Gravity and electroweak vacuum stability beyond the fixed background approximation.

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Outline

Introduction and Motivation

- High precision calculations exist for flat space decay rate. (Isidori et al.[1],Buttazzo et al.[2]).
- Gravitational corrections are potentially significant (Isidori et al.[3]).
- Implications for Inflation. (Kobakhidze and Specer-Smith [4], Herranen et al.[5], Espinosa et al.[6].
- Black holes can seed vacuum decay (Burda, Gregory, Moss[7])

Aims of Our Research

- Numerical study of the effect of gravitational backreaction on vacuum decay.
- Study the effect of non-minimal coupling.

Review of Tunnelling

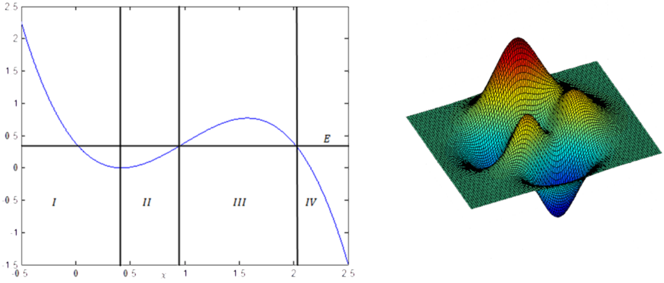


Figure 1: Possible tunnelling potentials

- Tunnelling in 1D: $T = \exp\left(-2\int_{x_1}^{x_2} \mathrm{d}x \sqrt{2(V(x)-E)}\right)$.
- For more degrees of freedom, controlled by dominant path:

$$S = \int \mathrm{d}^4x \left[\frac{1}{2}\nabla_\mu\phi\nabla_\mu\phi + V(\phi)\right] \tag{1}$$

- Decay rate: $\Gamma = A\exp(-(S - S_0))$ where S_0 is the action of the false vacuum solution, $\phi = 0$.

Flat space bounce properties

- Minimal action solutions are O(4) symmetric[8]
- Satisfy $\ddot{\phi} + \frac{3}{\chi}\dot{\phi} - V'(\phi) = 0$, with boundary conditions $\dot{\phi}(0) = 0, \phi(\chi \to \infty) \to 0$ required for finite action.
- Field space fluctuations have a single decreasing action direction ("negative mode").
- Associated to an imaginary contribution to vacuum energy, hence decay.
- Solutions with more negative modes are generally subleading.

