QFT

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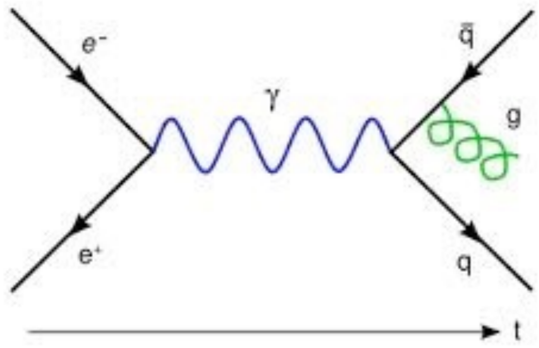
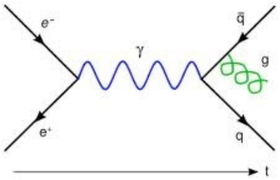
(Notes based on Dr A. Moss' lectures)



Lecture 9: Quantization of Dirac Field

QFT

Recap: Plane Wave Solutions of Dirac eqn



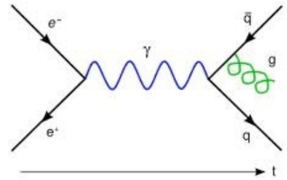
- Want to solve $(i\gamma^\mu \partial_\mu - m)\psi = 0$
- Make the ansatz $\psi = u(\mathbf{p})e^{-ip\cdot x}$
- In chiral representation Dirac equation becomes

$$(\gamma^\mu p_\mu - m)u(\mathbf{p}) = \begin{pmatrix} -m & p_\mu \sigma^\mu \\ p_\mu \bar{\sigma}^\mu & -m \end{pmatrix} u(\mathbf{p}) = 0$$

where $\sigma^\mu = (1, \sigma^i)$, $\bar{\sigma}^\mu = (1, -\sigma^i)$

- Use identity $(p\cdot\sigma)(p\cdot\bar{\sigma}) = p_0^2 - p_i p_j \sigma^i \sigma^j = p_0^2 - p_i p^i = m^2$
- Can easily check the solution is $u(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}$
- Here ξ is a two-component spinor

Plane Wave Solutions



- Also negative frequency solutions $\psi = v(\mathbf{p})e^{ip\cdot x}$

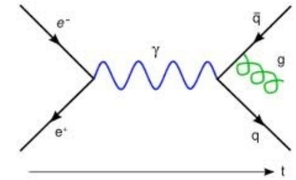
  with $v(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta \end{pmatrix}$

- Will be convenient to introduce a basis

$$\xi^{r\,\dagger}\xi^{s} = \delta^{rs} \qquad \eta^{r\,\dagger}\eta^{s} = \delta^{rs}$$

- For example $\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Dirac Field



- We will now quantize the Dirac field. Recall

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

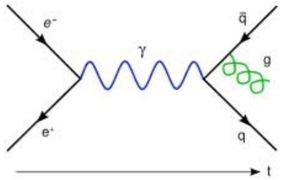
- Will proceed as for scalar field. Define conjugate momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\bar{\psi}\gamma^0 = i\psi^\dagger$$

- Note this does not involve the time derivative. True for equations of motion first order in derivatives - only need to specify ψ , $\psi^\dagger$ on initial time slice

- Four real degrees of freedom (will end up being spin up and spin down for particle and anti-particle)

Dirac Field



- Try imposing equal-time commutation relations

$$[\psi_\alpha(\mathbf{x},t),\psi_\beta(\mathbf{y},t)] = \left[\psi_\alpha^\dagger(\mathbf{x},t),\psi_\beta^\dagger(\mathbf{y},t)\right] = 0$$

$$\left[\psi_\alpha(\mathbf{x},t),\psi_\beta^\dagger(\mathbf{y},t)\right] = \delta_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y})$$

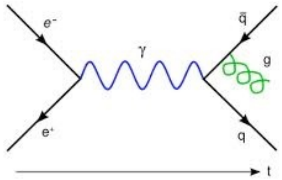
- No factor of *i* due to definition of conjugate momentum
- Expand field as sum of plane waves

$$\psi(x) = \sum_{s=1}^{2}\int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})}\left[v^s(\mathbf{k})c^{s\,\dagger}(\mathbf{k})e^{ik\cdot x} + u^s(\mathbf{k})b^s(\mathbf{k})e^{-ik\cdot x}\right]$$

$$\psi^\dagger(x) = \sum_{s=1}^{2}\int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})}\left[v^{s\,\dagger}(\mathbf{k})c^s(\mathbf{k})e^{-ik\cdot x} + u^{s\,\dagger}(\mathbf{k})b^{s\,\dagger}(\mathbf{k})e^{ik\cdot x}\right]$$

$$u^s(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \qquad v^s(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\eta^s \end{pmatrix}$$

