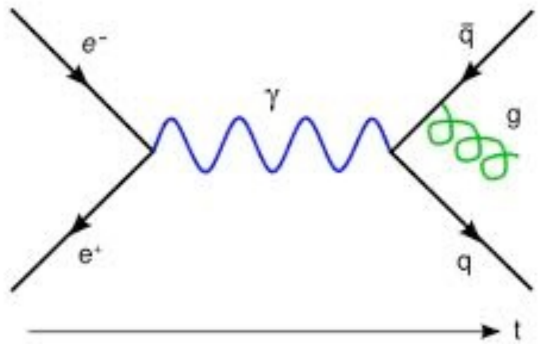
QFT

Dr Tasos Avgoustidis
(Notes based on Dr A. Moss' lectures)



Lecture 7: Interacting Fields - Wick's theorem & Feynman diagrams

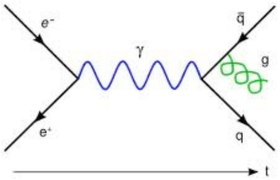
Lecture 6 not taken place due to the UCU industrial action:
https://www.ucu.org.uk/article/10408/UCU-announces-eight-days-of-strikes-starting-this-month-at-60-universities

Today we continue from where we left Lecture 5

(Wick's theorem)

QFT

Wick's Theorem
(Recap)



- Want to compute: $\langle f| T\left\{ H_I(x_1)\ldots H_I(x_n)\right\} |i\rangle$
- Will be convenient if we can move all annihilation operators to the right to act on $|i\rangle$
- Wick's Theorem tells us how to go from time-ordered to normal-ordered products:

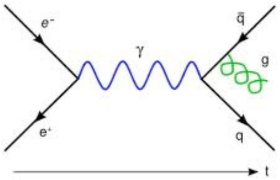
$$T[\phi_1 \ldots .\phi_n] =: \phi_1 \ldots .\phi_n : + : \text{all possible contractions} :$$

defined contraction: $\overbrace{\phi(x)\phi(y)} = \Delta_F(x-y)$

- For example 4 fields:

$$\begin{aligned} T[\phi_1\phi_2\phi_3\phi_4] = &: \phi_1\phi_2\phi_3\phi_4 : + \\ &\Delta_F(x_1-x_2) : \phi_3\phi_4 : + \Delta_F(x_1-x_3) : \phi_2\phi_4 : + \Delta_F(x_1-x_4) : \phi_2\phi_3 : + \\ &\Delta_F(x_2-x_3) : \phi_1\phi_4 : + \Delta_F(x_2-x_4) : \phi_1\phi_3 : + \Delta_F(x_3-x_4) : \phi_1\phi_2 : + \\ &\Delta_F(x_1-x_2)\Delta_F(x_3-x_4) + \Delta_F(x_1-x_3)\Delta_F(x_2-x_4) + \Delta_F(x_1-x_4)\Delta_F(x_2-x_3) \end{aligned}$$

Wick’s Theorem



- Similar story applies for complex scalar fields. The contractions (difference between time- and normal-ordered products) are

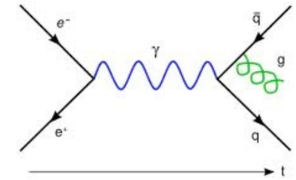
$$\overbrace{\psi(x)\psi^\dagger(y)} = \Delta_F(x-y) \quad \text{and} \quad \overbrace{\psi(x)\psi(y)} = \overbrace{\psi^\dagger(x)\psi^\dagger(y)} = 0$$

$$T\psi(x)\psi^\dagger(y) =: \psi(x)\psi^\dagger(y) : + \Delta_F(x-y)$$

$$T\psi(x)\psi(y) =: \psi(x)\psi(y) :$$

$$T\psi^\dagger(x)\psi^\dagger(y) =: \psi^\dagger(x)\psi^\dagger(y) :$$

Nucleon-Anti-Nucleon Scattering



- $\psi\bar{\psi} \to \psi\bar{\psi}$: Initial and final states contain a nucleon-anti-nucleon pair $|i\rangle = c^\dagger(\mathbf{p}_1)b^\dagger(\mathbf{p}_2)|0\rangle$ $\quad |f\rangle = c^\dagger(\mathbf{q}_1)b^\dagger(\mathbf{q}_2)|0\rangle$

