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A simple, robust and powerful test of the trend hypothesis

by

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A Simple, Robust and Powerful Test of The Trend Hypothesis

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Abstract

In this paper we develop a simple test procedure for a linear trend which does not require knowledge of the form of serial correlation in the data, is robust to strong serial correlation, and has a standard normal limiting null distribution under either $I(0)$ or $I(1)$ shocks. In contrast to other available robust linear trend tests, our proposed test achieves the Gaussian asymptotic local power envelope in both the $I(0)$ and $I(1)$ cases. For near-$I(1)$ errors our proposed procedure is conservative and a modification for this situation is suggested. An estimator of the trend parameter, together with an associated confidence interval, which is asymptotically efficient, again regardless of whether the shocks are $I(0)$ or $I(1)$, is also provided.

Keywords: Linear trend; strong serial correlation; asymptotic normality; power envelope; unit root tests; stationarity tests.

JEL Classifications: C22.

1 Introduction

The ability to detect the presence of a deterministic linear trend in an economic time series is an important issue in applied econometrics for a number of reasons. The effectiveness of both policy modelling and forecasting is, for example, reliant on correct

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identification of the trend function. Correctly specifying the trend function is also of crucial importance in the context of unit root and stationarity testing. It is, for example, well known that an un-modelled linear trend effects non-similar and inconsistent unit root tests (see Perron, 1988), while unnecessarily including a trend vastly reduces power to reject the unit root null under $I(0)$ (weakly dependent) errors (see, for example, Marsh, 2005). Similarly, it is trivial to show that an un-modelled linear trend causes stationarity test statistics, such as that of Kwiatkowski *et al.* (1990) [KPSS, hereafter], to diverge regardless of whether the errors are $I(0)$ or $I(1)$. Hypothesis testing to detect whether a trend is present is also of interest in its own right. An example is given by the Prebisch-Singer hypothesis, re-considered recently in Bunzel and Vogelsang (2005), which predicts that over time, the net barter terms of trade should be declining between countries that primarily export commodities and those that primarily export manufactured goods, implying the presence of a negative trend in the data. Moreover, since the slope of the trend function represents the average growth in the series (or rate of growth if the data are measured in logarithms) it is clearly of considerable empirical interest to be able to construct point estimates and confidence intervals for this quantity, allowing, for example, comparisons of growth rates across countries or regions.

In response to this, there has recently been a number of papers which suggest various procedures for identifying the presence of a trend in the presence of serially correlated shocks. These include, *inter alia*, Canjels and Watson (1997), Vogelsang (1998), Sun and Pantula (1999) and Bunzel and Vogelsang (2005). Formal testing of whether a time series contains a trend is greatly complicated by the fact that in practice it is not known whether the trend is embedded in an $I(0)$ or $I(1)$ series, that is, within a weakly or strongly autocorrelated series. If one knew that the shocks were $I(0)$ then one could test for the presence of a linear trend using levels data. Similarly, if it were known that the shocks were $I(1)$ then one could perform tests on the first differences of the data (growth rates). However, tests based on growth rates display very poor power properties relative to those based on levels when the shocks are in fact $I(0)$, as is discussed in a wider context in Vogelsang (1998). Moreover, the large sample null distributions of tests on the parameters of the trend function in levels data depend on whether the shocks are $I(0)$ or $I(1)$, as is discussed in Phillips and Durlauf (1988).

Of the trend function testing procedures cited above only those of Vogelsang (1998) and Bunzel and Vogelsang (2005) are robust, in the sense that, asymptotically, inference on the trend function is unaffected as to whether the data are $I(0)$ or $I(1)$. The pertinent feature of these approaches is that they avoid the size problems of trend function tests that, as in Sun and Pantula (1999), rely on the results of conventional pre-testing (using stationarity or unit root statistics), whereby the asymptotic Type I errors associated with such pre-tests means that the size of subsequent trend function tests cannot be fully controlled, even asymptotically.

In this paper we propose a new testing procedure that falls into the class of robust tests for the trend function. The statistic is based on taking a simple data-dependent weighted average of two trend test statistics, both conventional t-ratios, one that is appropriate when the data are generated by an $I(0)$ process and a second that is appropriate when the data are $I(1)$. Determined from an auxiliary statistic which consistently estimates the true order of integration of the data, the weights are designed to switch weight between the two trend statistics, depending on whether the data are generated by an $I(0)$ or $I(1)$ process. We show that the new weighted statistic has a standard normal limiting null distribution in both the $I(0)$ and $I(1)$ cases.

We compare the asymptotic local power of our new trend test with that of the preferred robust trend function testing procedure from Bunzel and Vogelsang (2005), namely that based on the *Dan-J* statistic, and show that it offers significantly improved performance over the *Dan-J* test, achieving the Gaussian asymptotic local power envelope in both the $I(0)$ and $I(1)$ cases. We also examine the asymptotic properties of our procedure when the data are generated by a near-$I(1)$ process. Here we find that, in common with the *Dan-J* test, our procedure is under-sized, which can lead to a reduction in power for small values of the local trend alternative. However, we show that a straightforward modification to our procedure largely offsets this problem.

In order to keep exposition simple we focus our attention in this paper on the empirically relevant problem of testing for a linear time trend in an economic series, under which Vogelsang (1998) and Bunzel and Vogelsang (2005) derive all the numerical results reported in their papers. However, it should be noted that the basic testing principle which we advocate is actually quite general and could easily be extended to include testing of more sophisticated trend functions, such as polynomial trend functions or breaks in trend.

The paper is organized as follows. Section 2 motivates our suggested approach to testing for a linear trend within a simplified, stylized model. A more general specification is then considered in Section 3, where the asymptotic properties of our proposed statistic are also established. In Section 4 we discuss issues relating to the practical implementation of our testing procedure. Section 5 reports numerical evidence on the asymptotic power properties of our new trend tests, along with finite sample size and power simulations. These are compared to those of the *Dan-J* test from Bunzel and Vogelsang (2005). In Section 6 we apply our tests and that of Bunzel and Vogelsang (2005) to time-series data on a number of real GDP series and also on the commodity prices series previously analysed for the presence of a downward trend by Bunzel and Vogelsang (2005). Section 7 concludes. A short appendix contains the proofs of our main results.

In what follows we use the following notation: $x := y$ ($x =: y$) to indicate that x is defined by y (y is defined by x); $\lfloor \cdot \rfloor$ to denote the integer part of the argument; $\xrightarrow{p}$ to denote convergence in probability; $\xrightarrow{d}$ to denote weak convergence, and $N(a, b)$ to denote a Gaussian distribution with mean a and variance b. Finally, reference to a variable being $O_p(T^k)$ is taken to hold in its strict sense, meaning that the variable is not $o_p(T^k)$.

2 Motivation for the Test Procedure

To fix ideas, we start with a highly simplified model and testing problem. Consider, therefore, the Gaussian $AR(1)$ model

$$\begin{aligned} y_t &= \alpha + \beta t + u_t, \quad t = 1, ..., T && (1) \\ u_t &= \rho u_{t-1} + \varepsilon_t, \quad t = 2, ..., T, \quad u_1 = \varepsilon_1, && (2) \end{aligned}$$

where ε_t is assumed to be $NIID(0, \sigma^2)$. We suppose that the $I(0)$ scenario for u_t is represented by $\rho = 0$ and the $I(1)$ scenario by $\rho = 1$, with no other possibilities assumed to exist for the present. Our interest centres on testing $H_0 : \beta = \beta_0$ against either a two-sided alternative, $H_1 : \beta \neq \beta_0$, or either of the two one-sided alternatives $H_1 : \beta > \beta_0$ or $H_1 : \beta < \beta_0$, but without assuming knowledge of whether u_t in (1) is $I(0)$ or $I(1)$. The case of leading empirical relevance is the no trend null hypothesis, given by $\beta_0 = 0$, although other values of β_0 may also be of practical interest, for example testing whether the growth rate in a particular country coincides with some hypothetical or desired growth rate. As is customary in this kind of testing problem, we partition H_1 into two scaled components $H_{1,0} : \beta = \beta_0 + \kappa T^{-3/2}$ when u_t is $I(0)$ and $H_{1,1} : \beta = \beta_0 + \kappa T^{-1/2}$ when u_t is $I(1)$, where κ is a finite constant, which provide the appropriate Pitman drifts on β under $I(0)$ and $I(1)$ errors, respectively. Notice that both $H_{1,1}$ and $H_{1,0}$ reduce to H_0 when $\kappa = 0$.

If u_t is *known* to be $I(0)$ then $u_t = \varepsilon_t$, $t = 1, ..., T$, and a test which rejects for large values [large positive or large negative values for a two-tailed test, large negative (positive) values for a lower- (upper-)tailed test] of the (centred) t-ratio, associated with the OLS estimator of β in the estimated model (1), is an optimal (uniformly most powerful in the case of one-sided alternatives and uniformly most powerful unbiased in the case of the two-sided alternative) test of H_0 against $H_{1,0}$, and is consistent against fixed alternatives. Letting $\hat{\alpha}$ and $\hat{\beta}$ denote the OLS estimators from (1), this t-ratio is therefore given by

$$\begin{aligned} z_0 &:= \frac{\hat{\beta} - \beta_0}{s_0}, && (3) \\ s_0 &:= \sqrt{\hat{\sigma}_u^2 / \sum_{t=1}^{T} (t - \bar{t})^2} \end{aligned}$$

where $\hat{\sigma}_u^2 := (T-2)^{-1} \sum_{t=1}^{T} \hat{u}_t^2$, $\hat{u}_t := y_t - \hat{\alpha} - \hat{\beta} t$. Completely standard results show that under H_0, $z_0 \xrightarrow{d} N(0,1)$, while under $H_{1,0}$, $z_0 \xrightarrow{d} \frac{\kappa}{\sigma\sqrt{12}} + N(0,1)$.

Correspondingly, if u_t is *known* to be $I(1)$ then the optimal test of H_0 against $H_{1,1}$ is based on the t-ratio associated with the (centred) OLS estimator of β in the model (1) estimated in first differences,

$$\Delta y_t = \beta + v_t, \quad t = 2, ..., T \tag{4}$$

where $v_t := \Delta u_t = \varepsilon_t$. Defining $T_* := T - 1$, the t-ratio is therefore given by

$$\begin{aligned} z_1 &:= \frac{\tilde{\beta} - \beta_0}{s_1}, \\ s_1 &:= \sqrt{\tilde{\sigma}_v^2 / T_*} \end{aligned} \tag{5}$$

where $\tilde{\beta}$ is the OLS estimator of β in (4), *viz*, $\tilde{\beta} := T_*^{-1} \sum_{t=2}^{T} \Delta y_t = T_*^{-1}(y_T - y_1)$, and $\tilde{\sigma}_v^2 := (T_* - 1)^{-1} \sum_{t=2}^{T} \tilde{v}_t^2$, $\tilde{v}_t := \Delta y_t - \tilde{\beta}$. Once more, standard results show that under H_0, $z_1 \xrightarrow{d} N(0,1)$ and under $H_{1,1}$, $z_1 \xrightarrow{d} \frac{\kappa}{\sigma} + N(0,1)$. Again the test is consistent against fixed alternatives.

Now consider the behaviour of the statistic z_0 of (3) when u_t is in fact $I(1)$. It is entirely straightforward to establish that under both H_0 and $H_{1,1}$, z_0 is of $O_p(T^{1/2})$. That is, it diverges regardless of whether H_0 or $H_{1,1}$ is true. As for the behaviour of z_1 of (5) when u_t is $I(0)$, it is easy to show that under H_0 and $H_{1,0}$, z_1 is of $O_p(T^{-1/2})$ and, hence, converges in probability to zero, again regardless of whether H_0 or $H_{1,0}$ holds. The pertinent features of these findings are that z_0 does not control size under H_0 when u_t is $I(1)$ (its asymptotic size is unity), and z_1 does not control size when u_t is $I(0)$ (its asymptotic size is zero).

In view of the above results, and given that the order of integration of u_t is not known in practice, it is a fairly natural step to consider constructing a test procedure that employs some auxiliary routine which ensures that, asymptotically at least, the statistic z_0 of (3) is selected when u_t is $I(0)$ while z_1 of (5) is selected when u_t is $I(1)$. As this auxiliary routine should be ambivalent between H_0 and H_1, it needs to be based on the de-meaned and de-trended y_t from the fitted model (1) since these are then invariant to α and β.

Here we pursue an approach based on a data-dependent weighted average of z_0 of (3) and z_1 of (5) where the weights used are based on a consistent estimator of $d \in \{0, 1\}$, the (unknown) order of integration of u_t. The estimator of d which we propose is constructed from unit root and stationarity test statistics. In generic notation, let U denote some unit root statistic used for testing the $I(1)$ null that $\rho = 1$ against the $I(0)$ alternative, which corresponds to $\rho = 0$ in the present simplified context. Similarly, let S denote some stationarity statistic for testing the $I(0)$ null that $\rho = 0$ against the $I(1)$ alternative $\rho = 1$. We assume that these statistics have the following low-level properties.

Assumption 1. *Let the statistics U and S satisfy the following conditions: (i) If u_t is $I(0)$, then U diverges at a rate $O_p(T^{\delta_U})$, $\delta_U > 0$, while $S = O_p(1)$; (ii) If u_t is $I(1)$, then $U = O_p(1)$, while S diverges at a rate $O_p(T^{\delta_S})$, $\delta_S > 0$.*

We also suppose that a function of U and S is available which satisfies the following conditions.

