QFT

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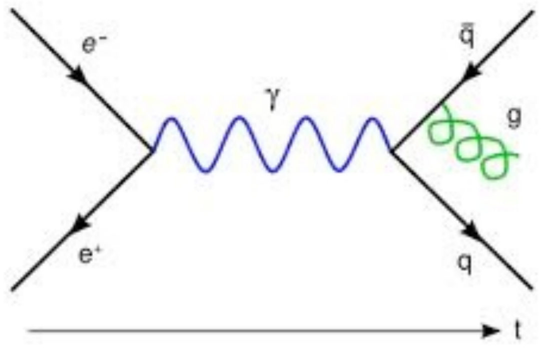
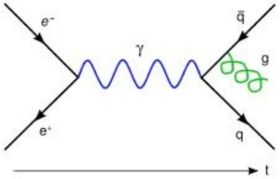
(Notes based on Dr A. Moss' lectures)



Lecture 8: Spinors & Dirac Equation

QFT

Lorentz Group
(Recap)



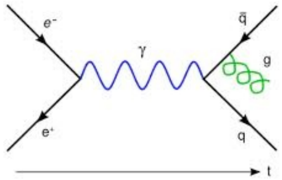
- Consider transformation $\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \epsilon w^{\mu}{}_{\nu}$
- Using definition of LT $\Lambda^{\mu}{}_{\sigma}\eta^{\sigma\tau}\Lambda^{\nu}{}_{\tau} = \eta^{\mu\nu}$
- For terms linear in ϵ then $w^{\mu\nu} + w^{\nu\mu} = 0$
- For infinitesimal LT the matrix needs to be anti-symmetric. This has 6 degrees of freedom, corresponding to the 6 transformations of the Lorentz group
- Introduce basis of 6 anti-symmetric 4x4 matrices

$$(\mathcal{M}^{\rho\sigma})^{\mu\nu} = \eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}$$

- ρ, σ label which matrix, μ, ν the row/column of each matrix

QFT

Lorentz Group



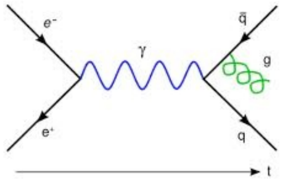
- Lower one index $(\mathcal{M}^{\rho\sigma})^{\mu}{}_{\nu} = \eta^{\rho\mu}\delta^{\sigma}{}_{\nu} - \eta^{\sigma\mu}\delta^{\rho}{}_{\nu}$
- Matrices are now no-longer antisymmetric on μ, ν
- Infinitesimal boosts:

$$(\mathcal{M}^{01})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (\mathcal{M}^{02})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (\mathcal{M}^{03})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

- Infinitesimal rotations:

$$(\mathcal{M}^{12})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (\mathcal{M}^{13})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (\mathcal{M}^{23})^{\mu}{}_{\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Lorentz Group



- Can write any infinitesimal LT in terms of this basis

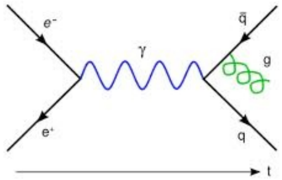
$$w^{\mu}{}_{\nu} = \frac{1}{2}\Omega_{\rho\sigma}(\mathcal{M}^{\rho\sigma})^{\mu}{}_{\nu}$$

- Here $\Omega_{\rho\sigma}$ are six real numbers specifying the LT
- Any finite LT can be written as $\Lambda^{\mu}{}_{\nu} = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}(\mathcal{M}^{\rho\sigma})^{\mu}{}_{\nu}\right)$
- The six basis matrices obey the Lie algebra

$$[\mathcal{M}^{\rho\sigma}, \mathcal{M}^{\tau\nu}] = \eta^{\sigma\tau}\mathcal{M}^{\rho\nu} - \eta^{\rho\tau}\mathcal{M}^{\sigma\nu} + \eta^{\rho\nu}\mathcal{M}^{\sigma\tau} - \eta^{\sigma\nu}\mathcal{M}^{\rho\tau}$$