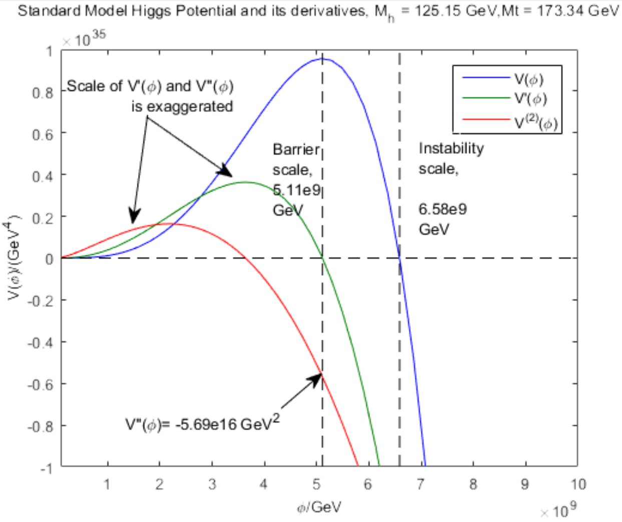
Bounces in the Standard Model

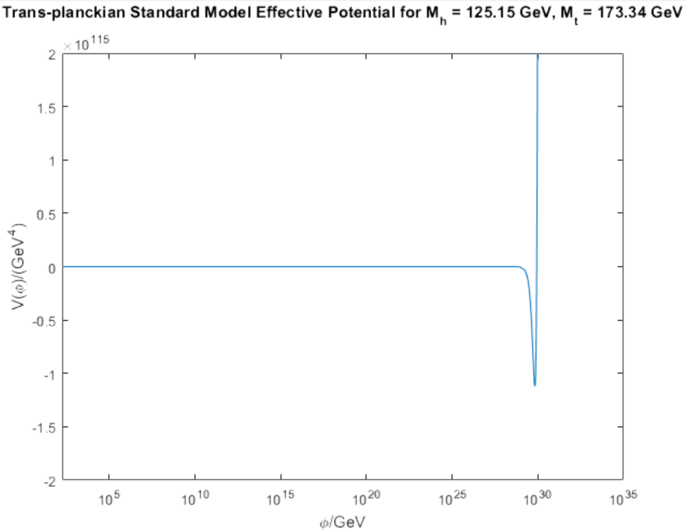
Lee-Weinberg bounce. Assume $V = -|\lambda|\frac{\phi^4}{4}$; solution is:

$$\phi(\chi) = \sqrt{\frac{2}{|\lambda|}}\frac{2R}{R^2 + \chi^2}, S = \frac{8\pi^2}{3|\lambda|}. \quad (2)$$

- R arbitrary. In the Standard Model, dominant bounce has $R \approx \mu_{min}^{-1}$, the scale where λ is minimised.
- For $m_t = 173.34\text{GeV}$ and $m_h = 125.15\text{GeV}$, $\mu_{min} = 2.79 \times 10^{17}\text{GeV}$, just below the Planck scale[1].
- It is reasonable to expect gravitational corrections to be important.Perturbative estimations of their size appears to confirm this[3].

Bounces in the Standard Model




Trans-planckian Standard Model Effective Potential for M_h = 125.15 GeV, M_t = 173.34 GeV
×10^115
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
V(φ)/(GeV^4)
10^5
10^10
10^15
10^20
10^25
10^30
10^35
φ/GeV

Bounces in the Presence of Gravity

- Curved space bounces minimise:

$$S = \int \mathrm{d}^4x\sqrt{|g|}[\frac{1}{2}\nabla_\mu\phi\nabla_\mu\phi + V(\phi) - \frac{M_p^2}{2}(1 - \frac{\xi\phi^2}{M_p^2})R]. \tag{3}$$

- Where ξ is the possible non-minimal coupling. Minimum action solutions are assumed to be O(4) symmetric, giving a (Euclidean) metric ansatz:

$$\mathrm{d}s^2 = \mathrm{d}\chi^2 + a^2(\chi)\mathrm{d}\Omega_{n-1}^2, \tag{4}$$

- Neglecting ξ terms, the instanton equations are:

$$\dot{a}^2 = 1 - \frac{a^2}{3M_p^2}\left(-\frac{\dot{\phi}^2}{2} + V(\phi)\right) \tag{5}$$

$$\ddot{\phi} + \frac{3\dot{a}}{a}\dot{\phi} - V'(\phi) = 0 \tag{6}$$

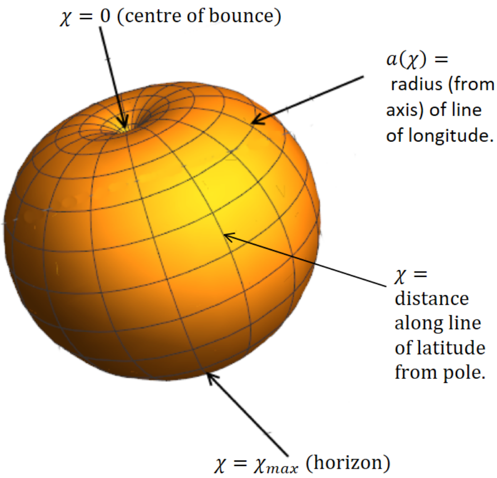


Figure 2: Example Euclidean space, de-Sitter distorted by a bounce. $a(\chi)$ here is *not* the usual scale factor of cosmology, but the radius of a sub-sphere (or sub-de-Sitter).

Interpretation:

- Boundary conditions: $\dot{\phi}(0) = \dot{\phi}(\chi_{max}) = 0, a(0) = 0$ where $\chi_{max} > 0$ defined by $a(\chi_{max}) = 0$.
- Looking for a decay *rate* so need a notion of time.
- Formulate the problem on a static patch, and then switch to $O(4)$ symmetric co-ordinates.
- Metric (Lorentzian) on static patch:

$$\mathrm{d}s_n^2 = \mathrm{d}\chi^2 + a^2(\chi)(-(1-r^2)\mathrm{d}t^2 + (1-r^2)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{n-3}^2) \quad (7)$$

- In de-Sitter space, $a(\chi) = \frac{1}{H}\sin(H\chi)$
- χ is the radial distance from the origin.