Assumption 2. *Let $\lambda(U, S)$ be some function on $[0, 1]$ for which: (i) If u_t is $I(0)$, $\lambda(U, S) = o_p(1)$; (ii) If u_t is $I(1)$, $\lambda(U, S) = 1 + o_p(T^{-1/2})$.*

Remark 2.1: Observe that, under Assumption 1, $\lambda(U,S) \xrightarrow{p} 0$ when u_t is $I(0)$, while $\lambda(U,S) \xrightarrow{p} 1$ when u_t is $I(1)$, as desired. Examples of statistics U and S which satisfy Assumption 1 and a simple function $\lambda(U,S)$ which satisfies Assumption 2 will be provided later in Section 4, with a specific recommendation for $\lambda(U,S)$ given in Remark 4.4. □

Our approach to testing for a linear trend is then based on a simple weighted average of z_0 of (3) and z_1 of (5) of the form

$$z_\lambda := \{1-\lambda(U,S)\}z_0 + \lambda(U,S)z_1. \tag{6}$$

As noted above, if u_t is $I(0)$ then z_1 is of $O_p(T^{-1/2})$, and, hence, from (6) it follows that under Assumptions 1 and 2

$$\begin{aligned} z_\lambda &= \{1+o_p(1)\}z_0 + o_p(1)O_p(T^{-1/2}) \\ &= z_0 + o_p(1) \end{aligned}$$

while if u_t is $I(1)$, then z_0 is of $O_p(T^{1/2})$ and, hence, we have, under Assumptions 1 and 2, that

$$\begin{aligned} z_\lambda &= o_p(T^{-1/2})O_p(T^{1/2}) + \{1+o_p(T^{-1/2})\}z_1 \\ &= z_1 + o_p(1). \end{aligned}$$

A consequence of these results is that z_λ will converge in probability to z_0 when u_t is $I(0)$ but will converge in probability to z_1 when u_t is $I(1)$. As such, the appropriate optimal test will be applied asymptotically in each of the $I(0)$ and $I(1)$ situations by using z_λ. In addition, since both z_0 and z_1 have standard normal limiting distributions under H_0, the same is clearly also true of the weighted statistic, z_λ of (6). Finally, it is trivial to show that z_λ will also be consistent against fixed alternatives of the form given in H_1.

Remark 2.2: The asymptotic standard normality of z_λ under both $I(0)$ and $I(1)$ shocks allows us to construct approximate confidence bounds for β which hold regardless of whether the shocks are $I(0)$ or $I(1)$. For z_λ evaluated at $\beta_0 = \beta$, we therefore have that

$$\lim_{T\to\infty} \Pr(-c_{\alpha/2} < z_\lambda < c_{\alpha/2}) = \alpha \tag{7}$$

where $\Pr(x > c_\alpha) = \alpha$, for $x \sim N(0,1)$. Substituting for z_λ in (7), and rearranging yields that

$$\lim_{T\to\infty} \Pr\left(\frac{\{1-\lambda(U,S)\}\hat{\beta}s_1 + \lambda(U,S)\tilde{\beta}s_0 - c_{\alpha/2}s_0s_1}{\{1-\lambda(U,S)\}s_1 + \lambda(U,S)s_0} < \beta < \frac{\{1-\lambda(U,S)\}\hat{\beta}s_1 + \lambda(U,S)\tilde{\beta}s_0 + c_{\alpha/2}s_0s_1}{\{1-\lambda(U,S)\}s_1 + \lambda(U,S)s_0} \right) = \alpha.$$

Consequently, an approximate $(1-\alpha)\%$ two-sided confidence interval for β is given by:

$$\hat{\beta}_\lambda \pm c_{\alpha/2} \frac{s_0 s_1}{\{1-\lambda(U,S)\}s_1 + \lambda(U,S)s_0} \tag{8}$$

where

$$\hat{\beta}_\lambda := \frac{\{1-\lambda(U,S)\}\hat{\beta}s_1 + \lambda(U,S)\tilde{\beta}s_0}{\{1-\lambda(U,S)\}s_1 + \lambda(U,S)s_0}, \tag{9}$$

while an approximate $(1-\alpha)\%$ one-sided upper (lower) confidence interval can be constructed using

$$\hat{\beta}_\lambda + (-)c_\alpha s_0 s_1 \left(\{1-\lambda(U,S)\}s_1 + \lambda(U,S)s_0\right)^{-1}. \tag{10}$$

It is easily established from the foregoing results and the BLUE properties of $\hat{\beta}$ when u_t is $I(0)$ and of $\tilde{\beta}$ when u_t is $I(1)$, that $\hat{\beta}_\lambda$ is a consistent and asymptotically efficient estimator of the trend parameter β regardless of whether u_t is $I(0)$ or $I(1)$; see, for example, Rao (1973,p.319).[1] Moreover, (8) and (10) are approximately uniformly most accurate confidence regions, by virtue of the corresponding asymptotic optimality properties of the test based on z_λ, discussed above.

3 An Autocorrelation Robust Procedure and its Asymptotic Properties

Having motivated our new testing procedure within a simplified framework, we now generalize the approach and modify our tests accordingly before establishing their large sample properties. To that end we now assume that u_t in (1) satisfies the following assumption.

Assumption 3. *The stochastic process* $\{u_t\}$ *of (1) is such that*

$$\begin{aligned} u_t &= \rho u_{t-1} + \varepsilon_t, \;\; t = 2, ..., T, \;\; u_1 = O_p(1) && (11) \\ \varepsilon_t &= c(L)e_t, \;\; c(L) = \sum_{i=0}^{\infty} c_i L^i && (12) \end{aligned}$$

with, for some finite constants $\bar{c}_1$ *and* $\bar{c}_2$*: (i)* $c(1)^2 > \bar{c}_1 > 0$*; (ii)* $\sum_{i=0}^{\infty} i|c_i| < \bar{c}_2 < \infty$*, and (iii)* $\{e_t\}$ *is a martingale difference sequence with unit conditional variance and* $\sup_t E(e_t^4) < \infty$*.*

Remark 3.1: Assumption 3 essentially coincides with Assumption 1 of Bunzel and Vogelsang (2005,p.382). As in Bunzel and Vogelsang (2005), we follow the standard

[1] An alternative consistent and asymptotically efficient estimator of β is given by the simple weighted average $\{1-\lambda(U,S)\}\hat{\beta} + \lambda(U,S)\tilde{\beta}$. However, unreported finite sample Monte Carlo simulations suggested that this estimator was less efficient than $\hat{\beta}_\lambda$.

approach in the literature and present large sample results for our statistics which are based on pointwise convergence. In order to overcome potential problems of near-observational equivalence between $I(0)$ and $I(1)$ specifications for u_t, in the sense of Faust (1996) and Pötscher (2002), it would be necessary to establish uniform convergence results. This would involve the need to place additional restrictions on ε_t, including a strengthening of the summability condition in (ii) of Assumption 3 (cf. Phillips and Xiao, 1999) to one which also imposes an eventual rate of decay on the $\{c_i\}$. One possibility is provided by condition $\mathcal{A}(K,p)$ of Faust (1996,p.728) with $p > 2$.[2] Faust also replaces condition *(iii)* of Assumption 3 with the requirement that $\{e_t\}$ is $IID(0,1)$ with finite fourth moments, although this restriction does not seem crucial. □

For the present we assume that the dominant autoregressive root ρ in (11) satisfies $\rho = 1$ (such that u_t is $I(1)$) or $|\rho| < 1$ (such that u_t is $I(0)$). Later in this section we will generalize our analysis to also allow for near unit root behaviour in u_t. Notice that, under Assumption 3, the long run variance of ε_t is given by $\omega_\varepsilon^2 := \lim_{T\to\infty} T^{-1}E(\sum_{t=1}^T \varepsilon_t)^2 = c(1)^2$. Moreover, in the $I(0)$ case the long run variance of u_t is given by $\omega_u^2 := \lim_{T\to\infty} T^{-1}E(\sum_{t=1}^T u_t)^2 = c(1)^2/(1-\rho)^2$.

Because we now allow for weak dependence in ε_t through Assumption 3, the asymptotic null distributions of z_0 and z_1 will no longer be pivotal. Consequently, it is necessary to work with non-parametrically autocorrelation-corrected analogues of z_0 and z_1. Specifically, in what follows we redefine

$$z_0 := \frac{\hat\beta - \beta_0}{s_0} \quad \text{and} \quad z_1 := \frac{\tilde\beta - \beta_0}{s_1} \tag{13}$$

with

$$s_0 := \sqrt{\hat\omega_u^2 / \sum_{t=1}^T (t-\bar t)^2} \quad \text{and} \quad s_1 := \sqrt{\tilde\omega_v^2/T_*} \tag{14}$$

where $\hat\beta$ and $\tilde\beta$ are as defined in Section 2, and $\hat\omega_u^2$ and $\tilde\omega_v^2$ are the long run variance estimators

$$\hat\omega_u^2 := \hat\gamma_0 + 2\sum_{j=1}^{T-1} h(j/\ell)\hat\gamma_j, \quad \hat\gamma_j := T^{-1}\sum_{t=j+1}^T \hat u_t \hat u_{t-j} \tag{15}$$

and

$$\tilde\omega_v^2 := \tilde\gamma_0 + 2\sum_{j=1}^{T_*-1} h(j/\ell)\tilde\gamma_j, \quad \tilde\gamma_j := T_*^{-1}\sum_{t=j+2}^T \tilde v_t \tilde v_{t-j}, \tag{16}$$

where the residuals $\{\hat u_t\}_{t=1}^T$ and $\{\tilde v_t\}_{t=2}^T$ are as defined in Section 2. In both (15) and (16) the kernel function, $h(\cdot)$, and the bandwidth parameter, ℓ, are assumed to satisfy Assumptions A3 and either A4 or A4' of Jansson (2002,pp.1450,1452), respectively.

[2] Condition $\mathcal{A}(K,p)$ requires that $|c_i| < ki^{-p}$ with $p > 1$ for all $i > K$, where k is a positive constant. In our case, setting $p > 2$ implies (ii) of Assumption 3.

In Theorem 1 we now detail the asymptotic properties of z_0 and z_1 of (13) when u_t is $I(0)$. Corresponding results when u_t is $I(1)$ are subsequently given in Theorem 2.

Theorem 1 *Let Assumption 3 hold and assume u_t is $I(0)$. Then, under $H_{1,0}$: (i) $z_0 \xrightarrow{d} \frac{\kappa}{\omega_u\sqrt{12}} + N(0,1)$; and (ii) $z_1 \xrightarrow{p} 0$.*

Remark 3.2: From the results in Theorem 1 it is seen that when u_t is $I(0)$ and H_0 holds, z_0 has a standard normal limiting null distribution, whereas z_1 converges in probability to zero. Moreover, when u_t is $I(0)$, it is seen that z_0 attains the Gaussian asymptotic local power envelope for this testing problem.

Theorem 2 *Let Assumption 3 hold and assume u_t is $I(1)$. Then, under $H_{1,1}$: (i) $z_0 = o_p(T^{1/2})$; and (ii) $z_1 \xrightarrow{d} \frac{\kappa}{\omega_\varepsilon} + W(1)$ where $W(r)$ is a standard Brownian motion on $[0,1]$.*

Remark 3.3: Noting that $W(1) \sim N(0,1)$, we observe from Theorem 2 that when u_t is $I(1)$ and H_0 holds, z_1 has a standard normal limiting null distribution. Moreover, when u_t is $I(1)$, observe that z_1 achieves the Gaussian asymptotic local power envelope for this testing problem. □

Now suppose that U is a unit root test for testing the $I(1)$ null that $\rho = 1$ against the $I(0)$ alternative $|\rho| < 1$ and S is a stationarity test for testing the $I(0)$ null that $|\rho| < 1$ against the $I(1)$ alternative $\rho = 1$, both tests being based on the de-trended residuals, $\hat{u}_t$. Also, suppose that U and S satisfy Assumption 1 and that Assumption 2 holds. Our non-parametrically corrected analogue of (6) is then given by

$$z_\lambda := \{1 - \lambda(U,S)\}z_0 + \lambda(U,S)z_1 \tag{17}$$

where z_0 and z_1 are as defined in (13). The following Corollary of Theorems 1 and 2 details the large sample behaviour of z_λ.

Corollary 1 *Let Assumptions 1-3 hold: (i) If u_t is $I(0)$ and $H_{1,0}$ is true, then $z_\lambda = z_0 + o_p(1) \xrightarrow{d} \frac{\kappa}{\omega_u\sqrt{12}} + N(0,1)$; and (ii) If u_t is $I(1)$ and $H_{1,1}$ is true, $z_\lambda = z_1 + o_p(1) \xrightarrow{d} \frac{\kappa}{\omega_\varepsilon} + N(0,1)$.*

Remark 3.4: The results in Corollary 1 show that if u_t is $I(0)$, z_λ is asymptotically equivalent to z_0, while if u_t is $I(1)$, z_λ is asymptotically equivalent to z_1. Consequently, z_λ achieves the Gaussian asymptotic local power envelope regardless of whether u_t is $I(0)$ or $I(1)$. Moreover, the limiting distribution of z_λ under H_0 is standard normal, again irrespective of whether u_t is $I(0)$ or $I(1)$. Again it is entirely straightforward to show that z_λ is consistent against fixed alternatives of the form given in H_1, regardless of whether u_t is $I(0)$ or $I(1)$.

Remark 3.5: Notice that in contrast to all other robust trend tests currently available, p-values can easily be constructed for the z_λ test due to the fact that it has a standard normal limiting null distribution regardless of whether u_t is $I(0)$ or $I(1)$. The finite-sample accuracy of the normal approximation involved here will be explored numerically in Section 5. Notice also that, unlike our proposed test based on z_λ, the feasible *Dan-J* test of Bunzel and Vogelsang (2005) has data-dependent critical values for a given significance level. Unlike other robust trend tests, the functional form of the z_λ test statistic does not depend on the choice of significance level. However, it is important to stress that in order to compute z_λ in practice one must make a specific choice for the function $\lambda(U, S)$ and, hence, for the statistics U and S. Specific recommendations are provided in Section 4.

