Towards a manifestly diffeomorphism invariant Exact Renormalization Group

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Contents

Introduction

- Einstein's General Relativity (GR) is a perturbatively non-renormalizable field theory.
- The field is the spacetime metric.
- The asymptotic safety conjecture suggests that there may exist a non-perturbative, non-trivial ultraviolet fixed point for gravity.
- Renormalization Group (RG) flow can be seen intuitively as describing physics at different scales of length by changing the resolution.
- My work develops a manifestly diffeomorphism invariant Exact RG.
- This is an approach that avoids gauge fixing, giving results independent of coordinates.

GR notation

The Riemann tensor, representing spacetime curvature, is written in our sign convention as

$$R^a{}_{bcd} = 2\partial_{[c}\Gamma^a{}_{b]d} + 2\Gamma^a{}_{[c|f}\Gamma^f{}_{|b]d}$$

where $A_{[ab]} = \frac{1}{2}(A_{ab} - A_{ba})$

We use the Levi-Civita connection (torsion-free metric connection)

$$\Gamma^a{}_{bc} = \frac{g^{ad}}{2}(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})$$

We have the Ricci tensor in our sign convention as $R_{ab} = R^c{}_{acb}$

Then the Ricci scalar by $R = g^{ab}R_{ab}$

Thus the Einstein field equation is

$$R_{ab} - \frac{g_{ab}}{2}R + \Lambda g_{ab} = 8\pi G T_{ab}$$

Diffeomorphism transformations

Consider a general coordinate transformation

$$x'^{\mu} = x^{\mu} - \xi^{\mu}(x)$$

For a covariant derivative, *D*, metrics transform as

$$g'_{\mu\nu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma}(x) = g_{\mu\nu}(x) + 2g_{\lambda(\alpha} D_{\beta)} \xi^{\lambda} + \xi \cdot D g_{\alpha\beta} \,.$$

So metric perturbations transform as

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) + \xi \cdot D g_{\alpha\beta} + 2g_{\lambda(\alpha} D_{\beta)} \xi^{\lambda} \,.$$

A general covariant tensor transforms via the Lie derivative:

$$\pounds_{\xi} T_{\alpha_1 \cdots \alpha_n} = \xi^{\lambda} D_{\lambda} T_{\alpha_1 \cdots \alpha_n} + \sum_{i=1}^{n} T_{\alpha_1 \cdots \lambda \cdots \alpha_n} D_{\alpha_i} \xi^{\lambda},$$

We will later find it useful to generalize this further to objects with two position arguments.

Kadanoff blocking

In the Ising model, Kadanoff blocking is the process of grouping microscopic spins together to form macroscopic “blocked” spins via a majority rule.

The continuous version integrates out the high-energy modes of a field to give a renormalized field, used in a renormalized action.

The blocking functional, *b*, is defined via the effective Boltzmann factor:

$$e^{-S[\varphi]} = \int \mathcal{D}\varphi_0 \; \delta\left[\varphi - b\left[\varphi_0\right]\right] e^{-S_{\text{bare}}[\varphi_0]}.$$

There are an infinite number of possible Kadanoff blockings, but a simple linear example is

$$b[\varphi_0](x) = \int_y B(x-y)\varphi_0(y),$$

where the kernel, *B*, contains an infrared cutoff function.

The partition function must be invariant under change of cutoff scale, *Λ*, this will be ensured by construction, i.e.

$$\mathcal{Z} = \int \mathcal{D}\varphi \; e^{-S[\varphi]} = \int \mathcal{D}\varphi_0 \; e^{-S_{\text{bare}}[\varphi_0]}.$$

Kadanoff blocking demands a suitable notion of locality that requires us to work exclusively in Euclidean signature.

RG Flow Equation

Differentiate the effective Boltzmann factor w.r.t. "RG time":

$$\Lambda \frac{\partial}{\partial \Lambda} e^{-S[\varphi]} = -\int_x \frac{\delta}{\delta \varphi(x)} \int \mathcal{D}\varphi_0 \; \delta\left[\varphi - b\left[\varphi_0\right]\right] \Lambda \frac{\partial b[\varphi_0](x)}{\partial \Lambda} e^{-S_{\mathrm{bare}}[\varphi_0]}.$$

This can be rewritten in terms of the "rate of change of the blocking functional", Ψ(x), as

$$\Lambda \frac{\partial}{\partial \Lambda} e^{-S[\varphi]} = \int_x \frac{\delta}{\delta \varphi(x)} \left(\Psi(x) e^{-S[\varphi]} \right).$$

