(y)] = \frac{i}{2}\int \frac{d^3k}{(2\pi)^3}\left(e^{-ik\cdot(x-y)} + e^{ik\cdot(x-y)}\right)$$

$$[\hat{\phi}(x), \hat{\phi}(y)] = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})}\left(e^{-ik\cdot(x-y)} - e^{ik\cdot(x-y)}\right)$$

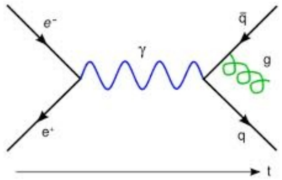
- These are c-number functions (classical numbers)

  (but note this statement is only true in the free theory)

- Can show they vanish for space-like separations

(Chose t=0 and do a rotation or relabel p: first is a δ-function, second is zero)

Propagators



- Compute the amplitude of particle created at *y* to propagate to *x.* Define the propagator

$$D(x-y) \equiv \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} e^{-ik\cdot(x-y)}$$

- Propagator decays outside light-cone but is non-vanishing! Rewrite commutator

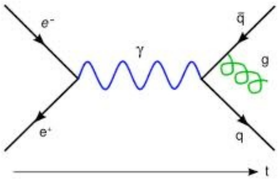
$$[\hat{\phi}(x), \hat{\phi}(y)] = D(x-y) - D(y-x)$$

- Interpretation: Particle can travel in space-like direction from *y* to *x*, but can also travel from *x* to *y*. The amplitudes for these two processes cancel

(In fact one is a particle and the other an antiparticle but it is not obvious for real scalar)

QFT

Feynman Propagator



- An important quantity in interacting theories is the Feynman propagator

$$\Delta_F(x-y) = \langle 0|T\hat{\phi}(x)\hat{\phi}(y)|0\rangle$$

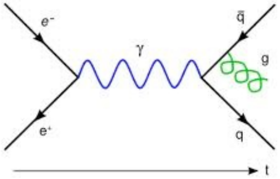
- Here *T* stands for time ordering, such that all operators at later times are placed to the left

$$T\hat{\phi}(x)\hat{\phi}(y) = \begin{cases} \hat{\phi}(x)\hat{\phi}(y), & \text{if } x^0 > y^0 \\ \hat{\phi}(y)\hat{\phi}(x), & \text{if } y^0 > x^0 \end{cases}$$

- Useful to turn this into a four-dimensional integral rather than fixing $E(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$. Find:

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} e^{-ik\cdot(x-y)}$$

Residue Theorem



- Let Γ be anticlockwise closed contour. If $f(z)$ is analytic except for a finite number of singular points z_i in the interior of Γ then

$$\int_{\Gamma} f(z)dz = 2\pi i \sum_{i}^{n} b_i$$

- Here b_i is the residue of $f(z)$ at point z_i. The residue is defined as coefficient c_1 of the Laurent expansion around z_i

$$f(z) = \sum_{n=0}^{\infty} (z - z_i)^n + \frac{c_1}{z - z_i} + \frac{c_2}{(z - z_i)^2} + \dots$$

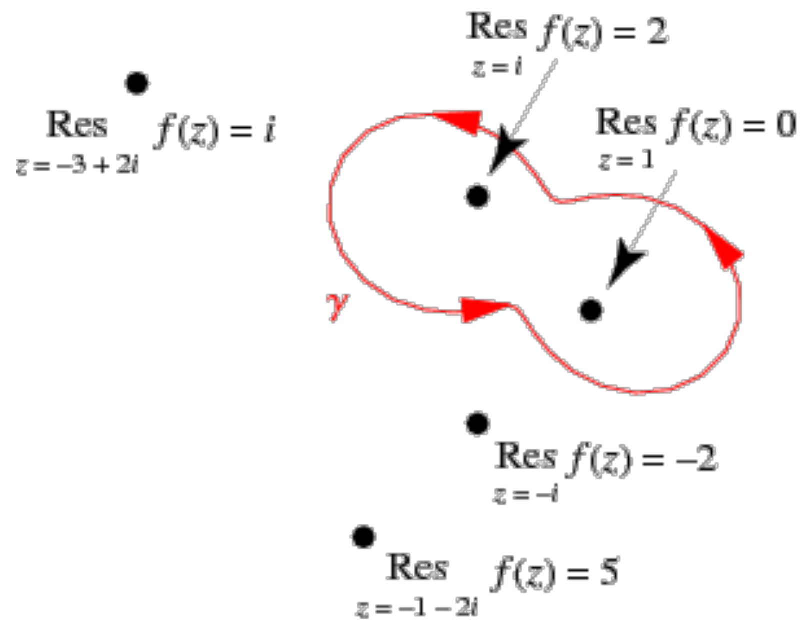
Residue Theorem

- For example

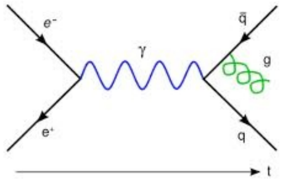
$$f(z) = \frac{3}{(z-1)^2} + \frac{2}{z-i} - \frac{2}{z+i} + \frac{i}{z+3-2i} + \frac{5}{z+1+2i}$$