- Contribution to S-matrix at $O(g^2)$

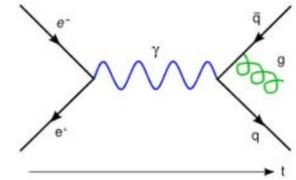
$$\frac{(-ig)^2}{2}\langle 0|\int d^4x\, d^4y\, b(\mathbf{q}_2)c(\mathbf{q}_1)T\left\{\psi^\dagger(x)\,\psi(x)\,\phi(x)\psi^\dagger(y)\,\psi(y)\,\phi(y)\right\}c^\dagger(\mathbf{p}_1)b^\dagger(\mathbf{p}_2)|0\rangle$$

- Expand time-ordered product using Wick's Theorem. Convince yourself the only term that contributes is

$$:\psi^\dagger(x)\,\psi(x)\psi^\dagger(y)\,\psi(y):\Delta_F^\phi(x-y)$$

- Need strings of $b^\dagger c^\dagger bc$ for overlap with initial and final states. 4 such terms, which double up under $x \leftrightarrow y$

Nucleon-Anti-Nucleon Scattering



- Find

$$i(-ig)^2 \int \frac{d^4x\, d^4y\, d^4k}{(2\pi)^4} \frac{e^{ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon} \left[e^{-i(p_1-q_1)\cdot x} e^{-i(p_2-q_2)\cdot y} + e^{-i(p_1+p_2)\cdot x} e^{i(q_1+q_2)\cdot y} \right]$$

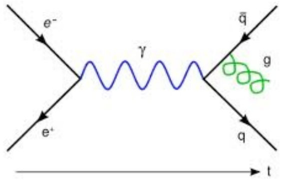
- Can now integrate to obtain amplitude

$$i(-ig)^2 \left[\frac{1}{s - m^2 + i\epsilon} + \frac{1}{t - m^2 + \cancel{i\epsilon}} \right] (2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2)$$

$$s = (p_1 + p_2)^2 \qquad t = (p_1 - q_1)^2$$

- These are ‘s-channel’ and t-channel’ interactions - each has a simple interpretation using Feynman diagrams
- Other scattering processes such as $\psi\psi \to \psi\psi$ and $\bar{\psi}\bar{\psi} \to \bar{\psi}\bar{\psi}$ come from different strings of operators

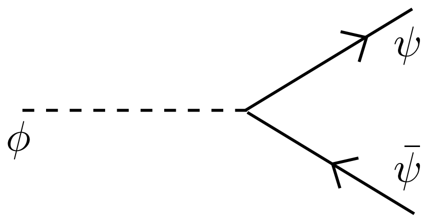
Feynman Diagrams



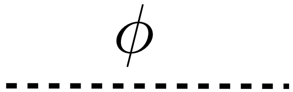
- Computing scattering amplitudes with Wick's Theorem is rather tedious
- Feynman diagrams provide a nice way of pictorially representing the expansion of $\langle f|S|i\rangle$
- We are interested in the terms $\langle f|S-1|i\rangle$ (i.e. not the trivial process where no interaction occurs)
- Draw an external line for each particle in the initial and final state (choose dotted lines for mesons, solid lines for nucleons)
- Add an arrow to nucleons to denote charge (incoming arrow for ψ in initial state)

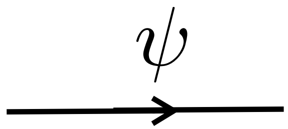
Feynman Rules

- For our scalar Yukawa theory join lines by trivalent vertices



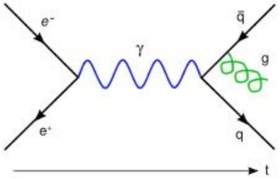
- Add momenta k to each line
- For each vertex $(-ig)(2\pi)^4\delta^4\left(\sum_i k_i\right)$ where momenta are into vertex
- For each internal line integrate the propagator



