# QFT

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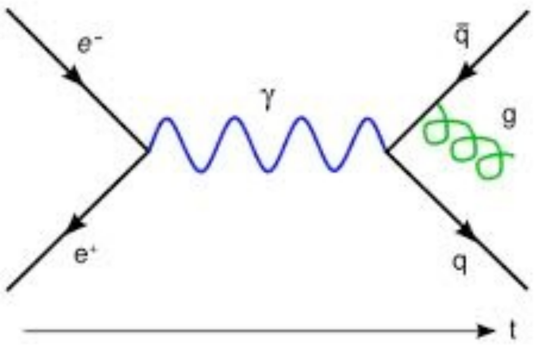
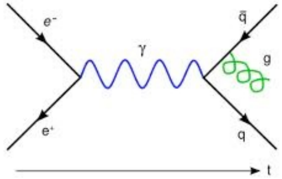
(Notes based on Dr A. Moss' lectures)



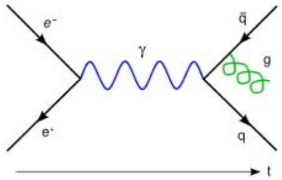

## Lecture 5: Interacting Fields

# Interacting Fields




- Will always discuss quantized fields - drop the hats off operators
- Will consider small perturbations to free theory Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$
- Hamiltonian density of interaction $\mathcal{H}_I = -\mathcal{L}_I$
- E.g. real scalar field $\mathcal{L}_I = -\sum_{n \geq 3} \frac{\lambda_n}{n!} \phi^n$
- What conditions do we require on $\lambda_n$ so the additional terms are small perturbations?

# Interacting Fields

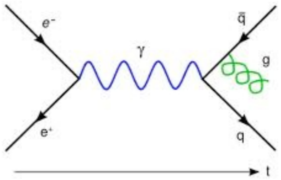



- Dimensional analysis leads to

$$[\mathcal{L}] = 4 \quad [\phi] = 1 \quad [m] = 1 \quad [\lambda_n] = 4 - n$$

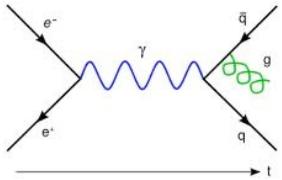
- $n = 3 \quad [\lambda_3] = 1$: Dimensionless parameter is $\lambda_3/E$. These terms are called *relevant* since they are most important at low energies. Since $E>m$, just need $\lambda<m$.
- $n = 4 \quad [\lambda_4] = 0$: Dimensionless parameter is $\lambda_4$. These terms are called *marginal* and are important if $\lambda_4 \sim O(1)$
- $n \geq 5 \quad [\lambda_n] < 0$: Dimensionless parameter is $\lambda_n E^{n-4}$. These terms are called *irrelevant* and are important at high energies

# Interacting Fields




- Irrelevant couplings can cause problems at high energies - lead to non-renormalizable theories
- Doesn't mean quantum theory is useless, but it is incomplete above some energy scale
- Theory still perfectly good as an Effective Field Theory at low energies - decoupling
- We will only consider theories with relevant/marginal couplings and which are weakly interacting

# $\phi^4$ theory




- Lagrangian and interaction Hamiltonian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \qquad \mathcal{H}_I = \frac{\lambda}{4!}\phi^4 \qquad \lambda << 1$$

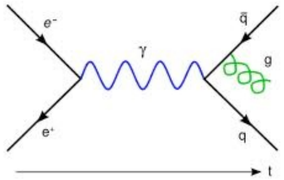
- Interaction term contains:

  $a^\dagger(\mathbf{k})a^\dagger(\mathbf{k})a^\dagger(\mathbf{k})a^\dagger(\mathbf{k})$ , $a^\dagger(\mathbf{k})a^\dagger(\mathbf{k})a^\dagger(\mathbf{k})a(\mathbf{k})$, etc

  and so can create and destroy particles

- Particle number is no longer conserved $[H, N] \neq 0$

# Scalar Yukawa Theory




- Lagrangian and interaction Hamiltonian:

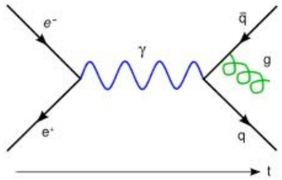
$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \partial_\mu\psi^\star\partial^\mu\psi - \frac{1}{2}m^2\phi^2 - M^2\psi^\star\psi - g\psi^\star\psi\phi$$

$$\mathcal{H}_I = g\psi^\star\psi\phi \qquad g << M, m$$

- Individual particle numbers not conserved

- Symmetry ensuring number of $\psi$ particles minus anti-particles (denoted $\bar{\psi}$ ) is conserved