- Here the row/column index is suppressed. This equation encapsulates the properties of the Lorentz group. We are interested in other matrices which satisfy this algebra

Spinor Representation



- Interested in finding other representations of the Lorentz group
- The Clifford algebra is defined as $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}1$
- γ^μ with $\mu = 0, 1, 2, 3$ are a set of 4 matrices, so

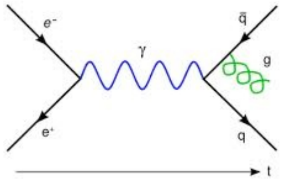
$$(\gamma^0)^2 = 1 \qquad (\gamma^i)^2 = -1 \qquad \gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu \quad \nu \neq \mu$$

- Simplest representation is 4x4 matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

- Where σ^i are the Pauli matrices

Spinor Representation



- There is a "unique" (up to a similarity transformation) irreducible representation of the Clifford algebra. These γ^μ matrices define the chiral (or Weyl) rep

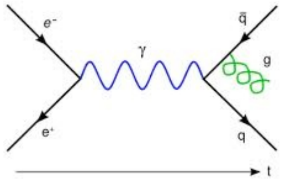
- Consider the commutator of two γ^μ

$$S^{\rho\sigma} = \frac{1}{4}[\gamma^\rho, \gamma^\sigma] = \frac{1}{2}\gamma^\rho\gamma^\sigma - \frac{1}{2}\eta^{\rho\sigma}$$

- Can show these form a representation of the Lorentz group such that

$$[S^{\rho\sigma}, S^{\tau\nu}] = \eta^{\sigma\tau}S^{\rho\nu} - \eta^{\rho\tau}S^{\sigma\nu} + \eta^{\rho\nu}S^{\sigma\tau} - \eta^{\sigma\nu}S^{\rho\tau}$$

Dirac Spinor



- Introduce a Dirac spinor, a complex valued object with 4 components which transforms as

$$\psi^\alpha(x) \to S[\Lambda]^\alpha{}_\beta \, \psi^\beta(\Lambda^{-1}x)$$

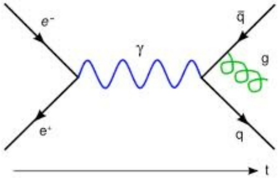
- Here $\alpha = 1, 2, 3, 4$ labels the row/column of the $S^{\mu\nu}$ matrices and

$$\Lambda = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}\mathcal{M}^{\rho\sigma}\right) \qquad S[\Lambda] = \exp\left(\frac{1}{2}\Omega_{\rho\sigma}S^{\rho\sigma}\right)$$

- Particular LT specified by $\Omega_{\rho\sigma}$ - these are the same for both Λ and $S[\Lambda]$

- Lets look at $S[\Lambda]$ in the chiral representation

Dirac Spinor



- For <u>rotations</u> $S^{ij} = \frac{1}{4}[\gamma^i, \gamma^j] = -\frac{i}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$

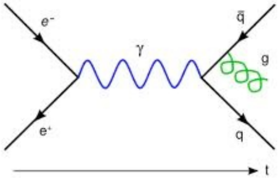
- Writing rotation as $\Omega_{ij} = -\epsilon_{ijk}\varphi^k$ $S[\Lambda] = \begin{pmatrix} e^{i\varphi\cdot\sigma/2} & 0 \\ 0 & e^{i\varphi\cdot\sigma/2} \end{pmatrix}$

- For a rotation of $\varphi = (0, 0, 2\pi)$ $S[\Lambda] = \begin{pmatrix} e^{i\pi\sigma^3} & 0 \\ 0 & e^{i\pi\sigma^3} \end{pmatrix} = -1$

- This means that under 2π rotations $\psi^\alpha(x) \to -\psi^\alpha(x)$ which is not what happens to a vector - different rep

- For rotations in the chiral representation $S[\Lambda]$ is unitary, i.e. $S[\Lambda]^\dagger S[\Lambda] = 1$

