# Flow of central charge from the dilaton effective action

## Geometrizing the RG flow

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**Based on:**

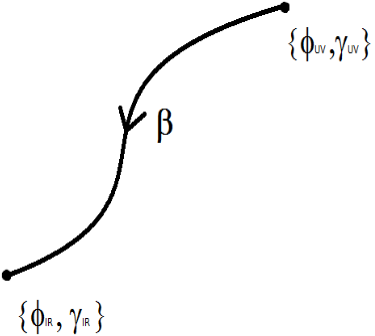
VP, R. Zwicky, *N=1 Euler anomaly flow from dilaton effective action*, JHEP01(2016)041

VP, R. Zwicky, in preparation

# Overview

1. RG flows and anomalies
   - The trace anomaly
   - a-theorem
2. Field theory with the dilaton
   - Conformal compensator
   - Dilaton effective action
3. Geometry of the flow and flow of the geometry
   - Flow of the geometry
   - Example: Free field
   - Example: $\mathcal{N} = 1$ SYM
   - Outlook
4. Conclusions

# RG flows- some generalities




**Wilsonian RG**

- Integrating out UV degrees of freedom
- Beta functions interpolate between fixed points

**Asymptotic freedom**

- UV theory composed of free fields -calculable
- IR theory contains composite d.o.f - non-perturbative
- The transition is not clear!

# Anomalies as window to IR

- Classical symmetries are broken by quantum effects
- Corresponding Noether currents have anomalous divergence:

$$\partial^{\mu} j_{\mu} = c\, \partial \mathbf{O}$$

- The anomaly $c$ is usually related to type/number of d.o.f

**Example**: Chiral anomaly when gauging a global symmetry current $j^5$

$$\langle \partial^{\mu} j^5_{\mu} \rangle \propto c\, \mathbf{F}_{\mu\nu} \tilde{\mathbf{F}}^{\mu\nu}$$

- $c_{UV}$ is calculated from massless triangle graphs - depends on the fermion content of the UV theory
- t'Hooft anomaly matching equates $c_{UV}$ with $c_{IR}$

# Trace anomaly

- Gauge the conformal symmetry by coupling the theory to a background metric $\mathrm{g}_{\mu\nu}$, which transforms

$$\mathrm{g}_{\mu\nu} \to e^{-2\alpha(x)}\mathrm{g}_{\mu\nu}$$

- Conformal anomaly is given in terms of the trace of E-M tensor $(\partial \cdot j_D = T^{\mu}_{\mu})$

$$\langle T^{\mu}_{\mu} \rangle = a\, E_4 + b\, W^2 + c\, R^2 + c' \Box R$$

- $a, b, c, c'$ are the central charges 4D QFT
- Unlike the chiral anomaly case central charges run
- The Euler central charge $a$ 'counts' d.o.f along RG flow $\to$ $a-$theorem

# The $a$-theorem

## (weak) $a$-theorem

$$\Delta a = a_{UV} - a_{IR} \geq 0$$

- Effective number of d.o.f decreases along the RG flow
- Non-perturbatively proven in 4D **[Komargodski,Schwimmer '11]**
- Puts constraints on asymptotics of RG flows **[Luty,Polchinski, Rattazzi '12]**

## (strong) $a$-theorem

$$\dot{\tilde{a}} = \beta_i \beta_j \chi^{ij}$$
$$\tilde{a}_{UV,IR} = a_{UV,IR}$$

- Gradient flow of the $\tilde{a}$ function **[Jack, Osborn '90]**
- Used to calculate beta functions to high order in PT **[Jack, Poole '14]**

# Adding the dilaton

- A theory on a flat background without a mass scale is classically conformally invariant
- For a renormalizable theory conformal transformations act on the RG scale

$$\mu \to e^{-\alpha}\mu$$

- Running couplings $g_i(\mu)$ break the conformal invariance

$$T^{\mu}_{\mu} = \beta_i O_i$$

- Invariance is restored by adding external compensator field $\mu \to e^{\tau}\mu$ that transforms as

$$\tau \to \tau + \alpha$$

- The dilaton $\tau(x)$ is a background field $\to$ couplings become local objects $g_i(e^{\tau}\mu)$

# Dilaton effective action

- Need to quantize the theory coupled to the dilaton with the action

$$S_\tau \equiv S_{\mathrm{W}}(g(\mu e^\tau), \mu e^\tau, \phi)$$

- Interpretation of such theory is not clear
- The dilaton EA $W_\tau$ can be calculated using background field methods (similar to pion EA in ChPT)

$$e^{W_\tau} = \int [\mathcal{D}\phi]_\mu e^{-S_\tau} = \int [\mathcal{D}\phi]_\mu e^{-S_{\mathrm{W}}(g(\mu e^\tau), \mu e^\tau, \phi)}$$