Remark 3.6: Confidence intervals for β can once more be constructed as in (8) and (10) with s_0 and s_1 appropriately re-defined as in (14). Notice also from the foregoing results that the estimator $\hat{\beta}_\lambda$ of (9) given in Remark 2.2 remains an asymptotically efficient estimator of the trend parameter β under the weaker conditions placed on $\{\varepsilon_t\}$ in this section, again regardless of whether u_t is $I(0)$ or $I(1)$. This is because of the well-known asymptotic optimality property of the OLS estimators $\hat{\beta}$ and $\tilde{\beta}$ when u_t is $I(0)$ and when u_t is $I(1)$, respectively; see Grenander and Rosenblatt (1957). $\square$

Thus far we have assumed that $\rho = 1$ or $|\rho| < 1$, thereby excluding the possibility of a near unit root in u_t. We now consider extending our analysis of strong autocorrelation to include the case of near-$I(1)$ processes which obtain for $\rho = \rho_T = 1 - c/T$, $0 \le c < \infty$, in (1). When u_t is near-$I(1)$ it is easily shown that the result in Theorem 2 (i) continues to hold. The result in Theorem 2 (ii), however, generalizes to

$$z_1 \xrightarrow{d} \frac{\kappa}{\omega_\varepsilon} + J_c(1) \tag{18}$$

where $J_c(r)$ is the standard Ornstein-Uhlenbeck process, $J_c(r) = \int_0^r e^{-(r-s)c} dW(s) \sim N(0, \pi_{c,r})$, where $\pi_{c,r} := (2c)^{-1}(1 - e^{-2rc})$, and $W(r)$ is a standard Brownian motion. Consequently, $z_1 \xrightarrow{d} \frac{\kappa}{\omega_\varepsilon} + N(0, \pi_{c,1})$. Likewise, the result in Corollary 1 (ii) is amended to $z_\lambda \xrightarrow{d} \frac{\kappa}{\omega_\varepsilon} + N(0, \pi_{c,1})$, provided U and S still satisfy Assumption 2 under near-$I(1)$ errors. Observing that $\pi_{0,1} = 1$ (since $J_0(r) = W(r)$) and that $\pi_{c,1}$ is a monotonically decreasing function of c, it is seen that employing standard normal critical values for z_λ (which are appropriate for $c = 0$) will result in a conservative test under H_0 when $c > 0$, with a consequent reduction in power under $H_{1,1}$. It is straightforward to show that z_λ remains consistent against fixed alternatives, of the form given in H_1, when $c > 0$.

If $\pi_{c,1}$ were consistently estimable, we could appropriately re-scale z_λ in order to restore its asymptotic distribution to standard normal. Unfortunately, this is not possible since, as noted by Bunzel and Vogelsang (2005,p.387), c cannot be consistently estimated. It is, however, interesting to investigate whether there is any simple means of reducing the impact of the value of $c > 0$ on the behaviour of the test.

To this end, we consider the non-negative quantity

$$R_\delta := \left(\frac{\tilde{\omega}_v^2}{T^{-1}\hat{\sigma}_u^2}\right)^\delta, \quad \delta > 0,$$

where $\hat{\sigma}_u^2 = (T-2)^{-1}\sum_{t=1}^T \hat{u}_t^2$. In the near-$I(1)$ case it is straightforward to show that

$$R_\delta \xrightarrow{d} \left\{\int_0^1 N_c(r)^2 dr\right\}^{-\delta} \tag{19}$$

where $N_c(r)$ denotes the continuous time residual from the projection of $J_c(r)$ onto the space spanned by $\{1, r\}$. The limit distribution of R_δ is pivotal and its mean and variance are both increasing functions of c. It is therefore to be expected that multiplying z_1 by R_δ will, to some extent, offset the fact that $J_c(1)$ has a variance which diminishes with c. We therefore suggest replacing z_1 and, hence, z_λ with the modified statistics thereof:

$$z_1^{m\,\delta} := \gamma_{\xi,\delta} R_\delta z_1$$

and

$$z_\lambda^{m\,\delta} := \{1 - \lambda(U,S)\}z_0 + \lambda(U,S)z_1^{m\,\delta}$$

where $\gamma_{\xi,\delta}$ is some finite positive constant.

Under Assumption 3 and $H_{1,1}$ in the near-$I(1)$ case we obtain from (18), (19) and an application of the continuous mapping theorem that

$$z_1^{m\,\delta} \xrightarrow{d} \gamma_{\xi,\delta}\left\{\int_0^1 N_c(r)^2 dr\right\}^{-\delta}\left\{\frac{\kappa}{\omega_\varepsilon} + J_c(1)\right\} =: \eta_\xi(c,\delta,\kappa) \tag{20}$$

As is clear from (20), the null ($\kappa = 0$) limiting distribution of $z_1^{m\,\delta}$ is not standard normal for any c. However, using a technique popularised by Vogelsang (1998), the constant $\gamma_{\xi,\delta}$ can be chosen such that under H_0 and when $c = 0$, the critical value of $\eta_\xi(0,\kappa,\delta)$ at a given significance level ξ coincides with that of a standard normal variate. The adopted scale factor R_δ has the advantage that $\{\int_0^1 N_0(r)^2 dr\}^{-\delta}W(1)$ has a symmetric distribution which entails that the value of $\gamma_{\xi,\delta}$ will coincide for upper and lower tails.

For the modified test procedure, we obtain from (20) that under Assumptions 1-3 in the near-$I(1)$ case that under $H_{1,1}$,

$$z_\lambda^{m\,\delta} = z_1^{m\,\delta} + o_p(1) \xrightarrow{d} \eta_\xi(c,\kappa,\delta)$$

so that under H_0 and $c = 0$, the ξ level standard normal critical value again applies in the limit. In the $I(0)$ case, part (i) of Corollary 1 also holds for $z_\lambda^{m\,\delta}$; that is,

$$z_\lambda^{m\,\delta} = z_0 + o_p(1) \xrightarrow{d} \frac{\kappa}{\omega_u\sqrt{12}} + N(0,1).$$

Consequently, when u_t is $I(0)$ standard normal (asymptotic) critical values remain appropriate under H_0 and, moreover, $z_\lambda^{m\,\delta}$ still attains the Gaussian asymptotic local power envelope.

Notice that z_λ statistic of (17) coincides with the modified statistic, $z_\lambda^{m\,\delta}$ of (20), for $\delta = 0$ and $\gamma_{\xi,\delta} = 1$. The values $\delta = 1, 2$ can therefore be considered to represent two increasing levels of modification. We experimented with other values for δ and found that for values of δ smaller than one the correction had insufficient effect to offset the conservative behaviour of the test in the near-I(1) case, while for values larger than two near-$I(1)$ corrections were bought only at the expense of a significant power reduction in the pure $I(1)$ case. Accordingly we shall focus on the values $\delta = 1, 2$ in what follows. Values for $\gamma_{\xi,1}$ and $\gamma_{\xi,2}$, obtained by simulation, are given for various significance levels in Table 1.

4 Practical Implementation of the Test Procedure

In this section we discuss the practicalities of our proposed test procedure. Provided the choices for U, S and $\lambda(U, S)$ satisfy the asymptotic properties of Assumptions 1 and 2 respectively, their actual specification is otherwise essentially arbitrary. Clearly different specifications of U, S and $\lambda(U, S)$ will yield differing finite sample behaviour in z_λ (and $z_\lambda^{m\,\delta}$), since normality and optimality are only asymptotic characteristics of the behaviour of z_λ and our theoretical framework does not provide guidance on specific choices for U, S and $\lambda(U, S)$. In what follows, therefore, we provide recommended specifications that performed well in terms of a finite sample size/power trade-off in unreported Monte Carlo simulations. Our proposed choices, used throughout the remainder of the paper, are also relatively straightforward to compute, in keeping with our emphasis on a procedure that is simple to apply.

Consider first the choice of a suitable statistic U. It is well-known that the conditions placed on U by Assumption 1 are satisfied by standard unit root test statistics, such as the OLS- or local GLS-detrended (augmented) Dickey-Fuller statistic. We employ the local GLS-detrended augmented Dickey-Fuller t-test of Elliott *et al.* (1996), i.e., DF-GLS$^\tau$, as our choice for U in what follows. Precisely, DF-GLS$^\tau$ is the usual t-ratio for testing $\rho^* = 0$ in the regression equation

$$\Delta\tilde{u}_t = \rho^*\tilde{u}_{t-1} + \sum_{j=1}^{p}\phi_j\Delta\tilde{u}_{t-j} + \tilde{e}_t, \quad t = p+2, ..., T \tag{21}$$

where $\tilde{u}_t$ are the local GLS de-trended residuals obtained from the regression of $\mathbf{y}_{\bar{c}} := (y_1, y_2 - \bar{c}y_1, ..., y_T - \bar{c}y_{T-1})'$ on $\mathbf{Z}_{\bar{c}} := (z_1, z_2 - \bar{c}z_1, ..., z_T - \bar{c}z_{T-1})'$, where $z_t = (1,\ t)'$ and $\bar{c} = -13.5$; cf. Elliott *et al.* (1996). The number of lagged difference terms, p, included in (21) is determined by application of the autocorrelation-robust MAIC procedure of Ng and Perron (2001), setting the maximum lag length at $p_{\max} = \lfloor 12(T/100)^{1/4}\rfloor$. Notice that DF-GLS$^\tau$ is exact invariant to α and β.[3]

[3]The use of DF-GLS with MAIC involves parametric model selection, unlike the individual trend

As regards S, again it is well-known that standard (trend-) stationarity test statistics, such as that of KPSS, satisfy Assumption 1. In what follows we will therefore adopt as our choice for S the KPSS statistic

$$\hat{\eta}_\tau := \frac{\sum_{t=1}^{T}\left(\sum_{i=1}^{t}\hat{u}_i\right)^2}{T^2\hat{\omega}_u^2}$$

where the long run variance estimator $\hat{\omega}_u^2$ is as defined in (15). In what follows, in the context of both $\hat{\omega}_u^2$ and $\tilde{\omega}_v^2$ of (16), we will use the quadratic spectral kernel with Newey and West (1994) automatic bandwidth selection adopting a non-stochastic prior bandwidth of $\lfloor 4(T/100)^{2/25}\rfloor$. Again, $\hat{\eta}_\tau$ is exact invariant to α and β.

Remark 4.1: It is well-known that DF-GLS$^\tau$ and $\hat{\eta}_\tau$ statistics also satisfy Assumption 1 (ii) under near-$I(1)$ errors; see Elliott *et al.* (1996) and Müller (2005), respectively. Consequently, therefore, we are able to provide representations for the limiting null distributions and asymptotic local power functions associated with our feasible z_λ and $z_\lambda^{m\,\delta}$ statistics in the near-$I(1)$ setting, which are as given in Section 3. This is in contrast to Bunzel and Vogelsang (2005) who are unable to provide such a representation for their feasible *Dan-J* statistic; *op. cit.*, p.388.

Remark 4.2: Observe that since Assumption 1 only requires certain orders in probability, it is not actually necessary for DF-GLS$^\tau$ and $\hat{\eta}_\tau$ to be autocorrelation-corrected to obtain pivotal limit distributions under their respective null or alternative hypotheses. Whether or not they are autocorrelation-corrected will likely have consequences for the finite sample performance of z_λ and $z_\lambda^{m\,\delta}$, and it would seem expedient to employ the usual corrections, as above. □

We now turn to specifying a function which satisfies Assumption 2. Such a function is not difficult to obtain: a simple example (a slight generalization of which we adopt in what follows) is given by the exponential function

$$\lambda(U,S) = \exp\left(-\frac{|U|}{|S|}\right)$$

In the $I(0)$ case we have that

$$\lambda(U,S) = \exp\left(-\frac{|O_p(T^{\delta_U})|}{|O_p(1)|}\right)$$

and so $\lambda(U,S) \xrightarrow{p} 0$. Turing to the $I(1)$ case, we have that

$$\lambda(U,S) = \exp\left(-\frac{|O_p(1)|}{|O_p(T^{\delta_S})|}\right)$$

statistics z_0 and z_1. However, it would be entirely straightforward to replace U with a test based on non-parametric methods (cf., *inter alia*, Phillips and Perron, 1988), should a practitioner find that more appealing.

which converges in probability to unity faster than any $1 + O_p(T^{-\gamma})$ variate with finite $\gamma > 0$.

Remark 4.3: Notice that Assumption 2 (ii) is in fact slightly stronger than is now required because when u_t is $I(1)$ or near-$I(1)$, z_0 is of $o_p(T^{1/2})$, unlike in the Gaussian IID model considered initially in Section 2 where z_0 was of $O_p(T^{1/2})$.

Remark 4.4: Based on our specific choices of DF-GLS$^\tau$ and $\hat{\eta}_\tau$ for U and S, respectively, we found numerically that the best performance overall finite sample performance was obtained using

$$\lambda(U,S) = \exp\left(-0.00025\left(\frac{\text{DF-GLS}^\tau}{\hat{\eta}_\tau}\right)^2\right),$$

which clearly satisfies Assumption 2, and it is this particular function on which all subsequent finite sample results are based. As predicted by the large sample distribution theory in Section 3, we found that variations in the finite sample behaviour of our tests arising from the precise choice of $\lambda(U,S)$ diminished as the sample size was increased.

5 Comparisons with Other Tests

In this section we consider the performance of the newly proposed z_λ and $z_\lambda^{m\,\delta}$ tests and assess the results relative to the performance of the *Dan-J* test recommended by Bunzel and Vogelsang (2005). Bunzel and Vogelsang (2005) propose a further feasible test, which they denote *Dan-BG*, but advise against its use in practice: corresponding results for this test are therefore not reported but are obtainable on request.

Bunzel and Vogelsang's *Dan-J* test statistic is of the form

$$\textit{Dan-J} = z_0' \exp(-c_\xi J)$$

where z_0' is z_0 as defined in (13) but with the implicit long run variance estimator, $\hat{\omega}_u^2$, constructed using the Daniell kernel with a data-dependent bandwidth. Specifically, the bandwidth is given by $\max(\hat{b}_{\text{opt}}T, 2)$, where $\hat{b}_{\text{opt}} = b_{\text{opt}}(\hat{c})$. Here, $\hat{c} = T(1-\hat{\rho})$ with $\hat{\rho}$ obtained by OLS estimation of (2), and $b_{\text{opt}}(.)$ is a step function given in Bunzel and Vogelsang (2005). In the expressions for *Dan-J*, the z_0' statistic is scaled by a function of the J unit root test statistic of Park (1990) and Park and Choi (1988). The constant c_ξ is chosen so that, at a given significance level, ξ, a particular test has the same critical value under both $I(0)$ and $I(1)$ errors. The value of c_ξ depends on $\hat{b}_{\text{opt}}$; Bunzel and Vogelsang (2005) provide a response surface for determining c_ξ for a given significance level, and $\hat{b}_{\text{opt}}$. The critical values for the test also depend on $\hat{b}_{\text{opt}}$, and again a response surface is provided by the authors for a variety of significance levels.

We now examine the asymptotic power and finite sample size and power properties of the tests. All of the reported trend tests are conducted against a one-sided (upper tail) alternative.