Dirac Spinor



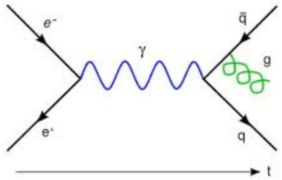
- For <u>boosts</u> $$S^{0i} = \frac{1}{4}[\gamma^0, \gamma^i] = \frac{1}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

- Writing boost as $\Omega_{i0} = \chi_i$ $$S[\Lambda] = \begin{pmatrix} e^{\chi\cdot\sigma/2} & 0 \\ 0 & e^{-\chi\cdot\sigma/2} \end{pmatrix}$$

- For boosts in the chiral representation $S[\Lambda]$ is not unitary, i.e. $S[\Lambda]^\dagger S[\Lambda] \neq 1$

- In general there are no finite dimensional unitary representations of the Lorentz group

Chiral Spinors



- The chiral representation of the Lorentz group is reducible. It decomposes into two irreducible representations

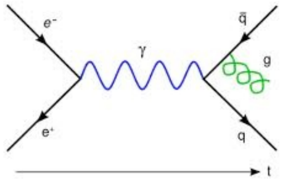
$$\psi = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$

- 2 component objects $u_\pm$ are called Weyl spinors

- Under rotations $u_\pm \to u_\pm e^{i\varphi\cdot\sigma/2}$

- Under boosts $u_\pm \to u_\pm e^{\pm\varphi\cdot\sigma/2}$

Dirac Action

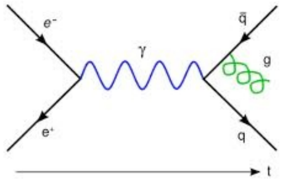


- Want an action which is Lorentz invariant
- Define adjoint in usual way $\psi^\dagger(x) = (\psi^\star)^T(x)$
- Try and form a Lorentz scalar from $\psi^\dagger\psi$ with the spinor index summed over
- Under LT

$$\psi(x) \to S[\Lambda]\,\psi(\Lambda^{-1}x) \qquad \psi^\dagger(x) \to \psi^\dagger(\Lambda^{-1}x)\,S[\Lambda]^\dagger$$

- Therefore $\psi^\dagger\psi$ is not a Lorentz scalar since $S[\Lambda]$ is not unitary

Dirac Action



- If we choose a representation of the Clifford algebra which satisfies $(\gamma^0)^\dagger = \gamma^0 \quad (\gamma^i)^\dagger = -\gamma^i$ then $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^\dagger$

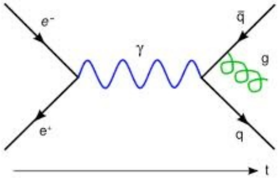
- Can show this gives $S[\Lambda]^\dagger = \gamma^0 S[\Lambda]^{-1}\gamma^0$

- With this in mind define the Dirac conjugate $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$

- Can form Lorentz invariant objects from Dirac spinor and its conjugate, e.g. scalars and vectors

$$\bar{\psi}(x)\psi(x) = \bar{\psi}(\Lambda^{-1}x)\psi(\Lambda^{-1}x)$$
$$\bar{\psi}(x)\gamma^\mu\psi(x) = \Lambda^\mu{}_\nu\,\bar{\psi}(\Lambda^{-1}x)\gamma^\nu\psi(\Lambda^{-1}x)$$

Dirac Equation



- Can construct a Lorentz invariant action

$$S = \int d^4x \, \bar{\psi}(x) \, (i\gamma^\mu \partial_\mu - m) \, \psi(x)$$

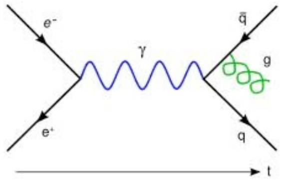
- After quantization this theory will describe particles of mass m and spin-1/2
- Varying with respect to $\bar{\psi}$ gives the Dirac equation

$$(i\gamma^\mu \partial_\mu - m) \, \psi(x) = 0$$

- First order in derivatives but Lorentz invariant
- Mixes up components of spinor but can show each individually solves Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m) \, \psi = 0$$