This is now a general form for an RG flow equation for a single scalar field. If instead we choose a gauge field, we have (suppressing indices inside arguments)

$$\Lambda \frac{\partial}{\partial \Lambda} e^{-S[A]} = \int_x \frac{\delta}{\delta A_\mu(x)} \left(\Psi_\mu(x) e^{-S[A]} \right).$$

For gravity, we have

$$\Lambda \frac{\partial}{\partial \Lambda} e^{-S[g]} = \int_x \frac{\delta}{\delta g_{\mu\nu}(x)} \left(\Psi_{\mu\nu}(x) e^{-S[g]} \right).$$

Polchinski flow equation

We can specialize to the Polchinski flow equation for a single scalar field by setting

$$\Psi(x) = \frac{1}{2}\int_y \dot{\Delta}(x,y)\frac{\delta\Sigma}{\delta\varphi(y)}.$$

The effective propagator, $\Delta = c(p^2/\Lambda^2)p^{-2}$, contains an ultraviolet cutoff profile, *c*.

The seed action, $\hat{S}$, which appears in $\Sigma = S - 2\hat{S}$,

contains the regularized kinetic term but otherwise can be chosen freely,

$$\hat{S} = \frac{1}{2}\int_x \partial_\mu\varphi\; c^{-1}\left(-\frac{\partial^2}{\Lambda^2}\right)\partial_\mu\varphi + \cdots$$

This freedom comes from the infinite number of possible Kadanoff blockings. The Polchinski flow equation can be neatly expressed as

$$\dot{S} = \frac{1}{2}\frac{\delta S}{\delta\varphi}\cdot\dot{\Delta}\cdot\frac{\delta\Sigma}{\delta\varphi} - \frac{1}{2}\frac{\delta}{\delta\varphi}\cdot\dot{\Delta}\cdot\frac{\delta\Sigma}{\delta\varphi},$$

where $f\cdot W\cdot g := \int_x f(x)W\left(-\frac{\partial^2}{\Lambda^2}\right)g(x).$

Gauge invariant flow equation

The generalization of the Polchinski flow equation to a single gauge field looks very similar:

$$\dot{S} = \frac{1}{2}\frac{\delta S}{\delta A_\mu} \cdot \{\dot{\Delta}\} \cdot \frac{\delta \Sigma}{\delta A_\mu} - \frac{1}{2}\frac{\delta}{\delta A_\mu} \cdot \{\dot{\Delta}\} \cdot \frac{\delta \Sigma}{\delta A_\mu}.$$

The crucial difference to the scalar case is that the kernel must be covariantized to maintain manifest gauge invariance.

There are an infinite number of ways to do this, a simple way is to replace the partial derivatives with covariant derivatives,

$D_\mu := \partial_\mu - iA_\mu,$ so the kernel now has an expansion in $-\frac{D^2}{\Lambda^2}.$

We continue to call Δ the "effective propagator", although instead of inverting the tree-level 2-point function, it maps it onto the transverse projector:

$$\Delta S_{0\,\mu\nu}^{\;AA} = \delta_{\mu\nu} - p_\mu p_\nu / p^2.$$

Diffeomorphism invariant flow equation

The background-independent generalization of the Polchinski flow equation to gravity is

$$\dot{S} = \int_x \frac{\delta S}{\delta g_{\mu\nu}(x)} \int_y K_{\mu\nu\rho\sigma}(x,y) \frac{\delta \Sigma}{\delta g_{\rho\sigma}(y)} - \int_x \frac{\delta}{\delta g_{\mu\nu}(x)} \int_y K_{\mu\nu\rho\sigma}(x,y) \frac{\delta \Sigma}{\delta g_{\rho\sigma}(y)},$$

The kernel, which transforms as a two-argument generalization of a tensor, is

$$K_{\mu\nu\rho\sigma}(x,y) = \frac{1}{\sqrt{g}} \delta(x-y) \left(g_{\mu(\rho} g_{\sigma)\nu} + j g_{\mu\nu} g_{\rho\sigma} \right) \dot{\Delta}.$$

The de Witt supermetric parameter, *j*, determines how modes propagate in the flow equation. For the "kinetic term" to be a regularized Einstein-Hilbert form, *j* = -1/2.

In this case, we have $$\dot{\Delta} = -\frac{2}{\Lambda^2} c' \left(-\frac{\nabla^2}{\Lambda^2} \right),$$

which is related to the "effective propagator", Δ, in the fixed-background description.