Dirac Field



- Operators $b^{s\,\dagger}(\mathbf{k})$ create particles associated with $u^s(\mathbf{k})$
- Operators $c^{s\,\dagger}(\mathbf{k})$ create particles associated with $v^s(\mathbf{k})$

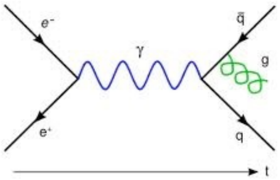
- The equal-time commutation relations then imply

$$\left[b^r(\mathbf{k_1}), b^{s\,\dagger}(\mathbf{k_2})\right] = (2\pi)^3 2E(\mathbf{k_1})\delta^{rs}\delta^3(\mathbf{k_1} - \mathbf{k_2})$$

$$\left[c^r(\mathbf{k_1}), c^{s\,\dagger}(\mathbf{k_2})\right] = -(2\pi)^3 2E(\mathbf{k_1})\delta^{rs}\delta^3(\mathbf{k_1} - \mathbf{k_2})$$

- All other commutators vanish. The minus sign will be disastrous for our theory

Hamiltonian



- Lets now compute the Hamiltonian of the theory

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \bar{\psi}(-i\gamma^i\partial_i + m)\psi \qquad H = \int d^3x\,\mathcal{H}$$

$$(-i\gamma^i\partial_i + m)\psi(x) = \sum_{s=1}^{2}\int \frac{d^3k}{2(2\pi)^3}\gamma^0\left[-v^s(\mathbf{k})c^{s\,\dagger}(\mathbf{k})e^{ik\cdot x} + u^s(\mathbf{k})b^s(\mathbf{k})e^{-ik\cdot x}\right]$$

- Make use of inner product relations

$$u^{r\,\dagger}(\mathbf{k})u^s(\mathbf{k}) = v^{r\,\dagger}(\mathbf{k})v^s(\mathbf{k}) = 2E(\mathbf{k})\delta^{rs}$$

$$u^{r\,\dagger}(\mathbf{k})v^s(-\mathbf{k}) = v^{r\,\dagger}(\mathbf{k})u^s(-\mathbf{k}) = 0$$

- Find $H = \int \frac{d^3k}{2(2\pi)^3}\sum_{s=1}^{2}\left(b^{s\,\dagger}(\mathbf{k})b^s(\mathbf{k}) - c^s(\mathbf{k})c^{s\,\dagger}(\mathbf{k})\right)$

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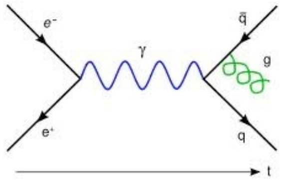
Hamiltonian



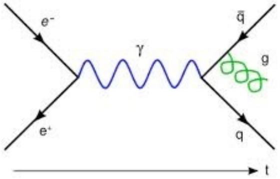
- After normal ordering (will keep in the vacuum contribution for now)

$$H = \int \frac{d^3k}{2(2\pi)^3} \sum_{s=1}^{2} \left(b^{s\,\dagger}(\mathbf{k}) b^s(\mathbf{k}) - c^{s\,\dagger}(\mathbf{k}) c^s(\mathbf{k}) + 2E(\mathbf{k})(2\pi)^3 \delta^3(0) \right)$$

- Minus sign is disastrous - Hamiltonian is unbounded below

- What we missed: Spin 1/2 particles are fermions - by applying commutation relations we are missing the minus sign under the exchange of two particles

Fermionic Quantization



- To have states obeying Fermi-Dirac statistics we need to impose equal time anti-commutation relations

$$\{\psi_\alpha(\mathbf{x},t),\psi_\beta(\mathbf{y},t)\} = \left\{\psi^\dagger_\alpha(\mathbf{x},t),\psi^\dagger_\beta(\mathbf{y},t)\right\} = 0$$