5.1 Asymptotic Performance

Figures 1 and 2 report, for $I(0)$ and (near-) $I(1)$ errors respectively, the asymptotic size and power of the z_λ and $z_\lambda^{m\delta}$ tests for $\delta = 1, 2$, along with results for the *Dan-J* test of Bunzel and Vogelsang (2005). In the near-$I(1)$ case, we set $\rho = 1 - c/T$ in (2) and consider the local-to-unity settings $c \in \{0, 5, 10, 15\}$, $c = 0$ giving the basic $I(1)$ case. The results were obtained by direct simulation of the limiting distributions of our tests, approximating the Wiener processes using $NIID(0, 1)$ random variates, with the integrals approximated by normalized sums of 1000 steps. For the *Dan-J* test, because c is not consistently estimated using $\hat{c}$, Bunzel and Vogelsang (2005) only provide a limiting distribution for *Dan-J* when it is assumed that c is known in the calculation of $\hat{b}_{\text{opt}}$. That is, when $\hat{b}_{\text{opt}} = b_{\text{opt}}(\hat{c})$ is replaced by $b_{\text{opt}}(c)$. Although this strictly means that their asymptotic results are based on the limiting behaviour of an infeasible test, for tractability here we also calculate the limit distribution of *Dan-J* using $b_{\text{opt}}(c)$. All tests are run at the asymptotic 0.05 level appropriate for $c = 0$; that is, using a critical value of 1.645 for all tests other than *Dan-J*.

Results are reported for the the null hypothesis $\beta = \beta_0$ and local alternatives $H_{1,0} : \beta = \beta_0 + \kappa T^{-3/2}$ and $H_{1,1} : \beta = \beta_0 + \kappa T^{-1/2}$ (i.e. for $I(0)$ and (near-) $I(1)$ errors respectively). We consider the range of values $\kappa \in [0, 20]$ for $I(0)$ errors and $\kappa \in [0, 8]$ for (near-) $I(1)$ errors, in each case using a grid with 100 steps. Here and throughout the paper, simulations were programmed in Gauss 6.0 using 50,000 Monte Carlo replications.

Figure 1 confirms that the power plots for z_λ, z_λ^{m1} and z_λ^{m2} are all equivalent in the $I(0)$ case and that these offer a power improvement, albeit very modest, over *Dan-J* which does not achieve the Gaussian asymptotic power envelope. Part (a) of Figure 2 provides results pertaining to the pure $I(1)$ case, $c = 0$. Here the power ranking amongst the tests is quite unambiguous: z_λ, z_λ^{m1}, z_λ^{m2} and finally *Dan-J*. That z_λ is, and by a very clear margin, the most powerful test here reflects its optimality (in that it achieves the Gaussian asymptotic power envelope) in this setting. In parts (b)-(d) of Figure 2 the effects of near-integration in the $I(1)$ process can clearly be seen. In contrast to Figure 1 and Figure 2 part (a), all of the tests become under-sized when $\kappa/\omega_\varepsilon = 0$, particularly so z_λ. The effect of this is to reduce its power to detect small values of $\kappa/\omega_\varepsilon$, with the power profile of z_λ resembling a step function around the point 1.645. However, the modification embodied in z_λ^{m1} leads to a substantial improvement in power for small $\kappa/\omega_\varepsilon$, and that in z_λ^{m2} even more so. The trade-off here is that, as noted for part (a) above, z_λ^{m1} and z_λ^{m2} have lower power than z_λ when $c = 0$. Notice that throughout parts (b)-(d) of Figure 2, z_λ^{m2} is almost always more powerful than the *Dan-J* test. Only when both tests have power below 0.15 is *Dan-J* the more powerful. In summary, if only $I(0)$ and $I(1)$ processes are being considered, then the best performing test is z_λ. If performance with near-$I(1)$ processes is also considered to be of importance, then z_λ^{m2} is arguably the better choice, at least from the perspective of the near-$I(1)$ asymptotic theory.

5.2 Finite Sample Performance

We now turn to a consideration of the finite sample behaviour of the z_λ, z_λ^{m1}, z_λ^{m2} and *Dan-J* tests. Specifically, we report simulated rejection frequencies of these tests when applied to data generated according to the following data generating process,

$$\begin{aligned} y_t &= \beta t + u_t, & t = 1, ..., T & \\ (1-\rho L)u_t &= (1-\theta L)\varepsilon_t, & t = 2, ..., T, & \quad u_1 = 0 \end{aligned} \tag{22}$$

with $\varepsilon_t \sim NIID(0,1)$. We focus attention on testing the null hypothesis of the absence of a linear trend, $\beta = 0$, against the alternative hypothesis of a positive trend, $\beta > 0$, although it should be stressed that identical results would obtain for the more general problem of testing $\beta = \beta_0$ against $\beta > \beta_0$.

Table 2 reports finite sample empirical sizes ($\beta = 0$) for the above tests, again using the 0.05 level asymptotic critical values appropriate for $c = 0$. Experiments are reported for the design parameters $\rho = 1 - c/T$, $c \in \{0, 5, 10, 15, T\}$, $\theta \in \{0, \pm 0.4, \pm 0.8\}$ and sample sizes $T \in \{100, 200\}$. Notice that for $c = T$, u_t is a pure $MA(1)$ process. From the results in Table 2 we see that z_λ is modestly over-sized when $c = 0$, suggesting that the normal approximation for the limiting distribution of z_λ is somewhat inaccurate for the pure $I(1)$ case in finite samples, although this problem appears to be ameliorated to some extent by increasing the sample size, as predicted by the asymptotic theory. It is also clear that z_λ^{m1} and, especially, z_λ^{m2} are largely free from size distortion problems when $c = 0$. All three of these tests tend to be under-sized across the other values of c, though less so when $c = T$. Interestingly, none of their sizes appear particularly sensitive to the choice of θ. The size properties of *Dan-J* are certainly more sensitive to the choice of c and θ. It is generally quite considerably over-sized when $\theta = 0.8$ (apart from when $c = T$, in which case it is badly under-sized). Our asymptotic results predict that all the tests are undersized when $c \in \{5, 10, 15\}$, so it would appear that, when $\theta = 0.8$ for *Dan-J*, rather larger finite sample sizes than those considered here are probably required before the asymptotic results become a decent predictor of finite sample performance.

Figures 3-6 present size-adjusted (such that for each parameterisation considered the tests were run at a nominal exact 0.05 level) power curves for the four tests in experiments with parameter settings $\rho = 1 - c/T$, $c \in \{0, 5, 10, 15\}$, $\theta \in \{0, \pm 0.4\}$ and $T \in \{100, 200\}$, in each case plotted across a range of values of β (varying with the choice of c). The tests have been size-adjusted to control for the differing finite sample size properties of the tests noted above. Our basic z_λ test comfortably dominates all other tests on finite sample size-adjusted power. This superiority is most pronounced in the pure $I(1)$ case, $c = 0$, as might be expected given the (asymptotic) optimality of z_λ here. The modified z_λ^{m1} test everywhere outperforms the z_λ^{m2} test, and also dominates *Dan-J* in all cases apart from when $T = 100$, $\theta = 0.4$ and $c = 15$ (Figure 6 part (e)), where the two power curves intersect. The z_λ^{m2} test outperforms *Dan-J* except when $\theta = 0.4$ and $c = 10, 15$. Notice also from Figures 3-6 that for each test, and as predicted by the large sample distribution theory, the power curves vary over the different values of θ considered with power being higher the larger is θ, other things being equal.

In order to investigate whether the finite sample power superiority of z_λ over all the other tests when $c > 0$ is attributable to the fact that we are reporting size-adjusted results, in Figure 7 we present finite sample power curves for tests run using the 0.05 level asymptotic critical values appropriate for $c = 0$. Results are reported for here for $c \in \{5, 10, 15\}$ for the central case of $\theta = 0$. With the exception of z_λ, all of the tests' power curves display reasonably similar patterns and rankings to those seen in the corresponding asymptotic power curves in parts (b)-(d) of Figure 2. For z_λ, however, we see that the step-function-like behaviour in its asymptotic local power function noted above is far from apparent even for $c = 15$ and $T = 200$. Consequently, the asymptotic theory for $I(0)$ errors appears to provide a better predictor for the finite sample behaviour of the z_λ test under near-$I(1)$ errors. Overall, z_λ appears to display the best power of the four tests. Although *Dan-J* displays slightly higher power for small values of β (most notably when $T = 200$ and $c = 15$), for larger values of β, z_λ tends to display significantly higher power than *Dan-J*.

On the basis of these results, we recommend the use of the z_λ test in practice. This test appears overall to have the best finite sample power properties, at least for the models we have examined, amongst available robust trend tests and attains the Gaussian asymptotic power envelope under either $I(0)$ or $I(1)$ errors. Our preferred z_λ test has the additional advantage of being computationally simpler than the other tests, especially when *Dan-J* is considered, and is run using standard normal critical values without any requirement for significance level-specific adjustments.

6 Empirical Examples

We first examine evidence for the presence of a trend in the logarithms of seasonally adjusted real GDP for a set of twelve countries: Australia, Canada, France, Germany, Japan, Italy, Netherlands, South Africa, Spain, Switzerland, UK and US. The data are observed quarterly over the period 1980:1–2005:2 ($T = 102$) and were obtained from the Office for National Statistics for the UK, and International Financial Statistics for all other countries. The time series (with adjusted intercepts) are plotted in Figure 8. The four tests considered in this paper: z_λ, z_λ^{m1}, z_λ^{m2} and *Dan-J* are applied to the series assuming a one-sided (upper-tail) alternative, and the results are presented in Table 3. All of the series in Figure 8 display very similar strong upward trending behaviour across the sample period. This is confirmed by the z_λ test which comfortably rejects the hypothesis of no trend in favour of a positive trend in all twelve series at the 0.05 level. In fact, the individual p-values for these outcomes are uniformly smaller than 0.0005. Because of the significance level-specific adjustments inherent in all the other tests, we do not report p-values, but at the 0.05 level, z_λ^{m1} finds eleven rejections; z_λ^{m2} eight and *Dan-J* four rejections. The DF-GLS$^\tau$ and DF-GLS$^\mu$ (the latter constructed as for DF-GLS$^\tau$ except that $\tilde{u}_t$ in (21) are replaced by the residuals from the regression of $\mathbf{y}_{\bar{c}}$ on $\mathbf{Z}_{\bar{c}}$ for $z_t = 1$ and $\bar{c} = -7$) tests of Elliott *et al.* (1996) which we also report show that, regardless of whether a constant or constant and trend is specified, the unit root hypothesis cannot be rejected for any of the series. As demonstrated by our Monte Carlo results, this is one of the situations where our proposed z_λ test is expected to strongly dominate the other tests on power, and in particular the *Dan-J* test. The trend test outcomes for these data appear reflective of this. The annualised percentage growth rates we report, calculated using the efficient estimator of (9), i.e. $400\hat{\beta}_\lambda$, and shown with a 95% confidence interval, are mostly around the 2-3% level, consistent with generally accepted figures over this period and indicative of the strong trend behaviour detected by the newly proposed tests.

As a further comparison of the z_λ, z_λ^{m1} and z_λ^{m2} tests with the *Dan-J* test of Bunzel and Vogelsang (2005), we also apply these tests to the series employed in that paper, namely the logarithm of the net barter terms of trade series constructed by Grilli and Yang (1998) and extended by Lutz (1999). The data are annual from 1900–1995 ($T = 96$) and are plotted in Figure 9. The Prebisch-Singer hypothesis postulates that such a series should exhibit a downward trend over time, and we consequently apply the tests against a one-sided (lower-tail) alternative, conducting the tests at the $\xi = 0.10$, 0.05 and 0.01 levels. The results are reported in Table 4. The z_λ^{m1}, z_λ^{m2} and *Dan-J* tests all reject the hypothesis of no trend in favour of a negative trend at the 0.05 level, while the z_λ test rejects the null at the 0.01 level. Notice that local GLS de-trending according to a constant and trend rather than just a constant, as appears appropriate from the trend tests, enables us to emphatically reject the unit root hypothesis in favour of trend stationarity for the net terms of trade data. The central result of a significant downward trend provides evidence in favour of the Prebisch-Singer hypothesis, with the percentage growth rate (again calculated using the estimator of (9), i.e. $100\hat{\beta}_\lambda$) found to be -0.625%, with a 95% confidence interval of -1.091% to -0.159%.

7 Conclusions

In this paper we have developed computationally simple linear trend tests which do not require knowledge of the form of serial correlation in the data and are robust to strong serial correlation. Our proposed test procedure is based on a simple data-dependent weighted average of two conventional t-ratios, one appropriate for when the data are generated by an $I(0)$ process and the other when the data are $I(1)$. We have demonstrated that our proposed test has a standard normal limiting null distribution and outperforms other robust trend tests in terms of asymptotic local power, achieving the Gaussian power envelope, in both $I(0)$ and $I(1)$ environments. In the presence of a near unit root, our proposed test, in common with other robust procedures, was shown to be conservative, a modification for which was proposed. The finite sample size and power properties of our new test and its modified variants were also shown, for the cases considered, to be superior overall to the recommended *Dan-J* robust trend test of Bunzel and Vogelsang (2005).

We applied our new tests together with the *Dan-J* test of Bunzel and Vogelsang (2005) to real GDP data from a variety of developed countries and also to the net barter terms of trade series previously analysed in Bunzel and Vogelsang (2005). All of the tests were able to reject the no trend null for the terms of trade data. For the real GDP data, where the data display a quite apparent positive trend, our preferred z_λ test is able to reject the no trend null in favour of positive growth rates for all twelve of the series, while the *Dan-J* test rejects for only four of the twelve series. This outcome is not wholly unexpected given that the presence of an autoregressive unit root cannot be rejected for any of these series.