Brief comments on *j*

The value of j determines the balance of modes in the flow equation, take for example

$$j \to \infty,$$

which is the case where the kernel only keeps the index structure that traces both sides. This choice ensures that only the conformal mode propagates in the flow equation. To see this, let's bring the conformal factor outside the metric:

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} e^{\sigma}.$$

We find that we can rewrite the flow equation as merely a flow equation for the conformal factor:

$$\frac{\delta S}{\delta \sigma} = g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}},$$

$$\dot{S} \sim \int_x \frac{\delta S}{\delta \sigma} \frac{\dot{\Delta}}{\sqrt{g}} \frac{\delta \Sigma}{\delta \sigma} - \int_x \frac{\delta}{\delta \sigma} \frac{\dot{\Delta}}{\sqrt{g}} \frac{\delta \Sigma}{\delta \sigma}.$$

Conversely, in D dimensions, $j = -1/D$ prevents the pure trace mode of the metric from propagating in the flow equation.

Fixed-background form

If we fix a Euclidean background metric, we can define the graviton field as the perturbation to that background:

$$h_{\mu\nu}(x) := g_{\mu\nu}(x) - \delta_{\mu\nu}.$$

The position representation is related to a momentum representation via a Fourier transform:

$$h_{\mu\nu}(x) = \int đp\, \mathrm{e}^{-ip\cdot x} h_{\mu\nu}(p), \quad \text{where} \quad đp := \frac{d^D p}{(2\pi)^D}.$$

The action is defined as a series expansion in the perturbation:

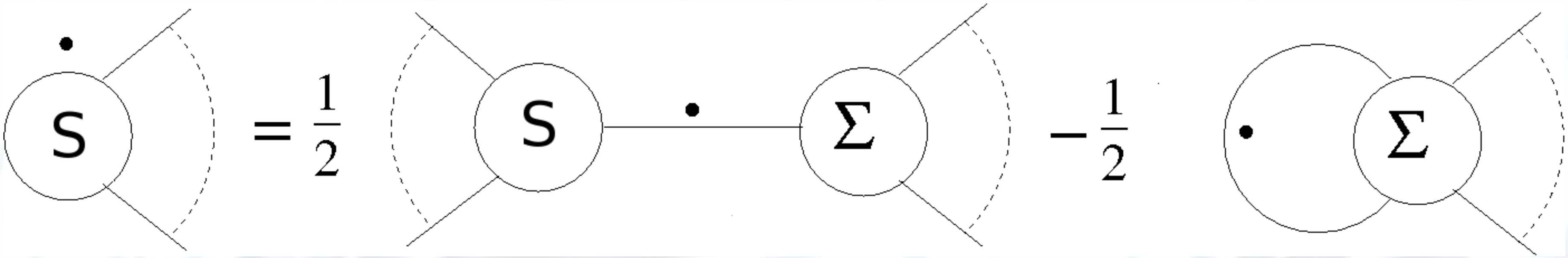
$$\begin{aligned} S \;&=\; \int đp\, \delta(p) \mathcal{S}^{\mu\nu}(p) h_{\mu\nu}(p) + \frac{1}{2} \int đp\, đq\, \delta(p+q) \mathcal{S}^{\mu\nu\rho\sigma}(p,q) h_{\mu\nu}(p) h_{\rho\sigma}(q) \\ &\quad + \frac{1}{3!} \int đp\, đq\, đr\, \delta(p+q+r) \mathcal{S}^{\mu\nu\rho\sigma\alpha\beta}(p,q,r) h_{\mu\nu}(p) h_{\rho\sigma}(q) h_{\alpha\beta}(r) + \cdots \end{aligned}$$

In this picture, we are able to define an “effective propagator”, $\Delta := \frac{1}{p^2} c\left(\frac{p^2}{\Lambda^2}\right).$

As we will see, gravity has two linearly independent transverse projectors. If we choose the linear combination with Einstein-Hilbert structure, the 2-point function is simply that transverse projector times Δ.

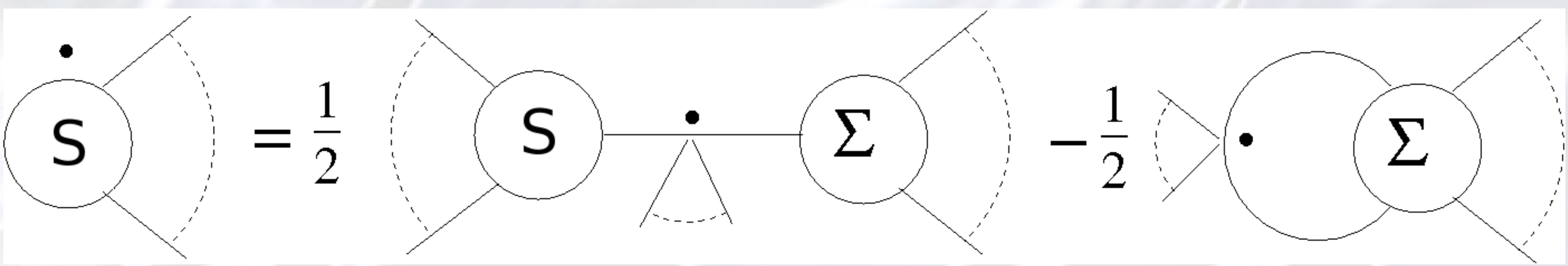
Diagrammatic view

We can visualize the *n*-point expansion of the flow equation for scalar field theory diagrammatically as follows:



$$\dot{S} = \frac{1}{2}\frac{\delta S}{\delta \varphi}\cdot \dot{\Delta} \cdot \frac{\delta \Sigma}{\delta \varphi} - \frac{1}{2}\frac{\delta}{\delta \varphi}\cdot \dot{\Delta} \cdot \frac{\delta \Sigma}{\delta \varphi}$$

In the gauge and gravity cases, the kernel also has an *n*-point expansion due to covariantization:



Also remember that the gravity case has extra factors in the kernel:

$$K_{\mu\nu\rho\sigma}(x,y) = \frac{1}{\sqrt{g}}\delta(x-y)\left(g_{\mu(\rho}g_{\sigma)\nu} + j g_{\mu\nu}g_{\rho\sigma}\right)\dot{\Delta}.$$

Loop expansion of the action

To preserve manifest gauge invariance, wavefunction renormalization must be avoided. For gauge theories, this is achieved by writing the coupling, *g*, as an overall scaling factor:

$$S[A](g) = \frac{1}{4g^2} tr \int_x F_{\mu\nu} c^{-1}\left(-\frac{D^2}{\Lambda^2}\right) F_{\mu\nu} + \cdots$$

The effective action is then written as a loopwise expansion that is also an expansion in powers of *g:*

$$S = \frac{1}{g^2} S_0 + S_1 + g^2 S_2 + \cdots \quad \text{and} \quad \Sigma_g = g^2 S - 2\hat{S}.$$

The β-functions also have a similar expansion: $\beta := \Lambda\partial_\Lambda g = \beta_1 g^3 + \beta_2 g^5 + \cdots$

If our gravity propagator has an Einstein-Hilbert structure, we have a similar setup:

$$S = \frac{1}{16\pi G}\int_x \sqrt{g}\left(-R + \frac{1}{\Lambda^2} R_{\mu\nu} d\left(-\frac{\nabla^2}{\Lambda^2}\right) R^{\mu\nu} - \frac{1}{2\Lambda^2} R d\left(\frac{-\nabla^2}{\Lambda^2}\right) R + \cdots\right),$$

where *d* is a function related to the inverse cutoff.

$$S = \frac{1}{\tilde{\kappa}} S_0 + S_1 + \tilde{\kappa} S_2 + \tilde{\kappa}^2 S_3 \cdots \quad \text{and} \quad \Sigma_{\tilde{\kappa}} = \tilde{\kappa} S - 2\hat{S}.$$

$$\beta := \Lambda\partial_\Lambda \tilde{\kappa} = \beta_1 + \beta_2\tilde{\kappa} + \beta_3\tilde{\kappa}^2 + \cdots$$

Background-independent expansion of the action

We can expand out a fixed-point action at tree-level by starting with a simple ansatz, such as:

$$S_0 = -\int_x \sqrt{g}R, \quad \Sigma_0 = \int_x \sqrt{g}R.$$

Knowing the form of the functional derivative:

$$\frac{\delta}{\delta g_{\mu\nu}(y)}\int_x \sqrt{g}R = \sqrt{g(y)}\left(R_{\mu\nu}(y) - \frac{1}{2}g_{\mu\nu}(y)R(y)\right),$$

and using the tree-level part of the flow equation:

$$\dot{S}_0 = \int_x \frac{\delta S}{\delta g_{\mu\nu}}\frac{1}{\sqrt{g}}\left(g_{\mu(\rho}g_{\sigma)\nu} + j g_{\mu\nu}g_{\rho\sigma}\right)\dot{\Delta}\frac{\delta\Sigma}{\delta g_{\rho\sigma}},$$

we get $$\dot{S}_0 = -\int_x \sqrt{g}\left(R_{\mu\nu}\dot{\Delta}R_{\mu\nu} + jR\dot{\Delta}R\right).$$

These new terms are integrated back w.r.t. RG time to give corrections our ansatz.
The original *R* term is reproduced as an "integration constant". The iterations build the fixed-point action by adding local operators of increasingly high dimension in 1/Λ.