Bounces in the Presence of Gravity

- χ_{max} finite $\rightarrow$compact Euclidean space $\rightarrow$periodic Euclidean time $\rightarrow$tunnelling at non-zero temperature.[9].
- $T = T_{GH} = \frac{H}{2\pi}$ for de-Sitter space.

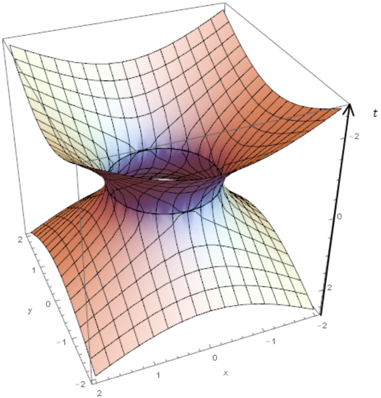


Figure 3: De Sitter space.

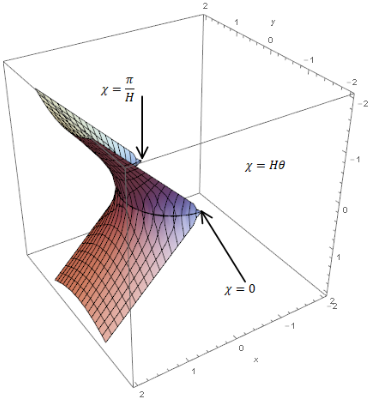


Figure 4: Static Patch of de Sitter space.

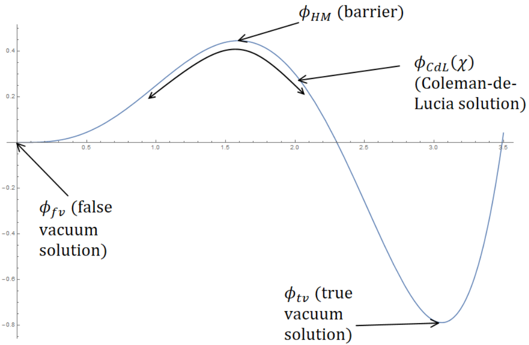


Figure 5: Various solutions in the presence of gravity. The true and false vacuum, and Hawking-Moss solutions are constant, while the CdL solution varies with χ.

The Hawking-Moss Solution

- Analytic solution, $\phi(\chi) = \text{const.} = \phi_{HM}$. Field sits at the top of the barrier.

- Action is $S = 24\pi^2 M_p^4 \left(\frac{1}{V_0} - \frac{1}{V(\phi_{HM})}\right)$ where $V_0 = V(\phi_{fv})$.

- If $\Delta V(\phi_{HM}) = V(\phi_{HM}) - V_0 \ll V_0$ then this is approximately $S = \frac{4\pi}{3}\left(\frac{1}{H_0}\right)^3 \frac{\Delta V(\phi_{HM})}{\left(\frac{H_0}{2\pi}\right)}$, where H_0 is the background Hubble rate, $H_0^2 = \frac{V_0}{3M_p^2}$.

- Decay rate of $\Gamma \approx \exp\left(-\frac{\text{Energy in one Hubble Volume}}{\text{Gibbons Hawking Temperature}}\right)$ - a Boltzmann distribution. Hawking-Moss is a thermal transition.

- Count negative eigenvalues to see if a valid bounce. Radial part of eigenvalue equation:

$$\frac{\mathrm{d}^2 R_{nl}}{\mathrm{d}\chi^2} + 3H_{HM}\cot H_{HM}\chi\frac{\mathrm{d}R_{nl}}{\mathrm{d}\chi} - \frac{H_{HM}^2}{\sin^2 H_{HM}\chi}l(l+2)R_{nl} - (V''(\phi_{HM}) - \lambda)R_{nl} = 0, \tag{8}$$