# Dyson's Formula

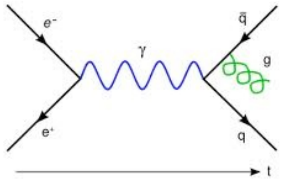


- Recall in interaction picture $i\frac{\partial}{\partial t}|\psi(t)\rangle_I = H_I(t)|\psi(t)\rangle_I$
- Write solution in terms of unitary time evolution operator such that $U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3)$ and $U(t,t) = 1$

$$|\psi(t)\rangle_I = U(t,t_0)|\psi(t_0)\rangle_I$$

- Requires that $i\frac{dU}{dt} = H_I(t)U$
- If $H_I(t)$ were a function $U(t,t_0) = \exp\left(-i\int_{t_0}^{t} H_I(t')dt'\right)$
- Since it is an operator there are ordering issues (apparent when expanding the exponential in powers of $H_I(t)$ )

# Dyson’s Formula




- To solve integrate time evolution equation

$$U(t,t_0) = 1 - i\int_{t_0}^{t} H_I(t_1)U(t_1,t_0)dt_1$$

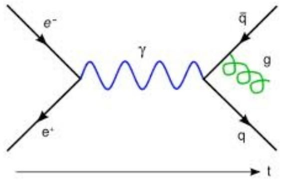
- Still depends on evolution operator. Substitute new expression into integrand ( $t_2 < t_1 < t$ )

$$U(t,t_0) = 1 - i\int_{t_0}^{t} H_I(t_1)dt_1 - \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2)\ldots$$

- Can be written as time-ordered exponential

$$U(t,t_0) = T\exp\left(-i\int_{t_0}^{t} H_I(t')dt'\right) \qquad T\hat{\phi}(x)\hat{\phi}(y) = \begin{cases} \hat{\phi}(x)\hat{\phi}(y), & \text{if } x^0 > y^0 \\ \hat{\phi}(y)\hat{\phi}(x), & \text{if } y^0 > x^0 \end{cases}$$

# S-Matrix

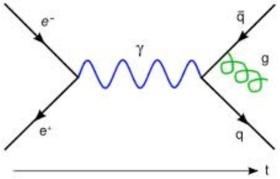



*Assume initial & final states are eigenstates of free theory*

- Initial state $|i\rangle$ at $t = -\infty$ and final state $|f\rangle$ at $t = +\infty$ are eigenstates of free Hamiltonian $H_0$
- At $t = \pm\infty$ particles are far separated and don't feel the effects of each other
- Initial and final states are eigenstates of number operator with $[H_0, N] = 0$, but crucially $[H_I, N] \neq 0$
- Particles briefly interact. Probability of going from $|i\rangle$ to $|f\rangle$

$$\lim_{t_\pm \to \pm\infty} \langle f|\, U(t_+, t_-)\, |i\rangle \equiv \langle f|\, S\, |i\rangle \qquad S = T\exp\left(-i\int_{-\infty}^{\infty} H_I(t')dt'\right)$$

# Scalar Yukawa Theory

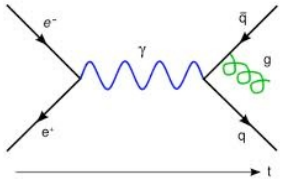


- Lets look at our scalar Yukawa theory

$$H_I = g \int d^3x \, \psi^\dagger(x)\psi(x)\phi(x)$$

- Interaction Hamiltonian contains
  - $\phi \sim a + a^\dagger$ which can create/destroy $\phi$ particles (call these mesons)
  - $\psi \sim b + c^\dagger$ which can create $\bar{\psi}$ and destroy $\psi$ (call these nucleons)
  - $\psi^\dagger \sim b^\dagger + c$ which can create $\psi$ and destroy $\bar{\psi}$

# Meson Decay




- $\phi \to \psi\bar{\psi}$: Initial state contains single meson of momentum $p$
  Final state a nucleon-anti-nucleon pair of $q_1$ and $q_2$

$$|i\rangle = a^\dagger(\mathbf{p})|0\rangle \qquad |f\rangle = b^\dagger(\mathbf{q}_1)c^\dagger(\mathbf{q}_2)|0\rangle$$

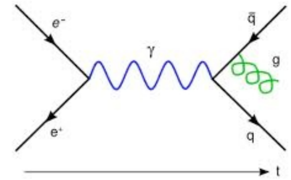
- To leading order

$$\langle f|S|i\rangle = -ig\langle 0| \int d^4x\, c(\mathbf{q}_2)b(\mathbf{q}_1)\psi^\dagger(x)\,\psi(x)\,\phi(x)a^\dagger(\mathbf{p})|0\rangle$$

- Expand out $\phi \sim a + a^\dagger$. Only term which contributes is:

$$\langle f|S|i\rangle = -ig\langle 0| \int d^4x\, c(\mathbf{q}_2)b(\mathbf{q}_1)\psi^\dagger(x)\,\psi(x) \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} a(\mathbf{k})a^\dagger(\mathbf{p})e^{-ik\cdot x}|0\rangle$$