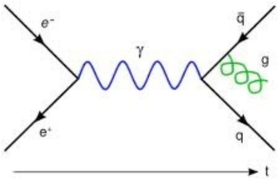
$$\left\{\psi_\alpha(\mathbf{x},t),\psi^\dagger_\beta(\mathbf{y},t)\right\} = \delta_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y})$$

- These are equivalent to

$$\left\{b^r(\mathbf{k_1}),b^{s\,\dagger}(\mathbf{k_2})\right\} = \left\{c^r(\mathbf{k_1}),c^{s\,\dagger}(\mathbf{k_2})\right\} = (2\pi)^3 2E(\mathbf{k_1})\delta^{rs}\delta^3(\mathbf{k_1}-\mathbf{k_2})$$

with all other anti-commutators vanishing

Hamiltonian



- After normal ordering (again keeping the vacuum contribution)

$$H = \int \frac{d^3k}{2(2\pi)^3} \sum_{s=1}^{2} \left(b^{s\,\dagger}(\mathbf{k}) b^s(\mathbf{k}) + c^{s\,\dagger}(\mathbf{k}) c^s(\mathbf{k}) - 2E(\mathbf{k})(2\pi)^3 \delta^3(0) \right)$$

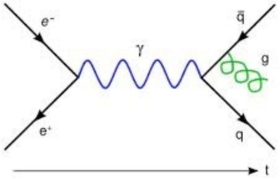
- Hamiltonian is now bounded from below

- Note although we typically throw away the vacuum contribution it has the opposite sign to bosons. These could in principle partially cancel!

  (cf. Cosmological Constant Problem and

  attempts to address it through Supersymmetry)

Particle States



- Define vacuum to satisfy $b^s(\mathbf{k})|0\rangle = c^s(\mathbf{k})|0\rangle = 0$

- Can check following *commutation* relations

$$[H, b^s(\mathbf{k})] = -E(\mathbf{k})b^s(\mathbf{k}) \qquad [H, b^{s\,\dagger}(\mathbf{k})] = E(\mathbf{k})b^{s\,\dagger}(\mathbf{k})$$

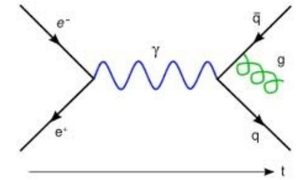
$$[H, c^s(\mathbf{k})] = -E(\mathbf{k})c^s(\mathbf{k}) \qquad [H, c^{s\,\dagger}(\mathbf{k})] = E(\mathbf{k})c^{s\,\dagger}(\mathbf{k})$$

- Use to construct tower of energy eigenstates
- Define one-particle states by $b^{s\,\dagger}(\mathbf{k})|0\rangle$ and $c^{s\,\dagger}(\mathbf{k})|0\rangle$
- Index *s* tells us the spin
- Note that if $|\mathbf{k}_1, s_1; \mathbf{k}_2, s_2\rangle = b^{s_1\,\dagger}(\mathbf{k}_1)b^{s_2\,\dagger}(\mathbf{k}_2)|0\rangle$ then

$$|\mathbf{k}_1, s_1; \mathbf{k}_2, s_2\rangle = -|\mathbf{k}_2, s_2; \mathbf{k}_1, s_1\rangle$$

- Particles obey Fermi-Dirac statistics

Causality



- We will show that (suppressing indices)

$$\{\psi(x), \bar{\psi}(y)\} = 0 \qquad (x-y)^2 < 0$$

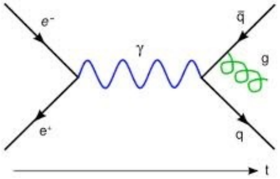
- Define the propagator (4x4 matrix)

$$S_+(x-y) = \langle 0|\psi(x)\bar{\psi}(y)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} e^{-ik\cdot(x-y)} \sum_{s=1}^{2} u^s(\mathbf{k})\bar{u}^s(\mathbf{k})$$

- Use outer product relation $\sum_{s=1}^{2} u^s(\mathbf{k})\bar{u}^s(\mathbf{k}) = \gamma^\mu k_\mu + m$

$$\boxed{S_+(x-y) = (i\gamma\cdot\partial_x + m)D(x-y)}, \; D(x-y) = \int \frac{d^3k}{(2\pi)^3 2E(\mathbf{k})} e^{-ik\cdot(x-y)}$$