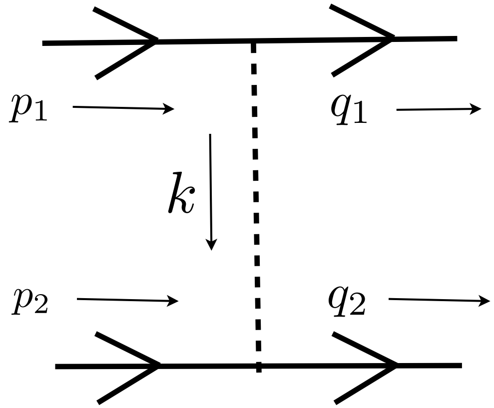


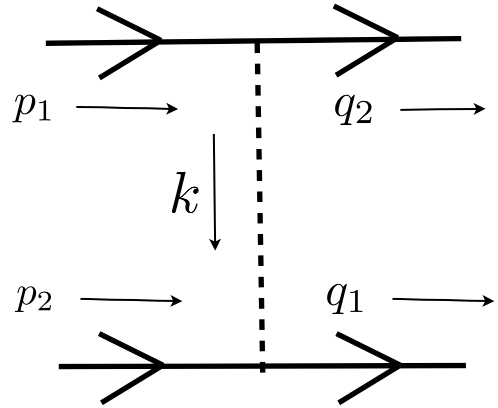
$$\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon} \qquad \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-M^2+i\epsilon}$$

Nucleon Scattering



$$\psi\psi \to \psi\psi$$





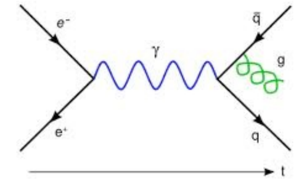
$$i(-ig)^2\left[\frac{1}{t-m^2+\cancel{i\epsilon}}+\frac{1}{u-m^2+\cancel{i\epsilon}}\right](2\pi)^4\delta^4(p_1+p_2-q_1-q_2)$$

- t and u are Mandelstam variables:

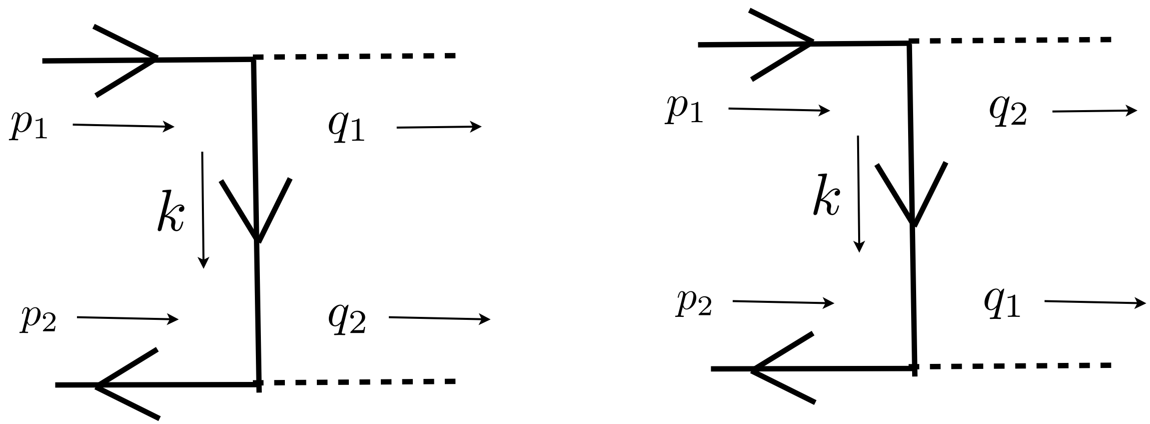
$$t=(p_1-q_1)^2=(p_2-q_2)^2 \qquad u=(p_1-q_2)^2=(p_2-q_1)^2$$