Ward identities for the action

Recall the Lie derivative for a metric perturbation in position representation:

$$\pounds_\xi h_{\alpha\beta} = 2(\delta + h)_{\lambda(\alpha}\partial_{\beta)}\xi^\lambda + \xi \cdot \partial h_{\alpha\beta}\,.$$

We can write the variation in momentum representation as

$$i\delta h_{\alpha\beta}(p) = 2p_{(\alpha}\xi_{\beta)} + \int đp'\, đk'\, \delta(p - p' - k')\left(2p'_{(\alpha}h_{\beta)\lambda}(k')\xi^\lambda(p') + \xi(p') \cdot k' h_{\alpha\beta}(k')\right)\,.$$

The Ward identities follow from requiring diffeomorphism invariance of the action:

$$\begin{aligned} 2p_{1\mu_1}\mathcal{S}^{\mu_1\nu_1\cdots\mu_n\nu_n}(p_1,\cdots,p_n) \quad = \quad & -\sum_{i=2}^{n}\pi_{2i}\Big\{p_2^{\nu_1}\mathcal{S}^{\mu_2\nu_2\cdots\mu_n\nu_n}(p_1+p_2,p_3,\cdots,p_n) \\ & -2p_{1\alpha}\delta^{\nu_1(\nu_2}\mathcal{S}^{\mu_2)\alpha\mu_3\nu_3\cdots\mu_n\nu_n}(p_1+p_2,p_3,\cdots,p_n)\Big\}, \end{aligned}$$

where momentum conservation is assumed: $\sum_{i=1}^{n} p_i = 0.$

We can obtain Ward identities for momentum-independent parts of the action by differentiating w.r.t. one of the momenta, then tending all momenta to zero:

$$2\mathcal{S}^{\mu_1\nu_1\cdots\mu_n\nu_n}(\underline{0}) = \delta^{\mu_1\nu_1}\mathcal{S}^{\mu_2\nu_2\cdots\mu_n\nu_n}(\underline{0}) - 2\sum_{i=2}^{n}\pi_{2i}\,\delta^{\nu_1(\nu_2}\mathcal{S}^{\mu_2)\mu_1\mu_3\nu_3\cdots\mu_n\nu_n}(\underline{0})\,.$$

Deriving Ward identities for the kernel

The kernel diffeomorphism transforms as the two-argument generalization of a tensor:

$$\pounds_\xi K_{\mu\nu\rho\sigma}(x,y) = \xi(x)\cdot\partial_x K_{\mu\nu\rho\sigma}(x,y) + \xi(y)\cdot\partial_y K_{\mu\nu\rho\sigma}(x,y)$$
$$+2K_{\lambda(\mu|\rho\sigma}(x,y)\partial_{x|\nu)}\xi^\lambda(x) + 2K_{\mu\nu\lambda(\rho|}(x,y)\partial_{y|\sigma)}\xi^\lambda(y),$$

The kernel transforms in momentum representation like

$$i\delta K_{\mu\nu\rho\sigma}(q,r) = -\xi(p')\cdot(p'+q)K_{\mu\nu\rho\sigma}(p'+q,r) - \xi(p')\cdot(p'+r)K_{\mu\nu\rho\sigma}(q,p'+r)$$
$$+2\xi^\lambda(p')p'_{(\mu}K_{\nu)\lambda\rho\sigma}(p'+q,r) + 2\xi^\lambda(p')p'_{(\rho|}K_{\mu\nu|\sigma)\lambda}(q,p'+r).$$

The kernel can be written in momentum representation as an expansion in metric perturbations:

$$K_{\mu\nu\rho\sigma}(q,r) = \mathcal{K}_{\mu\nu\rho\sigma}(q,r) + \int đp_1\ \delta(p_1+q+r)\mathcal{K}^{\alpha_1\beta_1}_{\mu\nu\rho\sigma}(p_1,q,r)h_{\alpha_1\beta_1}(p_1) + \cdots$$

Taking into account also the transformation of the metric perturbations, we can derive an overall set of Ward identities for the kernel...

Ward identities for the kernel

The result is

$$
\begin{aligned}
&2p'_\gamma \mathcal{K}^{\gamma\delta\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{\mu\nu\rho\sigma}(p', p_1, \cdots, p_n, q, r) = \\
&-(p'+q)^\delta \mathcal{K}^{\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{\mu\nu\rho\sigma}(p_1, \cdots, p_n, q+p', r) \\
&-(p'+r)^\delta \mathcal{K}^{\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{\mu\nu\rho\sigma}(p_1, \cdots, p_n, q, r+p') \\
&+2\delta^{\lambda\delta} p'_{(\mu|} \mathcal{K}^{\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{|\nu)\lambda\rho\sigma}(p', p_1, \cdots, p_n, q+p', r) \\
&+2\delta^{\lambda\delta} p'_{(\rho|} \mathcal{K}^{\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{\mu\nu|\rho)\lambda}(p', p_1, \cdots, p_n, q, r+p') \\
&-\sum_{i=1}^{n} \pi_{i1} \left\{ p_1^\delta \mathcal{K}^{\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{\mu\nu\rho\sigma}(p'+p_1, p_2, \cdots, p_n, q, r) \right. \\
&\left. +2p'_\lambda \delta^{\delta(\alpha_1} \mathcal{K}^{\beta_1)\lambda\alpha_2\beta_2\cdots\alpha_n\beta_n}{}_{\mu\nu\rho\sigma}(p'+p_1, p_2, \cdots, p_n, q, r) \right\}.
\end{aligned}
$$