Weyl Equation



- Let's decompose the Dirac Lagrangian into chiral spinors

$$\mathcal{L} = (u_+^\dagger, u_-^\dagger) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left[i \begin{pmatrix} 0 & \partial_0 + \sigma^i \partial_i \\ \partial_0 - \sigma^i \partial_i & 0 \end{pmatrix} - m \right] \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$

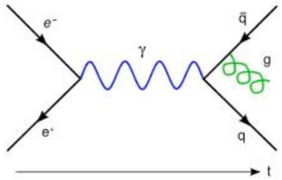
$$\mathcal{L} = i u_-^\dagger \sigma^\mu \partial_\mu u_- + i u_+^\dagger \bar{\sigma}^\mu \partial_\mu u_+ - m(u_-^\dagger u_+ + u_+^\dagger u_-)$$

where $\sigma^\mu = (1, \sigma^i) \quad \bar{\sigma}^\mu = (1, -\sigma^i)$

- For a massless fermion the chiral spinors decouple and they satisfy the Weyl equations of motion

$$i\sigma^\mu \partial_\mu u_- = 0 \qquad i\bar{\sigma}^\mu \partial_\mu u_+ = 0$$

γ^5

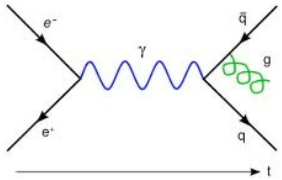


- The Lorentz transformation matrices $S[\Lambda]$ came out block diagonal in the chiral representation
- How do we define chiral spinors in a general representation of the Clifford algebra?
- Introduce the fifth gamma matrix $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$
- This satisfies $\{\gamma^5, \gamma^\mu\} = 0 \quad (\gamma^5)^2 = 1$
- Define a projection operator $P_\pm = \frac{1}{2}(1 \pm \gamma^5)$

$$P_\pm^2 = P_\pm \qquad P_+ P_- = 0$$

- Define chiral spinors by $\psi_\pm = P_\pm \psi$
- In chiral representation $\psi_+ = \begin{pmatrix} u_+ \\ 0 \end{pmatrix} \quad \psi_- = \begin{pmatrix} 0 \\ u_- \end{pmatrix}$

Symmetries

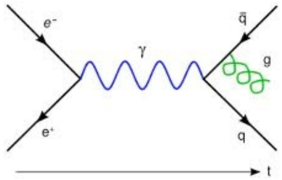


- The Dirac Lagrangian $\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$ enjoys a number of symmetries
- For space-time translations spinor transforms $\delta\psi = \epsilon^\mu \partial_\mu \psi$
- Lagrangian depends on $\partial_\mu \psi$ not $\partial_\mu \bar{\psi}$
- Recall previous definition of energy-momentum tensor

$$T^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial^\nu \phi_a - \eta^{\mu\nu} \mathcal{L}$$

- Conserved currents arise when equations of motion are satisfied - can set $\mathcal{L}$ to zero
- For Dirac Lagrangian obtain $T^{\mu\nu} = i\bar{\psi}\gamma^\mu \partial^\nu \psi$

Symmetries



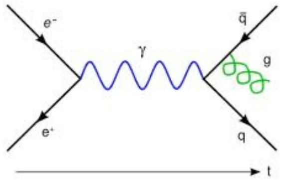
- Under LT $\psi^\alpha(x) \to S[\Lambda]^\alpha{}_\beta \, \psi^\beta(\Lambda^{-1}x)$
- Work infinitesimally $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + w^\mu{}_\nu$

$$\psi^\alpha(x) = \left[\delta^\alpha{}_\beta + \frac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\alpha{}_\beta + \ldots\right]\left[\psi^\beta(x) - w^\mu{}_\nu x^\nu \partial_\mu \psi^\beta(x) + \ldots\right]$$

$$\delta\psi^\alpha = -w^\mu{}_\nu x^\nu \partial_\mu \psi^\alpha + \frac{1}{2}\Omega_{\rho\sigma}(S^{\rho\sigma})^\alpha{}_\beta \psi^\beta$$