$$s=(p_1+p_2)^2=(q_1+q_2)^2$$

Nucleon-Meson Scattering



$$\psi\bar{\psi} \to \phi\phi$$

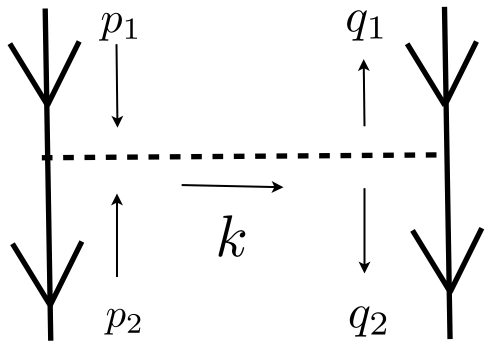


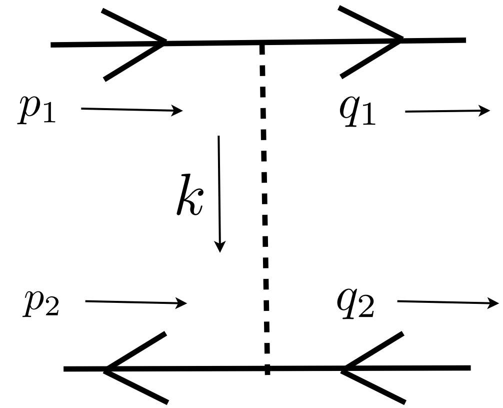
$$i(-ig)^2\left[\frac{1}{t - M^2 + \cancel{i\epsilon}} + \frac{1}{u - M^2 + \cancel{i\epsilon}}\right](2\pi)^4\delta^4(p_1 + p_2 - q_1 - q_2)$$

- Here the exchange particle is the nucleon rather than the meson
- Exchange particles do not satisfy the usual energy dispersion relation ($k^2 = m^2$ for mesons and $k^2 = M^2$ for nucleons) - we call them *virtual* particles

Nucleon-Anti-Nucleon Scattering

$\psi\bar{\psi} \to \psi\bar{\psi}$

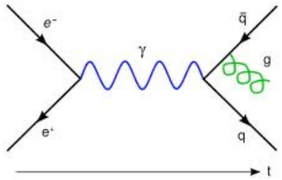




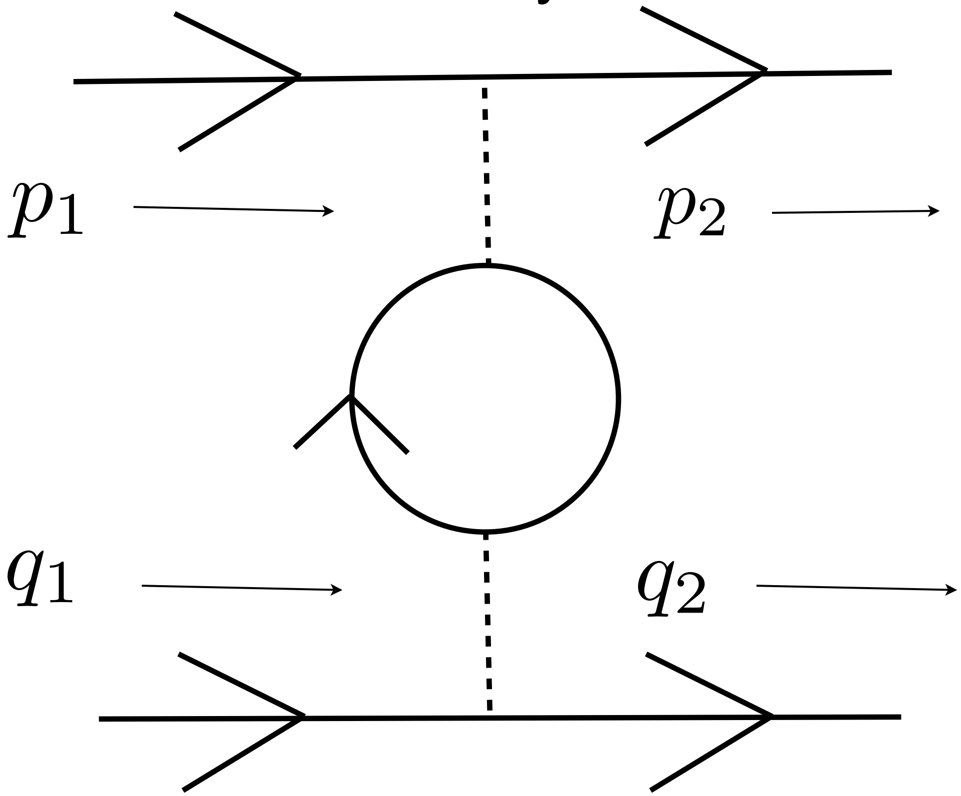
$$i(-ig)^2 \left[\frac{1}{s - m^2 + i\epsilon} + \frac{1}{t - m^2 + \cancel{i\epsilon}} \right] (2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2)$$

- If $m > 2M$ the s-channel term can diverge. However, the meson is unstable for this mass.