The momentum-independent part satisfies

$$
\begin{aligned}
\mathcal{K}^{\gamma\delta\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{\mu\nu\rho\sigma}(\underline{0}) \quad = \quad & -\frac{1}{2}\delta^{\gamma\delta} \mathcal{K}^{\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{\mu\nu\rho\sigma}(\underline{0}) \\
& +\delta^{\lambda(\gamma}\delta^{\delta)}{}_{(\mu|} \mathcal{K}^{\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{|\nu)\lambda\rho\sigma}(\underline{0}) \\
& +\delta^{\lambda(\gamma}\delta^{\delta)}{}_{(\rho|} \mathcal{K}^{\alpha_1\beta_1\cdots\alpha_n\beta_n}{}_{\mu\nu|\sigma)\lambda}(\underline{0}) \\
& -\sum_{i=1}^{n} \pi_{i1} \left\{ \delta^{(\gamma|(\alpha_1} \mathcal{K}^{\beta_1)|\delta)\cdots\alpha_n\beta_n}{}_{\mu\nu\rho\sigma}(\underline{0}) \right\}.
\end{aligned}
$$

Expansion of the metric determinant

The action has a factor of $\sqrt{g}$. The kernel has a factor of $1/\sqrt{g}$.

We can use the momentum-independent Ward identities to determine the expansion of these factors in metric perturbations, or we can do it more directly:

$$\sqrt{\det(g_{\mu\nu})}^{\,l} = e^{\frac{l}{2}(\ln(\delta_{\mu\nu}+h_{\mu\nu}))},$$

$$\sqrt{g}^{\,l} = 1 + l\frac{h}{2} - l\frac{h_{\mu\nu}h^{\mu\nu}}{4} + l^2\frac{h^2}{8} + l\frac{h_{\mu\nu}h^{\mu\rho}h^{\nu}{}_{\rho}}{6} - l^2\frac{h_{\mu\nu}h^{\mu\nu}h}{8} + l^3\frac{h^3}{48} + \cdots$$

A “cosmological constant” term would enter the action simply as $\int_x \sqrt{g} \times \text{constant}$

Thus a “cosmological constant” term would introduce a corresponding 1-point function because we are expanding around a Euclidean background rather than the more natural de Sitter background.

This causes all *n*-point functions to receive corrections from the corresponding (n+1)-point functions. We will not include a cosmological constant in what follows.

n-point expansion of the kernel

We wish to calculate the *n*-point functions of the kernel, which is complicated by the factor of

$$\dot{\Delta} = \frac{1}{\Lambda^N} f\left(\frac{-\nabla^2}{\Lambda^2}\right),$$

where *N* is the dimension of the (rescaled) Lagrangian density. To get the linear part in metric perturbations, we first calculate the result, *H*, of using just the linear term in the derivative expansion:

$$(-\nabla^2)(p,r)T^{\rho\sigma}(-r) = H^{\alpha\beta}{}_{\gamma\delta}{}^{\rho\sigma}(p,r)T^{\gamma\delta}(-r)h_{\alpha\beta}(p)\,.$$

We then generalize to any order in the derivative expansion:

$$(-\nabla^2)^{n+1}(p,r)T^{\rho\sigma}(-r) = \sum_{i=0}^{n}([p-r]^2)^{n-i}H^{\alpha\beta}{}_{\gamma\delta}{}^{\rho\sigma}(p,r)(r^2)^i T^{\gamma\delta}(-r)h_{\alpha\beta}(p)\,.$$

The sum is over terms in a geometric series, which goes like

$$\sum_{i=0}^{n}([p-r]^2)^{n-i}(r^2)^i = \frac{(p-r)^{2(n+1)} - r^{2(n+1)}}{(p-r)^2 - r^2}.$$

Finally, we add up all the terms in the function, *f*,

$$f(-\nabla^2)(p,r)T^{\rho\sigma}(-r) = \frac{\left(f\left((p-r)^2\right) - f(r^2)\right)}{(p-r)^2 - r^2} H^{\alpha\beta}{}_{\gamma\delta}{}^{\rho\sigma}(p,r)T^{\gamma\delta}(-r).$$