# Meson Decay




- Commute $a(\mathbf{k})$ past $a^{\dagger}(\mathbf{p})$ and integrate

$$\langle f|S|i\rangle = -ig\langle 0|\int d^4x\, c(\mathbf{q_2})b(\mathbf{q_1})\psi^{\dagger}(x)\,\psi(x)e^{-ip\cdot x}|0\rangle$$

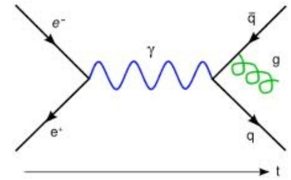
- Similarly expand $\psi \sim b + c^{\dagger}$ and $\psi^{\dagger} \sim b^{\dagger} + c$

$$\langle f|S|i\rangle = -ig\langle 0|\int \frac{d^4x\, d^3k_1\, d^3k_2}{(2\pi)^6 4E(\mathbf{k_1})E(\mathbf{k_2})}\, c(\mathbf{q}_2)b(\mathbf{q}_1)b^{\dagger}(\mathbf{k}_1)c^{\dagger}(\mathbf{k}_2)e^{i(k_1+k_2-p)\cdot x}|0\rangle$$

- Integrate to find $\langle f|S|i\rangle = -ig(2\pi)^4\delta^4(q_1+q_2-p)$

- Delta function puts constraints on decays. Boost to frame with $p=(m,0,0,0)$ then $\mathbf{q}_1 = -\mathbf{q}_2$ and $m = 2\sqrt{M^2+\mathbf{q}^2}$

# Wick's Theorem

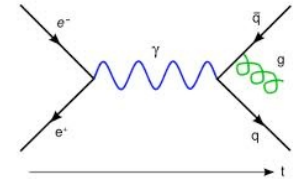



- At general order want to compute $\langle f | T\{H_I(x_1)\ldots H_I(x_n)\} | i\rangle$
- Things will be a lot more convenient if we can move all annihilation operators to the right to act on $|i\rangle$
- Wick's Theorem tells us how to go from time-ordered to normal-ordered products
- Consider scalar field $\phi(x) = \phi^+(x) + \phi^-(x)$ with:

$$\phi^+(x) = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} a(\mathbf{k}) e^{-ik\cdot x}$$

$$\phi^-(x) = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} a^\dagger(\mathbf{k}) e^{ik\cdot x}$$

# Wick's Theorem



- When $x^0 > y^0$

$$T\phi(x)\phi(y) = \phi(x)\phi(y) = \phi^+(x)\phi^+(y) + \phi^-(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^-(y) + [\phi^+(x), \phi^-(y)]$$

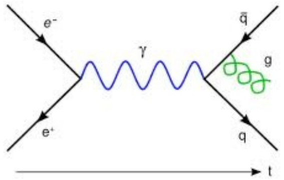
- Commutator is equal to the (Feynman) propagator. Recall

$$\Delta_F(x-y) = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} e^{-ik\cdot(x-y)} = D(x-y)$$

- Time ordered product is therefore

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + D(x-y)$$

# Wick's Theorem




- Case when $y^0 > x^0$

$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + D(y-x)$$

$$\Delta_F(x-y) = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} e^{-ik\cdot(y-x)} = D(y-x)$$

- Putting together for 2 fields:

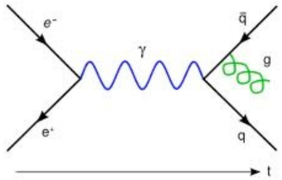
$$T\phi(x)\phi(y) =: \phi(x)\phi(y) : + \Delta_F(x-y)$$

- NB:

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik\cdot(x-y)}$$

QFT

# Wick's Theorem



- Define contraction of pair in string of operators $\ldots \overbrace{\phi(x_1)\ldots\phi(x_2)}\ldots$ to mean replace with Feynman propagator, i.e.

$$\overbrace{\phi(x)\phi(y)} = \Delta_F(x-y)$$

- For any string of operators $T[\phi(x_1)\ldots.\phi(x_n)] \equiv T[\phi_1\ldots.\phi_n]$ Wick's Theorem states

$$T[\phi_1\ldots.\phi_n] =: \phi_1\ldots.\phi_n : + : \text{all possible contractions} :$$

- For example 4 fields:

$$\begin{aligned} T[\phi_1\phi_2\phi_3\phi_4] = &: \phi_1\phi_2\phi_3\phi_4 : + \\ & \Delta_F(x_1-x_2) : \phi_3\phi_4 : + \Delta_F(x_1-x_3) : \phi_2\phi_4 : + \Delta_F(x_1-x_4) : \phi_2\phi_3 : + \\ & \Delta_F(x_2-x_3) : \phi_1\phi_4 : + \Delta_F(x_2-x_4) : \phi_1\phi_3 : + \Delta_F(x_3-x_4) : \phi_1\phi_2 : + \\ & \Delta_F(x_1-x_2)\Delta_F(x_3-x_4) + \Delta_F(x_1-x_3)\Delta_F(x_2-x_4) + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3) \end{aligned}$$