Although we have outlined our testing procedure through the problem of testing for the presence of a linear trend, as we noted in the Introduction, our approach is in fact much more general. For the usual linear regression model $y = X\beta + \varepsilon$, provided the individual (deterministic) regressors satisfy, for example, Assumption 2 of Bunzel and Vogelsang (2005,p.382), we can apply the same principle to testing any set of linearly independent linear restrictions on the elements of β of the usual $R\beta = r$ form. This is achieved simply by replacing the t-ratios used in this paper with the appropriate (maximum) likelihood ratio statistics and constructing the unit root (U) and stationarity (S) test statistics used in the switching function, $\lambda(U, S)$, using the residuals from the appropriate (either OLS or local-GLS) regression of y on X. Such tests will share the robustness properties of the tests discussed in this paper to (strong) serial correlation in the disturbances, and will have standard limiting χ^2 null distributions and asymptotic optimality properties under both $I(0)$ and $I(1)$ errors.

Another approach to circumventing the problems of model uncertainty inherent in a pre-testing methodology is that of Bayesian model averaging, allowing for uncertainty with regard to both the presence of a trend and the order of integration jointly (see, for example, Karlsson and Salabasis, 2004). In comparison to a Bayesian framework, the trend function hypothesis tests proposed in this paper appeal to asymptotic theory for their justification, but at the gain of not imposing very specific regularity conditions on the process innovations (such as normality), and obviating the need for the specification of priors, which can of course be problematic. The newly proposed test has the added appeal (at least from a practitioner's viewpoint) that it is computationally straightforward, requiring little or nothing in the way of bespoke software, unlike an implementation of the Markov-chain Monte Carlo algorithm needed for Bayesian model averaging. On the other hand, our procedure is potentially not as informative about the full range of model specifications as Bayesian model averaging, since it provides robust inference on the trend function alone.

Finally, a potentially worthwhile extension of our approach, at least from a macroeconomic aspect, would be to test for a linear trend in a multivariate system; for example, in a regression model with $I(1)$ variables where co-integration between these variables may, or may not, pertain. This is an area of research which is currently under consideration by the authors.

Appendix

PROOF OF THEOREM 1.

(i) Multiplying the numerator and denominator of z_0 of (13) by $T^{3/2}$ we have that

$$z_0 = \frac{T^{3/2}(\hat{\beta} - \beta_0)}{\sqrt{\frac{\hat{\omega}_u^2}{T^{-3}\sum_{t=1}^{T}(t-\bar{t})^2}}}. \tag{A.1}$$

Consider first the numerator of (A.1). Substituting for the OLS estimator of β we obtain that

$$\begin{aligned} T^{3/2}(\hat{\beta} - \beta_0) &= \kappa + \frac{T^{-3/2}\sum tu_t - (T^{-2}\sum t)(T^{-1/2}\sum u_t)}{T^{-3}\sum t^2 - (T^{-2}\sum t)^2} \\ &\xrightarrow{d} \kappa + \omega_u 12\left\{\frac{1}{2}W(1) - \int_0^1 W(r)dr\right\} \end{aligned}$$

where $W(r)$ is a standard Brownian motion process defined via $\omega_u^{-1}T^{-1/2}\sum_{t=1}^{\lfloor Tr \rfloor} u_t \xrightarrow{d} W(r)$. Integration by parts establishes the result that

$$\frac{1}{2}W(1) - \int_0^1 W(r)dr = \int_0^1 \left(s - \frac{1}{2}\right) dW(s)$$

the right member of which is normal with mean zero and variance

$$\int_0^1 \left(s - \frac{1}{2}\right)^2 ds = \frac{1}{12}.$$

Consequently,

$$T^{3/2}(\hat{\beta} - \beta_0) \xrightarrow{d} \kappa + \omega_u\sqrt{12}N(0,1). \tag{A.2}$$

Turning to the denominator of (A.1), under the conditions placed on $\hat{\omega}_u^2$ in section 3 we have that $\hat{\omega}_u^2 \xrightarrow{p} \omega_u^2$, since u_t is $I(0)$. It is then trivially seen that

$$\frac{\hat{\omega}_u^2}{T^{-3}\sum_{t=1}^{T}(t-\bar{t})^2} \xrightarrow{p} 12\omega_u^2 \tag{A.3}$$

and the stated result then follows from (A.2) and (A.3) *via* an application of the continuous mapping theorem.

(ii) Recall that $\tilde{\beta} = T_*^{-1}(y_T - y_1)$. Consequently, under $H_{1,0}$, $\tilde{\beta} = \beta_0 + \kappa T^{-3/2} + T_*^{-1}(u_T - u_1)$, and, hence,

$$z_1 = \frac{T_*^{1/2}\kappa T^{-3/2} + T_*^{-1/2}(u_T - u_1)}{\sqrt{\tilde{\omega}_v^2}}. \tag{A.4}$$

For the numerator of (A.4) observe that

$$
\begin{aligned}
T_*^{1/2}\kappa T^{-3/2} + T_*^{-1/2}(u_T - u_1) &= O(T^{-1}) + O_p(T^{-1/2}) \\
&= O_p(T^{-1/2}).
\end{aligned}
$$

Turning to the denominator of (A.4), observe that $\tilde{\omega}_v^2 \xrightarrow{p} \omega_v^2 = \lim_{T\to\infty} T^{-1}E(\sum_{t=2}^T v_t)^2 = 0$, since $v_t = \Delta u_t$ is over-differenced when u_t is $I(0)$. However, the fastest rate of convergence of any long run variance estimator satisfying the conditions laid out in section 3 is slower than $O_p(T^{-1/2})$ and, hence, $\sqrt{\tilde{\omega}_v^2}$ converges in probability to zero at a rate slower than $O_p(T^{-1/4})$. Consequently, z_1 is of $o_p(1)$.

Proof of Theorem 2.

(i) Multiplying the numerator and denominator of z_0 of (13) by $T^{1/2}$ we have that

$$
z_0 = \frac{T^{1/2}(\hat{\beta} - \beta_0)}{\sqrt{\frac{T^{-2}\hat{\omega}_u^2}{T^{-3}\sum_{t=1}^T (t-\bar{t})^2}}}. \tag{A.5}
$$

The numerator of (A.5) is such that

$$
T^{1/2}(\hat{\beta} - \beta_0) = \kappa + \frac{T^{-5/2}\sum t u_t - (T^{-2}\sum t)(T^{-3/2}\sum u_t)}{T^{-3}\sum t^2 - (T^{-2}\sum t)^2}
$$

which is easily seen to be of $O_p(1)$. Turning to the denominator of (A.5), it is well-known that $\hat{\omega}_u^2$ diverges to $+\infty$ at a rate faster than $O_p(T)$ (see, for example, Kwiatkowski *et al.*, 1992, p.168) and, hence, $\sqrt{T^{-2}\hat{\omega}_u^2}$ converges in probability to zero at a rate slower than $O_p(T^{-1/2})$. Consequently, z_0 is of $o_p(T^{1/2})$.

(ii) Under $H_{1,1}$, $\tilde{\beta} = \beta_0 + \kappa T^{-1/2} + T_*^{-1}(u_T - u_1)$, and so

$$
z_1 = \frac{T_*^{1/2}\kappa T^{-1/2} + T_*^{-1/2}(u_T - u_1)}{\sqrt{\tilde{\omega}_v^2}}.
$$

Since $T_*^{-1/2}(u_T - u_1) = T^{-1/2}u_T + o_p(1)$, we find that for the numerator

$$
T_*^{1/2}\kappa T^{-1/2} + T_*^{-1/2}(u_T - u_1) \xrightarrow{d} \kappa + \omega_\varepsilon W(1)
$$

where $W(r)$ is a standard Brownian motion defined *via* $\omega_\varepsilon^{-1}T^{-1/2}\sum_{t=1}^{\lfloor Tr \rfloor}\varepsilon_t \xrightarrow{d} W(r)$. Noting that $v_t = \varepsilon_t$ when u_t is $I(1)$, we therefore have that $\tilde{\omega}_v^2 \xrightarrow{p} \omega_\varepsilon^2$ under the conditions placed on $\tilde{\omega}_v^2$ in section 2. The stated result then follows directly using an application of the continuous mapping theorem.

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Table 1. Asymptotic $\gamma_{\xi,\delta}$ values for the $z_{\lambda}^{m\delta}$ tests

ξ	$\delta = 1$	$\delta = 2$
0.100	0.04953	0.00204
0.050	0.04411	0.00149
0.025	0.03952	0.00115
0.010	0.03462	0.00085
0.005	0.03292	0.00071

Table 2. Empirical sizes of nominal 0.05-level tests

		$T = 100$				$T = 200$			
c	θ	z_{λ}	z_{λ}^{m1}	z_{λ}^{m2}	*Dan-J*	z_{λ}	z_{λ}^{m1}	z_{λ}^{m2}	*Dan-J*
0	-0.8	0.114	0.076	0.057	0.041	0.097	0.070	0.055	0.046
	-0.4	0.118	0.080	0.061	0.043	0.099	0.072	0.056	0.047
	0.0	0.117	0.079	0.060	0.051	0.098	0.070	0.055	0.052
	0.4	0.118	0.067	0.050	0.107	0.097	0.064	0.047	0.084
	0.8	0.100	0.043	0.039	0.305	0.088	0.032	0.027	0.311
5	-0.8	0.012	0.013	0.013	0.019	0.008	0.010	0.012	0.020
	-0.4	0.014	0.015	0.015	0.019	0.010	0.012	0.014	0.021
	0.0	0.017	0.017	0.017	0.023	0.010	0.012	0.013	0.024
	0.4	0.019	0.016	0.015	0.064	0.012	0.012	0.012	0.051
	0.8	0.030	0.023	0.023	0.203	0.016	0.013	0.012	0.234
10	-0.8	0.011	0.011	0.012	0.022	0.008	0.009	0.011	0.024
	-0.4	0.012	0.013	0.014	0.022	0.010	0.012	0.014	0.024
	0.0	0.017	0.018	0.017	0.030	0.011	0.013	0.014	0.031
	0.4	0.019	0.016	0.016	0.072	0.012	0.013	0.013	0.062
	0.8	0.033	0.029	0.029	0.138	0.018	0.016	0.016	0.197
15	-0.8	0.012	0.013	0.013	0.025	0.009	0.011	0.013	0.027
	-0.4	0.014	0.014	0.015	0.025	0.012	0.014	0.016	0.028
	0.0	0.021	0.021	0.021	0.039	0.015	0.016	0.017	0.040
	0.4	0.021	0.019	0.018	0.077	0.015	0.015	0.016	0.071
	0.8	0.038	0.036	0.036	0.088	0.021	0.019	0.019	0.161
T	-0.8	0.030	0.028	0.028	0.031	0.031	0.030	0.030	0.037
	-0.4	0.037	0.035	0.035	0.033	0.034	0.033	0.033	0.039
	0.0	0.031	0.030	0.030	0.032	0.027	0.026	0.027	0.041
	0.4	0.038	0.036	0.036	0.012	0.031	0.031	0.031	0.033
	0.8	0.018	0.014	0.014	0.000	0.035	0.034	0.034	0.001

Table 3. Application of tests to real GDP

Country	z_λ	z_λ^{m1}	z_λ^{m2}	*Dan-J*	DF-GLS$^\mu$	DF-GLS$^\tau$	Growth rate (c.i.) %
Australia	10.355**	10.414**	8.831**	24.193**	1.507	−2.092	3.294 (±0.624)
Canada	6.853**	5.654**	4.088**	11.261	1.315	−2.169	2.736 (±0.782)
France	13.955**	12.955**	11.316**	11.747**	0.807	−2.155	1.999 (±0.281)
Germany	3.428**	1.939**	0.911	3.342	0.371	−1.174	2.101 (±1.201)
Japan	5.128**	1.349	0.317	0.004	1.006	−1.236	2.492 (±0.953)
Italy	6.048**	3.312**	1.466	3.121	0.379	−1.018	1.691 (±0.548)
Netherlands	8.607**	7.846**	6.518**	6.663	0.057	−2.040	2.312 (±0.526)
South Africa	3.935**	2.237**	1.028	2.529	1.214	−1.589	2.034 (±1.013)
Spain	8.383**	5.795**	3.588**	3.199	0.414	−1.466	2.958 (±0.692)
Switzerland	4.632**	3.377**	2.134**	7.641	0.480	−1.915	1.457 (±0.616)
UK	8.024**	8.562**	7.808**	32.228**	0.280	−1.619	2.449 (±0.598)
US	22.002**	22.781**	22.318**	65.633**	0.251	−2.715	3.101 (±0.276)

Note: ** denotes rejection at the 0.05-level

Table 4. Application of tests to net barter terms of trade

sig. level	z_λ	z_λ^{m1}	z_λ^{m2}	*Dan-J*	DF-GLS$^\mu$	DF-GLS$^\tau$	Growth rate (c.i.) %
0.10	−2.629*	−2.374*	−2.165*	−3.453*	−1.186	−3.757*	−0.625 (±0.391)
0.05	−2.629**	−2.326**	−2.102**	−2.445**	−1.186	−3.757**	−0.625 (±0.466)
0.01	−2.629***	−2.241	−2.029	−0.811	−1.186	−3.757***	−0.625 (±0.612)

Note: *, ** and *** denote rejection at the 0.10-, 0.05-, and 0.01-levels respectively