- where $H_{HM}^2 = \frac{V(\phi_{HM})}{3M_p^2}$, $n, l = 0, 1, 2, \ldots$. Resulting eigenvalues are:

$$\lambda_{nl} = V''(\phi_{HM}) + [n+l]([n+l]+3)H_{HM}^2, \tag{9}$$

- Define $N = n + l$. $V''(\phi_{HM}) < 0$ so $N = 0$ is always negative. Extra negative eigenvalue for $N = 1$ if:

$$V''(\phi_{HM}) + 4H_{HM}^2 < 0 \implies \frac{8\pi G_N V(\phi_{HM})}{3} < -\frac{V''(\phi_{HM})}{4}. \quad (10)$$

- At this critical threshold, the $l = 0, n = 1$ mode obeys:

$$\ddot{R}_{10} + 3H_{HM}\cot(H_{HM}\chi)\dot{R}_{10} + 4H_{HM}^2 R_{10} = 0, \quad (11)$$

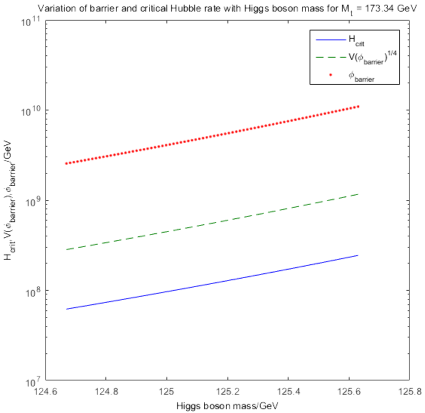
- Same as Coleman-de-Lucia linearised fluctuations about top of barrier:

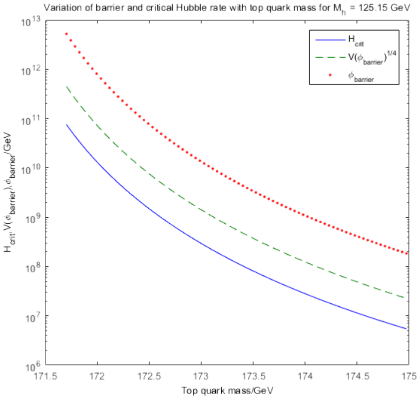
$$\ddot{\delta\phi} + 3H_{HM}\cot(H_{HM}\chi)\dot{\delta\phi} - V''(\phi_{HM})\delta\phi = 0. \quad (12)$$

- $N = 1$ direction points towards CdL solution - which is thus lower action below this threshold.
- Expect that for $H < H_{crit}$, CdL instantons dominate vacuum decay, and for $H > H_{crit}$, Hawking-Moss instantons dominate.
- If background Hubble rate is $H_0^2 = \frac{V_0}{3M_p^2}$ then the critical threshold for H_0 is:

$$H_{crit}^2 = -\frac{V''(\phi_{HM})}{4} - \frac{\Delta V(\phi_H M)}{3M_p^2}. \tag{13}$$

The Hawking-Moss Solution





Computing Thick-wall Bubbles

Numerical Setup

Equations to solve:

$$\dot{a}^2 = 1 - \frac{a^2}{3M_p^2}\left(-\frac{\dot{\phi}^2}{2} + V(\phi) - \frac{6\xi\dot{a}}{a}\phi\dot{\phi}\right) \tag{14}$$

$$\ddot{\phi} + \frac{3\dot{a}}{a}\dot{\phi} - V'(\phi_{HM}) - \xi R(\phi,\dot{\phi},\ddot{\phi})\phi = 0 \tag{15}$$

$$\dot{\phi}(0) = \dot{\phi}(\chi_{max}) = 0, a(0) = 0, \chi_{max} > 0 \text{ defined by } a(\chi_{max}) = 0. \tag{16}$$

Two approximations which fail for the Standard Model:

- Fixed background approximation. Assume that a is dominated by the background spacetime and ignore any backreaction.
- Thin wall approximation. Assume that gradient terms do not contribute significantly except in a thin region, the 'bubble wall'.

Thin Wall Approximation

The Thin wall approximation splits the bounce solution into three regions:

- The bubble interior, where $\phi \approx \phi_{tv}$. Gives a negative contribution to the action as $V < 0$.
- The bubble wall, where ϕ has a gradient. Gives a positive contribution to the action.
- The bubble exterior, where $\phi \approx \phi_{fv}$. Gives little contribution.
- Size of bubble determined by interplay of 'surface tension' (supresses bubbles) and 'bubble interior' (encourages bubble growth).
- Not valid if energy difference between vacua large [10], such as in the Standard Model.