- Remember $w^\mu{}_\nu = \frac{1}{2}\Omega_{\rho\sigma}(\mathcal{M}^{\rho\sigma})^\mu{}_\nu$, $(\mathcal{M}^{\rho\sigma})^\mu{}_\nu = \eta^{\rho\mu}\delta^\sigma{}_\nu - \eta^{\sigma\mu}\delta^\rho{}_\nu$
- This means that $w_{\mu\nu} = \Omega_{\mu\nu}$
- Obtain $\delta\psi^\alpha = -w^{\mu\nu}\left[x_\nu \partial_\mu \psi^\alpha - \frac{1}{2}(S_{\mu\nu})^\alpha{}_\beta \psi^\beta\right]$

Symmetries



- Now apply Noether's theorem (again setting $\mathcal{L}=0$) to find conserved current

$$j^{\mu} = -w^{\rho\nu}\left[x_{\nu}T^{\mu}{}_{\rho} - i\bar{\psi}\gamma^{\mu}S_{\rho\nu}\psi\right]$$

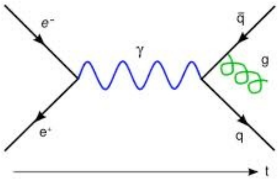
- Left choice of $w^{\mu\nu}$ explicit. Strip it off to give 6 different currents

$$(\mathcal{J}^{\mu})^{\rho\sigma} = x^{\rho}T^{\mu\sigma} - x^{\sigma}T^{\mu\rho} - i\bar{\psi}\gamma^{\mu}S^{\rho\sigma}\psi$$

  which satisfy $\partial_{\mu}(\mathcal{J}^{\mu})^{\rho\sigma} = 0$

- After quantization the final term will be responsible for providing single particle states with internal angular momentum

Symmetries

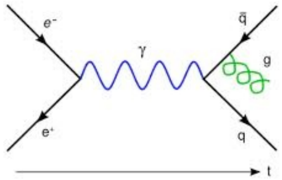


- Dirac Lagrangian is invariant under rotating phase of spinor $\psi \rightarrow e^{-i\alpha}\psi$ or $\delta\psi = -i\alpha\psi$
- This gives rise to a conserved vector current $j_V^\mu = \bar{\psi}\gamma^\mu\psi$
- When m=0 Lagrangian has an extra internal symmetry

$$\psi \rightarrow e^{i\alpha\gamma^5}\psi \qquad \bar{\psi} \rightarrow \bar{\psi}e^{i\alpha\gamma^5}$$

- This gives rise to a conserved axial current $j_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$
- This conserved quantity does not survive the quantization process - an example of anomaly

Plane Wave Solutions



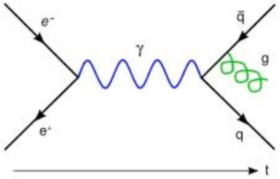
- Want to solve $(i\gamma^\mu \partial_\mu - m)\psi = 0$
- Make the ansatz $\psi = u(\mathbf{p})e^{-ip\cdot x}$
- In chiral representation Dirac equation becomes

$$(\gamma^\mu p_\mu - m)u(\mathbf{p}) = \begin{pmatrix} -m & p_\mu \sigma^\mu \\ p_\mu \bar{\sigma}^\mu & -m \end{pmatrix} u(\mathbf{p}) = 0$$

where $\sigma^\mu = (1, \sigma^i) \quad \bar{\sigma}^\mu = (1, -\sigma^i)$

- Use identity $(p\cdot\sigma)(p\cdot\bar{\sigma}) = p_0^2 - p_i p_j \sigma^i \sigma^j = p_0^2 - p_i p^i = m^2$
- Can easily check the solution is $u(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}$
- Here ξ is a two-component spinor

Plane Wave Solutions



- Also negative frequency solutions $\psi = v(\mathbf{p})e^{ip\cdot x}$

with $v(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta \end{pmatrix}$

- Will be convenient to introduce a basis

$$\xi^{r\,\dagger}\xi^{s} = \delta^{rs} \qquad \eta^{r\,\dagger}\eta^{s} = \delta^{rs}$$

- For example $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$