Transverse 2-point structures

The kinetic term for a massless field gives us transverse 2-point functions. We wish to use diffeomorphism invariance to constrain what 2-point functions we have. We start with the most general structure with two derivatives:

$$S^{(2)}_{2-\mathrm{momenta}} = \int đp \, \left(a_1 h p^2 h + a_2 h_{\alpha\beta} p^2 h^{\alpha\beta} + a_3 h p_\alpha p_\beta h^{\alpha\beta} + a_4 h^{\alpha\beta} p_\alpha p_\gamma h_\beta{}^\gamma \right).$$

We require the linearized diffeomorphism transformation to be zero:

$$\begin{aligned} 0 \quad = \quad & 4a_1 h p^2 p \cdot \xi + 4a_2 h^{\alpha\beta} p^2 p_\alpha \xi_\beta + 2a_3 h p^2 p \cdot \xi \\ & + 2a_3 h^{\alpha\beta} p_\alpha p_\beta p \cdot \xi + 2a_4 h^{\alpha\beta} p^2 p_\alpha \xi_\beta + 2a_4 h^{\alpha\beta} p_\alpha p_\beta p \cdot \xi. \end{aligned}$$

This gives us one unique structure, which corresponds to the Einstein-Hilbert action:

$$a_1 = -a_2 = -a_3/2 = a_4/2.$$

But what if we allow for four or more derivatives? The most general structure with four derivatives is

$$\begin{aligned} S^{(2)}_{4-\mathrm{momenta}} \quad = \quad & \int đp \, \left(b_1 h^{\alpha\beta} p^4 h_{\alpha\beta} + b_2 h p^4 h + b_3 h^{\alpha\beta} p^2 p_\alpha p_\beta h \right. \\ & \left. + b_4 h^{\alpha\beta} p^2 p_\alpha p_\gamma h_\beta{}^\gamma + b_5 h^{\alpha\beta} p_\alpha p_\beta p_\gamma p_\delta h^{\gamma\delta} \right). \end{aligned}$$

Transverse 2-point functions

Again, we set the linearized diffeomorphism transformation to zero:

$$\begin{aligned} 0 &= 4b_1 h^{\alpha\beta} p^4 p_\alpha \xi_\beta + 4b_2 h p^4 p\cdot\xi + 2b_3 h^{\alpha\beta} p^2 p_\alpha p_\beta p\cdot\xi \\ & 2b_3 h p^4 p\cdot\xi + 2b_4 h^{\alpha\beta} p^4 p_\alpha \xi_\beta + 2b_4 h^{\alpha\beta} p^2 p_\alpha p_\beta p\cdot\xi \\ & +4b_5 h^{\alpha\beta} p^2 p_\alpha p_\beta p\cdot\xi. \end{aligned}$$

This requires that $b_5 = b_1 + b_2, b_4 = -2b_1, \text{and } b_3 = -2b_2,$

giving us two linearly independent structures

$$2S_a^{(2)} = a\left(\frac{p^2}{\Lambda^2}\right)\left(h^{\mu\nu}p^4 h_{\mu\nu} - 2h^{\mu\nu}p^2 p_\mu p_\rho h_\nu{}^\rho + h^{\mu\nu}p_\mu p_\nu p_\rho p_\sigma h^{\rho\sigma}\right),$$

$$2S_b^{(2)} = b\left(\frac{p^2}{\Lambda^2}\right)\left(hp^4 h - 2h^{\mu\nu}p^2 p_\mu p_\nu h + h^{\mu\nu}p_\mu p_\nu p_\rho p_\sigma h^{\rho\sigma}\right).$$

The two transverse 2-point functions can be written in a factorized form:

$$\mathcal{S}_a^{\mu\nu\rho\sigma}(-p,p) = a\left(\frac{p^2}{\Lambda^2}\right)\left(p^2\delta^{(\mu|(\rho} - p^{(\mu|}p^{(\rho}\right)\left(p^2\delta^{\sigma)|\nu)} - p^{\sigma)}p^{|\nu)}\right),$$

$$\mathcal{S}_b^{\mu\nu\rho\sigma}(-p,p) = b\left(\frac{p^2}{\Lambda^2}\right)\left(p^2\delta^{\mu\nu} - p^\mu p^\nu\right)\left(p^2\delta^{\rho\sigma} - p^\rho p^\sigma\right).$$

Interpreting the 2-point functions

Increasing the number of derivatives further does not give us any new structures.