Thin Wall Approximation

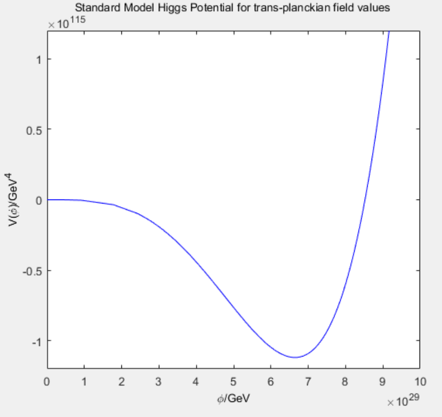


Figure 6: Standard Model potential for very large field values. Note the vast difference in scale between the true minimum and the barrier, which is not visible on this scale.

Qualitative interpreation - volume distorted by curvature, chaging surface area to volume ratio:

- If $R < 0$, then $V < \frac{4\pi r^3}{3}$ $\rightarrow$need to nucleate larger bubbles$\rightarrow$supresses decay.
- If $R > 0$, then $V > \frac{4\pi r^3}{3}$ $\rightarrow$can nucleate smaller bubbles$\rightarrow$enhances decay.

The Fixed Background Approximation.

- Assume that the background dominates.
- $a = \frac{1}{H_0}\sin(H_0\chi)$ and $\chi_{max} = \frac{\pi}{H_0}$.
- Single ode:

$$\ddot{\phi} + 3H_0\cot(H_0\chi)\dot{\phi} - V'(\phi) = 0. \tag{17}$$

- Solve by overshoot/undershoot method, introduced by Coleman[11]. Guess ϕ_0:
  - If ϕ_0 is too close to the barrier, then it will undershoot - $\dot{\phi}$ crosses zero before $\chi = \frac{\pi}{H_0}$.
  - If ϕ_0 is too far from the barrier it will overshoot - $\dot{\phi}$ does not reach zero before $\chi = \frac{\pi}{H_0}$.

- True solution is somewhere inbetween$\rightarrow$bisect until ϕ_0 found.
- See Balek and Demetrian for detailed discussion[12].
- Not valid for the Standard Model. Gravitational backreaction is significant[3].

Thick Wall Bubbles with Backreaction

- Thick wall bubbles - gradient terms significant, include a evolution.
- Problem 1 - $\frac{1}{a}$ terms diverge at the boundaries.
- Taylor expand for small $\chi, \chi - \chi_{max}$. Choose initial step $\delta\chi$ such that leading order term dominates.

- Problem 2 - χ_{max} is unknown before computing the solution.
- Define a new independent variable, $t = \frac{\chi}{\chi_{max}}$. Add a new equation $\dot{\chi_{max}} = 0$.
- Fix χ_{max} by Newton-Raphson such that solutions integrated from both poles match in the middle.

- This is faster than bisection, which has only linear convergence.
- However, it can fail if the initial guess is far from the correct guess.
- Hybrid - use bisection where Newton-Raphson fails.
- Problem 3 - interpolation too inaccurate.
- Add extra equation for S:

$$S(t) = 2\pi^2 \int_0^{t\chi_{max}} \mathrm{d}\chi a^3(\chi)(\frac{1}{2}\dot{\phi}^2 + V(\phi) - \frac{M_p^2}{2}(1 - \xi\frac{\phi^2}{M_p^2})R) \quad (18)$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 2\pi^2\chi_{max}a^3(\chi)(\frac{1}{2}\dot{\phi}^2 + V(\phi) - \frac{M_p^2}{2}(1 - \xi\frac{\phi^2}{M_p^2})R). \quad (19)$$

- Boundary conditions $S(0) = 0, S(1) = S$. Fix S, χ_{max} by Newton-Raphson, until $\phi, \dot{\phi}, a, S$ all match.