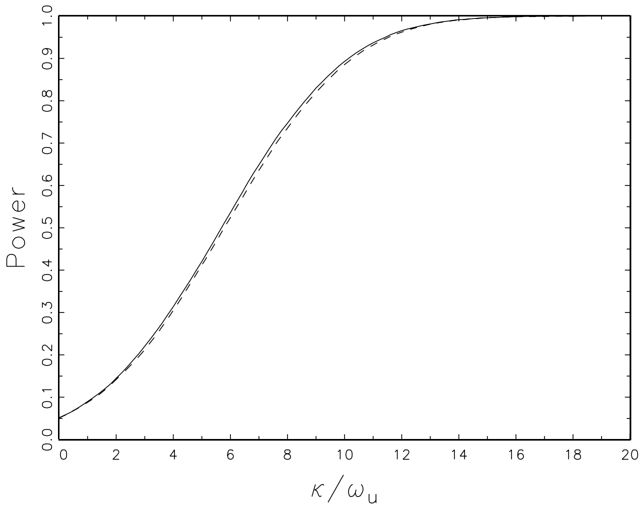
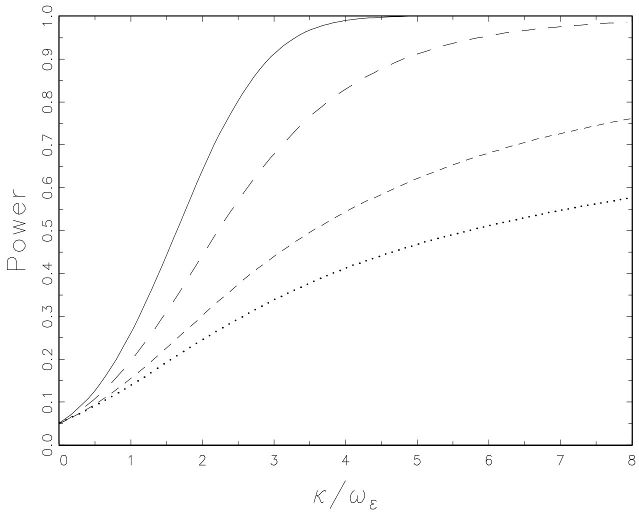
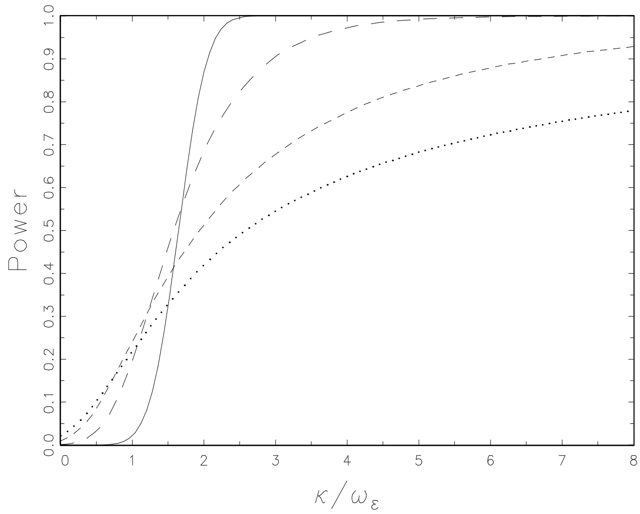


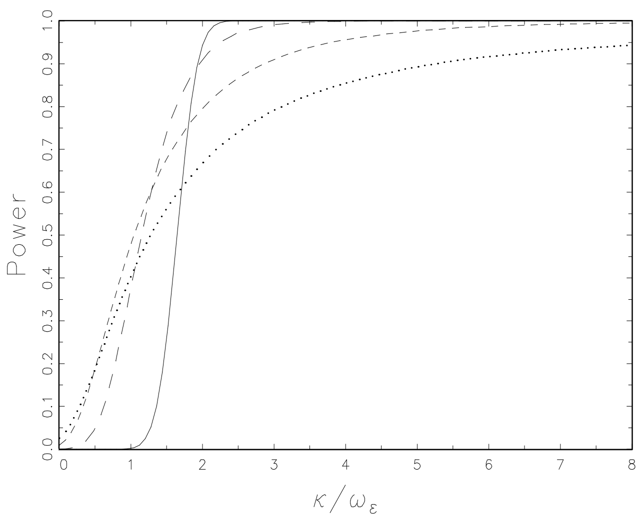
Figure 1. Asymptotic size and power: I(0) errors, z_λ, z_λ^{m1}, z_λ^{m2}: —— , *Dan-J*: - - -



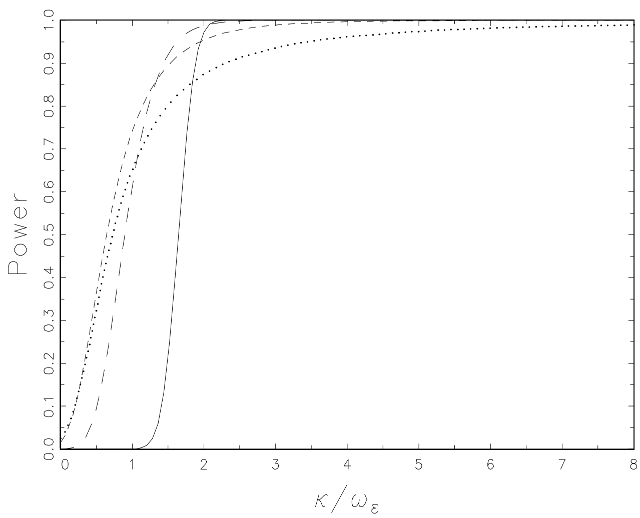
(a) $c = 0$



(b) $c = 5$

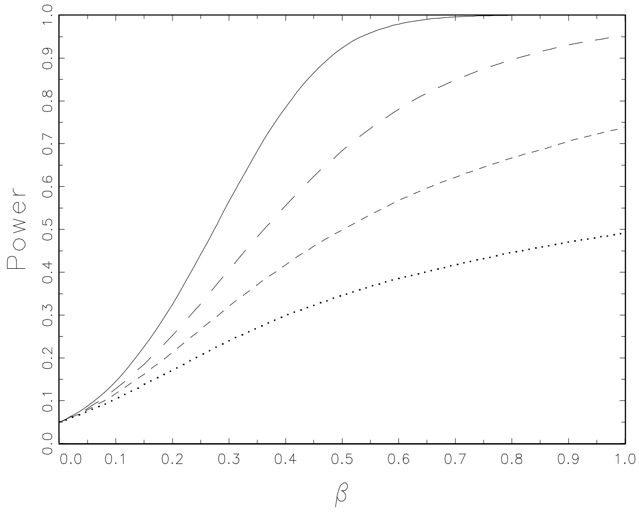


(c) $c = 10$

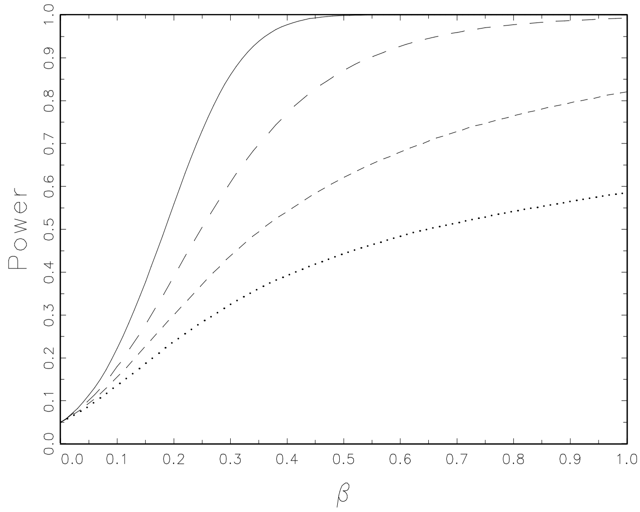


(d) $c = 15$

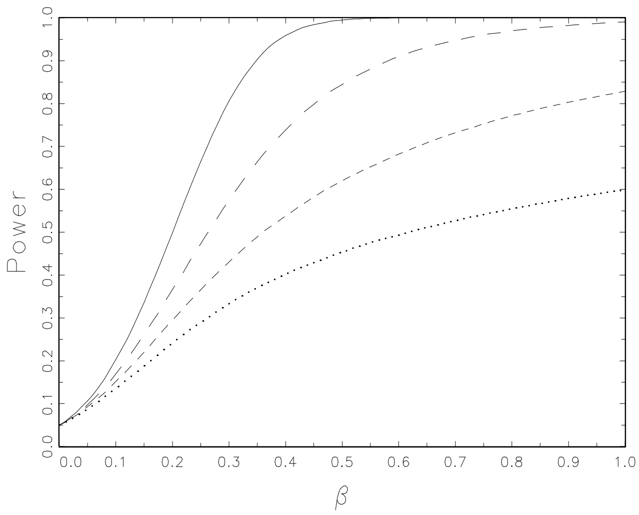
Figure 2. Asymptotic size and power: I(1) errors, z_λ: —— , z_λ^{m1}: – – , z_λ^{m2}: - - - , *Dan-J*: $\cdots$



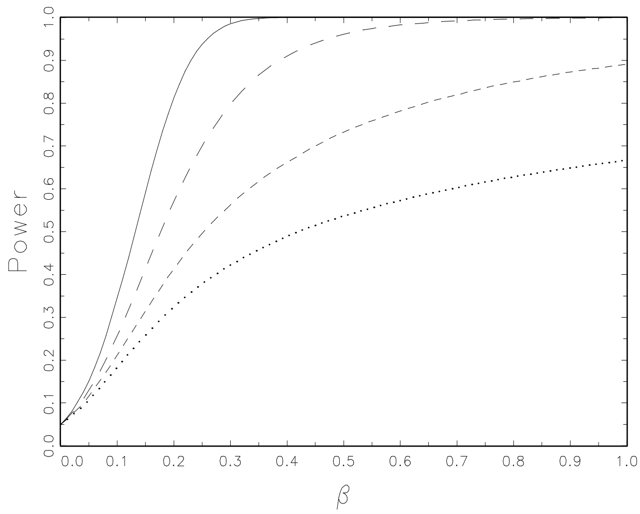
(a) $T = 100$, $\theta = -0.4$



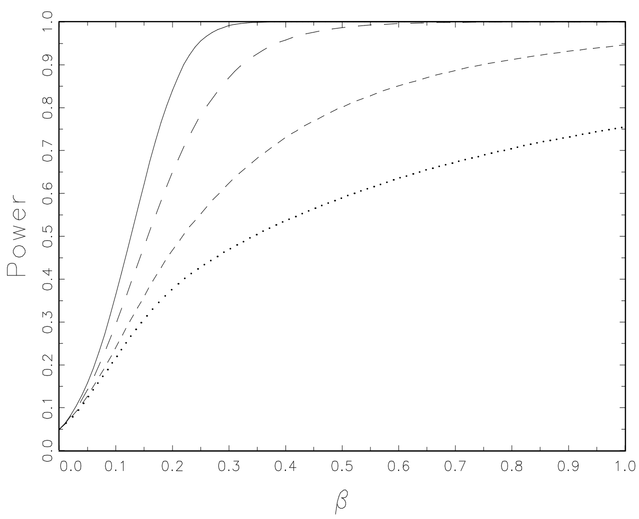
(b) $T = 200$, $\theta = -0.4$



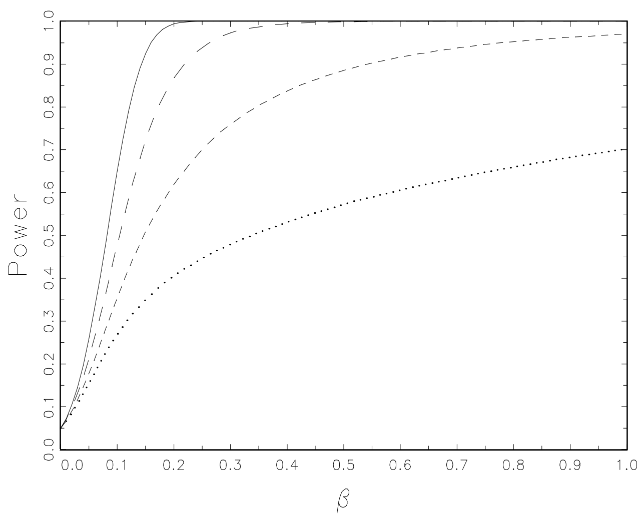
(c) $T = 100$, $\theta = 0$



(d) $T = 200$, $\theta = 0$

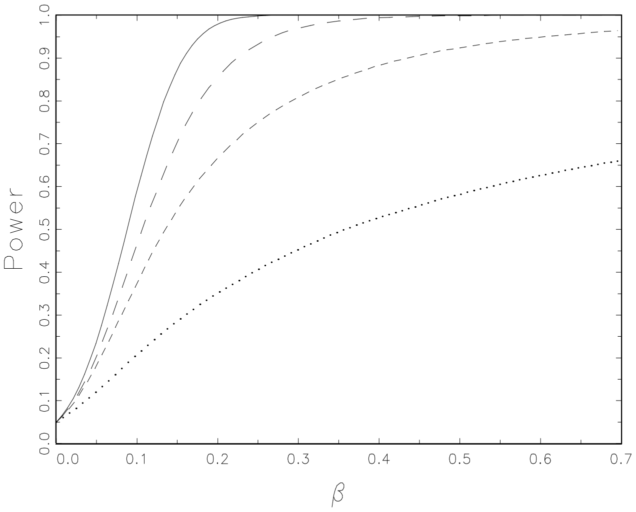


(e) $T = 100$, $\theta = 0.4$

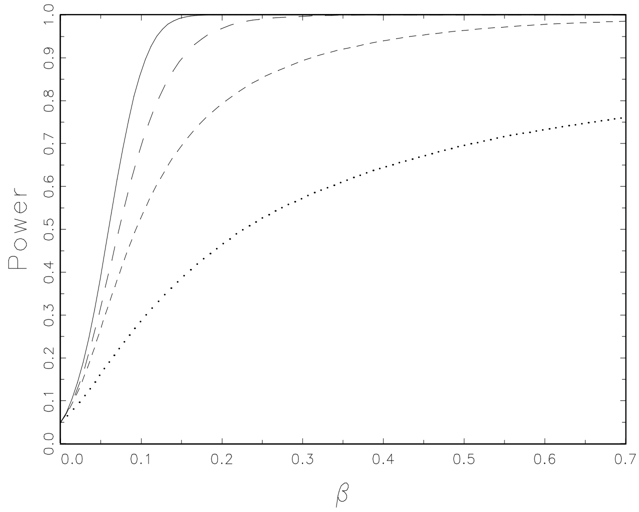


(f) $T = 200$, $\theta = 0.4$

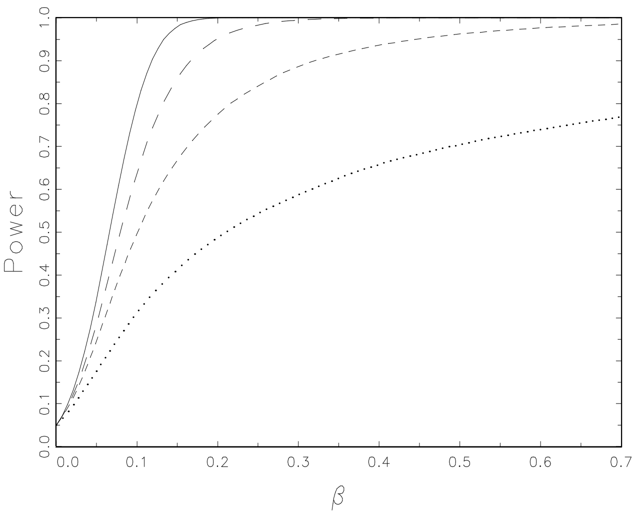
Figure 3. Finite sample size-adjusted power: $c = 0$, z_λ: ——, z_λ^{m1}: $--$, z_λ^{m2}: - - -, *Dan-J*: $\cdots$