- $W_\tau$ is a generating functional of connected correlators of $O_i$ where $g(\mu e^\tau)$ acts as a source

# Relation to the $a$-theorem

- It can be shown that in a flat background the term

$$S_{\mathrm{WZ}} = \int d^4x\, 2(2\Box\tau(\partial\tau)^2 - (\partial\tau)^4)$$

is the Wess-Zumino action corresponding to the Euler density $E_4$.
**[Schwimmer, Theisen '10]**

- One can use this to relate the flow of dilaton EA to the Euler central charge **[Komargodski '10]**

$$\Delta W_\tau \equiv \int_{g^*_{\mathrm{IR}}}^{g^*_{\mathrm{UV}}} dg\, \partial_g W_\tau = W_\tau(g^*_{\mathrm{UV}}) - W_\tau(g^*_{\mathrm{IR}}) = -\Delta a\, S_{\mathrm{WZ}} + \ldots$$

- Equivalently, to find $\Delta a$ we need to find $\dot{W}_\tau = \beta\partial_g W_\tau$ and project on $S_{\mathrm{WZ}}$

# Geometrizing the RG flow

**[Prochazka, Zwicky '15]**

- Assumption: RG running is contained within the metric

$$\tilde{g}_{\rho\lambda} = e^{-2s(\tau)}\delta_{\rho\lambda}$$

- Introducing the dilaton localises the RG scale $\rightarrow$ curved geometry
- $W_\tau$ can be equated with the gravitational action of this background
- One can derive an RG equation

## Geometrical flow of the DEA

$$\dot{W}_\tau + \int d^4x \sqrt{\tilde{g}}\, \langle T^\mu_\mu \rangle_{\tilde{g}} = 0$$

where

$$\langle T^\mu_\mu \rangle_{\tilde{g}} = \tilde{a}\widetilde{E}_4 + \tilde{c}\widetilde{R}^2$$

is the trace anomaly in the background $\tilde{g}$

# Example: Free scalar field

- Consider a toy model of scalar field focusing solely on kinetic term

$$S_{\mathrm{W}}(\mu) = \int d^4x\, Z(\mu)\delta^{\rho\lambda}\partial_\rho\phi\partial_\lambda\phi$$

- Introduce the dilaton via

$$Z(\mu) \to Z_\tau = Z(\mu e^\tau)$$

- This amounts to Weyl transformation of the (flat) metric

$$\delta_{\rho\lambda} \to Z_\tau \delta_{\rho\lambda}$$

- The new metric now contains the information on the RG flow

$$\tilde{g}_{\rho\lambda} = e^{-2s(\tau)}\delta_{\rho\lambda}$$

where $s(\tau) = -\frac{1}{2}\ln Z(\mu e^{\tau(x)})$

# Example: Free scalar field

- The trace anomaly can be evaluated exactly for this case

$$\sqrt{\tilde{g}}\langle T^{\mu}_{\mu}\rangle_{\tilde{g}} = a^{\text{free}}_{(0)}\sqrt{\tilde{g}}\widetilde{E}_4$$

  where $a^{\text{free}}_{(0)} = \frac{1}{5760\pi^2}$

- The RG scale dependence resides in the

$$\sqrt{\tilde{g}}\widetilde{E}_4 = -8(\frac{1}{2}\Box(\partial s)^2 - \partial\cdot(\partial s(\Box s - (\partial s)^2)))$$

- Since $s(\tau) = -\frac{1}{2}\ln Z(\mu e^{\tau(x)})$, the spacetime derivatives turn into scale derivatives

$$\partial_\rho s = -\frac{1}{2}\frac{\partial \ln Z(\mu e^{\tau})}{\partial(\mu e^{\tau})}\partial_\rho(\mu e^{\tau}) = -\frac{1}{2}\gamma\,\partial_\rho\tau\ , \quad \gamma = \frac{\partial \ln Z(\mu)}{\partial \ln \mu}$$