The Problem of Delicate Cancellations

- Action comptation involves delicate cancellation. E.g. for Hawking-Moss:

$$\begin{aligned}\Delta S_{HM} &= S_{HM} - S_0 \\ &= \frac{24\pi^2 M_p^4}{V_0} - \frac{24\pi^2 M_p^4}{V_0 + \Delta V(\phi_H M)} \\ &= 24\pi^2 M_p^4 \left(\frac{1}{V_0} - \frac{1}{V_0}(1 - \frac{\Delta V(\phi_{HM})}{V_0} + O\left(\left(\frac{\Delta V(\phi_{HM})}{V_0}\right)^2\right)) \right) \\ &= \frac{24\pi^2 M_p^4 \Delta V(\phi_{HM})}{V_0^2}(1 + O\left(\frac{\Delta V(\phi_{HM})}{V_0}\right)). \qquad (20)\end{aligned}$$

- But $\frac{M_p^4}{V_0}$ is large. CdL is similar - but disastrous if any small errors in a.

The Problem of Delicate Cancellations

- Fix by splitting $a(\chi) = \frac{1}{H}\sin(H\chi) + \delta a(\chi)$, $H = \frac{\pi}{\chi_{max}}$. Cancel large parts of S_0 explicitly.

- Round-off error is reduced, improving accuracy of S calculation.

The Problem of Delicate Cancellations

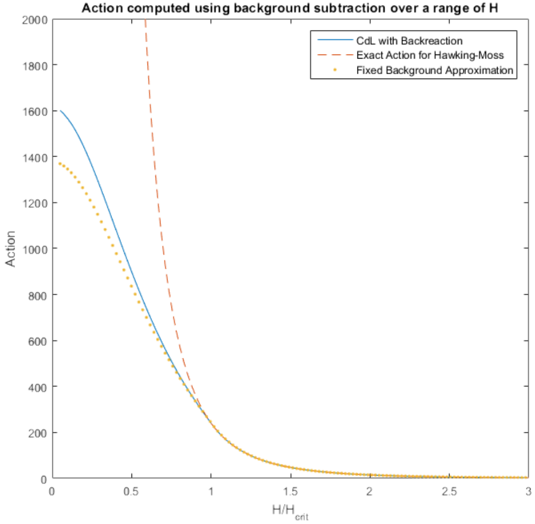


Figure 7: Action of Hawking-Moss and CdL solutions for a toy model, demonstrating the effect of bacreaction.

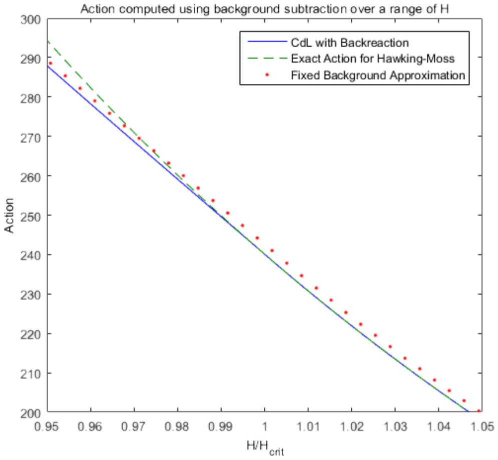


Figure 8: Action in the vicinty of the critical Hubble rate, for various solution.

The Problem of Differing Scales

- The most severe problem is that the Standard Model potential varies over a wide range of scales:
  - Minimum of λ, $\mu_{min} \approx 10^{17}$ GeV.
  - Barrier scale, $\approx 10^{10}$ GeV.
  - True minimum, $\approx 10^{30}$ GeV.
- Bounce solutions are very narrow spikes - precision impossible without very small step sizes and very many steps.
- But high precision is *essential*:
  - Overshoot/undershoot very sensitive to initial ϕ_0.

- Decay exponent formed from delicate cancellation of false vacuum and bounce actions.

- Currently beyond the limit of double precision numbers. But possible in principle with multi-precision arithmetic.

Future Directions

- We are developing a high precision modification of the odeint c++ library[13], optimised for our hybrid overshoot/undershoot search, and implementing multi-precision variables.
- We aim to invesitgate the effect of non-minimal coupling, ξ, on the decay rate.
- And ultimately try to develop methods for computing the functional determinant prefactor in the decay rate, $\Gamma = A\exp(-\Delta S)$, to investigate the effect of graviton loops.

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Questions?