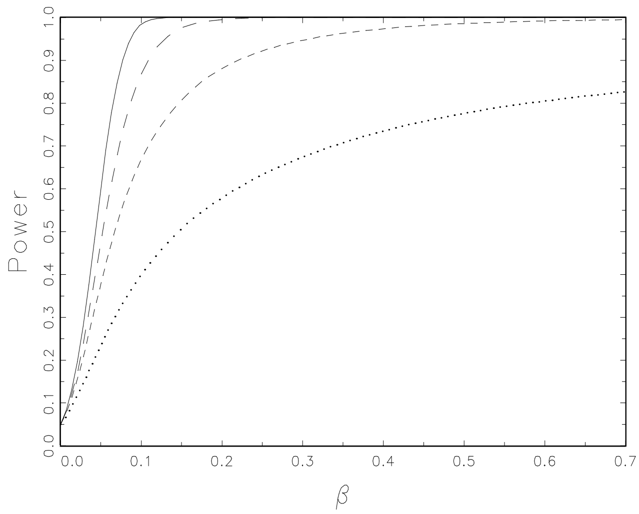
(a) $T = 100$, $\theta = -0.4$



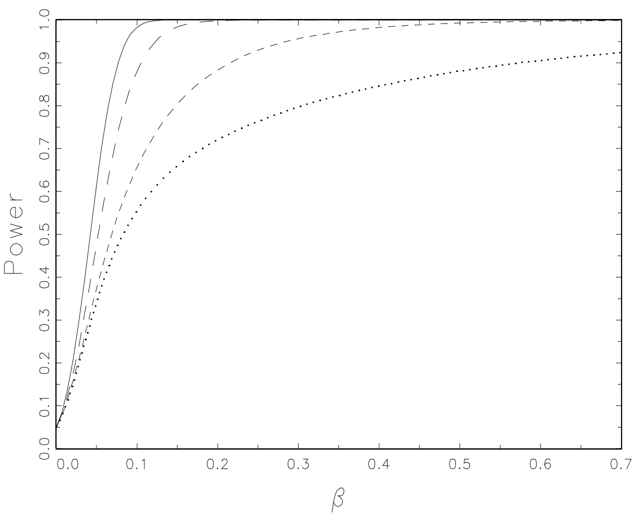
(b) $T = 200$, $\theta = -0.4$



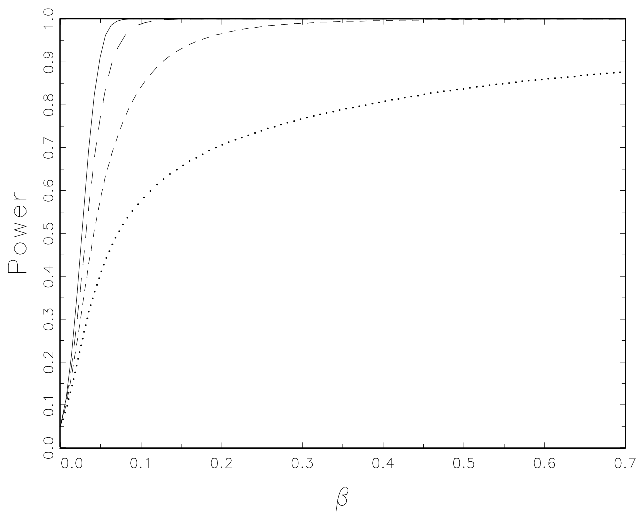
(c) $T = 100$, $\theta = 0$



(d) $T = 200$, $\theta = 0$

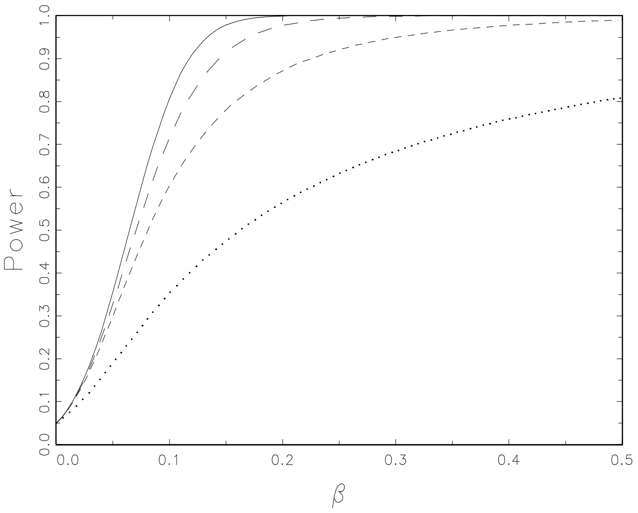


(e) $T = 100$, $\theta = 0.4$

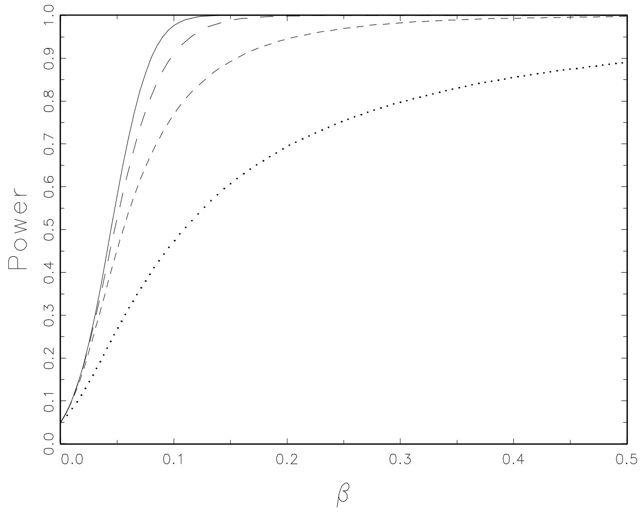


(f) $T = 200$, $\theta = 0.4$

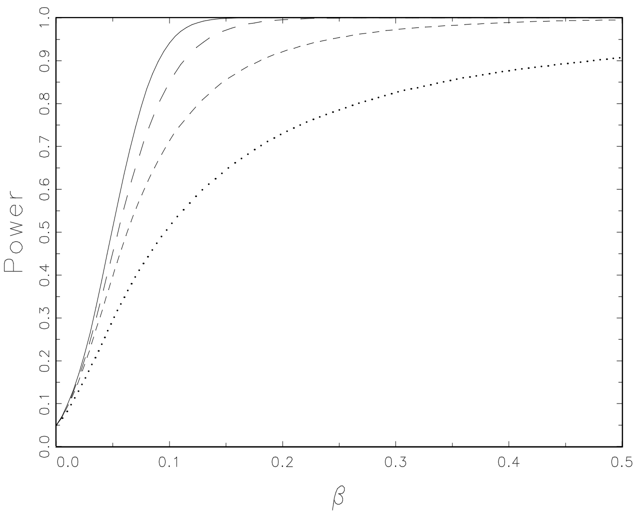
Figure 4. Finite sample size-adjusted power: $c = 5$, z_λ: —— , z_λ^{m1}: $--$, z_λ^{m2}: - - - , *Dan-J*: $\cdots$



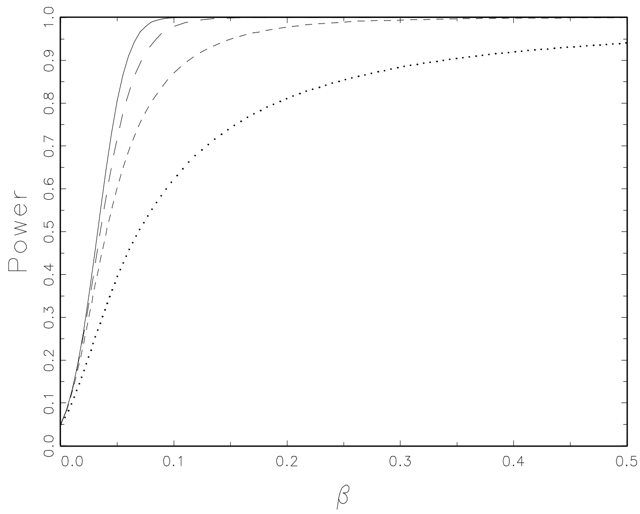
(a) $T = 100$, $\theta = -0.4$



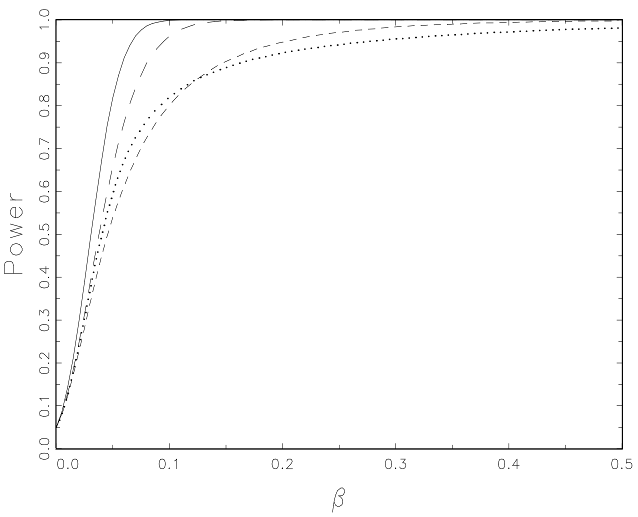
(b) $T = 200$, $\theta = -0.4$



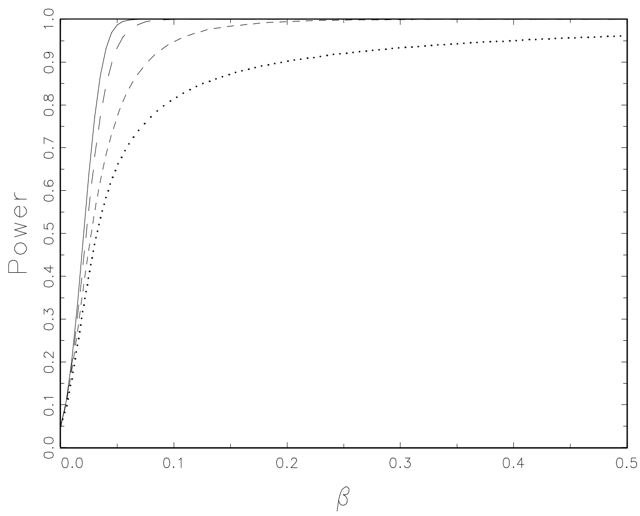
(c) $T = 100$, $\theta = 0$



(d) $T = 200$, $\theta = 0$

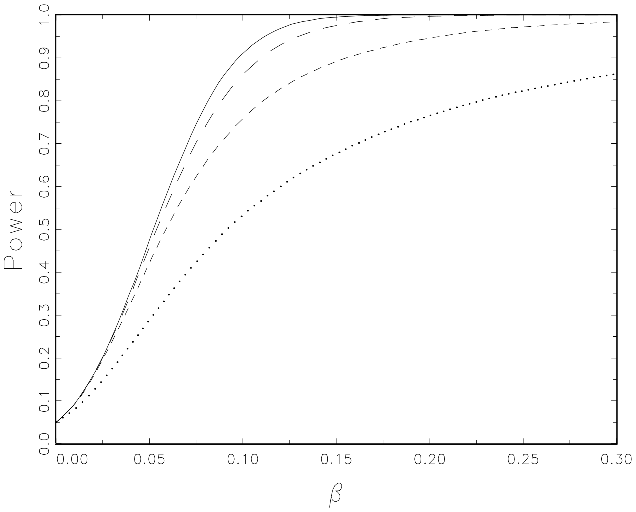


(e) $T = 100$, $\theta = 0.4$

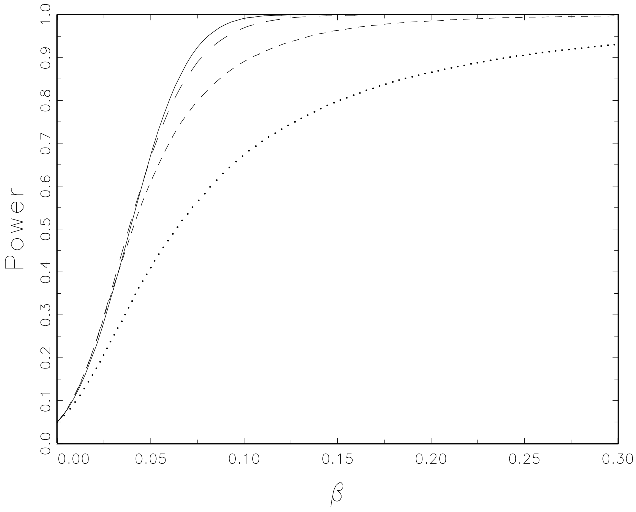


(f) $T = 200$, $\theta = 0.4$

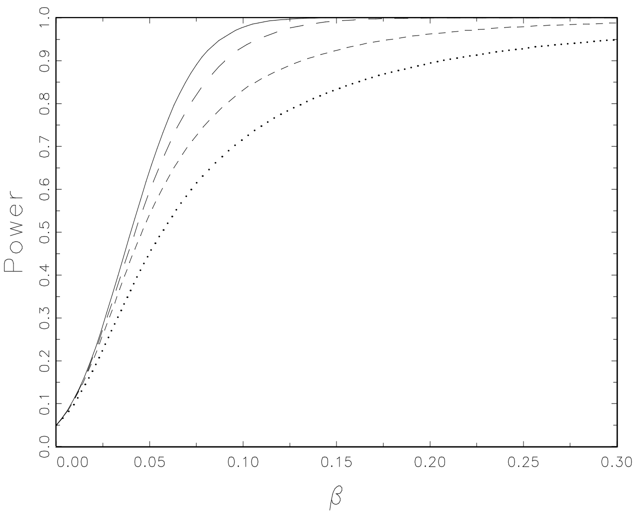
Figure 5. Finite sample size-adjusted power: $c = 10$, z_λ: —— , z_λ^{m1}: $--$, z_λ^{m2}: - - - , *Dan-J*: $\cdots$