- Consider the contour integral shown to the right

$$\int_\Gamma f(z)dz = 2\pi i(2+0) = 4\pi i$$



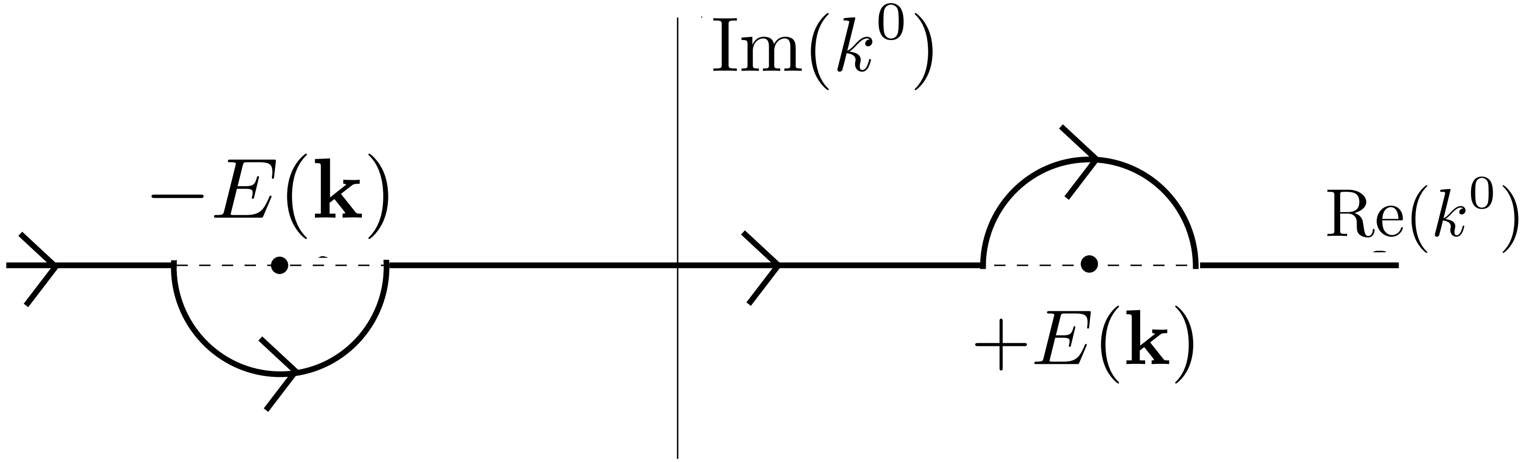
Feynman Propagator



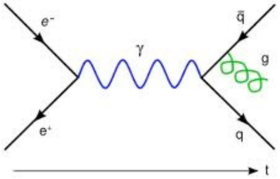
- From $k^2 - m^2 = (k^0)^2 - \mathbf{k}^2 - m^2 = (k^0)^2 - E(\mathbf{k})^2$ then

$$\Delta_F(x-y) = \int \frac{d^3k}{(2\pi)^3} \int \frac{dk^0}{2\pi} \frac{i}{(k^0 - E(\mathbf{k}))(k^0 + E(\mathbf{k}))} e^{-ik^0\cdot(x^0-y^0)} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}$$

- Integrating over k^0 find poles at $k^0 = \pm E(\mathbf{k})$



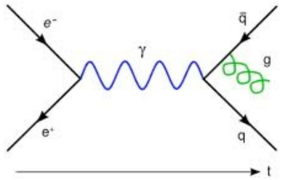
Feynman Propagator



- Case when $x^0 > y^0$
  - Close contour in lower half of plane (integrand vanishes at $k^0 \to -i\infty$)
  - Pole at $k^0 = +E(\mathbf{k})$
  - Residue of $\dfrac{1}{k^2 - m^2}$ is $+\dfrac{1}{2E(\mathbf{k})}$
  - Result for line integral is $-\dfrac{2\pi i}{2E(\mathbf{k})}$ (minus from clockwise contour)
  - Propagator

$$\Delta_F(x-y) = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} e^{-ik\cdot(x-y)} = D(x-y)$$

Feynman Propagator



- Case when $y^0 > x^0$
  - Close contour in upper half of plane (integrand vanishes at $k^0 \to i\infty$)
  - Pole at $k^0 = -E(\mathbf{k})$
  - Residue is now $-\frac{1}{2E(\mathbf{k})}$
  - Result is $-\frac{2\pi i}{2E(\mathbf{k})}$ (minus from pole)
  - Propagator

$$\Delta_F(x-y) = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} e^{-ik\cdot(y-x)} = D(y-x)$$