$a = 2,\ b = 0$ corresponds to $\int_x \sqrt{g} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma},$

$a = 0,\ b = 2$ corresponds to $\int_x \sqrt{g} R^2,$

$a = b = \frac{1}{2}$ corresponds to $\int_x \sqrt{g} R_{\mu\nu} R^{\mu\nu},$

$a = -b = \frac{1}{2p^2}$ corresponds to $-\int_x \sqrt{g} R.$

The quadratic terms are related by the Gauss-Bonnet topological invariant:

$$\frac{1}{32\pi^2} \int_M \left(R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right).$$

2-point flow equation: first example

Let us try an effective action that starts at quadratic order in Riemann tensors:

$$S = \int_x \sqrt{g}\left(R_{\alpha\beta}c^{-1}R^{\alpha\beta} + sRc^{-1}R + \cdots\right),$$

where *c* is the cutoff function and *s* is a value to be determined. The 2-point function is

$$\mathcal{S}^{\alpha\beta\gamma\delta}(p,-p) = \frac{1}{2}c^{-1}\mathcal{S}_a^{\alpha\beta\gamma\delta}(p,-p) + \left(\frac{1}{2}+2s\right)c^{-1}\mathcal{S}_b^{\alpha\beta\gamma\delta}(p,-p),$$

which solves the 2-point flow equation provided that $\dot{(c^{-1})} = -p^4c^{-2}\dot{\Delta}$,

and $s\dot{(c^{-1})} = -p^4c^{-2}\left(1+4j+8(1+3j)s+12(1+3j)s^2\right).$

This is solved by $\Delta\left(\frac{p^2}{\Lambda^2}\right) = \frac{1}{p^4}c\left(\frac{p^2}{\Lambda^2}\right),$

and $s = -\frac{1}{3}\ \text{or}\ -\frac{1}{4}\left(\frac{1+4j}{1+3j}\right),$

leaving the de Witt supermetric parameter, *j*, arbitrary.

2-point flow equation: second example

Let us now consider an effective action that begin with the Einstein-Hilbert term:

$$S = \int_x \sqrt{g} \left(-R + \frac{1}{\Lambda^2} R_{\mu\nu} d\left(\frac{-\nabla^2}{\Lambda^2}\right) R^{\mu\nu} + \frac{j}{\Lambda^2} R d\left(\frac{-\nabla^2}{\Lambda^2}\right) R + \cdots \right).$$

We choose *j* = -1/2 so that the entire expression uses the same transverse projector, the flow equation then requires only that

$$\Lambda\partial_\Lambda \left(\frac{d}{\Lambda^2}\right) = -p^2 \left(\frac{d^2}{\Lambda^4} + \frac{2d}{p^2\Lambda^2} + \frac{1}{p^4}\right).$$

Alternatively, *j* = -1/3 would have given us the same constraint, but then the 2- and (4+)-derivative terms would use different transverse projectors.

We can then write the "effective propagator" as

$$\Delta^{-1} = p^2 + d\frac{p^4}{\Lambda^2} = p^2 c^{-1}\left(\frac{p^2}{\Lambda^2}\right).$$

So we have the familiar form $\Delta = \frac{1}{p^2} c\left(\frac{p^2}{\Lambda^2}\right).$

Additional regularization for loops

The manifestly gauge invariant ERG for *SU(N)* requires additional regularization at 1-loop level using covariant Pauli-Villars fields. The elegant way to do this is with *SU(N|N)* regularization.

The field is promoted to a supermatrix of bosonic components, *A*, and fermionic components, *B*:

$$\mathcal{A}_\mu = \begin{pmatrix} A^1_\mu & B_\mu \\ \bar{B}_\mu & A^2_\mu \end{pmatrix} + \mathcal{A}^0_\mu \mathcal{I}, \quad \text{and} \quad D_\mu = \partial_\mu - i\mathcal{A}_\mu .$$

The action is built in a similar way:

$$S = \frac{1}{4g^2} str \int \mathcal{F}_{\mu\nu} c^{-1}\left(-\frac{D^2}{\Lambda^2}\right)\mathcal{F}_{\mu\nu} + \cdots$$

where $$str \begin{pmatrix} X^{11} & X^{12} \\ X^{21} & X^{22} \end{pmatrix} = tr_1 X^{11} - tr_2 X^{22} .$$

This supersymmetry is spontaneously broken by a super-Higgs mechaism with a mass at order *Λ*, so that the physical *SU(N)* can be recovered at low energy.

A similar procedure would be required for the 1-loop manifestly diffeomorphism invariant ERG for gravity.

Summary

- The manifestly diffeomorphism invariant ERG is based on a generalization of the Polchinski flow equation for gravity.
- There is no need for gauge fixing or BRST ghosts. All results are independent of coordinates.
- It has both fixed-background and background-independent versions.
- We have developed the formalism at tree-level and can exactly solve the flow equation for n-point vertices iteratively, starting at the 2-point level.
- Loop corrections will require additional regularization that could be similar to $SU(N|N)$ regularization in the manifestly gauge invariant ERG.