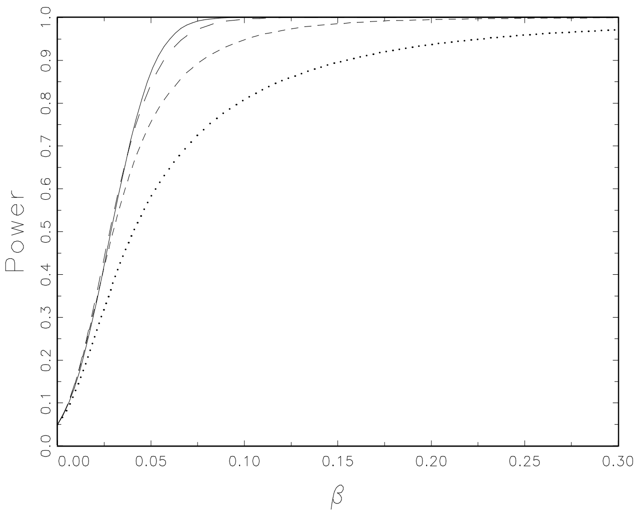
(a) $T = 100$, $\theta = -0.4$



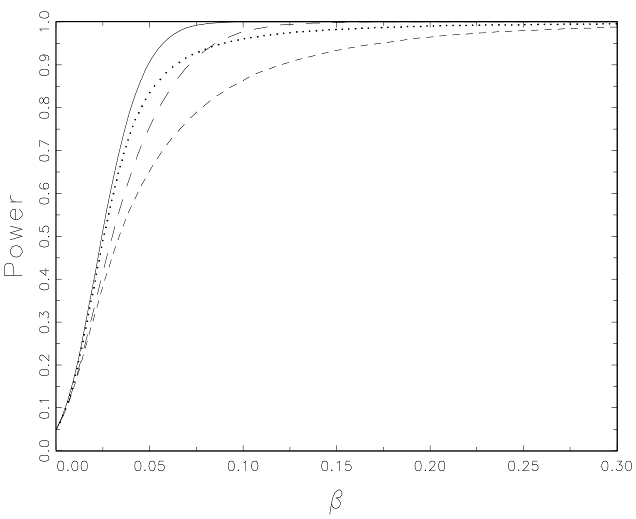
(b) $T = 200$, $\theta = -0.4$



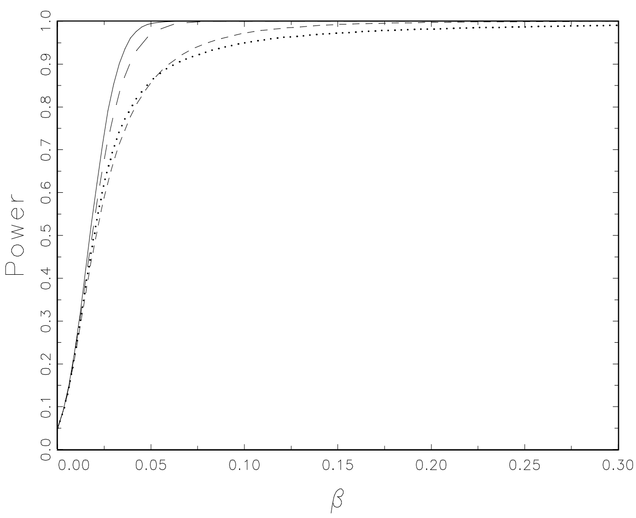
(c) $T = 100$, $\theta = 0$



(d) $T = 200$, $\theta = 0$

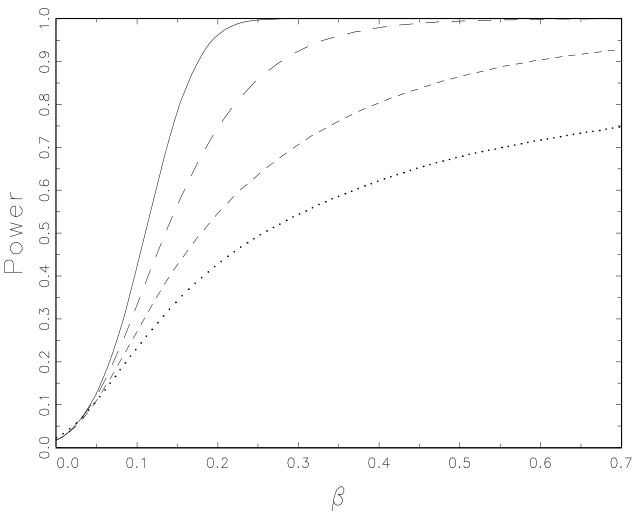


(e) $T = 100$, $\theta = 0.4$

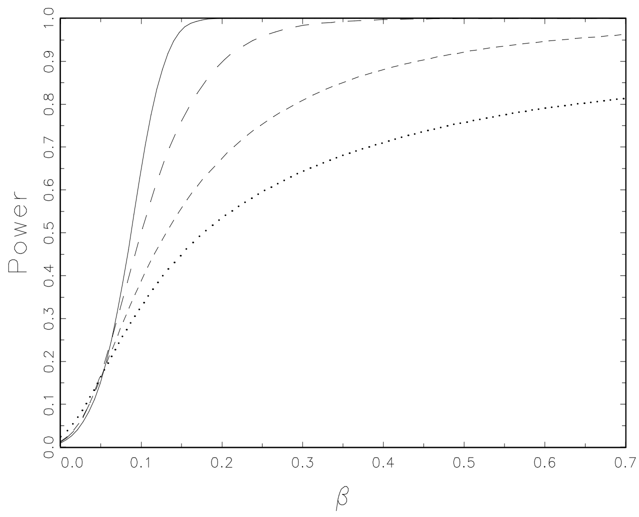


(f) $T = 200$, $\theta = 0.4$

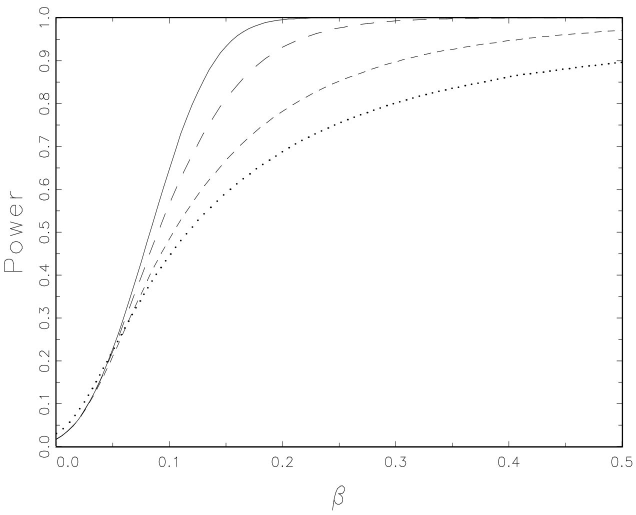
Figure 6. Finite sample size-adjusted power: $c = 15$, z_λ: —— , z_λ^{m1}: $--$, z_λ^{m2}: - - - , *Dan-J*: $\cdots$



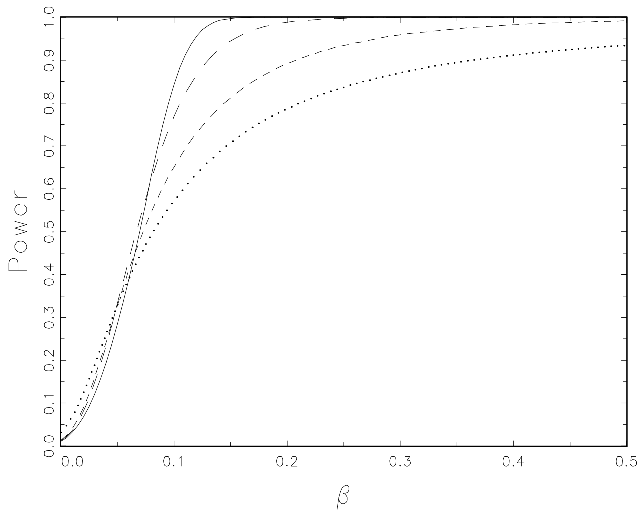
(a) $T = 100$, $c = 5$



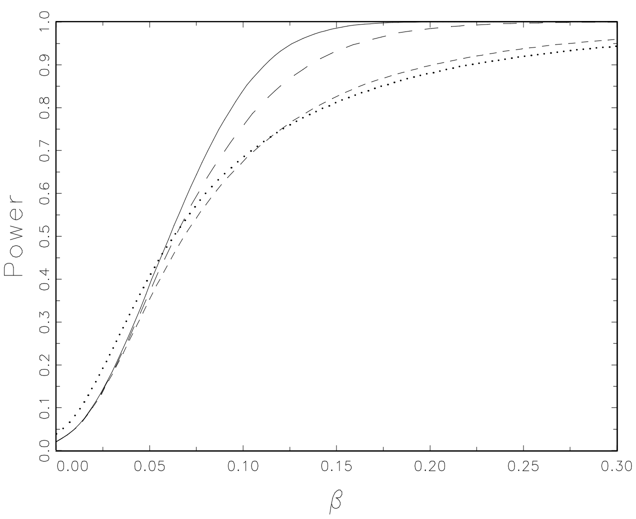
(b) $T = 200$, $c = 5$



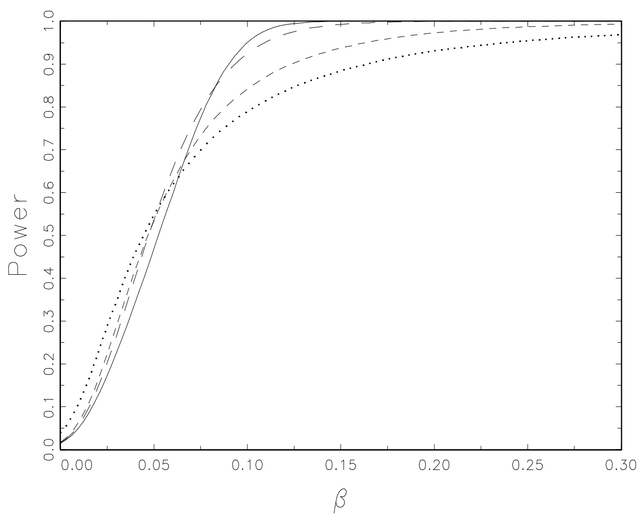
(c) $T = 100$, $c = 10$



(d) $T = 200$, $c = 10$

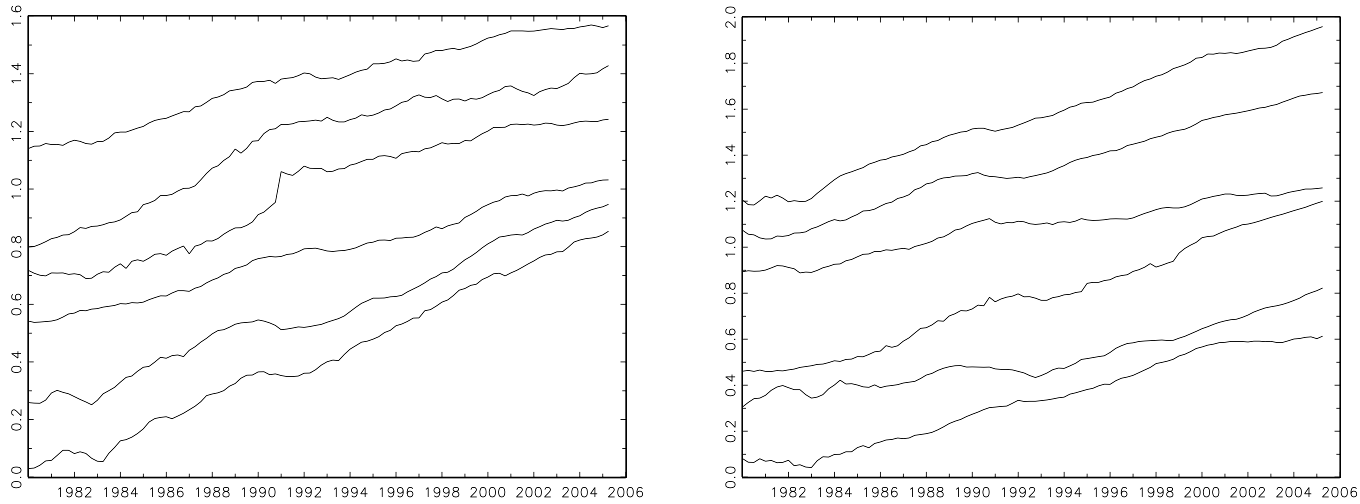


(e) $T = 100$, $c = 15$



(f) $T = 200$, $c = 15$

Figure 7. Finite sample empirical power: $\theta = 0$, z_λ: —— , z_λ^{m1}: – – , z_λ^{m2}: - - - , *Dan-J*: $\cdots$



(a) Australia, Canada, France, Germany, Japan, Italy (bottom to top)

(b) Netherlands, South Africa, Spain, Switzerland, UK, US (bottom to top)

Figure 8. Real GDP in logarithms (intercept adjusted), 1980:1–2005:2

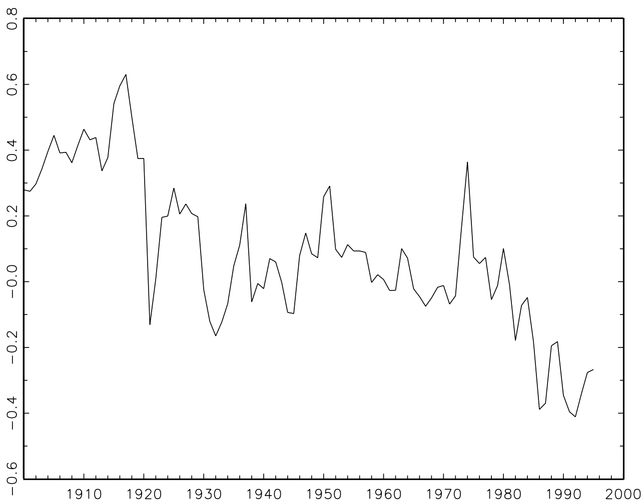


Figure 9. Net barter terms of trade in logarithms, 1900–1995