Causality



- Similarly find

$$S_-(x-y) = \langle 0|\bar{\psi}(x)\psi(y)|0 = (i\gamma\cdot\partial_x - m)D(x-y)$$

- We can now write the anti-commutator

$$\{\psi(x), \bar{\psi}(y)\} = (i\gamma\cdot\partial_x + m)\left[D(x-y) - D(y-x)\right]$$

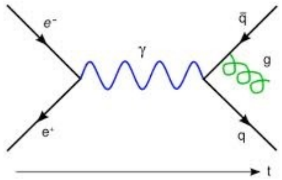
- (Compare to bosonic case)

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x)$$

- It is now the anti-commutator that vanishes for space-like separations!

- Causality? This is enough to guarantee causality as observables are bilinears of fermionic fields.

Feynman Propagator



- Define the Feynman propagator

$$S_F(x-y) = \langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \begin{cases} \langle 0|\psi(x)\bar{\psi}(y)|0\rangle, & \text{if } x^0 > y^0 \\ \langle 0|-\bar{\psi}(y)\psi(x)|0\rangle, & \text{if } y^0 > x^0 \end{cases}$$

- Note the important minus sign in the time-ordered product

- Can show $S_F(x-y) = (i\gamma\cdot\partial_x + m)\Delta_F(x-y)$ where $\Delta_F(x-y)$ is the Feynman propagator in the bosonic case

- This has the 4-momentum integral expression

$$S_F(x-y) = \int \frac{d^4k}{(2\pi)^4}\frac{i(\gamma^\mu k_\mu + m)}{k^2 - m^2 + i\epsilon}e^{-ik\cdot(x-y)}$$

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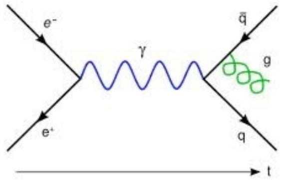
Wick's Theorem



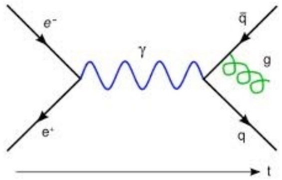
- Fermionic operators anti-commute inside time-ordering T

- The same is true for normal ordering, e.g. $:cc^\dagger: = -c^\dagger c$

- Following same procedure as in bosonic case define the contraction of two operators to be the difference between time and normal ordering

$$\overbrace{\psi(x)\bar{\psi}(y)} = T(\psi(x)\bar{\psi}(y)) - :\psi(x)\bar{\psi}(y): = S_F(x-y)$$

Yukawa Model

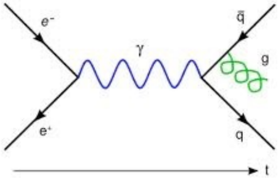


- Let's revisit the toy Yukawa model but now describe the nucleons as fermions

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \bar{\psi}\left(i\gamma^\mu\partial_\mu - M\right)\psi - g\phi\bar{\psi}\psi$$

- Note that now: $[\psi] = 3/2$, $[M] = 1$, $[g] = 0$

Nucleon-Anti-Nucleon Scattering



- $\psi\bar{\psi} \to \psi\bar{\psi}$: Initial and final state contains a nucleon-anti-nucleon pair $|i\rangle = b^{s_1\,\dagger}(\mathbf{p}_1)c^{s_2\,\dagger}(\mathbf{p}_2)|0\rangle$, $|f\rangle = b^{r_1\,\dagger}(\mathbf{q}_1)c^{r_2\,\dagger}(\mathbf{q}_2)|0\rangle$

- Contribution to S-matrix at $O(g^2)$

$$\frac{(-ig)^2}{2}\langle 0| \int d^4x\, d^4y\, c^{r_2}(\mathbf{q}_2)b^{r_1}(\mathbf{q}_1)T\left\{\bar{\psi}(x)\,\psi(x)\,\phi(x)\bar{\psi}(y)\,\psi(y)\,\phi(y)\right\} b^{s_1\,\dagger}(\mathbf{p}_1)c^{s_2\,\dagger}(\mathbf{p}_2)|0\rangle$$

- As in bosonic case only term which contributes in time-ordered product is

$$:\bar{\psi}(x)\,\psi(x)\bar{\psi}(y)\,\psi(y): \Delta_F^{\phi}(x-y)$$

- Have to be careful with spinor indices - calculation is quite tedious (try it!)