- This gives us the flow of DEA through the Geometric RGE

# Overview of $\mathcal{N} = 1$ SYM

- Low energy action for $\mathcal{N} = 1$ SYM with flavour symmetry $SU(N_f) \times SU(N_f)$ and gauge group $SU(N_c)$ **[Shifman, Vainshtein '86]**

$$S_{\mathrm{W}}(\mu) = \int d^6 z \left( \frac{1}{g^2(\mu')} - \frac{b_0}{8\pi^2} \ln \frac{\mu'}{\mu} \right) \mathrm{tr} W^2 + \mathrm{h.c.} +$$
$$\frac{1}{8} Z(\mu, \mu') \sum_f \left[ \int d^8 z \Phi_f^\dagger e^{-2V} \Phi_f + \int d^8 z \tilde{\Phi}_f^\dagger e^{-2V} \tilde{\Phi}_f \right]$$

- Can be interpreted in Wilsonian sense as RG transformation from $\mu'$ to $\mu$ **[Arkani-Hamed, Murayama '00]**
- Asymptotically free theory flows to an IR fixed point (conformal window)
- Konishi anomaly arises under field rescaling of matter superfields

$$(\Phi_f, \tilde{\Phi}_f) \to e^{-\alpha} (\Phi_f, \tilde{\Phi}_f)$$

- This results in an anomalous contribution to the action

$$\alpha \frac{N_f}{8\pi^2} \mathrm{tr} W^2 + \mathrm{h.c.}$$

# Geometrizing the flow of SYM

- Perform a matter field rescaling with parameter $\left(\frac{\mu'}{\mu}\right)^{\gamma_*/2}$ that removes $\mu$-dependence from the gauge field term

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- Running of the theory is parametrised by a coefficient in front of the matter term with

$$\widetilde{Z}(\mu, \mu') \equiv Z(\mu, \mu') \left( \frac{\mu'}{\mu} \right)^{\gamma_*}, \gamma_* = -\frac{b_0}{N_f}$$

- As before this parameter can be absorbed in the background by means of a (super)Weyl transformation
- Introducing the dilaton $\mu \to \mu e^\tau$ we get a curved space theory with coupling defined at $\mu'$

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# Calculating $\Delta a$

- Renormalization has been absorbed in the geometry $\rightarrow$ the trace anomaly will take the form:

$$\sqrt{\tilde{g}}\langle T^{\mu}_{\mu}\rangle_{\tilde{g}} = \tilde{a}(g(\mu'))\sqrt{\tilde{g}}\widetilde{E}_4 + \tilde{c}(g(\mu'))\sqrt{\tilde{g}}\widetilde{R}^2$$

- Red quantities are evaluated at the UV scale $\mu'$
- Blue quantities include the RG evolution to $\mu$ (beta function) $\rightarrow$ only depend on $g(\mu)$
- The trace anomaly can't depend on the arbitrary scale $\mu' \rightarrow$ we can match $a, c$ to their far UV (constant) values

$$\tilde{a} \rightarrow a_{UV} = 15 N_f N_c \frac{1}{5760\pi^2}$$
$$\tilde{c} \rightarrow c_{UV} = 0$$

- One can readily calculate the nonperturbative quantity $\Delta a|_{\mathcal{N}=1}$ using the free field result and geometric RG equation

$$\Delta a|_{\mathcal{N}=1} = a^{\text{free}}_{(0)} \frac{15}{2}(-\gamma_*^3 + 3\gamma_*^2)$$

# The result

**$a_{UV} - a_{IR}$ for $\mathcal{N} = 1$ SYM**

$$\Delta a|_{\mathcal{N}=1} = a_{(0)}^{\text{free}} \frac{15}{2}(-\gamma_*^3 + 3\gamma_*^2)$$

- Result agrees with the non-perturbative computation in **[Anselmi, Freedman, Grisaru, Johansen '98]** $\rightarrow$ non-trivial test of the formalism
- Analogy with the t'Hooft UV-IR matching
- Extension to different gauge groups is straight-forward
- Nonperturbative result relies on exactness of the Konishi anomaly

# Outlook

- Application to non-SUSY gauge theories (in preparation)
- Extension to multiple couplings $\rightarrow$ multi-metric gravity
- Connection with holography ('Geometric RG flows' **[Jackson, Pourhasan, Verlinde '13]**)
- Possible applications to cosmology?

# Conclusions

- Studied connection between gravity and renormalization of gauge theories
  - Generating functional with local couplings = gravitational action for the corresponding background
  - Coefficients of gravitational terms related to UV dynamics of the theory
  - RG running is contained within the terms themselves
- Derived a geometrical RG equation for the flow of dilaton effective action
- Related the flow of DEA to the flow of Euler anomaly building up on the work of Komargodski
- Used the formalism to rederive the nonpertrubative formula for Euler central charge of $\mathcal{N} = 1$ SYM in conformal window