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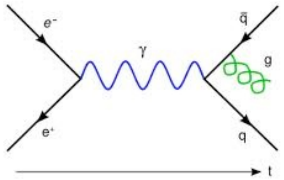
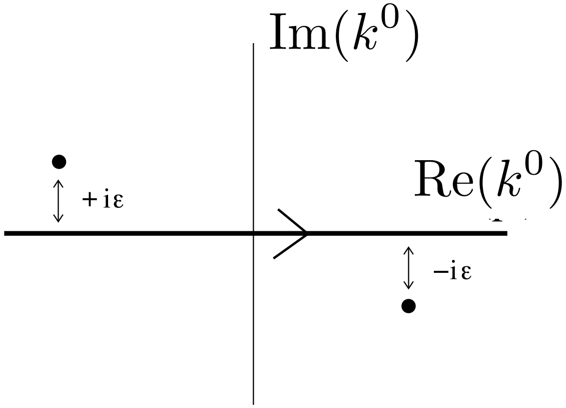
Feynman Propagator



- There is an equivalent way of writing the Feynman propagator

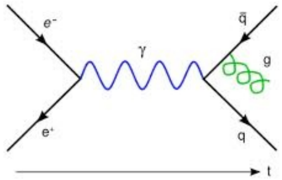
$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)}$$

- This shifts the poles slightly off the real axis, so we can integrate over the real k^0 component



- Equivalent to contour integration

Complex Scalar Fields



- Consider complex scalar field with Lagrangian

$$\mathcal{L} = \partial_\mu \psi^\star \partial^\mu \psi - M^2 \psi^\star \psi$$

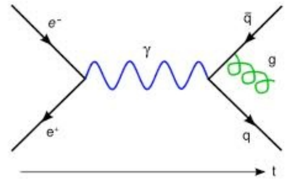
- Invariant under global transformation $\psi \to e^{i\alpha}\psi$
- Associated Noether current $j^\mu = i(\partial^\mu \psi^\star)\psi - i\psi^\star(\partial^\mu \psi)$
- Treat ψ and $\psi^\star$ as independent variables. Equations of motion

$$\partial_\mu \partial^\mu \psi + M^2 \psi = 0$$

$$\partial_\mu \partial^\mu \psi^\star + M^2 \psi^\star = 0$$

- Classical field momentum $\pi = \dfrac{\partial \mathcal{L}}{\partial \dot{\psi}} = \dot{\psi}^\star$

Complex Scalar Fields



- Expand field operator in terms of plane waves

$$\hat{\psi}(x) = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} \left[\hat{c}^\dagger(\mathbf{k}) e^{ik\cdot x} + \hat{b}(\mathbf{k}) e^{-ik\cdot x} \right]$$

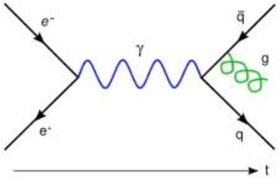
$$\hat{\psi}^\dagger(x) = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} \left[\hat{c}(\mathbf{k}) e^{-ik\cdot x} + \hat{b}^\dagger(\mathbf{k}) e^{ik\cdot x} \right]$$

- Equal-time commutation relations

$$\left[\hat{\psi}(\mathbf{x},t), \hat{\pi}(\mathbf{y},t) \right] = i\delta^3(\mathbf{x}-\mathbf{y})$$ (same for $\hat{\psi}^\dagger, \hat{\pi}^\dagger$)

$$\left[\hat{\psi}(\mathbf{x},t), \hat{\pi}^\dagger(\mathbf{y},t) \right] = \left[\hat{\psi}(\mathbf{x},t), \hat{\psi}(\mathbf{y},t) \right] = \left[\hat{\psi}(\mathbf{x},t), \hat{\psi}^\dagger(\mathbf{y},t) \right] = \ldots = 0$$

Complex Scalar Fields



- Can show these are equivalent to (all other combinations commute)

$$\left[\hat{b}(\mathbf{k_1}), \hat{b}^\dagger(\mathbf{k_2})\right] = \left[\hat{c}(\mathbf{k_1}), \hat{c}^\dagger(\mathbf{k_2})\right] = (2\pi)^3 2E(\mathbf{k_1})\delta^3(\mathbf{k_1} - \mathbf{k_2})$$

- Quantizing complex scalar field leads to two creation operators - interpreted as particles and anti-particles, both of mass *M* and spin-zero

- After normal ordering conserved charge

$$\hat{Q} = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} \left(\hat{c}^\dagger(\mathbf{k})\hat{c}(\mathbf{k}) - \hat{b}^\dagger(\mathbf{k})\hat{b}(\mathbf{k})\right)$$

- $[\hat{H}, \hat{Q}] = 0$ ensuring charge is conserved in quantum theory (number of particles minus anti-particles)