Higher Order Terms



- Higher order terms can easily be considered, e.g. $O(g^4)$



However, such diagrams involving loop momenta are often divergent → Renormalisation (not covered in this course)

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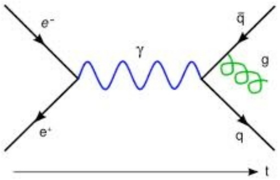
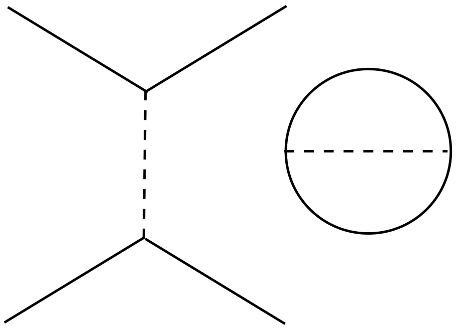
Connected and Amputated Diagrams

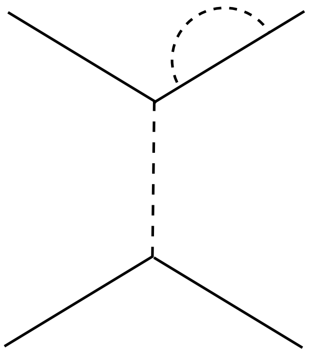


We have assumed that the initial and final states are eigenstates of the free theory Hamiltonian. This is not quite true! However, it can be dealt with as follows:

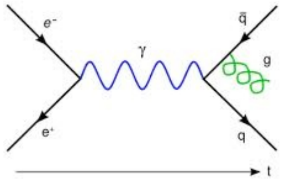
- We consider only connected diagrams, where every part is connected to external leg. Related to the fact that the true vacuum of the interacting theory is not the same as that of free theory



- Do not consider diagrams with loops on external legs. Related to the fact that one-particle states of the interacting theory are not the same as those of the free theory

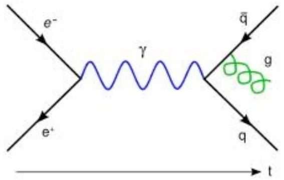


Lorentz Group



- So far we have only considered scalar fields
- Under a Lorentz transformation $x^\mu \to (x')^\mu = \Lambda^\mu{}_\nu x^\nu$ these transform as $\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$. The Λ^{-1} is because we are doing an *active* transformation
- Scalar fields give rise to spin-0 particles
- To describe particles with spin (i.e. they have some intrinsic angular momentum) look at fields which have non-trivial transformations under the Lorentz group
- E.g. a vector field $A^\mu(x) \to \Lambda^\mu{}_\nu \, A^\nu(\Lambda^{-1}x)$. This gives rise to spin-1 particles

Lorentz Group



- In general a field can transform as

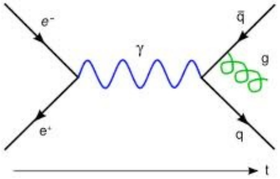
$$\phi^a(x) \to D[\Lambda]^b{}_a \, \phi^b(\Lambda^{-1}x)$$

- Here $D[\Lambda]^b{}_a$ is a matrix which depends on the Lorentz transformation (LT) we are considering. It is a *representation* of the Lorentz group
- It has the same properties as the Lorentz group, i.e.

$$D[\Lambda_1]D[\Lambda_2] = D[\Lambda_1\Lambda_2] \qquad D[\Lambda^{-1}] = D[\Lambda]^{-1}$$

- Want to find all possible representations such that these properties are true
- Look at infinitesimal transformations and study Lie Algebra

Lorentz Group



- Consider transformation $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \epsilon w^{\mu}{}_{\nu}$
- Using definition of LT $\Lambda^{\mu}{}_{\sigma}\eta^{\sigma\tau}\Lambda^{\nu}{}_{\tau} = \eta^{\mu\nu}$
- For terms linear in ϵ then $w^{\mu\nu} + w^{\nu\mu} = 0$
- For infinitesimal LT the matrix needs to be anti-symmetric. This has 6 degrees of freedom, corresponding to the 6 transformations of the Lorentz group
- Introduce basis of 6 anti-symmetric 4x4 matrices

$$(\mathcal{M}^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}$$

- ρ, σ label which matrix, μ, ν the row/column of each matrix